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Research article

H_∞ deployment of nonlinear multi-agent systems with Markov switching topologies over a finite-time interval based on T–S fuzzy PDE control

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Abstract: The deployment of multi-agent systems (MASs) is widely used in the fields of unmanned agricultural machineries, unmanned aerial vehicles, intelligent transportation, etc. To make up for the defect that the existing PDE-based results are overly idealistic in terms of system models and control strategies, we study the PDE-based deployment of clustered nonlinear first-order and second-order MASs over a finite-time interval (FTI). By designing special communication protocols, the collective dynamics of numerous agents are modeled by simple fist-order and second-order PDEs. Two practical factors, external disturbance and Markov switching topology, are considered in this paper to better match actual situations. Besides, T–S fuzzy technology is used to approximate the unknown nonlinearity of MASs. Then, by using boundary control scheme with collocated measurements, two theorems are obtained to ensure the finite-time H_∞ deployment of first-order and second-order agents, respectively. Finally, numerical examples are provided to illustrate the effectiveness of the proposed approaches.

Keywords: nonlinear MASs; H_∞ deployment; finite-time interval; fuzzy PDE-based control; Markov switching topologies
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1. Introduction

Scholars have extensively studied the cooperative control of multi-agent systems (MASs) because of their significant applications in unmanned agricultural machineries, intelligent transportation, environmental monitoring, and so on [1–4]. As a special cooperative control, deployment of MASs, whose purpose is to drive agents rearrange their positions into target spatial configurations, has recently attracted many researchers [5]. In most of existing works, the interconnected agents are usually modeled by ordinary differential equations (ODEs), which is efficient when the number of agents is small. However, with increasing complexities of practical requirements, the scale of MASs is also rapidly expanding, which leads to difficulties in the control design and theoretical analysis of ODE-based methods [6].

In order to make up for the shortcomings of ODE-based methods for large-scale MASs, some scholars have recently tried to design new methods based on partial differential equations (PDEs) for the deployment of large-scale MASs. In PDE-based approaches, a high-dimensional ODE system can be reduced to a single PDE, and this procedure is independent with the number of agents if the number is large enough [7–9]. A methodology was introduced in [10] for deployment of a large-scale MAS in 3-D space, where the agents' collective dynamics were modeled by complex-valued reaction-diffusion 2-D PDEs in polar coordinates, and the issue of stabilizing PDEs through boundary control on a disk had been successfully resolved. By supposing that agents could only obtain their positions related to only one neighbor, [11] developed a hyperbolic PDE-based approach to multi-agent deployment and proved that $L2$-stability implied the stability of the MASs for numerous enough agents. Considering network imperfections that transmission delay, variable sampling and quantization, [12] investigated the deployment of first-order agents (FOAs) and second-order agents (SOAs) over desired curves, for which a static output-feedback controller was designed and a nonlinear heat equation and a damped wave equation were constructed to represent the collective dynamics of FOAs and SOAs, respectively. In [13], the exponential deployment of second-order multiple mobile agents was investigated in 2D or 3D space, where the agents dynamics were modeled as a strongly damped wave equation (a kind of second-order PDEs) by assuming that an informed agent could measure its position related to desired curve and velocity, and other agents could obtain the local information of desired curves as well as positions and velocities related to closest neighbors.

In the context of multi-agent cooperative control, there are three crucial practical considerations that need to be addressed: Transient performance of agents, external disturbances, and topological uncertainties. First, as is known to us all, steady-state performance is what most MASs must pay attention to, but transient performance cannot be ignored either, because during the process from initial operation to stability of the system, if transient performance, such as overshoot, buffeting degree and convergence rate, cannot meet the requirements of the actual system, then the required steady-state performance cannot be achieved, and catastrophic consequences will be caused [14–16]. Second, external disturbance is a problem that actual systems have to face, because the diversification and unpredictability of the actual environment will lead to the appearance of external disturbance, if it is not dealt with, and the performance of the system will be more or less affected, whether for the steady-state performance or transient performance [17–19]. Third, in MASs, because agents are connected through the communication network that is often affected by external attacks, hardware failures, magnetic field disturbance, etc., the topologies will be randomly switched and changed, and determine the collective dynamics of MASs to a large extent [20]. The literature has extensively demonstrated that the topological switching can be effectively regulated by Markov chains [21–23].

Although PDE-based multi-agent deployment has produced some interesting results, compared to the ODE-based research, the one of PDE-based methods is in its infancy. The most obvious evidence of this is that none of the three important issues mentioned above are reflected in the existing PDE-based results, which is also an significant motivation of this paper.

Based on the above discussion, in this paper, for the PDE-based deployment of nonlinear MASs, our purpose is to introduce the Markov switching topology and external disturbance into MASs for more practicability, and construct a novel PDE-based framework that consists of finite-time control, H_∞ control, and boundary control, such that the target of finite-time H_∞ deployment of nonlinear MASs can be achieved. In short, major contributions of this paper can be summarized as follows:

- By designing a suitable communication protocol, and introducing T-S fuzzy method and continuum technique, the collective dynamic models of FOAs and SOAs in fuzzy PDE forms are constructed. As a result, the dynamics of large-scale nonlinear MASs can be expressed by a simple PDE, which could simplify analysis process and avoid high-dimensional problems caused by ODE-based approaches.
- The existing researches of PDE-based multi-agent deployment are carried out in infinite time domain, such as [11–13], they only pay attention to the steady-state performance of systems and ignore the transient performance of systems. To compensate for this shortcoming, inspired by [16,24], the deployment problem of FOAs and SOAs in FTI is studied based on a PDE method, and the effective controller design criteria are obtained under boundary control scheme.
- In the process of model construction and control design, this paper considers two practical factors that affect the performance of MASs: Topology switching and external disturbance. Inspired by [16, 17, 19, 23], we introduce Markov switching rules to describe the switching phenomena of topologies, and design H_∞ control strategy to eliminate the influence of external disturbance. Compared with existing results on PDE-based deployment, the model and control strategy in this paper are more consistent with the requirements of actual systems.

Organizations: This paper mainly consists of the following five sections: The research background, motivations, and contributions are introduced in Section 1. Section 2 describes the considered FOAs and SOAs, and shows the PDE modeling process of their collective dynamics. Moreover, the design of controllers, and necessary definitions and a lemma are also included in Section 2. Section 3 gives two finite-time H_∞ deployment criteria and their detailed proofs for FOAs and SOAs under the designed communication and control schemes. Section 4 provides two numerical examples for Theorems 1 and 2 to verify the effectiveness of the developed approaches. Finally, our main work is summarized in Section 5.

2. Problem formulation

Notations: In this paper, $\mathbb{R}$ is the real number set; $\mathbb{R}^n$ is n-dimensional real-number space; C^1 and C^2 represent first-order and second-order continuous function clusters, respectively; $*$ is the omission of symmetric elements in a symmetric matrix; $\mathbb{E}(\cdot)$ is the operator of mathematical expectation; $\mathcal{H}^2(0, L)$ is a real Hilbert space, where for a square integrable function $g(x) : [0, L] \to \mathbb{R}$, its inner product induced norm is calculated by $\|g(x)\|_{\mathcal{H}^2} = \sqrt{\int_0^L g^2(x)dx}$; For a function of two variables $e(t, x) \in C^2$, we denote $e_t(t, x) = \frac{\partial e(t,x)}{\partial t}$, $e_x(t, x) = \frac{\partial e(t,x)}{\partial x}$, $e_{xx}(t, x) = \frac{\partial^2 e(t,x)}{\partial x^2}$.

In this paper, we consider a group of N agents that is governed by first-order or second-order dynamics. These agents could move in space $\mathbb{R}^n$, ($n \in \{2, 3\}$), and initially locate on a C^1 curve $\Gamma_0 : [0, L] \to \mathbb{R}^n$. Our objective is to deploy the agents onto the desired C^1 or C^2 curve $\Gamma : [0, L] \to \mathbb{R}^n$.

Assumption 1: The distributions of agents on spatial curves are uniform, that is, we choose N points on the curves, denoted by $\Gamma_0(\vartheta), \Gamma_0(2\vartheta), ..., \Gamma_0(N\vartheta)$ and $\Gamma(\vartheta), \Gamma(2\vartheta), ..., \Gamma(N\vartheta)$ with $\vartheta = L/N$. The curves Γ_0 and Γ are without intersections.

Remark 1: The curves Γ_0 and Γ are continues in $\mathbb{R}^2$ or $\mathbb{R}^3$, which means that the agent formation we considered is general, because distribution of agents in space does not affect the existence of a continuous curve Γ_0 or Γ that can connect them consecutively.

Assumption 2: For the sake of simplicity, we suppose that the desired curve remains static over time and there are no obstacles, either stationary or in motion, present within the operational workspace. Furthermore, we assume that the agents have negligible volume and operate within an extensive workspace.

Assumption 3: The agents could obtain the local information of desired curves, and they could access to their positions relative to their nearest neighbors. For the second-order agents, their velocities can be also obtained by themselves. Moreover, the designed informed agent is capable of measuring its absolute position.

Define $\{\varrho_t\}$ as a continuous-time Markov process, which has right continuous trajectories and takes values in a finite set $\mathcal{M} = 1, 2, ..., M$. The transition probability matrix $\Lambda \triangleq \{\chi_{\delta\varpi}\}$ is governed by

$$\Pr\{\varrho_{t+\Delta} = \varpi \,|\varrho_t = \delta\} = \begin{cases} \chi_{\delta\varpi}\Delta + o(\Delta), & \delta \neq \varpi, \\ 1 + \chi_{\delta\delta}\Delta + o(\Delta), & \delta = \varpi, \end{cases}$$

where $\Pr\{\varrho_{t+\Delta} = \varpi \,|\varrho_t = \delta\}$ is the probability of occurrence of $\varrho_{t+\Delta} = \varpi$ under the condition of $\varrho_t = \delta$; $\Delta > 0$, $\lim\limits_{\Delta\to 0} o(\Delta)/\Delta = 0$; $\chi_{\delta\varpi} \geq 0$ for $\delta \neq \varpi$ is the transition rate and $\chi_{\delta\delta} = -\sum\limits_{\varpi\in\mathcal{M},\delta\neq\varpi} \chi_{\delta\varpi}$.

2.1. *For first-order agents (FOAs)*

In this case, the dynamics of each agent is described as

$$\frac{dm_i^j(t)}{dt} = v_i^j(t) + f^j(m_i^j(t)) + \omega_i^j(t), \tag{2.1}$$

where the subscript i represents the serial number of agents, and $i = 1, 2, ..., N$; The superscript j represents the serial number of spatial axis, and $j = 1, 2, 3$; m_i^j is the absolute position of ith agent on jth axis; v_i^j is the communication protocol related to ith agent to be designed; $f^j(m_i)$ is a nonlinear function that is sufficiently smooth; $\omega_i^j(t)$ represents the external disturbance to the ith agent and satisfies $\int_0^T (\omega_i^j(t))^2 dt \leq \mathcal{W}$ with $T > 0$.

By using the local sector nonlinearity method, the T-S fuzzy model with η fuzzy rules accurately representing the system (2.1) can be represented below.

Model Rule ε: IF $\xi_{i1}^j(t)$ is $F_{\varepsilon 1}^j$ and ... and $\xi_{ip}^j(t)$ is $F_{\varepsilon p}^j$, THEN,

$$\frac{dm_i^j(t)}{dt} = v_i^j(t) + \gamma_\varepsilon^j m_i^j(t) + \omega_i^j(t), \tag{2.2}$$

where premise variables $\xi_i(t) = [\xi_{i1}(t), \xi_{i2}(t), ..., \xi_{ip}(t)]$ are functions of $m_i^j(t)$ and $\varepsilon = 1, 2, ..., \eta$; $F_{\varepsilon p}^j$ is the subordinating degree function; γ_ε^j is a known constant. Then, the combination of product fuzzy reasoning, weighted average defuzzification, and singleton fuzzier methods allows for a comprehensive description of the overall fuzzy dynamics of system (2.3) as follows:

$$\frac{dm_i^j(t)}{dt} = v_i^j(t) + \sum_{\varepsilon=1}^{\eta} h_\varepsilon^j(\xi_i(t))\gamma_\varepsilon^j m_i^j(t) + \omega_i^j(t), \tag{2.3}$$

where

$$h_\varepsilon^j(\xi_i(t)) = \beta_\varepsilon^j(\xi_i(t))/\sum_{\varepsilon=1}^{\eta}\beta_\varepsilon^j(\xi_i(t)), \quad \beta_\varepsilon^j(\xi_i(t)) = \prod_{w=1}^{p} F_{\varepsilon w}^j(\xi_{iw}(t)),$$

and $\beta_\varepsilon^j(\xi_i(t)) \geq 0$, $\varepsilon = 1, 2, ..., \eta$, are implicit fuzzy sets; and $h_\varepsilon^j(\xi_i(t)) \geq 0$, $\sum_{\varepsilon=1}^{\eta} h_\varepsilon^j(\xi_i(t)) = 1$.

For brevity, the superscript j will be omitted in subsequent descriptions.

Remark 2: Most of papers on PDE-based deployment of nonlinear MASs suppose that the nonlinear function $f(\cdot)$ satisfies Lipschitz condition or other inequality conditions, which undoubtedly has limitations. Here, we explore the incorporation of T-S fuzzy rules to address nonlinearity, with the sole requirement being its sufficient smoothness. This operation could relax the restriction on the nonlinear function. Moreover, previous results on PDE-based deployment are obtained based on ideal external conditions, and the external disturbance $\omega_i(t)$ is considered for the first time in this paper. Therefore, the MASs considered in this paper are more general and practical than those in previous PDE-based studies, such as [7–13].

Based on Assumptions 1, 2, and 3, we first design the following Markovian communication protocol of agents:

$$\begin{cases} v_1(t) = \dfrac{a_\delta}{\vartheta^2}[m_2(t) - m_1(t)] - \dfrac{a_\delta}{\vartheta^2}[\Gamma(2\vartheta) - \Gamma(\vartheta)] - \displaystyle\sum_{\varepsilon=1}^{\eta} h_\varepsilon(\xi_1(t))\gamma_\varepsilon\Gamma(\vartheta), \\ v_i(t) = \dfrac{a_\delta}{\vartheta^2}[m_{i+1}(t) - 2m_i(t) + m_{i-1}(t)] - \dfrac{a_\delta}{\vartheta^2}[\Gamma(i\vartheta + \vartheta) - 2\Gamma(i\vartheta) + \Gamma(i\vartheta - \vartheta)] \\ \qquad - \displaystyle\sum_{\varepsilon=1}^{\eta} h_\varepsilon(\xi_i(t))\gamma_\varepsilon\Gamma(i\vartheta), \quad i = 2, 3, ..., N-1, \\ v_N(t) = \dfrac{a_\delta}{\vartheta^2}[m_{N-1}(t) - m_N(t)] - \dfrac{a_\delta}{\vartheta^2}[\Gamma((N-1)\vartheta) - \Gamma(N\vartheta)] - \displaystyle\sum_{\varepsilon=1}^{\eta} h_\varepsilon(\xi_N(t))\gamma_\varepsilon\Gamma(N\vartheta), \end{cases} \tag{2.4}$$

where a_δ is a designed switching topological weight, and δ represents the mode of Markov process at time t.

Define position error as $e_i(t) = m_i(t) - \Gamma(i\vartheta)$, then, integrating T–S fuzzy MASs (2.3) and communication protocol (2.4) generates the following error system:

$$\begin{cases} \dfrac{de_1(t)}{dt} = \dfrac{a_\delta}{\vartheta^2}[e_2(t) - e_1(t)] + \displaystyle\sum_{\varepsilon=1}^{\eta} h_\varepsilon(\xi_1(t))\gamma_\varepsilon e_1(t) + \omega_1(t), \\ \dfrac{de_i(t)}{dt} = \dfrac{a_\delta}{\vartheta^2}[e_{i+1}(t) - 2e_i(t) + e_{i-1}(t)] + \displaystyle\sum_{\varepsilon=1}^{\eta} h_\varepsilon(\xi_i(t))\gamma_\varepsilon e_i(t) + \omega_i(t), \quad i = 2, 3, ..., N-1, \\ \dfrac{de_N(t)}{dt} = \dfrac{a_\delta}{\vartheta^2}[e_{N-1}(t) - e_N(t)] + \displaystyle\sum_{\varepsilon=1}^{\eta} h_\varepsilon(\xi_N(t))\gamma_\varepsilon e_N(t) + \omega_N(t). \end{cases} \tag{2.5}$$

As suggested in [7] and [12], according to Euler dispersion rule, if the number of agents is sufficient, error system (2.5) can be regarded as the discretization of the follwing first-order PDE system:

$$e_t(t,x) = a_\delta e_{xx}(t,x) + \sum_{\varepsilon=1}^{\eta} h_\varepsilon(\xi(t,x))\gamma_\varepsilon e(t,x) + \omega(t,x), \tag{2.6}$$

where $x \in (0, L)$ represents the position of agents mapping to a coordinate axis, and $e(t,x) \in \mathcal{H}^2(0, L)$.

In order to realize the deployment of MAS (2.3) over a FTI, an anchor and an informed agent are set at both ends of the curve where agents is located, and their sequence numbers are 0 and $N + 1$, respectively. Dynamics of the anchor and informed agent are designed as

$$m_0(t) = \Xi(0), \tag{2.7}$$
$$m_{N+1}(t) = \vartheta u(t) + m_N(t), \tag{2.8}$$

where $u(t)$ is a fuzzy control scheme to be designed, which is given below.

Control Rule β**:** IF $\xi^j_{(N+1)1}(t)$ is $F^j_{\beta 1}$ and ... and $\xi^j_{(N+1)p}(t)$ is $F^j_{\beta p}$, THEN,

$$u(t) = k_{\delta\beta}\left[m_{N+1}(t) - \Gamma((N+1)\vartheta)\right], \tag{2.9}$$

where $\beta = 1, 2, ..., \eta$, $k_{\delta\beta} < 0$ is the controller gain to be determined. Similar to (2.3), the overall control law can be obtained as

$$u(t) = \sum_{\beta=1}^{\eta} h_\beta(\xi_{N+1}(t))k_{\delta\beta}\left[m_{N+1}(t) - \Gamma((N+1)\vartheta)\right]. \tag{2.10}$$

The proposal of (2.7–2.10) implies that boundary conditions of PDE system (2.6) are of the Neumann type, specifically,

$$e(t,0) = 0, \tag{2.11}$$
$$\left.\frac{\partial e(t,x)}{\partial x}\right|_{x=L} = \sum_{\beta=1}^{\eta} h_\beta(\xi(t,L))k_{\delta\beta} e(t,L). \tag{2.12}$$

2.2. For second-order agents (SOAs)

In this case, the dynamics of each agent is governed by

$$\frac{d^2 m_i(t)}{dt^2} = \tilde{v}_i(t) + f(m_i(t)) + \omega_i(t), \tag{2.13}$$

where $\tilde{v}_i$ is the communication protocol to be designed. The other symbols have same meanings as those defined in (2.1).

Similar to discussions for FOAs, the overall T-S fuzzy model with η fuzzy rules accurately representing the system (2.13) can be obtained as follows:

$$\frac{d^2 m_i(t)}{dt^2} = \tilde{v}_i(t) + \sum_{\varepsilon=1}^{\eta} h_\varepsilon(\xi_i(t))\gamma_\varepsilon m_i(t) + \omega_i(t). \tag{2.14}$$

The Markov switching communication protocol $\tilde{v}_i(t)$ is designed as

$$\begin{cases}\tilde{v}_1(t)=\frac{a_\delta}{\vartheta^2}\left[m_2(t)-m_1(t)\right]-\frac{a_\delta}{\vartheta^2}\left[\Gamma(2\vartheta)-\Gamma(\vartheta)\right]-\sum_{\varepsilon=1}^{\eta}h_\varepsilon(\xi_1(t))\gamma_\varepsilon\Gamma(\vartheta)-\alpha\frac{dm_1(t)}{dt},\\ \tilde{v}_i(t)=\frac{a_\delta}{\vartheta^2}\left[m_{i+1}(t)-2m_i(t)+m_{i-1}(t)\right]-\frac{a_\delta}{\vartheta^2}\left[\Gamma(i\vartheta+\vartheta)-2\Gamma(i\vartheta)+\Gamma(i\vartheta-\vartheta)\right]\\ \qquad-\sum_{\varepsilon=1}^{\eta}h_\varepsilon(\xi_i(t))\gamma_\varepsilon\Gamma(i\vartheta)-\alpha\frac{dm_i(t)}{dt},\quad i=2,3,...,N-1,\\ \tilde{v}_N(t)=\frac{a_\delta}{\vartheta^2}\left[m_{N-1}(t)-m_N(t)\right]-\frac{a_\delta}{\vartheta^2}\left[\Gamma((N-1)\vartheta)-\Gamma(N\vartheta)\right]\\ \qquad-\sum_{\varepsilon=1}^{\eta}h_\varepsilon(\xi_N(t))\gamma_\varepsilon\Gamma(N\vartheta)-\alpha\frac{dm_N(t)}{dt},\end{cases}\tag{2.15}$$

with which the error system $e_i(t)=m_i(t)-\Gamma(i\vartheta)$ can be derived as

$$\begin{cases}\frac{d^2e_1(t)}{dt^2}=\frac{a_\delta}{\vartheta^2}\left[e_2(t)-e_1(t)\right]+\sum_{\varepsilon=1}^{\eta}h_\varepsilon(\xi_1(t))\gamma_\varepsilon e_1(t)-\alpha\frac{de_1(t)}{dt}+\omega_1(t),\\ \frac{d^2e_i(t)}{dt^2}=\frac{a_\delta}{\vartheta^2}\left[e_{i+1}(t)-2e_i(t)+e_{i-1}(t)\right]+\sum_{\varepsilon=1}^{\eta}h_\varepsilon(\xi_i(t))\gamma_\varepsilon e_i(t)\\ \qquad-\alpha\frac{de_i(t)}{dt}+\omega_i(t),\quad i=2,3,...,N-1,\\ \frac{d^2e_N(t)}{dt^2}=\frac{a_\delta}{\vartheta^2}\left[e_{N-1}(t)-e_N(t)\right]+\sum_{\varepsilon=1}^{\eta}h_\varepsilon(\xi_N(t))\gamma_\varepsilon e_N(t)\ -\alpha\frac{de_N(t)}{dt}+\omega_N(t).\end{cases}\tag{2.16}$$

Then, as mentioned in [12] and [13], according to Euler dispersion rule, if the number of agents is sufficient, error system (2.16) can be regarded as the discretization of the follwing second-order PDE system:

$$e_{tt}(t,x)=a_\delta e_{xx}(t,x)+\sum_{\varepsilon=1}^{\eta}h_\varepsilon(\xi(t,x))\gamma_\varepsilon e(t,x)-\alpha e_t(t,x)+\omega(t,x),\quad x\in(0,L).\tag{2.17}$$

For the convenience of subsequent processing, define $\psi(t,x)=e_t(t,x)+\theta e(t,x)$, then the following first-order system equivalent to (2.17) can be derived:

$$\begin{cases}e_t(t,x)=\psi(t,x)-\theta e(t,x)\\ \psi_t(t,x)=a_\delta e_{xx}(t,x)+\sum_{\varepsilon=1}^{\eta}h_\varepsilon(\xi(t,x))\gamma_\varepsilon e(t,x)-(\alpha-\theta)e_t(t,x)+\omega(t,x).\end{cases}\tag{2.18}$$

Moreover, for SOAs, dynamics of the anchor and informed agent, numbered as 0 and $N+1$, are governed as

$$m_0(t)=\Xi(0),\tag{2.19}$$

$$m_{N+1}(t)=\vartheta u_1(t)+m_N(t),\quad \frac{dm_{N+1}(t)}{dt}=u_2(t),\tag{2.20}$$

where $u_1(t)$ and $u_2(t)$ are fuzzy control schemes that are designed as

Control Rule $\hat{\theta}$: IF $\xi_{(N+1)1}(t)$ is $F_{\hat{\theta}1}$ and ... and $\xi_{(N+1)p}(t)$ is $F_{\hat{\theta}p}$, THEN,

$$u_1(t) = k_{1\delta\hat{\theta}}\left[m_{N+1}(t) - \Gamma((N+1)\vartheta)\right], \tag{2.21}$$

Control Rule $\tilde{\theta}$: IF $\xi_{(N+1)1}(t)$ is $F_{\tilde{\theta}1}$ and ... and $\xi_{(N+1)p}(t)$ is $F_{\tilde{\theta}p}$, THEN,

$$u_2(t) = k_{2\delta\tilde{\theta}}\left[m_{N+1}(t) - \Gamma((N+1)\vartheta)\right], \tag{2.22}$$

where $\hat{\theta} = 1, 2, ..., \eta$ and $\tilde{\theta} = 1, 2, ..., \eta$; $k_{1\delta\hat{\theta}} < 0$ and $k_{2\delta\tilde{\theta}} < 0$ are controller gains to be determined. Similar to (2.3), the overall fuzzy control laws can be obtained as

$$u_1(t) = \sum_{\hat{\theta}=1}^{\eta} h_{\hat{\theta}}(\xi_{N+1}(t))k_{1\delta\hat{\theta}}\left[m_{N+1}(t) - \Gamma((N+1)\vartheta)\right], \tag{2.23}$$

$$u_2(t) = \sum_{\tilde{\theta}=1}^{\eta} h_{\tilde{\theta}}(\xi_{N+1}(t))k_{2\delta\tilde{\theta}}\left[m_{N+1}(t) - \Gamma((N+1)\vartheta)\right]. \tag{2.24}$$

The proposal of (2.19–2.24) implies that boundary conditions of second-order PDE system (2.17) are of the Neumann type, specifically,

$$e(t, 0) = 0, \tag{2.25}$$

$$\left.\frac{\partial e(t, x)}{\partial z}\right|_{z=L} = \sum_{\hat{\theta}=1}^{\eta} h_{\hat{\theta}}(\xi(t, L))k_{1\delta\hat{\theta}}e(t, L), \tag{2.26}$$

$$\frac{\partial e(t, L)}{\partial t} = \sum_{\tilde{\theta}=1}^{\eta} h_{\tilde{\theta}}(\xi(t, L))k_{2\delta\tilde{\theta}}e(t, L). \tag{2.27}$$

Remark 3: In some papers on PDE-based deployment of MASs, such as [7, 12, 13], the location information of informed agents (or leaders) needs to be sent to all other agents, which poses a challenge to the agents' communication ability. In order to save communication resources and reduce the difficulty of communication, the communication protocols (2.4) and (2.15) designed in this paper only need to send the location information of the informed agent to the nearest agent, which improves the realization of communication protocols.

Remark 4: The communication protocol $\tilde{v}_i(t)$ designed in (2.15) implies that compared with the communication ability of FOAs, the SOAs not only need to obtain the relative positions of their neighbors, but also need to obtain their own speed information, because the dynamics of SOAs are determined by their accelerations, so the speed will also cause a non-negligible impact on them [12, 13].

Remark 5: The principle of T–S fuzzy method in this paper and [25–27] is the same, but there are two differences: On the one hand, the nonlinear functions in [25–27] are fixed, so the corresponding premise variables and fuzzy sets are also fixed, while the nonlinear function considered in this paper is general, so the corresponding premise variables and fuzzy sets are variable. However, it should be pointed out that in our simulation, involving specific nonlinear functions, it is necessary to concretify the premise variables and fuzzy sets. Therefore, the T–S fuzzy methods in [25–27] could be regarded as special cases in this paper. On the other hand, the systems targeted by [25–27] can be viewed as a single concrete agent, and this paper considers the MASs, so this paper needs to design fuzzy rules for each agent. In addition, due to the PDE method adopted in this paper, in the subsequent processing, the premise variable are transformed into the spatiotemporal related, that is, $\xi(t,x)$, which is quite different from these two articles in terms of form and treatment.

Definition 1 [28]: Given three positive constants c_1, c_2 and T, for T–S fuzzy PDE systems (2.6) and (2.18), if the following condition is satisfied:

$$\mathbb{E}\,\|e(0,x)\|^2_{\mathcal{H}^2} \le c_1 \Longrightarrow \mathbb{E}\,\|e(t,x)\|^2_{\mathcal{H}^2} \le c_2,\quad \forall t \in (0,T],$$

where $0 < c_1 < c_2$, the PDE systems (2.6) and (2.18) can be mean-square stochastically bounded over a FTI with respect to (c_1, c_2, T).

In this paper, we suppose the outputs $o_i(t)$ of considered FOAs and SOAs are their absolute positions, that is $o_i(t) = m_i(t)$, then the measurement output errors are $e_i^o(t) = e_i(t)$, which maps to PDEs is $e^o(t,x) = e(t,x)$.

Definition 2 [29]: Given a scalar $\lambda > 0$, the PDE systems (2.6) and (2.18) are said to be mean-square stochastically H_∞ bounded over a FTI, if (i) the PDE systems (2.6) and (2.18) can be mean-square stochastically finite-time bounded in the sense of Definition 1, and (ii) the measurement output error $e^o(t,x)$ satisfies following condition under zero initial state:

$$\mathbb{E}\int_0^\infty\int_0^L \{[e^o(t,x)]^2 - \lambda^2[\omega(t,x)]^2\}dxdt < 0.$$

Lemma 1 [30]: For a scalar function $w(x) \in \mathcal{H}^2(0,L)$ and a $n \times n$ positive matrix X, we have

$$\int_0^L w^T(x)Xw(x)dx \le \frac{4L^2}{\pi^2}\int_0^L \frac{dw^T(x)}{dx}X\frac{dw(x)}{dx}dx$$

if $w(0) = 0$ or $w(L) = 0$.

3. Main results

3.1. H_∞ deployment of FOAs over a FTI

Theorem 1: With a positive scalar w_δ, $(\delta = 1,2,...M)$, and negative scalar $\psi_{\delta\beta}$, $(\beta = 1,2,...,\eta)$, and appropriate system parameters a_δ, γ_ε, $(\varepsilon = 1,2,...,\eta)$, τ and λ, under T–S fuzzy control scheme (2.10), the first-order PDE system (2.6) can be mean-square stochastically H_∞ bounded over a FTI with respect to (c_1, c_2, T) if the following linear matrix inequalities (LMIs) hold:

$$\Phi_{\delta\varepsilon\beta} = \begin{bmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ * & \varphi_{22} & \varphi_{23} \\ * & * & \varphi_{33} \end{bmatrix} < 0, \tag{3.1}$$

and

$$\frac{c_1\exp(\tau T)\max\limits_{\delta\in\mathcal{M}}(w_\delta) + \lambda^2\exp(\tau T)L\mathcal{W}}{\min\limits_{\delta\in\mathcal{M}}(w_\delta)} < c_2, \tag{3.2}$$

where

$$\varphi_{11} = \sum_{\varpi=1}^{M} \chi_{\delta\varpi} w_{\varpi} + 2w_\delta \gamma_\varepsilon - \frac{w_\delta a_\delta \pi^2}{2L^2} - \tau w_\delta + 1, \ \ \varphi_{12} = \frac{w_\delta a_\delta \pi^2}{2L^2}, \ \ \varphi_{13} = w_\delta,$$
$$\varphi_{22} = \frac{2a_\delta \psi_{\delta\beta}}{L} - \frac{w_\delta a_\delta \pi^2}{2L^2}, \ \ \varphi_{23} = 0, \ \ \varphi_{33} = -\lambda^2, \ \ \mathbb{E}\,\|e(0,x)\|^2_{\mathcal{H}^2} \le c_1,$$

and the controller gains can be calculated as $k_{\delta\beta} = \psi_{\delta\beta}/w_\delta$.
Proof. For the first-order PDE system (2.6), we construct the following Lyapunov function:

$$V_1(\delta,t) = \int_0^L w_\delta e^2(t,x)dx. \tag{3.3}$$

Then, based on [31–33] and (2.6), the weak differential of $V_1(\delta,t)$ can be obtained as

$$\begin{aligned} \mathcal{L}V_1(\delta,t) &= \int_0^L \sum_{\varpi=1}^{M} \chi_{\delta\varpi} w_{\varpi} e^2(t,x)dx + 2\int_0^L e(t,x) w_\delta e_t(t,x)dx \\ &= \int_0^L \sum_{\varpi=1}^{M} \chi_{\delta\varpi} w_{\varpi} e^2(t,x)dx + 2\int_0^L e(t,x) w_\delta [a_\delta e_{xx}(t,x) \\ &\quad + \sum_{\varepsilon=1}^{\eta} h_\varepsilon(\xi(t,x))\gamma_\varepsilon e(t,x) + \omega(t,x)]dx, \end{aligned} \tag{3.4}$$

where $\mathcal{L}$ is a weak differential operator.

For the item $2\int_0^L e(t,x)w_\delta a_\delta e_{xx}(t,x)dx$, using integration by parts and boundary conditions (2.11) and (2.12) yields

$$\begin{aligned} &2\int_0^L e(t,x) w_\delta a_\delta e_{xx}(t,x)dx = 2w_\delta a_\delta \int_0^L e(t,x)de_x(t,x) \\ &= 2w_\delta a_\delta e(t,x)e_x(t,x)|_0^L - 2w_\delta a_\delta \int_0^L (e_x(t,x))^2 dx \\ &\le 2w_\delta a_\delta \sum_{\beta=1}^{\eta} h_\beta(\xi(t,L))k_{\delta\beta} e^2(t,L) - 2w_\delta a_\delta \int_0^L e_x^2(t,x)dx. \end{aligned} \tag{3.5}$$

Define $\tilde{e}(t,x) = e(t,x) - e(t,L)$, then we have $\tilde{e}_x(t,x) = e_x(t,x)$ and $\tilde{e}(t,L) = 0$, which creates conditions for the use of Lemma 1, therefore, it can be obtained that

$$-2w_\delta a_\delta \int_0^L e_x^2(t,x)dx = 2w_\delta a_\delta \int_0^L \tilde{e}_x^2(t,x)dx \le -\frac{w_\delta a_\delta \pi^2}{2L^2} \int_0^L \tilde{e}^2(t,x)dx. \tag{3.6}$$

The above calculations show that the following inequality is correct:

$$\mathcal{L}V_1(\delta,t) - \tau V_1(\delta,t) + \int_0^L [e^2(t,x) - \lambda^2 \omega^2(t,x)]dx$$

$$
\begin{aligned}
&\leq \int_0^L \sum_{\varpi=1}^{M} \chi_{\delta\varpi} w_{\varpi} e^2(t,x)dx + 2\int_0^L e(t,x) w_\delta [\sum_{\varepsilon=1}^{\eta} h_\varepsilon(\xi(t,x))\gamma_\varepsilon e(t,x) + \omega(t,x)]dx \\
&\quad + 2w_\delta a_\delta \sum_{\beta=1}^{\eta} h_\beta(\xi(t,L)) k_{\delta\beta} e^2(t,L) - \tau \int_0^L w_\delta e^2(t,x)dx \\
&\quad - \frac{w_\delta a_\delta \pi^2}{2L^2} \int_0^L [e(t,x) - e(t,L)]^2 dx + \int_0^L [e^2(t,x) - \lambda^2 \omega^2(t,x)]dx \\
&= \int_0^L \sum_{\varepsilon=1}^{\eta} \sum_{\beta=1}^{\eta} h_\varepsilon(\xi(t,x)) h_\beta(\xi(t,L)) \zeta_1^T \Phi_{\delta\varepsilon\beta} \zeta_1 dx, \qquad (3.7)
\end{aligned}
$$

where $\zeta_2 = [e(t,x), e(t,L), \omega(t,x)]^T$, and $\Phi_{\delta\varepsilon\theta}$ has been defined in (3.1).

Therefore, if the condition (3.1) is satisfied, it is obvious that

$$
\mathcal{L}V_1(\delta,t) < \tau V_1(\delta,t) - \int_0^L [e^2(t,x) - \lambda^2 \omega^2(t,x)]dx. \qquad (3.8)
$$

Multiplying inequality (3.8) by $\exp(-\tau t)$, we have

$$
\begin{aligned}
\frac{d}{dt}[\exp(-\tau t) V_1(\delta,t)] &< -\exp(-\tau t)\int_0^L e^2(t,x)dx + \lambda^2 \exp(-\tau t) \int_0^L \omega^2(t,x)dx \\
&\leq \lambda^2 \exp(-\tau t) \int_0^L \omega^2(t,x)dx. \qquad (3.9)
\end{aligned}
$$

Since $\int_0^T \omega_i^2(t)dt \leq \mathcal{W}$, we get $\int_0^T \int_0^L \omega^2(t,x)dxdt \leq L\mathcal{W}$, and by integrating (3.9) between 0 and T, one can obtain readily that

$$
\begin{aligned}
\mathbb{E}V_1(\delta,t) &< \exp(\tau T)\mathbb{E}V_1(\varrho(0),0) + \lambda^2 \exp(\tau T) \int_0^T \int_0^L \exp(-\tau t)\omega^2(t,x)dxdt \\
&\leq \exp(\tau T)\mathbb{E}V_1(\varrho(0),0) + \lambda^2 \exp(\tau T) L\mathcal{W}. \qquad (3.10)
\end{aligned}
$$

According to the designed Lyapunov function (3.3), one can derive

$$
\mathbb{E}V_1(\delta,t) = \mathbb{E}\int_0^L w_\delta e^2(t,x)dx \geq \min_{\delta\in\mathcal{M}}(w_\delta)\mathbb{E}\,\|e(t,x)\|^2_{\mathcal{H}^2}. \qquad (3.11)
$$

If $\mathbb{E}\,\|e(0,x)\|^2_{\mathcal{H}^2} < c_1$, it is true that

$$
\mathbb{E}V_1(\varrho(0),0) = \mathbb{E}\int_0^L w(\varrho(0))e^2(0,x)dx \leq \max_{\delta\in\mathcal{M}}(w_\delta)\mathbb{E}\,\|e(0,x)\|^2_{\mathcal{H}^2} \leq c_1 \max_{\delta\in\mathcal{M}}(w_\delta), \qquad (3.12)
$$

and

$$
\mathbb{E}\,\|e(t,x)\|^2_{\mathcal{H}^2} \leq \frac{c_1 \exp(\tau T)\max_{\delta\in\mathcal{M}}(w_\delta) + \lambda^2 \exp(\tau T) L\mathcal{W}}{\min_{\delta\in\mathcal{M}}(w_\delta)} < c_2. \qquad (3.13)
$$

As a result, according to Definition 1, the PDE system (2.6) can be mean-square stochastically bounded over a FTI with respect to (c_1, c_2, T).

On the other hand, under zero initial state, integrate (3.9) between 0 and T and taking the expectation, we get

$$\begin{aligned}\mathbb{E}V_1(\delta,t) <& -\exp(\tau T)\mathbb{E}\int_0^T\int_0^L \exp(-\tau t)e^2(t,x)dxdt + \lambda^2\exp(\tau T)\mathbb{E}\int_0^T\int_0^L \exp(-\tau t)\omega^2(t,x)dxdt\\ \le& -\mathbb{E}\int_0^T\int_0^L e^2(t,x)dxdt + \lambda^2\exp(\tau T)\mathbb{E}\int_0^T\int_0^L \omega^2(t,x)dxdt, \end{aligned}\tag{3.14}$$

which means that

$$\begin{aligned}\mathbb{E}\int_0^T\int_0^L e^2(t,x)dxdt <& \lambda^2\exp(\tau T)\mathbb{E}\int_0^T\int_0^L \omega^2(t,x)dxdt\\ =& \tilde{\lambda}^2\mathbb{E}\int_0^T\int_0^L \omega^2(t,x)dxdt,\end{aligned}\tag{3.15}$$

where

$$\tilde{\lambda} = \lambda\sqrt{\exp(\tau T)}.$$

Therefore, the first-order PDE system (2.6) can be mean-square stochastically H_∞ bounded over a FTI under T–S fuzzy control scheme (2.10) according to Definition 2. This completes the proof of Theorem 1.

3.2. H_∞ deployment of SOAs over a FTI

Theorem 2: With a positive scalars w_δ, q_δ, $\tilde{\psi}_{\delta\hat{\theta}}$ and $\hat{\psi}_{\delta\hat{\theta}\tilde{\theta}}$, $(\delta = 1,2,...M; \hat{\theta} = 1,2,...,\eta; \tilde{\theta} = 1,2,...,\eta)$, arbitrary scalars ρ_ν, $(\nu = 1,2,3,4,5,6)$, and appropriate system parameters a_δ, γ_ε, $(\varepsilon = 1,2,...,\eta)$, θ, τ and λ, under T–S fuzzy control scheme (2.23) and (2.24), the second-order PDE system (2.18) can be mean-square stochastically H_∞ bounded over a FTI with respect to (c_1, c_2, T), if the following LMIs hold:

$$\Psi_{\delta\varepsilon\hat{\theta}\tilde{\theta}} = \begin{bmatrix} \mu_{11} & \mu_{12} & \mu_{13} & \mu_{14} & \mu_{15} & \mu_{16}\\ * & \mu_{22} & \mu_{23} & \mu_{24} & \mu_{25} & \mu_{26}\\ * & * & \mu_{33} & \mu_{34} & \mu_{35} & \mu_{36}\\ * & * & * & \mu_{44} & \mu_{45} & \mu_{46}\\ * & * & * & * & \mu_{55} & \mu_{56}\\ * & * & * & * & * & \mu_{66}\end{bmatrix} < 0, \tag{3.16}$$

and

$$\frac{c_1\exp(\tau T)\max\limits_{\delta\in\mathcal{M}}(w_\delta, q_\delta) + \lambda^2\exp(\tau T)L\mathcal{W}}{\min\limits_{\delta\in\mathcal{M}}(w_\delta)} < c_2 \tag{3.17}$$

where

$$\mu_{11}=\sum_{\varpi=1}^{M}\chi_{\delta\varpi}w_{\varpi}-2w_{\delta}\theta+\sum_{\varpi=1}^{M}\chi_{\delta\varpi}q_{\varpi}-\rho_3\theta,\;\;\mu_{12}=w_{\delta}+q_{\delta}\gamma_{\varepsilon}+\rho_1\theta+\rho_3,\;\;\mu_{13}=\rho_4\theta,\;\;\mu_{14}=\rho_5\theta,$$
$$\mu_{15}=-\rho_2\theta-\rho_3,\;\;\mu_{16}=\rho_6\theta,\;\;\mu_{22}=-2\rho_1,\;\;\mu_{23}=-\rho_4,\;\;\mu_{24}=-\rho_5,\;\;\mu_{25}=\rho_1+\rho_2-(\alpha-\theta),$$
$$\mu_{26}=q_{\delta}-\rho_6,\;\;\mu_{33}=\sum_{\varpi=1}^{M}\chi_{\delta\varpi}a_{\varpi}-2a_{\delta}q_{\delta}\theta,\;\;\mu_{34}=0,\;\;\mu_{35}=\rho_4,\;\;\mu_{44}=2a_{\delta}\hat{\psi}_{\delta\hat{\theta}\tilde{\theta}}+2a_{\delta}\theta\tilde{\psi}_{\delta\hat{\theta}},$$
$$\mu_{45}=\rho_5,\;\;\mu_{46}=0,\;\;\mu_{36}=0,\;\;\mu_{55}=-2\rho_2,\;\;\mu_{56}=\rho_6,\;\;\mu_{66}=-\lambda^2,$$
$$c_1\geq\max(\mathbb{E}\,\|e(0,x)\|^2_{\mathcal{H}^2},\mathbb{E}\,\|\psi(0,x)\|^2_{\mathcal{H}^2}),$$

and the controller gains can be calculated as $k_{1\delta\hat{\theta}}=\tilde{\psi}_{\delta\hat{\theta}}/q_{\delta}$ and $k_{2\delta\tilde{\theta}}=\hat{\psi}_{\delta\hat{\theta}\tilde{\theta}}/(q_{\delta}k_{1\delta\hat{\theta}})$.
Proof. For the second-order PDE system (2.18), we construct the following Lyapunov function:

$$V_2(\delta,t)=\int_0^L w_{\delta}e^2(t,x)dx+\int_0^L q_{\delta}\psi^2(t,x)dx+\int_0^L a_{\delta}e_x^2(t,x)dx. \tag{3.18}$$

Similarly, the weak differential of $V_2(\delta,t)$ can be obtained as follows:

$$\begin{aligned}\mathcal{L}V_2(\delta,t)=&\int_0^L\sum_{\varpi=1}^{M}\chi_{\delta\varpi}w_{\varpi}e^2(t,x)dx+2\int_0^L e(t,x)w_{\delta}e_t(t,x)dx+\int_0^L\sum_{\varpi=1}^{M}\chi_{\delta\varpi}q_{\varpi}e^2(t,x)dx\\&+2\int_0^L\psi(t,x)q_{\delta}\psi_t(t,x)dx+\int_0^L\sum_{\varpi=1}^{M}\chi_{\delta\varpi}a_{\varpi}e_x^2(t,x)dx+2a_{\delta}\int_0^L e_x(t,x)e_{xt}(t,x)dx\\=&\int_0^L\sum_{\varpi=1}^{M}\chi_{\delta\varpi}w_{\varpi}e^2(t,x)dx+\int_0^L\sum_{\varpi=1}^{M}\chi_{\delta\varpi}q_{\varpi}e^2(t,x)dx+2\int_0^L e(t,x)w_{\delta}[\psi(t,x)-\theta e(t,x)]dx\\&+2\int_0^L\psi(t,x)q_{\delta}[a_{\delta}e_{xx}(t,x)+\sum_{\varepsilon=1}^{\eta}h_{\varepsilon}(\xi(t,x))\gamma_{\varepsilon}e(t,x)-(\alpha-\theta)e_t+\omega(t,x)]dx\\&+\int_0^L\sum_{\varpi=1}^{M}\chi_{\delta\varpi}a_{\varpi}e_x^2(t,x)dx+2a_{\delta}\int_0^L e_x(t,x)e_{xt}(t,x)dx.\end{aligned} \tag{3.19}$$

Variable translation $\psi(t,x)=e_t(t,x)+\theta e(t,x)$ means that

$$2q_{\delta}a_{\delta}\int_0^L\psi(t,x)e_{xx}(t,x)dx=2a_{\delta}q_{\delta}\int_0^L[e_t(t,x)+\theta e(t,x)]e_{xx}(t,x)dx, \tag{3.20}$$

for which we employ integration by parts and boundary conditions (2.25, 2.26, 2.27), and derive the following estimations:

$$\begin{aligned}&2a_{\delta}q_{\delta}\int_0^L e_t(t,x)e_{xx}(t,x)dx=2a_{\delta}q_{\delta}\int_0^L e_t(t,x)de_x(t,x)\\&=2a_{\delta}q_{\delta}e_t(t,x)e_x(t,x)|_0^L-2a_{\delta}q_{\delta}\int_0^l e_x(t,x)e_{tx}(t,x)dx\end{aligned}$$

$$\leq 2a_\delta q_\delta \sum_{\hat{\theta}=1}^{\eta} h_{\hat{\theta}}(\xi(t,L))k_{1\delta\hat{\theta}} \sum_{\tilde{\theta}=1}^{\eta} h_{\tilde{\theta}}(\xi(t,L))k_{2\delta\tilde{\theta}} e^2(t,L) - 2a_\delta q_\delta \int_0^L e_x(t,x)e_{tx}(t,x)dx, \tag{3.21}$$

and

$$\begin{aligned} 2a_\delta q_\delta \int_0^L \theta e(t,x)e_{xx}(t,x)dx = & \ 2a_\delta q_\delta \theta e(t,x)e_x(t,x)|_0^L - 2a_\delta q_\delta \theta \int_0^L e_x^2(t,x)dx \\ = & \ 2a_\delta q_\delta \theta \sum_{\hat{\theta}=1}^{\eta} h_{\hat{\theta}}(\xi(t,L))k_{1\delta\hat{\theta}} e^2(t,L) - 2a_\delta q_\delta \theta \int_0^L e_x^2(t,x)dx. \end{aligned} \tag{3.22}$$

In addition, it is obviously true that

$$\begin{aligned} & [\rho_1\psi(t,x) - \rho_2 e_t(t,x) - \rho_3 e(t,x) + \rho_4 e_x(t,x) + \rho_5 e(t,L) + \rho_6 \omega(t,x)] \\ & \times [-\psi(t,x) + e_t(t,x) + \theta e(t,x)]dx = 0. \end{aligned} \tag{3.23}$$

By integrating the above calculations, we derive that

$$\begin{aligned} & \mathcal{L}V_2(\delta,t) - \tau V_2(\delta,t) + \int_0^L [e^2(t,x) - \lambda^2\omega^2(t,x)]dx \\ \leq & \int_0^L \sum_{\varepsilon=1}^{\eta}\sum_{\hat{\theta}=1}^{\eta}\sum_{\tilde{\theta}=1}^{\eta} h_\varepsilon(\xi(t,x))h_{\hat{\theta}}(\xi(t,L))h_{\tilde{\theta}}(\xi(t,L))\zeta_2^T \Psi_{\delta\varepsilon\hat{\theta}\tilde{\theta}}\zeta_2 dz, \end{aligned}$$

where $\zeta_2 = [e(t,x), \psi(t,x), e_x(t,x), e(t,L), e_t(t,x), \omega(t,x)]^T$, and $\Psi_{\delta\varepsilon\hat{\theta}\tilde{\theta}}$ has been defined in (3.16).
Therefore, if the condition (3.16) is satisfied, we have

$$\mathcal{L}V_2(\delta,t) < \tau V_2(\delta,t) - \int_0^L [e^2(t,x) - \lambda^2\omega^2(t,x)]dx.$$

Similar to the process in the proof of Theorem 1, it is easy to obtain that

$$\mathbb{E}V_2(\delta,t) = \mathbb{E}\{\int_0^L w_\delta e^2(t,x)dx + \int_0^L q_\delta \psi^2(t,x)dx + \int_0^L a_\delta e_x^2(t,x)dx\} \geq \min_{\delta\in\mathcal{M}}(w_\delta)\mathbb{E}\,\|e(t,x)\|_{\mathcal{H}^2}^2,$$

and if $c_1 \geq \max(\mathbb{E}\,\|e(0,x)\|_{\mathcal{H}^2}^2, \mathbb{E}\,\|\psi(0,x)\|_{\mathcal{H}^2}^2)$, we have

$$\begin{aligned} \mathbb{E}V_2(\varrho(0),0) = & \ \mathbb{E}\{\int_0^L w(\varrho(0))e^2(0,x)dx + \int_0^L q(\varrho(0))\psi^2(0,x)dx\} \\ \leq & \max_{\delta\in\mathcal{M}}(w_\delta)\mathbb{E}\,\|e(0,x)\|_{\mathcal{H}^2}^2 + \max_{\delta\in\mathcal{M}}(q_\delta)\mathbb{E}\,\|\psi(0,x)\|_{\mathcal{H}^2}^2 \\ \leq & \ c_1 \max_{\delta\in\mathcal{M}}(w_\delta, q_\delta). \end{aligned}$$

As a result,

$$\mathbb{E}\,\|e(t,x)\|_{\mathcal{H}^2}^2 \leq \frac{c_1 \exp(\tau T)\max_{\delta\in\mathcal{M}}(w_\delta,q_\delta) + \lambda^2\exp(\tau T)L\mathcal{W}}{\min_{\delta\in\mathcal{M}}(w_\delta)} < c_2.$$

Then, the second-order PDE system (2.18) can be mean-square stochastically H_∞ bounded over a FTI under control scheme (2.23) and (2.24) according to Definition 2. This completes the proof of Theorem 2.

Remark 6: Compared to control strategies over infinite-time intervals [6–13], and control strategies guaranteeing finite-time convergence [14, 15, 25], the bounded control scheme in a FTI considered in this paper has its own unique advantages. The above two control strategies focus on the steady-state performance of systems, that is, they only care about whether the convergence is achieved and the convergence time, but do not care about the state of systems during the convergence process. This paper considers the boundedness of MASs in a FTI, that is, it is required to ensure that the system state does not exceed a certain threshold in the FTI, which is also very important in the actual MASs. For example, two agents will lose contact if their distance is too far. Therefore, it is necessary to ensure the boundedness of the system state in a FTI, which cannot be guaranteed by the other two control strategies. In this sense, the control strategy in this paper is more practical in many specific scenarios than the other two.

4. Simulations

Two numerical examples are provided in this section to verify the effectiveness of the proposed communication and control schemes for $N = 39$ FOAs and SOAs, respectively. The diagram of communication for FOAs or SOAs are shown in Figure 1.

First, the Markov switching rule that topological weights follow is defined. We suppose there are three modes in the considered Markov chain, that is, $\mathcal{M} = \{1, 2, 3\}$. The transition probability matrix is chosen as

$$\Lambda = \begin{bmatrix} -0.5 & 0.2 & 0.3 \\ 0.4 & -0.8 & 0.4 \\ 0.1 & 0.5 & -0.6 \end{bmatrix},$$

with which the mode switching rules are described as Figure 2.

The nonlinear function in MASs (2.1) and (2.13) is defined as $f(y) = 0.2y(1 - y)$. Then, inspired by [24, 34, 35], the membership functions $h_1(\xi)$ and $h_2(\xi)$ can be derived as

$$h_1(\xi) = \frac{\xi - \ell_{min}}{\ell_{max} - \ell_{min}}, \;\; h_2(\xi) = \frac{\ell_{max} - \xi}{\ell_{max} - \ell_{min}},$$

and $\gamma_1 = 0.2(1 - \ell_{max})$ and $\gamma_2 = 0.2(1 - \ell_{min})$, where ℓ_{min} and ℓ_{max} can be determined by system state. Moreover, the external disturbance are supposed as $w_i(t) = 0.1tanh(i\vartheta) + 0.05sin(t)$.

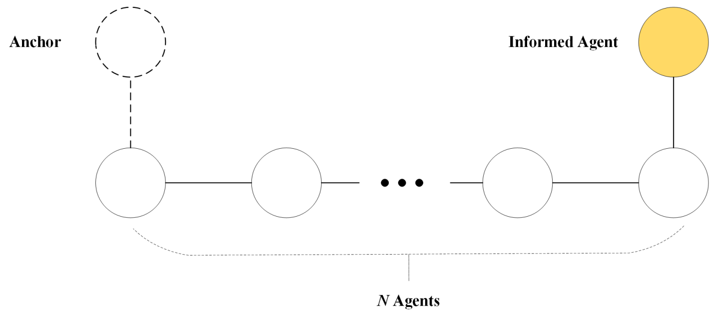

Figure 1. The communication diagram of agents.

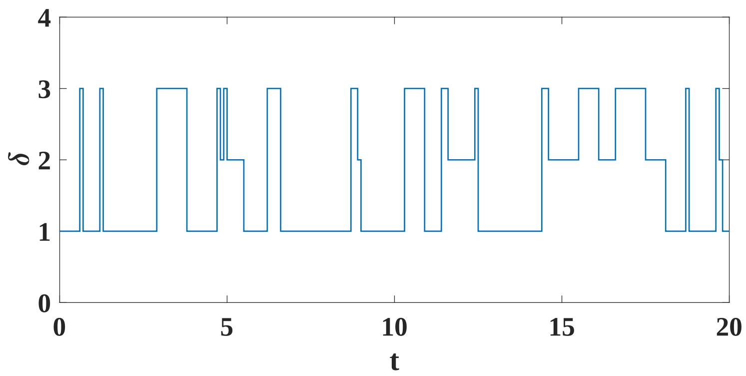

Figure 2. The diagram of mode switching.

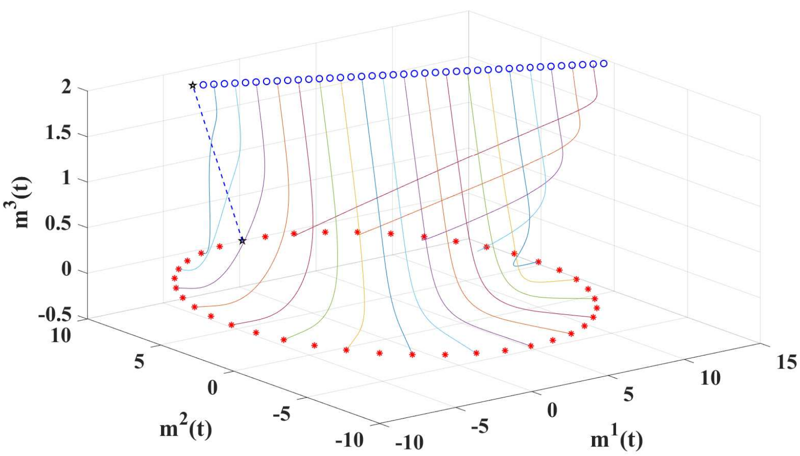

Figure 3. Deployment of FOAs under T–S fuzzy control scheme (2.10).

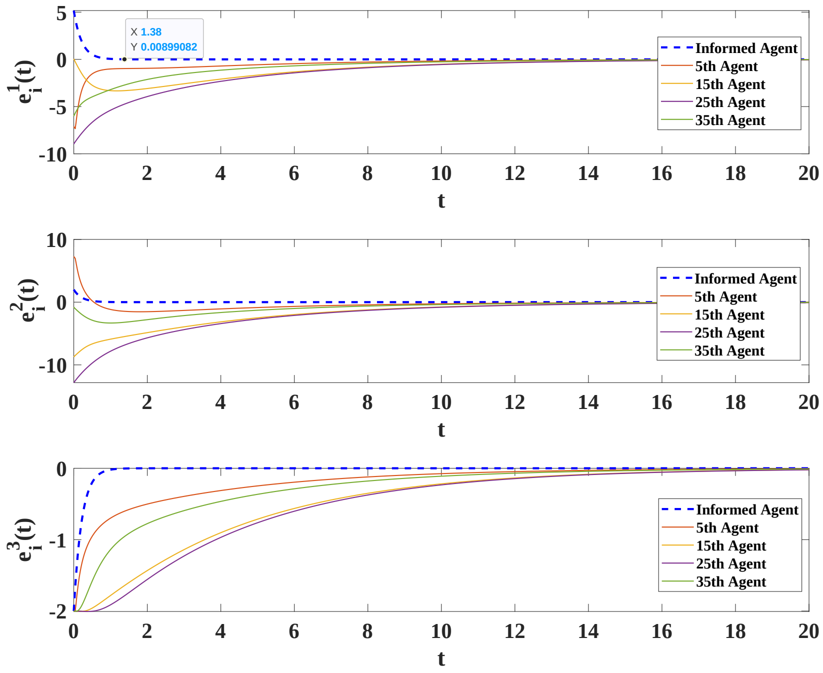

Figure 4. Position errors of several randomly selected FOAs in three spatial dimensions.

4.1. Simulation for FOAs

For $N = 39$ FOAs, we suppose their initial positions as $\Gamma_0(i\vartheta) = [10\sin(3.15i\vartheta), 10\cos(3.15i\vartheta), 0]$, and their desired positions are $\Gamma(i\vartheta) = [-10i\vartheta + 15, 4i\vartheta, 2]$ with $\vartheta = 0.05$ and $i = 1, 2, ..., N$.

For the communication protocol (2.4), we select $a_1 = 0.1$, $a_2 = 0.15$, and $a_3 = 0.2$, and define $\lambda = 2$ and $\tau = 2$. By solving LMI (2.18) proposed in Theorem 1, the controller gains can be obtained as $k_{11} = -6.4521$, $k_{12} = -4.2492$, $k_{21} = -3.2063$, $k_{22} = -5.5137$, $k_{31} = -5.2142$, and $k_{32} = -4.9611$, and the other unknown parameters are solved as $w_1 = 1.8806$, $w_2 = 1.7736$, and $w_3 = 1.7836$. Moreover, set $c_1 = 14.36$, $T = 20$, under condition (3.13), we can obtain $c_2 > 26.94$, such that first-order PDE system (2.6) is mean-square stochastically H_∞ bounded over a FTI under T–S fuzzy control scheme (2.10). The deployment of FOAs from Γ_0 to Γ are shown in Figure 3, from which we know that the finite-time H_∞ deployment can be reached with proposed approaches. Furthermore, to more clearly show the deployment of agents, position errors of several randomly selected FOAs in three spatial dimensions are shown in Figure 4.

Remark 7: It should be pointed out that the switching topologies considered in this paper mainly refer to the switching of topological weights, and there is no case of topology disappearance between agents, because the PDE method adopted in this paper must ensure the topological connectivity between neighbor agents; otherwise, the PDE method will no longer be applicable. a_δ represents the topological weight between adjacent agents, and δ represents the mode of the topology following the Markov chain. In the simulation, we assume that the topological weights switch between three modes, that is, $\delta = 1, 2, 3$, and the corresponding topological weights are $a_1 = 0.1$, $a_2 = 0.15$, and $a_3 = 0.2$, respectively. Therefore, simulations in this paper could show the topology switching phenomena.

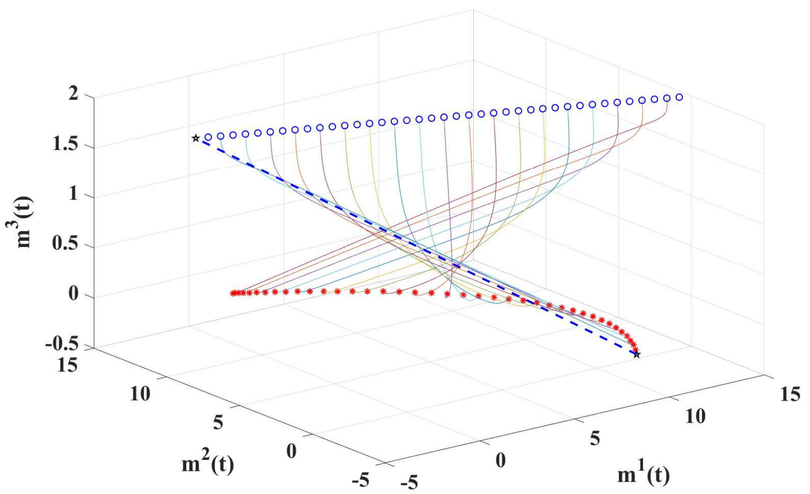

Figure 5. Deployment of SOAs under T–S fuzzy control scheme (2.23) and (2.24).

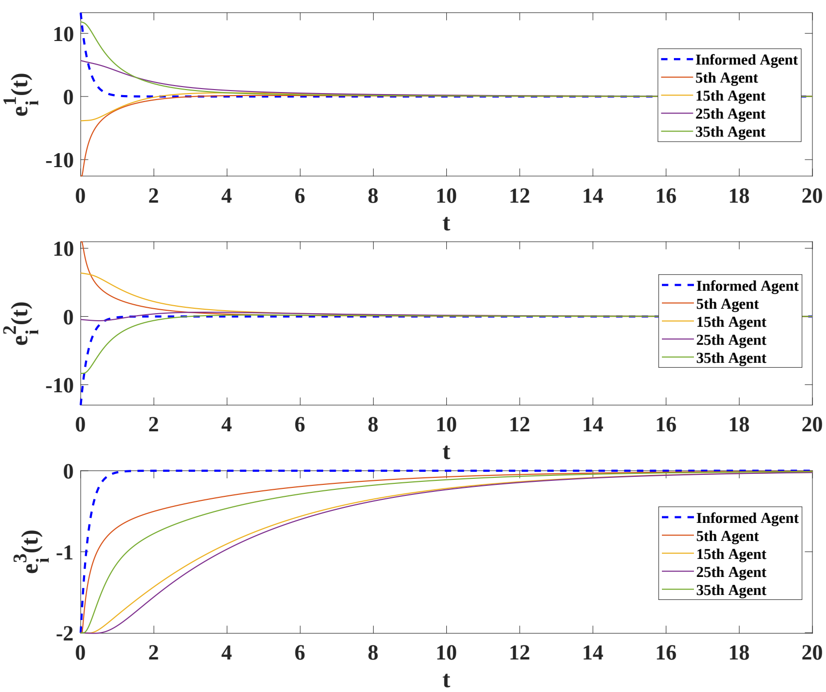

Figure 6. Position errors of several randomly selected SOAs in three spatial dimensions.

4.2. Simulation for SOAs

For $N = 39$ SOAs, we suppose their initial positions as $\Gamma_0(i\vartheta) = [10\sin^2(i\vartheta), 12\cos(i\vartheta), 0]$, and their desired positions are $\Gamma(i\vartheta) = [-10i\vartheta + 15, 4i\vartheta, 2]$ with $\vartheta = 0.05$ and $i = 1, 2, ..., N$.

For the communication protocol (2.15), we select $\alpha = 5.3$ and $\theta = 1.2$, and the other parameters are selected the same as those in the simulation for FOAs. By solving LMI (3.16) proposed in Theorem 2, with which the controller gains can be obtained as $k_{111} = -11.1612$, $k_{112} = -14.5069$, $k_{121} = -13.1211$, $k_{122} = -9.4445$, $k_{131} = -12.4737$, $k_{132} = -12.7194$, $k_{211} = -20.4924$, $k_{212} = -13.0284$, $k_{221} = -8.9102$, $k_{222} = -16.9199$, $k_{231} = -12.2412$, and $k_{232} = -17.0933$, and the other unknown parameters are solved as $w_1 = 10.1532$, $w_2 = 10.0100$, $w_3 = 10.6308$, $q_1 = 1.7697$, $q_2 = 1.2870$, $q_3 = 0.7850$, $\rho_1 = 5.2108$, $\rho_2 = 11.7772$, $\rho_3 = -12.9393$, and $\rho_4 = \rho_5 = \rho_6 = 0$. Moreover, set $c_1 = 13.54$, $T = 20$, under condition (3.13), we can obtain $c_2 > 32.16$, such that the second-order PDE system (2.18) can be mean-square stochastically H_∞ bounded over a FTI under T–S fuzzy control scheme (2.23) and (2.24). The deployment of SOAs from Γ_0 to Γ are shown in Figure 5, from which we know that the finite-time H_∞ deployment can be reached with proposed approaches. Furthermore, to more clearly show the deployment of agents, position errors of several randomly selected SOAs in three spatial dimensions are shown in Figure 6.

5. Conclusions

The H_∞ deployment problem of large-scale nonlinear FOAs and SOAs over a FTI is studied in this paper. First, by designing communication protocols with Markov switching topologies, collective dynamics of MASs can be modeled as first-order and second-order Markovian PDEs. Second, by introducing the T–S fuzzy technology, the T–S fuzzy controller is designed for informed agents, such that finite-time H_∞ deployment of FOAs and SOAs can be guaranteed. Third, we provide detailed proofs of main results and corresponding simulations to verify the rationality and effectiveness of proposed approaches. However, the Markov switching rule considered in this paper requires that the sojourn time must be exponentially distributed, which will affect the application scope of this paper in real systems to some extent. Therefore, inspired by [36–38], we will consider semi-Markov switching topologies of MASs in the future work. As a result, the sojourn time can be subject to other probability distributions.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest in this paper.

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