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Research article

An extension of Herstein's theorem on Banach algebra

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Abstract: Let $\mathcal{A}$ be a $(p+q)!$-torsion free semiprime ring. We proved that if $\mathcal{H}, \mathcal{D} : \mathcal{A} \to \mathcal{A}$ are two additive mappings fulfilling the algebraic identity $2\mathcal{H}(a^{p+q}) = \mathcal{H}(a^p)a^q + a^p\mathcal{D}(a^q) + \mathcal{H}(a^q)a^p + a^q\mathcal{D}(a^p)$ for all $a \in \mathcal{A}$, then $\mathcal{H}$ is a generalized derivation with $\mathcal{D}$ as an associated derivation on $\mathcal{A}$. In addition to that, it is also proved in this article that $\mathcal{H}_1$ is a generalized left derivation associated with a left derivation δ on $\mathcal{A}$ if they fulfilled the algebraic identity $2\mathcal{H}_1(a^{p+q}) = a^p\mathcal{H}_1(a^q) + a^q\delta(a^p) + a^q\mathcal{H}_1(a^p) + a^p\delta(a^q)$ for all $a \in \mathcal{A}$. Further, the legitimacy of these hypotheses is eventually demonstrated by examples and at last, an application of Banach algebra is presented.

Keywords: semiprime ring; generalized left derivation; algebraic identities; Banach algebra
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1. Introduction

$\mathcal{A}$ is an associative ring with identity e and $Z(\mathcal{A})$ represents the center of $\mathcal{A}$ throughout the paper. $Q_l(\mathcal{A}_C)$ is the left Martindale quotients ring and C represents its extended centroid. A ring $\mathcal{A}$ is known as q-torsion free if $qa = 0$ implies that $a = 0$ for a fixed integer $q > 1$ and for each $a \in \mathcal{A}$. $[a, b] = ab - ba$ is the commutator. A ring $\mathcal{A}$ is known as prime when $a\mathcal{A}b = \{0\}$ signifies that either $a = 0$ or $b = 0$ and is termed as a semiprime ring if $a\mathcal{A}a = \{0\}$ implies that $a = 0$. A mapping $\mathcal{D} : \mathcal{A} \to \mathcal{A}$ is a derivation if $\mathcal{D}$ is additive, and holds the condition $\mathcal{D}(ab) = \mathcal{D}(a)b + a\mathcal{D}(b)$ for every $a, b \in \mathcal{A}$, and it is a Jordan derivation if for all b in $\mathcal{A}$, it fulfills the condition $\mathcal{D}(b^2) = \mathcal{D}(b)b + b\mathcal{D}(b)$. If $\mathcal{D}$ is a derivation, then it will obviously be a Jordan derivation, but generally, the converse statement cannot be considered true.

A well known result due to Herstein [7], asserts that a Jordan derivation and a derivation are identical in a prime ring holding characteristic different from two. Cusack [4] revived the last statement of Herstein for a 2-torsion free semiprime ring. Herstein's theorem has been significantly extended by a number of researchers in several ways. Bresar [3] has proved the following result: Let $\mathcal{A}$ be a 2-torsion free semiprime ring and let $\mathcal{D}$ be an additive mapping on $\mathcal{A}$ satisfying the condition

$$\mathcal{D}(aba) = \mathcal{D}(a)ba + a\mathcal{D}(b)a + ab\mathcal{D}(a)$$

for all $a, b \in \mathcal{A}$. In this case, $\mathcal{D}$ is a derivation.

A Jordan triple derivation is an additive mapping $\mathcal{D}$ that maps an arbitrary ring $\mathcal{A}$ into itself and satisfies the above relation for every $a, b \in \mathcal{A}$. Any Jordan derivation on any 2-torsion free ring may be shown to be a Jordan triple derivation easily. This indicates that the above result of Bresar is a generalized version of Cusack's generalization of Herstein's theorem. Some related researches can be found in [5, 6, 10]. These researches represent a motivation for our investigation, and we have obtained an extension of Herstein's theorem.

From Zalar [15], an additive mapping $\mathcal{T} : \mathcal{A} \to \mathcal{A}$ is known as a left centralizer if $\mathcal{T}(ab) = \mathcal{T}(a)b$ for every a, b in $\mathcal{A}$. We say that $\mathcal{T}$ is a right centralizer if $\mathcal{T}(ab) = a\mathcal{T}(b)$ is true for all $a, b \in \mathcal{A}$. Particularly, $\mathcal{T}$ is a Jordan left (respectively, Jordan right) centralizer of $\mathcal{A}$ if $\mathcal{T}(b^2) = \mathcal{T}(b)b$ (respectively, $\mathcal{T}(b^2) = b\mathcal{T}(b)$) for all $b \in \mathcal{A}$. A mapping $\mathcal{H} : \mathcal{A} \to \mathcal{A}$, which is additive and satisfies $\mathcal{H}(ab) = \mathcal{H}(a)b + a\mathcal{D}(b)$ for all a, b in $\mathcal{A}$, is recognized as a generalized derivation associated with a derivation $\mathcal{D}$ on $\mathcal{A}$. Particularly, $\mathcal{H}$ is a generalized Jordan derivation if there exists a Jordan derivation $\mathcal{D}$ on $\mathcal{A}$ such that $\mathcal{H}(b^2) = \mathcal{H}(b)b + b\mathcal{D}(b)$ for all b in $\mathcal{A}$.

Every generalized derivation can be easily verified to be a generalized Jordan derivation, while the converse is typically untrue. If $\mathcal{H}$ is a generalized derivation associated with a derivation $\mathcal{D}$ on $\mathcal{A}$, then the algebraic identity $2\mathcal{H}(a^{p+q}) = \mathcal{H}(a^p)a^q + a^p\mathcal{D}(a^q) + \mathcal{H}(a^q)a^p + a^q\mathcal{D}(a^p)$ holds for each a in $\mathcal{A}$, but the converse of this mathematical statement is not true in general. In this paper, we find out under which condition the converse will also be true. More precisely, we proved that if $\mathcal{H}, \mathcal{D} : \mathcal{A} \to \mathcal{A}$ are two additive maps fulfilling the algebraic identity $2\mathcal{H}(a^{p+q}) = \mathcal{H}(a^p)a^q + a^p\mathcal{D}(a^q) + \mathcal{H}(a^q)a^p + a^q\mathcal{D}(a^p)$ for each a in $\mathcal{A}$, then $\mathcal{H}$ will be a generalized derivation having $\mathcal{D}$ as an associated derivation on $\mathcal{A}$, where $\mathcal{A}$ is a $(p + q)!$-torsion free semiprime ring.

We need the following lemma to complete the proof of Theorem 2.1.

Lemma 1.1. *[16] Every generalized Jordan derivation from a 2-torsion free semiprime ring with identity into itself is a generalized derivation.*

2. Generalized derivation

Let us start with the preceding theorem:

Theorem 2.1. *Suppose that $p, q \geq 1$ are any two fixed integers and $\mathcal{A}$ is a $(p + q)!$-torsion free semiprime ring. If $\mathcal{H}$ and $\mathcal{D}$ are two additive mappings from $\mathcal{A}$ to itself, which satisfy the algebraic equation $2\mathcal{H}(a^{p+q}) = \mathcal{H}(a^p)a^q + a^p\mathcal{D}(a^q) + \mathcal{H}(a^q)a^p + a^q\mathcal{D}(a^p)$ for every a in $\mathcal{A}$, then $\mathcal{H}$ will be a generalized derivation associated with a derivation $\mathcal{D}$ on $\mathcal{A}$.*

Proof. We have given that

$$2\mathcal{H}(a^{p+q}) = \mathcal{H}(a^p)a^q + a^p\mathcal{D}(a^q) + \mathcal{H}(a^q)a^p + a^q\mathcal{D}(a^p) \text{ for all } a \in \mathcal{A}. \tag{2.1}$$

Replacing a by e in the above equation, we obtain $\mathcal{D}(e) = 0$. The following expression is obtained by putting $a + me$ for a in (2.1), where m is any positive integer.

$2\mathcal{H}\Big(a^{p+q}+\binom{p+q}{1}a^{p+q-1}me+\cdots+\binom{p+q}{p+q-2}a^2m^{p+q-2}e+\binom{p+q}{p+q-1}am^{p+q-1}e+m^{p+q}e\Big)=\mathcal{H}\Big(a^p+\binom{p}{1}a^{p-1}me+\cdots+\binom{p}{p-2}a^2m^{p-2}e+\binom{p}{p-1}am^{p-1}e+m^pe\Big)\Big(a^q+\binom{q}{1}a^{q-1}me+\cdots+\binom{q}{q-2}a^2m^{q-2}e+\binom{q}{q-1}am^{q-1}e+m^qe\Big)+\Big(a^p+\binom{p}{1}a^{p-1}me+\cdots+\binom{p}{p-2}a^2m^{p-2}e+\binom{p}{p-1}am^{p-1}e+m^pe\Big)\mathcal{D}\Big(a^q+\binom{q}{1}a^{q-1}me+\cdots+\binom{q}{q-2}a^2m^{q-2}e+\binom{q}{q-1}am^{q-1}e+m^qe\Big)+\mathcal{H}\Big(a^q+\binom{q}{1}a^{q-1}me+\cdots+\binom{q}{q-2}a^2m^{q-2}e+\binom{q}{q-1}am^{q-1}e+m^qe\Big)\Big(a^p+\binom{p}{1}a^{p-1}me+\cdots+\binom{p}{p-2}a^2m^{p-2}e+\binom{p}{p-1}am^{p-1}e+m^pe\Big)+\Big(a^q+\binom{q}{1}a^{q-1}me+\cdots+\binom{q}{q-2}a^2m^{q-2}e+\binom{q}{q-1}am^{q-1}e+m^qe\Big)\mathcal{D}\Big(a^p+\binom{p}{1}a^{p-1}me+\cdots+\binom{p}{p-2}a^2m^{p-2}e+\binom{p}{p-1}am^{p-1}e+m^pe\Big)$, for all $a\in\mathcal{A}$.

Rewrite the above expression by using (2.1) as

$$mf_1(a,e)+m^2f_2(a,e)+\cdots+m^{p+q-1}f_{p+q-1}(a,e)=0,$$

where $f_i(a,e)$ are the coefficients of m^i's for each $i=1,2,\cdots,(p+q-1)$. Replacing m by $1,2,\cdots,(p+q-1)$, we obtain a system of $(p+q-1)$ linear homogeneous equations that provides a Vandermonde matrix,

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 2 & 2^2 & \cdots & 2^{p+q-1} \\ \cdots & & & \\ \cdots & & & \\ p+q-1 & (p+q-1)^2 & \cdots & (p+q-1)^{p+q-1} \end{bmatrix},$$

which yields that $f_i(a,e)=0$ for every a in $\mathcal{A}$ and for $i=1,2,\cdots,p+q-1$. In particular, for all $a\in\mathcal{A}$, we have the following for $i=p+q-1$:

$$\begin{aligned} 2\binom{p+q}{p+q-1}\mathcal{H}(a) &= \binom{q}{q-1}\mathcal{H}(e)a+\binom{p}{p-1}\mathcal{H}(a)+\binom{q}{q-1}\mathcal{D}(a) \\ &\quad+\binom{p}{p-1}\mathcal{H}(e)a+\binom{q}{q-1}\mathcal{H}(a)+\binom{p}{p-1}\mathcal{D}(a), \text{ for all } a\in\mathcal{A}. \end{aligned}$$

This implies that $2(p+q)\mathcal{H}(a)=(p+q)\mathcal{H}(e)a+(p+q)\mathcal{H}(a)+(p+q)\mathcal{D}(a)$, then torsion restriction of $\mathcal{A}$ gives that

$$\mathcal{H}(a)=\mathcal{H}(e)a+\mathcal{D}(a) \text{ for all } a \text{ in } \mathcal{A}. \tag{2.2}$$

Next, from $f_{p+q-2}(a,e)=0$ with $\mathcal{D}(e)=0$, we have

$$\begin{aligned} 2\binom{p+q}{p+q-2}\mathcal{H}(a^2) &= \binom{q}{q-2}\mathcal{H}(e)a^2+\binom{p}{p-1}\binom{q}{q-1}\mathcal{H}(a)a+\binom{p}{p-2}\mathcal{H}(a^2) \\ &\quad+\binom{q}{q-2}\mathcal{D}(a^2)+\binom{p}{p-1}\binom{q}{q-1}a\mathcal{D}(a)+\binom{p}{p-2}\mathcal{H}(e)a^2 \\ &\quad+\binom{q}{q-1}\binom{p}{p-1}\mathcal{H}(a)a+\binom{q}{q-2}\mathcal{H}(a^2)+\binom{p}{p-2}\mathcal{D}(a^2) \\ &\quad+\binom{q}{q-1}\binom{p}{p-1}a\mathcal{D}(a), \text{ for all } a\in\mathcal{A}. \end{aligned}$$

This implies that

$$\begin{aligned} (p+q)(p+q-1)\mathcal{H}(a^2) &= [\tfrac{q(q-1)}{2}+\tfrac{p(p-1)}{2}][\mathcal{H}(e)a^2+\mathcal{D}(a^2)]+pq\mathcal{H}(a)a+\tfrac{p(p-1)}{2}\mathcal{H}(a^2) \\ &\quad+pqa\mathcal{D}(a)+pq\mathcal{H}(a)a+\tfrac{q(q-1)}{2}\mathcal{H}(a^2)+qpa\mathcal{D}(a). \end{aligned}$$

Replacing a by a^2 in (2.2), we arrive at $\mathcal{H}(a^2)=\mathcal{H}(e)a^2+\mathcal{D}(a^2)$ for all a in $\mathcal{A}$. Using this relation in the above relation, we obtain

$$(p+q)(p+q-1)\mathcal{H}(a^2) \;=\; [q(q-1)+p(p-1)]\mathcal{H}(a^2)+2pq\mathcal{H}(a)a+2pqa\mathcal{D}(a).$$

After some calculations, we find

$$2qp\mathcal{H}(a^2) \;=\; 2qp[\mathcal{H}(a)a+a\mathcal{D}(a)].$$

Using torsion condition, we have

$$\mathcal{H}(a^2)=\mathcal{H}(a)a+a\mathcal{D}(a), \text{ for all } a\in\mathcal{A}. \tag{2.3}$$

Next, from (2.2),

$$\begin{aligned}\mathcal{D}(a^2) &= \mathcal{H}(a^2)-\mathcal{H}(e)(a^2)\\ &= \mathcal{H}(a)a+a\mathcal{D}(a)-\mathcal{H}(e)(a^2)\\ &= [\mathcal{H}(a)-\mathcal{H}(e)(a)]a+a\mathcal{D}(a)\\ &= \mathcal{D}(a)a+a\mathcal{D}(a).\end{aligned}$$

Hence, $\mathcal{D}$ is a Jordan derivation on $\mathcal{A}$. Therefore, from (2.3), we conclude that $\mathcal{H}$ is a generalized Jordan derivation on $\mathcal{A}$, so we obtain the required result by Lemma 1.1. □

The following example shows that the above result of the article is not redundant.

Example 2.1. *Consider a ring* $\mathcal{A}=\left\{\begin{pmatrix}\bar{i_1} & 0\\ 0 & \bar{i_2}\end{pmatrix} \mid \bar{i_1},\bar{i_2}\in 2\mathbb{Z}_8\right\}$, *and* $\mathbb{Z}_8$ *has its usual meaning. Define mappings* $\mathcal{H},\mathcal{D}:\mathcal{A}\to\mathcal{A}$ *by* $\mathcal{H}\left[\begin{pmatrix}\bar{i_1} & 0\\ 0 & \bar{i_2}\end{pmatrix}\right]=\begin{pmatrix}0 & 0\\ 0 & \bar{i_2}\end{pmatrix}$ *and* $\mathcal{D}\left[\begin{pmatrix}\bar{i_1} & 0\\ 0 & \bar{i_2}\end{pmatrix}\right]=\begin{pmatrix}\bar{i_1} & 0\\ 0 & 0\end{pmatrix}$. *It is obvious that* $\mathcal{H}$ *and* $\mathcal{D}$ *are not a generalized derivation or a derivation on* $\mathcal{A}$, *respectively, but* $\mathcal{H}$ *and* $\mathcal{D}$ *satisfy the algebraic conditions in Theorem 2.1, which shows that semiprimeness is an essential condition in the main theorem of the present section.*

3. Generalized left derivation

The present section is assigned to the study of another extension of a derivation, which is termed as a left derivation defined as: A mapping $\delta:\mathcal{A}\to\mathcal{A}$ is known as a left derivation (respectively, Jordan left derivation) if it is additive and fulfilling $\delta(ab)=a\delta(b)+b\delta(a)$ (respectively, $\delta(b^2)=2b\delta(b)$) for all $a,b\in\mathcal{A}$. We say that a mapping $\delta:\mathcal{A}\to\mathcal{A}$ is known as a right derivation (respectively, Jordan right derivation) if δ is additive and holds the condition $\delta(ab)=\delta(a)b+\delta(b)a$ (respectively, $\delta(b^2)=2\delta(b)b$) for each $a,b\in\mathcal{A}$. δ is a derivation if it is both a left and right derivation. Clearly, every right (respectively, left) derivation on a ring is a Jordan right (respectively, Jordan left) derivation, but the converse statement does not hold generally (for the reference, see [14]).

An additive mapping $\mathcal{H}_1$ from $\mathcal{A}$ to itself is a generalized left derivation (respectively, generalized Jordan left derivation) with an associated Jordan left deviation δ if the statement $\mathcal{H}_1(ab)=a\mathcal{H}_1(b)+b\delta(a)$ (respectively, $\mathcal{H}_1(b^2)=b\mathcal{H}_1(b)+b\delta(b)$) holds for every $a,b\in\mathcal{A}$. The concept of left derivations is encompassed by the theory of generalized left derivations. On the other hand, if we take $\delta=0$, a generalized left derivation follows the theory of right centralizer on $\mathcal{A}$.

In [2], the author's goal is to define a set of requirements that must be met for each generalized Jordan left derivation on a ring to be considered as a generalized left derivation. This led to the discovery of some new results, which can be regarded as a contribution to the theory of Jordan derivations in rings. Motivated by previously mentioned research, we plan to investigate the ensuing identities of generalized left derivation.

If $\mathcal{H}_1$ is a generalized left derivation and δ is an associated left derivation of $\mathcal{H}_1$ on $\mathcal{A}$, then $2\mathcal{H}_1(a^{p+q}) = a^q\mathcal{H}_1(a^p) + a^p\delta(a^q) + a^p\mathcal{H}_1(a^q) + a^q\delta(a^p)$ is satisfied for each a in $\mathcal{A}$. The converse statement of the aforesaid statement is true with some restrictions on $\mathcal{A}$. This section presents the converse of this statement. More precisely, we proved the following: Let $q, p \geq 1$ be any two fixed integers and $\mathcal{A}$ be a $(p+q)!$-torsion free semiprime ring. If two mappings $\mathcal{H}_1, \delta : \mathcal{A} \longrightarrow \mathcal{A}$ are additive and fulfilling the algebraic identity $2\mathcal{H}_1(a^{p+q}) = a^q\mathcal{H}_1(a^p) + a^p\delta(a^q) + a^p\mathcal{H}_1(a^q) + a^q\delta(a^p)$, $\forall a \in \mathcal{A}$, then $\mathcal{H}_1$ will be a generalized left derivation associated with a left derivation δ on $\mathcal{A}$:

To complete the proof of Theorem 3.1, we need the following lemma:

Lemma 3.1. *[1] Let $\mathcal{A}$ be a 2-torsion free semiprime ring and let $\mathcal{H}_1 : \mathcal{A} \longrightarrow \mathcal{A}$ be a generalized Jordan left derivation with associated Jordan left derivation $\delta : \mathcal{A} \longrightarrow \mathcal{A}$, then every generalized Jordan left derivation is a generalized left derivation on $\mathcal{A}$.*

Theorem 3.1. *Let $q, p \geq 1$ be any two fixed integers and $\mathcal{A}$ be a $(p+q)!$-torsion free semiprime ring. If two mappings $\mathcal{H}_1, \delta : \mathcal{A} \longrightarrow \mathcal{A}$ are additive and fulfilling the algebraic identity $2\mathcal{H}_1(a^{p+q}) = a^q\mathcal{H}_1(a^p) + a^p\delta(a^q) + a^p\mathcal{H}_1(a^q) + a^q\delta(a^p)$, $\forall a \in \mathcal{A}$, then $\mathcal{H}_1$ will be a generalized left derivation with associated left derivation δ on $\mathcal{A}$.*

Proof. Since

$$2\mathcal{H}_1(a^{p+q}) = a^q\mathcal{H}_1(a^p) + a^p\delta(a^q) + a^p\mathcal{H}_1(a^q) + a^q\delta(a^p)\ \forall a \in \mathcal{A}, \tag{3.1}$$

then replacing a by e, we get $\delta(e) = 0$. Next, put $a + ne$ in place of a to get

$2\mathcal{H}_1\Big(a^{p+q} + \binom{p+q}{1}a^{p+q-1}ne + \cdots + \binom{p+q}{p+q-2}a^2n^{p+q-2}e + \binom{p+q}{p+q-1}an^{p+q-1}e + n^{p+q}e\Big) = \Big(a^q + \binom{q}{1}a^{q-1}ne + \cdots + \binom{q}{q-2}a^2n^{q-2}e + \binom{q}{q-1}an^{q-1}e + n^qe\Big)\mathcal{H}_1\Big(a^p + \binom{p}{1}a^{p-1}ne + \cdots + \binom{p}{p-2}a^2n^{p-2}e + \binom{p}{p-1}an^{p-1}e + n^pe\Big) + \Big(a^p + \binom{p}{1}a^{p-1}ne + \cdots + \binom{p}{p-2}a^2n^{p-2}e + \binom{p}{p-1}an^{p-1}e + n^pe\Big)\delta\Big(a^q + \binom{q}{1}a^{q-1}ne + \cdots + \binom{q}{q-2}a^2n^{q-2}e + \binom{q}{q-1}an^{q-1}e + n^qe\Big) + \Big(a^p + \binom{p}{1}a^{p-1}ne + \cdots + \binom{p}{p-2}a^2n^{p-2}e + \binom{p}{p-1}an^{p-1}e + n^pe\Big)\mathcal{H}_1\Big(a^q + \binom{q}{1}a^{q-1}ne + \cdots + \binom{q}{q-2}a^2n^{q-2}e + \binom{q}{q-1}an^{q-1}e + n^qe\Big) + \Big(a^q + \binom{q}{1}a^{q-1}ne + \cdots + \binom{q}{q-2}a^2n^{q-2}e + \binom{q}{q-1}an^{q-1}e + n^qe\Big)\delta\Big(a^p + \binom{p}{1}a^{p-1}ne + \cdots + \binom{p}{p-2}a^2n^{p-2}e + \binom{p}{p-1}an^{p-1}e + n^pe\Big)$, where n is any positive integer.
Rewrite the above expression by using (3.1) as

$$nP_1(a,e) + n^2P_2(a,e) + \cdots + n^{p+q-1}P_{p+q-1}(a,e) = 0,$$

where $P_i(a,e)$ stand for the coefficients of n^i's for $i = 1, 2, \cdots, p+q-1$. If we replace n by $1, 2, \cdots, p+q-1$, then we find a system of $(p+q-1)$ linear homogeneous equations. It gives us a Vandermonde matrix ,

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 2 & 2^2 & \cdots & 2^{p+q-1} \\ \cdots & & & \\ \cdots & & & \\ p+q-1 & (p+q-1)^2 & \cdots & (p+q-1)^{p+q-1} \end{bmatrix},$$

which yields that $P_i(a,e) = 0$ for all $a \in \mathcal{A}$ and for $i = 1, 2, \cdots, p+q-1$. Particularly, $i = p+q-1$ gives the following:

$$\begin{aligned} 2\binom{p+q}{p+q-1}\mathcal{H}_1(a) &= \binom{q}{q-1}a\mathcal{H}_1(e) + \binom{p}{p-1}\mathcal{H}_1(a) + \binom{q}{q-1}\delta(a) \\ &\quad + \binom{p}{p-1}a\mathcal{H}_1(e) + \binom{q}{q-1}\mathcal{H}_1(a) + \binom{p}{p-1}\delta(a), \text{ for all } a \in \mathcal{A}. \end{aligned}$$

Using torsion restriction of $\mathcal{A}$, we arrive at

$$\mathcal{H}_1(a) = a\mathcal{H}_1(e) + \delta(a) \text{ for every } a \in \mathcal{A}. \tag{3.2}$$

Next, $P_{p+q-2}(a,e) = 0$ implies the following:

$$\begin{aligned} 2\binom{p+q}{p+q-2}\mathcal{H}_1(a^2) &= \binom{q}{q-2}a^2\mathcal{H}_1(e) + \binom{p}{p-1}\binom{q}{q-1}a\mathcal{H}_1(a) + \binom{p}{p-2}\mathcal{H}_1(a^2) \\ &\quad + \binom{q}{q-2}\delta(a^2) + \binom{p}{p-1}\binom{q}{q-1}a\delta(a) + \binom{p}{p-2}a^2\mathcal{H}_1(e) \\ &\quad + \binom{q}{q-1}\binom{p}{p-1}a\mathcal{H}_1(a) + \binom{q}{q-2}\mathcal{H}_1(a^2) + \binom{p}{p-2}\delta(a^2) \\ &\quad + \binom{q}{q-1}\binom{p}{p-1}a\delta(a), \text{ for all } a \in \mathcal{A}. \end{aligned}$$

This implies that

$$\begin{aligned} (p+q)(p+q-1)\mathcal{H}_1(a^2) &= \tfrac{q(q-1)}{2}a^2\mathcal{H}_1(e) + pqa\mathcal{H}_1(a) + \tfrac{p(p-1)}{2}\mathcal{H}_1(a^2) + \tfrac{q(q-1)}{2}\delta(a^2) \\ &\quad + pqa\delta(a) + \tfrac{p(p-1)}{2}a^2\mathcal{H}_1(e) + qpa\mathcal{H}_1(a) + \tfrac{q(q-1)}{2}\mathcal{H}_1(a^2) \\ &\quad + \tfrac{p(p-1)}{2}\delta(a^2) + qpa\delta(a), \text{ for every } a \in \mathcal{A}. \end{aligned}$$

That is,

$$\begin{aligned} 2(p+q)(p+q-1)\mathcal{H}_1(a^2) &= [q(q-1) + p(p-1)][a^2\mathcal{H}_1(e) + \delta(a^2)] + 4pqa\mathcal{H}_1(a) \\ &\quad + [q(q-1) + p(p-1)]\mathcal{H}_1(a^2) + 4pqa\delta(a), \end{aligned}$$

for every $a \in \mathcal{A}$. Replace a by a^2 in (3.2) and use the above relation to find the following

$$\begin{aligned} 2(p+q)(p+q-1)\mathcal{H}_1(a^2) &= 4pqa\mathcal{H}_1(a) + 4pqa\delta(a) \\ &\quad + 2[q(q-1) + p(p-1)]\mathcal{H}_1(a^2), \text{ for every } a \in \mathcal{A}. \end{aligned}$$

Simplifying the above expression and making use of torsion freeness of $\mathcal{A}$, we have

$$\mathcal{H}_1(a^2) = a\mathcal{H}_1(a) + a\delta(a), \text{ for every } a \in \mathcal{A}. \tag{3.3}$$

Consider (3.2), then

$$\begin{aligned} \delta(a^2) &= \mathcal{H}_1(a^2) - a^2\mathcal{H}_1(e) \\ &= a\mathcal{H}_1(a) + a\delta(a) - a^2\mathcal{H}_1(e) \\ &= a\delta(a) + a[\mathcal{H}_1(a) - a\mathcal{H}_1(e)] \\ &= 2a\delta(a). \end{aligned}$$

Hence, $\mathcal{H}_1$ is a generalized Jordan left derivation on $\mathcal{A}$ having an associated Jordan left derivation δ. Applying Lemma 3.1, we find the required conclusion. □

The implications of Theorem 3.1 lead to the following consequences:

Theorem 3.2. *Suppose that $q, p \geq 1$ are fixed integers and $\mathcal{A}$ is a $(p+q)!$-torsion free semiprime ring. If two mappings $\mathcal{H}_1, \delta : \mathcal{A} \to \mathcal{A}$ are additive and satisfying $2\mathcal{H}_1(a^{p+q}) = a^q\mathcal{H}_1(a^p) + a^p\delta(a^q) + a^p\mathcal{H}_1(a^q) + a^q\delta(a^p)$ for every a in $\mathcal{A}$, then*

(1) δ is a derivation of $\mathcal{A}$ and for each a, b in $\mathcal{A}$, $[\delta(a), b] = 0$,
(2) $\delta(\mathcal{A}) = Z(\mathcal{A})$,
(3) either $\mathcal{A}$ is commutative or $\delta = 0$ on $\mathcal{A}$,
(4) $\mathcal{H}_1$ will be a generalized derivation of $\mathcal{A}$,
(5) $\mathcal{H}_1(a) = aq$ for some $q \in Q_l(\mathcal{A}_C)$, $\forall a \in \mathcal{A}$.

Proof. (1) Since $2\mathcal{H}_1(a^{p+q}) = a^q\mathcal{H}_1(a^p) + a^p\delta(a^q) + a^p\mathcal{H}_1(a^q) + a^q\delta(a^p)$ for each a belong to $\mathcal{A}$, then making use of Theorem 3.1 and [1, Theorem 3.1], we get that δ is a derivation on $\mathcal{A}$ and $[\delta(a), b] = 0\ \forall a, b \in \mathcal{A}$.

(2) Given that $2\mathcal{H}_1(a^{p+q}) = a^q\mathcal{H}_1(a^p) + a^p\delta(a^q) + a^p\mathcal{H}_1(a^q) + a^q\delta(a^p)$ for every a in $\mathcal{A}$, applying Theorem 3.1, $\mathcal{H}_1$ will be a generalized left derivation associated with Jordan left derivation δ of $\mathcal{A}$. Therefore, using [13, Theorem 2], we conclude that $\delta(\mathcal{A}) = Z(\mathcal{A})$.

(3) Suppose that $\delta \neq 0$. From (1), δ is a derivation and $[\delta(w), b] = 0$ for every w and b in $\mathcal{A}$. For instance $[\delta(w), w] = 0$ for every w in $\mathcal{A}$. As $\delta \neq 0$, we say $\mathcal{A}$ is a commutative ring by utilizing [9, Theorem 2].

(4) Consider $2\mathcal{H}_1(a^{p+q}) = a^q\mathcal{H}_1(a^p) + a^p\delta(a^q) + a^p\mathcal{H}_1(a^q) + a^q\delta(a^p)$. In perspective of part (3) and Theorem 3.1, the ring $\mathcal{A}$ is commutative and δ is a derivation of $\mathcal{A}$. Hence, $\mathcal{H}_1$ is a generalized derivation of $\mathcal{A}$.

(5) Since $2\mathcal{H}_1(a^{p+q}) = a^q\mathcal{H}_1(a^p) + a^p\delta(a^q) + a^p\mathcal{H}_1(a^q) + a^q\delta(a^p)$ for all $a \in \mathcal{A}$, then from Theorem 3.1, $\mathcal{H}_1$ will be a generalized left derivation on $\mathcal{A}$. Again, if $\mathcal{A}$ is a noncommutative semiprime ring possessing 2-torsion freeness, then from (3), we have $\delta = 0$. Therefore, $\mathcal{H}_1$ will be a right centralizer of $\mathcal{A}$. Hence, using Proposition 2.10 of [1], there exists $q \in Q_l(\mathcal{A}_C)$ such that $\mathcal{H}_1(a) = aq$ for each $a \in \mathcal{A}$.

□

The subsequent example demonstrates that the noteworthy outcome presented in the article is not superfluous.

Example 3.1. *Define mappings $\mathcal{H}_1, \delta : \mathcal{A} \to \mathcal{A}$ by $\mathcal{H}_1\left[\begin{pmatrix} \bar{i_1} & \bar{i_2} \\ 0 & \bar{i_3} \end{pmatrix}\right] = \begin{pmatrix} 0 & \bar{i_2} \\ 0 & 0 \end{pmatrix}$ and $\delta\left[\begin{pmatrix} \bar{i_1} & \bar{i_2} \\ 0 & \bar{i_3} \end{pmatrix}\right] = \begin{pmatrix} 0 & 0 \\ 0 & \bar{i_3} \end{pmatrix}$, where $\mathcal{A} = \left\{\begin{pmatrix} \bar{i_1} & \bar{i_2} \\ 0 & \bar{i_3} \end{pmatrix} \mid \bar{i_1}, \bar{i_2}, \bar{i_3} \in 2\mathbb{Z}_8\right\}$. Obviously, $\mathcal{H}_1$ is not a generalized left derivation associated with δ (δ is not a Jordan left derivation) on $\mathcal{A}$, but $\mathcal{H}_1, \delta$ satisfy the algebraic conditions in Theorem 3.1. This demonstrates that semiprimeness is a necessary condition for Theorem 3.1 in this section.*

4. Application

The Singer-Wermer theorem, a well-known theorem in Banach algebra, states that any continuous derivation translates into its Jacobson radical on a commutative Banach algebra. Thomas [12] demonstrated that the Singer-Wermer theorem is still valid even when the derivation is not continuous.

The Singer-Wermer conjecture alludes to this generalization. The ensuing systematic observations are motivated by the parallel line of investigation. Given that every semisimple Banach algebra is semiprime, it will be useful to think about Theorem 3.1 in the context of semisimple Banach algebra. Consider the algebraic condition $2\mathfrak{G}(A^{p+q}) = A^q\mathfrak{G}(A^p) + A^p\mathfrak{g}(A^q) + A^p\mathfrak{G}(A^q) + A^q\mathfrak{g}(A^p)$ for all A on a semisimple Banach algebra $\mathfrak{B}$. To prove Theorem 4.1, we required the following results:

Result 4.1. *[8] Every linear derivation is continuous on a semi-simple Banach Algebra.*

Result 4.2. *[11] Any continuous linear derivation maps algebra into its radical on a commutative Banach algebra.*

Result 4.3. *[12] On commutative semi-simple Banach algebras, every linear derivation is zero.*

In light of the aforementioned findings, we arrive at the following theorem:

Theorem 4.1. *If $q, p \geq 1$ are two fixed integers and $\mathfrak{B}$ is a semi-simple Banach algebra, assuming that $\mathfrak{G}, \mathfrak{g} : \mathfrak{B} \to \mathfrak{B}$ are two additive mappings that satisfy $2\mathfrak{G}(A^{p+q}) = A^q\mathfrak{G}(A^p) + A^p\mathfrak{g}(A^q) + A^p\mathfrak{G}(A^q) + A^q\mathfrak{g}(A^p)$ for all $A \in \mathfrak{B}$, then $\mathfrak{g} = 0$ on $\mathfrak{B}$.*

Proof. Recall that every semi-simple Banach algebra is a semiprime ring, then all assumptions of the first part of Theorem 3.2 are satisfied. Therefore, we find a derivation on a semi-simple Banach algebra $\mathfrak{B}$, which is also linear. Thus, $\mathfrak{g} = 0$ from Theorem 4 of the reference [13]. □

5. Conclusions

This would be interesting and thought provoking to analyze the above results in the setting of elementary operators by using the tools of standard operator theory.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

1. S. Ali, On generalized left derivations in rings and Banach algebras, *Aequationes Math.*, **81** (2011), 209–226. https://doi.org/10.1007/s00010-011-0070-5

2. M. Ashraf, S. Ali, On generalized Jordan left derivations in rings, *B. Korean Math. Soc.*, **45**2 (2008), 253–261. https://doi.org/10.4134/BKMS.2008.45.2.253
3. M. Bresar, Jordan mappings of semiprime rings, *J. Algebra*, **127** (1089), 218–228. http://dx.doi.org/10.1016/0021-8693(89)90285-8
4. J. Cusack, Jordan derivations in rings, *P. Am. Math. Soc.*, **53** (1975), 321–324. https://doi.org/10.1090/S0002-9939-1975-0399182-5
5. B. L. M. Ferreira, W. Feng, Mixed $*$-Jordan-type derivations on $*$-algebras, *J. Algebra Appl.*, **22** (2023), 2350100. https://doi.org/10.1142/S0219498823501001
6. B. L. M. Ferreira, H. Guzzo, R. N. Ferreira, F. Wei, Jordan derivations of alternative rings, *Commun. Algebra*, **48** (2020), 717–723. https://doi.org/10.1080/00927872.2019.1659285
7. I. N. Herstein, Derivations in prime rings, *P. Am. Math. Soc.*, **8** (1957), 1104–1110. https://doi.org/10.1090/S0002-9939-1957-0095864-2
8. B. E. Johnson, A. M. Sinclair, Continuity of derivations and a problem of Kaplansky, *Am. J. Math.*, **90** (1968), 1068–1073. https://doi.org/10.2307/2373290
9. E. C. Posner, Derivations in prime rings, *P. Am. Math. Soc.*, 1957, 1093–1100. https://doi.org/10.1090/S0002-9939-1957-0095863-0
10. G. Scudo, A. Z. Ansari, Generalized derivations on Lie ideals and power values on prime rings, *Math. Slovaca*, **65** (2015), 975–980. https://doi.org/10.1515/ms-2015-0066
11. I. M. Singer, J. Wermer, Derivations on commutative normed spaces, *Math. Ann.*, **129** (1995), 435–460.
12. M. P. Thomos, The image of a derivation is contained in the radical, *Ann. Math.*, **128** (1988), 435–460. https://doi.org/10.2307/1971432
13. J. Vukman, On left Jordan derivations on rings and Banach algebras, *Aequationes Math.*, **75** (2008), 260–266. https://doi.org/10.1007/s00010-007-2872-z
14. S. M. A. Zaidi, M. Ashraf, S. Ali, On Jordan ideals and left (θ, θ)-derivation in prime rings, *Int. J. Math. Math. Sci.*, **37** (2004), 1957–1965. https://doi.org/10.1155/S0161171204309075
15. B. Zalar, On centralizers of semiprime rings, *Comment. Math. Univ. Ca.*, **32** (1991), 609–614.
16. J. Zhu, C. Xiong, Generalized derivations on rings and mappings of P-preserving kernel into range on Von Neumann algebras, *Acta. Math. Sin.*, **41** (1998), 795–800. https://doi.org/10.12386/A1998sxxb0133

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