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Research article

Laplacian spectrum of the unit graph associated to the ring of integers modulo pq

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Abstract: Let R be a ring and $U(R)$ be the set of unit elements of R. The unit graph $G(R)$ of R is the graph whose vertices are all the elements of R, defining distinct vertices x and y to be adjacent if and only if $x + y \in U(R)$. The Laplacian spectrum of $G(\mathbb{Z}_n)$ was studied when $n = p^m$, where p is a prime and m is a positive integer. Consequently, in this paper, we study the Laplacian spectrum of $G(\mathbb{Z}_n)$, for $n = p_1 p_2 ... p_k$, where p_i are distinct primes and $i = 1, 2, ..., k$.

Keywords: unit graph; Laplacian matrix; Laplacian spectrum; direct product; ring of integers modulo n
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1. Introduction

All graphs considered in this paper are finite, undirected, and may contain loops. Let G be a graph with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$ and edge set $E(G) = \{e_1, e_2, ..., e_m\}$. For $1 \leq i, j \leq n$, two vertices v_i and v_j in G are adjacent (or neighbors) in G if v_i and v_j are endpoints of an edge e of G, and we write $v_i \sim v_j$ if v_i is adjacent to v_j in G. The degree of a vertex v in G, denoted by $deg(v)$, is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex. The adjacency matrix of G is the $n \times n$ matrix $A(G) = (a_{ij})$, where $a_{ij} = 1$ or 0 according to whether $v_i \sim v_j$ is in G or not. The Laplacian matrix $L(G)$ of G is defined by $L(G) := D(G) - A(G)$, where $D(G) = Diag(d_1, d_2, ..., d_n)$ is the diagonal matrix such that d_i are degrees of vertices of G. $L(G)$ is a symmetric, real, and positive semidefinite matrix; all eigenvalues of $L(G)$ are real and nonnegative. For a simple graph G, the sum of the entries in each row of $L(G)$ is zero, and hence the smallest eigenvalue of $L(G)$ is 0. More literature about the Laplacian matrix of graphs can be seen in [7, 13].

The spectrum of a square matrix B, denoted by $\sigma(B)$, is the multiset of all the eigenvalues of B.

If $\mu_1, \mu_2, ..., \mu_t$ are distinct eigenvalues of B with respective multiplicities $m_1, m_2, ..., m_t$, then we shall denote the spectrum of B by

$$\sigma(B) = \left\{ \begin{array}{cccc} \mu_1 & \mu_2 & ... & \mu_t \\ m_1 & m_2 & ... & m_t \end{array} \right\}.$$

For a graph G, the Laplacian spectrum of G is the spectrum of $L(G)$, denoted by $\sigma_L(G)$. The Laplacian spectrum of graphs of rings has been widely studied in literature, see [3–5].

For a positive integer n, let $\mathbb{Z}_n$ denote the ring of integers modulo n. In this paper, the elements of the ring $\mathbb{Z}_n$ are referred to as $0, 1, 2$, and $n-1$. A nonzero element $x \in \mathbb{Z}_n$ is a unit in $\mathbb{Z}_n$ if x is relatively prime with n: $(x, n) = 1$. In 1990, the unit graph was first introduced by Grimaldi [8] for $\mathbb{Z}_n$ as follows: the unit graph $G(\mathbb{Z}_n)$ is the graph obtained by setting all the elements of $\mathbb{Z}_n$ to be vertices and defining distinct vertices x and y to be adjacent if and only if $x + y$ is a unit in $\mathbb{Z}_n$. He discussed certain basic properties of the structure of the unit graph $G(\mathbb{Z}_n)$ and studied the covering number, the degree of a vertex, the independence number, the Hamilton cycles, and the chromatic polynomial of the graph $G(\mathbb{Z}_n)$. More about the unit graph $G(\mathbb{Z}_n)$ can be seen in [16, 17]. Later, Ashrafi et al. [2] generalized the unit graph from $G(\mathbb{Z}_n)$ to $G(R)$ for an arbitrary ring R. They studied the chromatic index, diameter, girth, and planarity of $G(R)$. In addition, they defined the closed unit graph $\bar{G}(R)$ by dropping the word "distinct" from the definition of the unit graph $G(R)$, and stated that some of $\bar{G}(R)$'s vertices may have loops. Many properties of the unit graph $G(R)$ were investigated in [1, 11, 18].

The remaining parts of the paper are organized as follows: In Section 2, we present some preliminaries and deduce the Laplacian matrix of the direct product for graphs with loops. In Section 3, we obtain the Laplacian spectrum of the graphs $G(\mathbb{Z}_{pq})$ and $G(\mathbb{Z}_{2pq})$, where $p,\ q \neq 2$ are distinct primes. We deduce several consequences from these results, which include the determination of the Laplacian spectrum of $G(\mathbb{Z}_n)$ for $n = p_1 p_2 ... p_k$, where p_i are distinct primes and $i = 1, 2, ..., k$.

2. Laplacian matrix of the direct product of graphs with at most one loop at each vertex

Let us denote the graph with at most one loop at each vertex by $\bar{G}$ and let G be the simple graph corresponding to $\bar{G}$. For $v \in V(G)$, we denote by $N(v)$ the set of all neighbors of v in G. Note that, if v has a loop, then $v \in N(v)$ [15]. For graphs G and $\bar{G}$ we define the following matrices:

- $M(\bar{G}) = (m_{ij})$ is a diagonal matrix such that

$$m_{ij} = \begin{cases} 1 & \text{if } i = j \text{ and } v_i \text{ is vertex has a loop,} \\ 0 & \text{otherwise.} \end{cases}$$

  Clearly, $M(G) = 0$.
- $N(G) = (n_{ij})$ is a diagonal matrix defined by

$$n_{ij} = \begin{cases} |N(v_i)| & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

  So, $N(\bar{G}) = N(G) + M(\bar{G})$.

For any vertex v in G, we have $deg(v) = |N(v)|$, and hence $N(G) = D(G)$. Let $v \in V(\bar{G})$; if v has a loop, then $deg(v) = |N(v)| + 1$, and hence $D(\bar{G}) = N(\bar{G}) + M(\bar{G}) = D(G) + 2M(\bar{G})$. Every loop in $\bar{G}$ is represented by a 1 in the main diagonal of $A(\bar{G})$, so $A(\bar{G}) = A(G) + M(\bar{G})$. Now, $L(\bar{G}) = D(\bar{G}) - A(\bar{G})$, and using substitutions of $D(\bar{G})$ and $A(\bar{G})$, we get $L(\bar{G}) = L(G) + M(\bar{G})$.

From Figure 1, we can find the following:

$$M(\bar{G}(\mathbb{Z}_3)) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; N(G(\mathbb{Z}_3)) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ and } N(\bar{G}(\mathbb{Z}_3)) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Note that, $N(\bar{G}(\mathbb{Z}_3)) = N(G(\mathbb{Z}_3)) + M(\bar{G}(\mathbb{Z}_3))$.

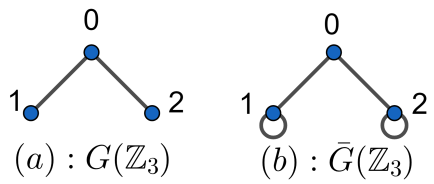

Figure 1. (a) The unit graph of the ring $\mathbb{Z}_3$. (b) The closed unit graph of the ring $\mathbb{Z}_3$.

In this section, we provide a formula for the Laplacian matrix of the direct product of graphs with at most one loop at each vertex. Let us first recall that the Kronecker product $A \otimes B$ of a $p \times q$ matrix $A = (a_{ij})$ by an $r \times s$ matrix B is defined as

$$A \otimes B = \begin{pmatrix} a_{11}B & \dots & a_{1q}B \\ \vdots & & \vdots \\ a_{p1}B & \dots & a_{pq}B \end{pmatrix}.$$

Other names for the Kronecker product are the tensor product or direct product. Let G_1 and G_2 be two graphs; the direct product of G_1 and G_2 is a graph, denoted by $G_1 \otimes G_2$, whose vertex set is $V(G_1) \times V(G_2)$, and for which vertices (x_1, y_1) and (x_2, y_2) are adjacent if x_1 and x_2 are adjacent in G_1, and y_1 and y_2 are adjacent in G_2.

Kaveh and Alinejad described the Laplacian matrix of $G_1 \otimes G_2$ in terms of the Laplacian matrices of G_1 and G_2, where G_1 and G_2 are simple graphs, as can be seen below [10].

Proposition 2.1. *Let G_1 and G_2 be simple graphs. Then, the Laplacian matrix of $G_1 \otimes G_2$ is*

$$L(G_1 \otimes G_2) = D(G_1) \otimes L(G_2) + L(G_1) \otimes D(G_2) - L(G_1) \otimes L(G_2).$$

The following lemma describes the degree matrix and the adjacency matrix of $\bar{G}_1 \otimes \bar{G}_2$.

Lemma 2.1. *Let $\bar{G}_1$ and $\bar{G}_2$ be graphs with at most one loop at each vertex. Then,*

(1) $D(\bar{G}_1 \otimes \bar{G}_2) = N(\bar{G}_1) \otimes N(\bar{G}_2) + M(\bar{G}_1) \otimes M(\bar{G}_2)$.
(2) $A(\bar{G}_1 \otimes \bar{G}_2) = (A(G_1) + M(\bar{G}_1)) \otimes (A(G_2) + M(\bar{G}_2))$.

Proof. (1) Suppose that $\bar{G}_1$ and $\bar{G}_2$ are two graphs with at most one loop at each vertex. If $(x, y) \in V(\bar{G}_1 \otimes \bar{G}_2)$, then $N((x, y)) = N(x) \times N(y)$ [12]. It follows that $N(\bar{G}_1 \otimes \bar{G}_2) = N(\bar{G}_1) \otimes N(\bar{G}_2)$. By using the fact that (x, y) has a loop in $\bar{G_1} \otimes \bar{G_2}$ if and only if x and y have a loop in $\bar{G_1}$ and $\bar{G_2}$, respectively [9], then $M(\bar{G_1} \otimes \bar{G_2}) = M(\bar{G_1}) \otimes M(\bar{G_2})$. Since $D(\bar{G_1} \otimes \bar{G_2}) = N(\bar{G_1} \otimes \bar{G_2}) + M(\bar{G_1} \otimes \bar{G_2})$, then

$$D(\bar{G_1} \otimes \bar{G_2}) = N(\bar{G_1}) \otimes N(\bar{G_2}) + M(\bar{G_1}) \otimes M(\bar{G_2}).$$

(2) For simple graphs G_1 and G_2 we have $A(G_1 \otimes G_2) = A(G_1) \otimes A(G_2)$ [6]. If we agree on the convention that a 1 diagonal entry in the adjacency matrix of $\bar{G_i}$, $i = 1, 2$, means a loop, whereas a 0 means no loop, then the adjacency matrix of $\bar{G_1} \otimes \bar{G_2}$ still corresponds to the Kronecker product of the adjacency matrices of $A(\bar{G_1})$ and $A(\bar{G_2})$. Thus,

$$\begin{aligned} A(\bar{G_1} \otimes \bar{G_2}) &= A(\bar{G_1}) \otimes A(\bar{G_2}) \\ &= (A(G_1) + M(\bar{G_1})) \otimes (A(G_2) + M(\bar{G_2})). \quad \square \end{aligned}$$

The following theorem generalizes the result about the Laplacian matrix of the direct product of simple graphs to the Laplacian matrix of the direct product of graphs with at most one loop at each vertex.

Theorem 2.1. *Let $\bar{G_1}$ and $\bar{G_2}$ be graphs with at most one loop at each vertex. Then, the Laplacian matrix of $\bar{G_1} \otimes \bar{G_2}$ is*

$$L(\bar{G_1} \otimes \bar{G_2}) = N(\bar{G_1}) \otimes L(G_2) + L(G_1) \otimes N(\bar{G_2}) - L(G_1) \otimes L(G_2) + M(\bar{G_1}) \otimes M(\bar{G_2}). \tag{2.1}$$

Proof. By using the fact that $L(\bar{G_1} \otimes \bar{G_2}) = D(\bar{G_1} \otimes \bar{G_2}) - A(\bar{G_1} \otimes \bar{G_2})$, Lemma 2.1, and applying the properties of the direct product of matrices, we obtain

$$\begin{aligned} L(\bar{G_1} \otimes \bar{G_2}) &= N(\bar{G_1}) \otimes N(\bar{G_2}) + M(\bar{G_1}) \otimes M(\bar{G_2}) \\ &\quad - [A(G_1) + M(\bar{G_1})] \otimes [A(G_2) + M(\bar{G_2})] \\ &= N(\bar{G_1}) \otimes N(\bar{G_2}) - A(G_1) \otimes A(G_2) \\ &\quad - A(G_1) \otimes M(\bar{G_2}) - M(\bar{G_1}) \otimes A(G_2). \end{aligned}$$

By using $A(G) = N(\bar{G}) - L(G) - M(\bar{G})$, we have

$$\begin{aligned} L(\bar{G_1} \otimes \bar{G_2}) &= N(\bar{G_1}) \otimes N(\bar{G_2}) - [N(\bar{G_1}) - L(G_1) - M(\bar{G_1})] \otimes [N(\bar{G_2}) - L(G_2) - M(\bar{G_2})] \\ &\quad - [N(\bar{G_1}) - L(G_1) - M(\bar{G_1})] \otimes M(\bar{G_2}) - M(\bar{G_1}) \otimes [N(\bar{G_2}) - L(G_2) - M(\bar{G_2})] \\ &= N(\bar{G_1}) \otimes N(\bar{G_2}) - N(\bar{G_1}) \otimes N(\bar{G_2}) + N(\bar{G_1}) \otimes L(G_2) + N(\bar{G_1}) \otimes M(\bar{G_2}) \\ &\quad + L(G_1) \otimes N(\bar{G_2}) - L(G_1) \otimes L(G_2) - L(G_1) \otimes M(\bar{G_2}) \\ &\quad + M(\bar{G_1}) \otimes N(\bar{G_2}) - M(\bar{G_1}) \otimes L(G_2) - M(\bar{G_1}) \otimes M(\bar{G_2}) \\ &\quad - N(\bar{G_1}) \otimes M(\bar{G_2}) + L(G_1) \otimes M(\bar{G_2}) + M(\bar{G_1}) \otimes M(\bar{G_2}) \\ &\quad - M(\bar{G_1}) \otimes N(\bar{G_2}) + M(\bar{G_1}) \otimes L(G_2) + M(\bar{G_1}) \otimes M(\bar{G_2}). \end{aligned}$$

Through basic cancellations, the result follows. □

Rezagholibeigia et al. [14] observed that $\bar{G}(R \times S) \cong \bar{G}(R) \otimes \bar{G}(S)$, where $R \times S$ is the direct product of the rings R and S. Motivated by their observations, we study the Laplacian matrix of $G(\mathbb{Z}_{pq}) \cong G(\mathbb{Z}_p \times \mathbb{Z}_q)$ as the Laplacian matrix of $\bar{G}(\mathbb{Z}_p) \otimes \bar{G}(\mathbb{Z}_q)$ after removing the loops of this graph by deleting the matrix $M(\bar{G}(\mathbb{Z}_p)) \otimes M(\bar{G}(\mathbb{Z}_q))$ from Eq (2.1). So, we have the following consequential result.

Corollary 2.1. *The Laplacian matrix of $G(\mathbb{Z}_{pq})$, where $p, q \neq 2$ are primes, is*

$$L(G(\mathbb{Z}_{pq})) = N(\bar{G}(\mathbb{Z}_p)) \otimes L(G(\mathbb{Z}_q)) + L(G(\mathbb{Z}_p)) \otimes N(\bar{G}(\mathbb{Z}_q)) - L(G(\mathbb{Z}_p)) \otimes L(G(\mathbb{Z}_q)).$$

3. Laplacian spectrum of $G(\mathbb{Z}_{pq})$

Recall that Euler's totient function, denoted by $\varphi(n)$, is the number of positive integers less than or equal to n that are relatively prime to n. If p_i, where $i = 1, 2, ..., k$, are primes, then $\varphi(p_i) = p_i - 1$ and $\varphi(p_1 p_2 ... p_k) = \varphi(p_1)\varphi(p_2)...\varphi(p_k)$. If $2 \notin U(R)$, then $x + x = 2x$ for all $x \in R$, and hence $2x \notin U(R)$. So, in $\bar{G}(R)$, no vertex has a loop. That means that $\bar{G}(R) = G(R)$, which is pointed out in [2]. If $2 \in U(R)$, then $x + x = 2x$ is a unit in R for all $x \in U(R)$ and hence x has a loop. So, in this case, the number of vertices in $\bar{G}(\mathbb{Z}_n)$, which has a loop, is $\varphi(n)$. The next result is derived from the previous discussion.

Proposition 3.1. *The Laplacian matrix of $G(\mathbb{Z}_{2p})$, where $p \neq 2$ is prime, is*

$$L(G(\mathbb{Z}_{2p})) = N(\bar{G}(\mathbb{Z}_2)) \otimes L(G(\mathbb{Z}_p)) + L(G(\mathbb{Z}_2)) \otimes N(\bar{G}(\mathbb{Z}_p)) - L(G(\mathbb{Z}_2)) \otimes L(G(\mathbb{Z}_p)).$$

Proof. Since $2 \notin U(\mathbb{Z}_2)$, $\bar{G}(\mathbb{Z}_2) = G(\mathbb{Z}_2)$, then $M(\bar{G}(\mathbb{Z}_2)) \otimes M(\bar{G}(\mathbb{Z}_p)) = 0$. So,

$$\begin{aligned} L(G(\mathbb{Z}_{2p})) &= L(G(\mathbb{Z}_2 \times \mathbb{Z}_p)) = L(\bar{G}(\mathbb{Z}_2 \times \mathbb{Z}_p)) \\ &= N(\bar{G}(\mathbb{Z}_2)) \otimes L(G(\mathbb{Z}_p)) + L(G(\mathbb{Z}_2)) \otimes N(\bar{G}(\mathbb{Z}_p)) \\ &\quad - L(G(\mathbb{Z}_2)) \otimes L(G(\mathbb{Z}_p)). \quad \square \end{aligned}$$

The following results will be used in the next part.

Proposition 3.2. *[2] Let R be a finite ring. Then, the following statements hold for the unit graph of R:*

(1) If $2 \notin U(R)$, then the unit graph $G(R)$ is a $|U(R)|$-regular graph;
(2) If $2 \in U(R)$, then for every $x \in U(R)$ we have $deg(x) = |U(R)| - 1$, and for every $x \in R \backslash U(R)$ we have $deg(x) = |U(R)|$.

Theorem 3.1. *[16] Suppose q is an odd prime and suppose that n is a positive integer. Then, $G(\mathbb{Z}_{q^n})$'s Laplacian spectrum is given by*

$$\sigma_L(G(\mathbb{Z}_{q^n})) = \begin{Bmatrix} 0 & q^n & q^n - q^{n-1} & q^n - 2q^{n-1} \\ 1 & \frac{q-1}{2} & q^n - q & \frac{q-1}{2} \end{Bmatrix}.$$

Note that in the above theorem, if $q = 2$, then $G(\mathbb{Z}_{2^n})$ is a complete bipartite graph* by Remark 3.6 [2]. So, $G(\mathbb{Z}_{2^n})$ is isomorphic to $K_{2^{n-1},2^{n-1}}$, and thus $\sigma_L(G(\mathbb{Z}_{2^n}))$, which is the well-known $\sigma_L(K_{2^{n-1},2^{n-1}})$, is given by

$$\begin{Bmatrix} 0 & 2^{n-1} & 2^n \\ 1 & 2(2^{n-1} - 1) & 1 \end{Bmatrix}.$$

Lemma 3.1. *Let p, q, p_i, where $i = 1, 2, ..., k$, be primes such that $p, p_i \neq 2$. Then,*

(1) $N(\bar{G}(\mathbb{Z}_q)) = \varphi(q)I$, where I is a $q \times q$ identity matrix.
(2) $N(\bar{G}(\mathbb{Z}_{2p})) = \varphi(p)I$, where I is a $2p \times 2p$ identity matrix.

*A complete bipartite graph $K_{m,n}$ is a graph that has its vertex set partitioned into two subsets of m and n vertices, respectively, with an edge between two vertices if and only if one vertex is in the first subset and the other vertex is in the second subset.

(3) If $n = p_1p_2...p_k$, $N(\bar{G}(\mathbb{Z}_n)) = \varphi(p_1)\varphi(p_2)...\varphi(p_k)I$, where I is an $n \times n$ identity matrix.
(4) If $n = 2p_1p_2...p_k$, $N(\bar{G}(\mathbb{Z}_n)) = \varphi(p_1)\varphi(p_2)...\varphi(p_k)I$, where I is an $n \times n$ identity matrix.

Proof. (1) If $q \neq 2$, then $2 \in U(\mathbb{Z}_q)$, and hence all units of $\mathbb{Z}_q$ have loops. The set of units in $\mathbb{Z}_q$ is $\mathbb{Z}_q - \{0\}$. By Proposition 3.2, for $v \in V(\bar{G}(\mathbb{Z}_q))$, such that $1 \leq v \leq q-1$, $|N(v)| = deg(v) + 1 = |U(\mathbb{Z}_q)| - 1 + 1$. If $v = 0$, $|N(v)| = |U(\mathbb{Z}_q)|$. Therefore, $|N(v)| = |U(\mathbb{Z}_q)| = \varphi(q)$ for all $v \in V(\bar{G}(\mathbb{Z}_q))$, and hence $N(\bar{G}(\mathbb{Z}_q)) = \varphi(q)I$. Now, if $q = 2$, then $N(0) = \{1\}$ and $N(1) = \{0\}$ in $\bar{G}(\mathbb{Z}_2)$, and hence $N(\bar{G}(\mathbb{Z}_2)) = I_2$.

(2) Since $2 \notin U(\mathbb{Z}_{2p})$, then $\bar{G}(\mathbb{Z}_{2p}) = G(\mathbb{Z}_{2p})$. By Proposition 3.2, $|N(v)| = |U(\mathbb{Z}_{2p})|$ for $v \in V(G(\mathbb{Z}_{2p}))$. So, $|N(v)| = \varphi(2p)$ for all $v \in V(\bar{G}(\mathbb{Z}_{2p}))$, and hence $N(\bar{G}(\mathbb{Z}_{2p})) = \varphi(p)I$.

(3) By Proposition 3.2, if $2 \in U(\mathbb{Z}_n)$, $|N(v)| = deg(v)+1 = |U(\mathbb{Z}_n)|-1+1$ for every unit v of $\mathbb{Z}_n$. Also, if v is non-unit in $\mathbb{Z}_n$, then $|N(v)| = deg(v) = |U(\mathbb{Z}_n)|$. So, $|N(v)| = |U(\mathbb{Z}_n)| = \varphi(n) = \varphi(p_1)\varphi(p_2)...\varphi(p_k)$ for all $v \in V(\bar{G}(\mathbb{Z}_n))$, and hence $N(\bar{G}(\mathbb{Z}_n)) = \varphi(p_1)\varphi(p_2)...\varphi(p_k)I$.

(4) The proof is similar to that of 3.1. □

Theorem 3.2. *If $p, q \neq 2$ are primes, then the Laplacian spectrum of $G(\mathbb{Z}_{pq})$ is*

$$\sigma_L(G(\mathbb{Z}_{pq})) = \begin{Bmatrix} 0 & [\varphi(p)-1]\varphi(q) & [\varphi(q)-1]\varphi(p) & \varphi(p)\varphi(q)-1 & \varphi(p)\varphi(q)+1 & \varphi(p)[\varphi(q)+1] & \varphi(q)[\varphi(p)+1] \\ 1 & \dfrac{\varphi(p)}{2} & \dfrac{\varphi(q)}{2} & \dfrac{\varphi(p)\varphi(q)}{2} & \dfrac{\varphi(p)\varphi(q)}{2} & \dfrac{\varphi(q)}{2} & \dfrac{\varphi(p)}{2} \end{Bmatrix}. \tag{3.1}$$

Proof. By using Corollary 2.1 and Lemma 3.1, we have

$$L(G(\mathbb{Z}_{pq})) = \varphi(p)I \otimes L(G(\mathbb{Z}_q)) + L(G(\mathbb{Z}_p)) \otimes \varphi(q)I - L(G(\mathbb{Z}_p)) \otimes L(G(\mathbb{Z}_q)).$$

Now, to determine the spectrum of $L(G(\mathbb{Z}_{pq}))$, we suppose that X_i and Y_j are eigenvectors of $L(G(\mathbb{Z}_p))$ and $L(G(\mathbb{Z}_q))$ according to the eigenvalues λ_i and μ_j, respectively. That is, $L(G(\mathbb{Z}_p))X_i = \lambda_i X_i$, $X_i \neq 0$ and $L(G(\mathbb{Z}_q))Y_j = \mu_j Y_j$, $Y_j \neq 0$. Thus,

$$\begin{aligned} L(G(\mathbb{Z}_{pq}))(X_i \otimes Y_j) &= [\varphi(p)I \otimes L(G(\mathbb{Z}_q)) + L(G(\mathbb{Z}_p)) \otimes \varphi(q)I \\ &\quad - L(G(\mathbb{Z}_p)) \otimes L(G(\mathbb{Z}_q))](X_i \otimes Y_j) \\ &= [\varphi(p)I \otimes L(G(\mathbb{Z}_q))](X_i \otimes Y_j) \\ &\quad + [L(G(\mathbb{Z}_p)) \otimes \varphi(q)I](X_i \otimes Y_j) \\ &\quad - [L(G(\mathbb{Z}_p)) \otimes L(G(\mathbb{Z}_q))](X_i \otimes Y_j) \\ &= \varphi(p)X_i \otimes \mu_j Y_j + \lambda_i X_i \otimes \varphi(q)Y_j - \lambda_i X_i \otimes \mu_j Y_j. \end{aligned}$$

So,

$$L(G(\mathbb{Z}_{pq}))(X_i \otimes Y_j) = [\varphi(p)\mu_j + \lambda_i\varphi(q) - \lambda_i\mu_j](X_i \otimes Y_j).$$

Therefore, the eigenvalues of $L(G(\mathbb{Z}_{pq}))$ are given by $\varphi(p)\mu_j + \lambda_i\varphi(q) - \lambda_i\mu_j$, where $1 \leq i \leq p$ and $1 \leq j \leq q$. By using Theorem 3.1, we have

$$\sigma_L(G(\mathbb{Z}_p)) = \left\{ \begin{array}{ccc} 0 & p-2 & p \\ 1 & \frac{p-1}{2} & \frac{p-1}{2} \end{array} \right\} \text{ and } \sigma_L(G(\mathbb{Z}_q)) = \left\{ \begin{array}{ccc} 0 & q-2 & q \\ 1 & \frac{q-1}{2} & \frac{q-1}{2} \end{array} \right\}.$$

So, the spectrum of $L(G(\mathbb{Z}_{pq}))$ consists of

$$\left\{ \varphi(p)\begin{Bmatrix} 0 \\ 1 \end{Bmatrix} + \varphi(q)\begin{Bmatrix} 0 & \varphi(p)-1 & \varphi(p)+1 \\ 1 & \frac{\varphi(p)}{2} & \frac{\varphi(p)}{2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}\begin{Bmatrix} 0 & \varphi(p)-1 & \varphi(p)+1 \\ 1 & \frac{\varphi(p)}{2} & \frac{\varphi(p)}{2} \end{Bmatrix} \right\}$$

$$\cup \left\{ \varphi(p)\begin{Bmatrix} \varphi(q)-1 \\ \frac{\varphi(q)}{2} \end{Bmatrix} + \varphi(q)\begin{Bmatrix} 0 & \varphi(p)-1 & \varphi(p)+1 \\ 1 & \frac{\varphi(p)}{2} & \frac{\varphi(p)}{2} \end{Bmatrix} - \begin{Bmatrix} \varphi(q)-1 \\ \frac{\varphi(q)}{2} \end{Bmatrix}\begin{Bmatrix} 0 & \varphi(p)-1 & \varphi(p)+1 \\ 1 & \frac{\varphi(p)}{2} & \frac{\varphi(p)}{2} \end{Bmatrix} \right\}$$

$$\cup \left\{ \varphi(p)\begin{Bmatrix} \varphi(q)+1 \\ \frac{\varphi(q)}{2} \end{Bmatrix} + \varphi(q)\begin{Bmatrix} 0 & \varphi(p)-1 & \varphi(p)+1 \\ 1 & \frac{\varphi(p)}{2} & \frac{\varphi(p)}{2} \end{Bmatrix} - \begin{Bmatrix} \varphi(q)+1 \\ \frac{\varphi(q)}{2} \end{Bmatrix}\begin{Bmatrix} 0 & \varphi(p)-1 & \varphi(p)+1 \\ 1 & \frac{\varphi(p)}{2} & \frac{\varphi(p)}{2} \end{Bmatrix} \right\}$$

$$= \begin{Bmatrix} 0 & \varphi(q)[\varphi(p)-1] & \varphi(q)[\varphi(p)+1] \\ 1 & \frac{\varphi(p)}{2} & \frac{\varphi(p)}{2} \end{Bmatrix} \cup \begin{Bmatrix} \varphi(p)[\varphi(q)-1] & \varphi(p)\varphi(q)-1 & \varphi(p)\varphi(q)+1 \\ \frac{\varphi(q)}{2} & (\frac{\varphi(p)}{2})(\frac{\varphi(q)}{2}) & (\frac{\varphi(p)}{2})(\frac{\varphi(q)}{2}) \end{Bmatrix}$$

$$\cup \begin{Bmatrix} \varphi(p)[\varphi(q)+1] & \varphi(p)\varphi(q)+1 & \varphi(p)\varphi(q)-1 \\ \frac{\varphi(q)}{2} & (\frac{\varphi(p)}{2})(\frac{\varphi(q)}{2}) & (\frac{\varphi(p)}{2})(\frac{\varphi(q)}{2}) \end{Bmatrix}.$$

Hence, the Laplacian spectrum of $G(\mathbb{Z}_{pq})$ is as in Eq (3.1). □

Theorem 3.3. *Let $p \neq 2$ be a prime. Then, the Laplacian spectrum of $G(\mathbb{Z}_{2p})$ is*

$$\sigma_L(G(\mathbb{Z}_{2p})) = \left\{ \begin{array}{cccc} 0 & \varphi(p)-1 & \varphi(p)+1 & 2\varphi(p) \\ 1 & \varphi(p) & \varphi(p) & 1 \end{array} \right\}.$$

Proof. By applying Proposition 3.1 and Lemma 3.1, we have

$$L(\bar{G}(\mathbb{Z}_2 \times \mathbb{Z}_p)) = I_2 \otimes L(G(\mathbb{Z}_p)) + L(G(\mathbb{Z}_2)) \otimes \varphi(p)I - L(G(\mathbb{Z}_2)) \otimes L(G(\mathbb{Z}_p)).$$

Then, proceeding similarly as with the proof of Theorem 3.2, the eigenvalues of $L(G(\mathbb{Z}_{2p}))$ are given by $\mu_j + \lambda_i\varphi(p) - \lambda_i\mu_j$, where λ_i, $i = 1, 2$, and μ_j, $1 \leq j \leq p$, are the eigenvalues of $L(G(\mathbb{Z}_2))$

and $L(G(\mathbb{Z}_p))$, respectively. By using Theorem 3.1 and the argument after it, the Laplacian spectrum of $G(\mathbb{Z}_{2p})$ is

$$\sigma_L(G(\mathbb{Z}_{2p})) = \left\{ \begin{array}{cccc} 0 & \varphi(p)-1 & \varphi(p)+1 & 2\varphi(p) \\ 1 & \varphi(p) & \varphi(p) & 1 \end{array} \right\}. \qquad \square$$

Corollary 3.1. *Let $p, q \neq 2$ be primes. Then, the Laplacian spectrum of $G(\mathbb{Z}_{2pq})$ is*

$$\sigma_L(G(\mathbb{Z}_{2pq})) = \begin{cases} 0 & [\varphi(p)-1]\varphi(q) \quad [\varphi(q)-1]\varphi(p) \quad \varphi(p)\varphi(q)-1 \quad \varphi(p)\varphi(q)+1 \\ 1 & \varphi(p) \quad \varphi(q) \quad \varphi(p)\varphi(q) \quad \varphi(p)\varphi(q) \end{cases}$$

$$\left. \begin{array}{ccc} \varphi(p)[\varphi(q)+1] & \varphi(q)[\varphi(p)+1] & 2\varphi(p)\varphi(q) \\ \varphi(q) & \varphi(p) & 1 \end{array} \right\}.$$

Proof. Since $\bar{G}(\mathbb{Z}_{2p} \times \mathbb{Z}_q) \cong \bar{G}(\mathbb{Z}_{2p}) \otimes \bar{G}(\mathbb{Z}_q)$, by using Corollary 2.1 and Lemma 3.1 we have

$$L(\bar{G}(\mathbb{Z}_{2p} \times \mathbb{Z}_q)) = \varphi(p)I \otimes L(G(\mathbb{Z}_q)) + L(G(\mathbb{Z}_{2p})) \otimes \varphi(q)I - L(G(\mathbb{Z}_{2p})) \otimes L(G(\mathbb{Z}_q)).$$

Approaching the proof in a similar manner as with Theorem 3.2, the eigenvalues of $L(G(\mathbb{Z}_{2pq}))$ are given by $\varphi(p)\mu_j + \lambda_i\varphi(q) - \lambda_i\mu_j$, where λ_i, $1 \leq i \leq 2p$, and μ_j, $1 \leq j \leq q$, are the eigenvalues of $L(G(\mathbb{Z}_{2p}))$ and $L(G(\mathbb{Z}_q))$, respectively. Thus, the result follows from Theorems 3.1 and 3.3. $\square$

Example 3.1. *To find the Laplacian spectrum of $G(\mathbb{Z}_{30})$, let $p = 3$ and $q = 5$. By using the above theorem, we get*

$$\sigma_L(G(\mathbb{Z}_{30})) = \begin{cases} 0 & (2-1)(5-1) \quad (4-1)(3-1) \quad (3-1)(5-1)-1 \quad (3-1)(5-1)+1 \\ 1 & 3-1 \quad 5-1 \quad (3-1)(5-1) \quad (3-1)(5-1) \end{cases}$$

$$\left. \begin{array}{ccc} (3-1)(4+1) & (5-1)(2+1) & 2(3-1)(5-1) \\ 5-1 & 3-1 & 1 \end{array} \right\}$$

$$= \left\{ \begin{array}{cccccccc} 0 & 4 & 6 & 7 & 9 & 10 & 12 & 16 \\ 1 & 2 & 4 & 8 & 8 & 4 & 2 & 1 \end{array} \right\}.$$

The following theorem gives the Laplacian spectrum of $G(\mathbb{Z}_n)$ if $n = p_1p_2...p_k$, where p_i are distinct primes and $i = 1, 2, ..., k$.

Theorem 3.4. *Let $p_i \neq 2$ be distinct primes and k be a positive integer, $1 \leq i, j \leq k$. Then:*

(1) If $n = p_1p_2...p_k$, the Laplacian spectrum of $G(\mathbb{Z}_n)$ is

$$\sigma_L(G(\mathbb{Z}_n)) = \begin{cases} 0 & [\varphi(p_i) \pm 1]\prod_{i\neq j}\varphi(p_j) & [\varphi(p_i)\varphi(p_j) \pm 1]\prod_{h\neq i,j}\varphi(p_h) & \cdots \\ 1 & \frac{\varphi(p_i)}{2} & \frac{\varphi(p_i)\varphi(p_j)}{2} & \cdots \end{cases} \tag{3.2}$$

$$\left.\begin{matrix} [\varphi(p_1)\varphi(p_2)...\varphi(p_{k-1}) \pm 1]\varphi(p_k) & [\prod_{1\leq i\leq k}\varphi(p_i)] \pm 1 \\ \frac{\prod_{1\leq i\leq k-1}\varphi(p_i)}{2} & \frac{\prod_{1\leq i\leq k}\varphi(p_i)}{2} \end{matrix}\right\}.$$

(2) If $n = 2p_1p_2...p_k$, the Laplacian spectrum of $G(\mathbb{Z}_n)$ is

$$\sigma_L(G(\mathbb{Z}_n)) = \begin{cases} 0 & [\varphi(p_i) \pm 1]\prod_{i\neq j}\varphi(p_j) & [\varphi(p_i)\varphi(p_j) \pm 1]\prod_{h\neq i,j}\varphi(p_h) & \cdots \\ 1 & \varphi(p_i) & \varphi(p_i)\varphi(p_j) & \cdots \end{cases} \tag{3.3}$$

$$\left.\begin{matrix} [\varphi(p_1)\varphi(p_2)...\varphi(p_{k-1}) \pm 1]\varphi(p_k) & [\prod_{1\leq i\leq k}\varphi(p_i)] \pm 1 & 2\prod_{1\leq i\leq k}\varphi(p_i) \\ \prod_{1\leq i\leq k-1}\varphi(p_i) & \prod_{1\leq i\leq k}\varphi(p_i) & 1 \end{matrix}\right\}.$$

Proof. (1) When $n = p$, by Theorem 3.1 we have

$$\sigma_L(G(\mathbb{Z}_p)) = \left\{\begin{matrix} 0 & \varphi(p) - 1 & \varphi(p) + 1 \\ 1 & \frac{\varphi(p)}{2} & \frac{\varphi(p)}{2} \end{matrix}\right\}.$$

Therefore, the result of Eq (3.2) is valid for $n = p$.

Now we consider the case where $i > 1$. We claim that for $n = p_1p_2...p_{k-1}$, the Laplacian spectrum of $G(\mathbb{Z}_n)$ is

$$\begin{cases} 0 & [\varphi(p_i) \pm 1]\prod_{i\neq j}\varphi(p_j) & [\varphi(p_i)\varphi(p_j) \pm 1]\prod_{h\neq i,j}\varphi(p_h) & \cdots \\ 1 & \frac{\varphi(p_i)}{2} & \frac{\varphi(p_i)\varphi(p_j)}{2} & \cdots \end{cases} \tag{3.4}$$

$$\left.\begin{matrix} [\varphi(p_1)\varphi(p_2)...\varphi(p_{k-2}) \pm 1]\varphi(p_{k-1}) & [\prod_{1\leq i\leq k-1}\varphi(p_i)] \pm 1 \\ \frac{\prod_{1\leq i\leq k-2}\varphi(p_i)}{2} & \frac{\prod_{1\leq i\leq k-1}\varphi(p_i)}{2} \end{matrix}\right\}.$$

Since $\mathbb{Z}_{p_1p_2...p_k} \cong \mathbb{Z}_{p_1p_2...p_{k-1}} \times \mathbb{Z}_{p_k}$, then $G(\mathbb{Z}_{p_1p_2...p_k}) \cong G(\mathbb{Z}_{p_1p_2...p_{k-1}} \times \mathbb{Z}_{p_k})$. Proceeding in a manner similar as with the proof of Theorem 3.2, the eigenvalues of $L(G(\mathbb{Z}_n))$ are given by $[\prod_{1\leq i\leq k-1}\varphi(p_i)]\mu_j + \varphi(p_k)\lambda_i - \lambda_i\mu_j$, where λ_i, $1 \leq i \leq p_1p_2...p_{k-1}$, and μ_j, $1 \leq j \leq p_k$, are the eigenvalues of $L(G(\mathbb{Z}_{p_1p_2...p_{k-1}}))$ and $L(G(\mathbb{Z}_{p_k}))$, respectively. By using Eq (3.4) and Theorem 3.1, we get Eq (3.2).

(2) Since $\mathbb{Z}_{2p_1p_2...p_k} \cong \mathbb{Z}_2 \times \mathbb{Z}_{p_1p_2...p_k}$, then $G(\mathbb{Z}_{2p_1p_2...p_k}) \cong G(\mathbb{Z}_2 \times \mathbb{Z}_{p_1p_2...p_k})$. Using the Laplacian spectrum of $G(\mathbb{Z}_2)$ and the Laplacian spectrum of $G(\mathbb{Z}_{p_1p_2...p_k})$ that is given by Eq (3.2), the result follows in a manner similar as with the proof of Theorem 3.2. □

4. Conclusions

In this study, we discussed the degree matrix and adjacency matrix of the direct product of graphs with at most one loop at each vertex, and then we deduced a formula for the Laplacian matrix of the direct product of graphs with at most one loop at each vertex. Based on $\bar{G}(\mathbb{Z}_p \times \mathbb{Z}_q) \cong \bar{G}(\mathbb{Z}_p) \otimes \bar{G}(\mathbb{Z}_q)$, we obtained $L(G(\mathbb{Z}_{pq}))$ by using $L(\bar{G}(\mathbb{Z}_p) \otimes \bar{G}(\mathbb{Z}_q))$ after removing the matrix $M(\bar{G}(\mathbb{Z}_p)) \otimes M(\bar{G}(\mathbb{Z}_q))$ which represented the loops. So, we determined the Laplacian spectrum of $G(\mathbb{Z}_{p_1p_2...p_k})$, where p_i are distinct primes and $i = 1, 2, ..., k$. We have future plans to compute the Laplacian spectrum of $G(\mathbb{Z}_{p_1^{r_1} p_2^{r_2} ... p_k^{r_k}})$, where p_i are distinct primes, r_i are positive integers, and $i = 1, 2, ..., k$.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest regarding the publishing of this paper.

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