Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(2): 3957–4013.
DOI: 10.3934/math.2024194
Received: 14 November 2023
Revised: 22 December 2023
Accepted: 27 December 2023
Published: 11 January 2024

Research article

Melnikov functions and limit cycle bifurcations for a class of piecewise Hamiltonian systems

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Abstract: This study evaluated the number of limit cycles for a class of piecewise Hamiltonian systems with two zones separated by two semi-straight lines. First, we obtained explicit expressions of higher Melnikov functions. Then we applied these expressions to find the upper bounds of the number of limit cycles bifurcated from a period annulus of a piecewise polynomial Hamiltonian system.

Keywords: piecewise smooth system; Melnikov function; limit cycle; bifurcation
Mathematics Subject Classification: 34C05, 34C07, 37G15

1. Introduction

In recent years, there has been a growing focus on the study of non-smooth systems, particularly piecewise near-Hamiltonian systems, see [4, 18, 19] and the references therein. Within the realm of bifurcation problems associated with piecewise smooth systems, a significant area of investigation involves determining the number of limit cycles or periodic solutions. This exploration can be considered an extension of the Hilbert's 16th problem. Researchers commonly employ two principal methods to analyze limit cycle bifurcations: the Melnikov function method [1, 2, 6, 7, 12, 15, 19, 23] and the averaging method [3, 4, 9, 13, 14, 16, 17]. It was proved in [5, 14] that the above two methods are equivalent in studying the number of limit cycles of planar C^∞ near-Hamiltonian systems or piecewise C^∞ near-integrable systems in two or higher dimensional spaces.

In 2010, Liu and Han [12] considered a piecewise near-Hamiltonian system of the form

$$\begin{cases}\dot{x} = H_y^+(x,y) + \epsilon f^+(x,y),\\ \dot{y} = -H_x^+(x,y) + \epsilon g^+(x,y),\end{cases} x > 0,$$

$$\begin{cases}\dot{x} = H_y^-(x,y) + \epsilon f^-(x,y),\\ \dot{y} = -H_x^-(x,y) + \epsilon g^-(x,y),\end{cases} x \le 0,$$

where $H_x^{\pm},\ H_y^{\pm},\ f^{\pm},\ g^{\pm} \in C^{\infty}$, and $\epsilon \geq 0$ is a small real parameter, and established a formula of the first order Melnikov function which was widely used in studying the number of limit cycles bifurcated from periodic orbits, see [8, 10, 21] for example. Recently, more general results have appeared for piecewise smooth systems with multiple zones [11, 18, 20, 24]. For instance, Tian and Han [18] studied the number of limit cycles bifurcated from a period annulus of a class of planar piecewise near-Hamiltonian systems with three different switching curves. The authors [11] investigated limit cycle bifurcations in piecewise near-Hamiltonian systems with multiple switching curves and obtained a formula of the first order Melnikov function. Yang, Yang and Yu [23] studied a planar piecewise Hamiltonian system with two zones separated by the two semi-straight lines and presented expressions of the first and second order Melnikov functions. In [22], Yang, Han and Huang considered a piecewise Hamiltonian system of the form

$$\dot{x} = H_y,\ \dot{y} = -H_x,\ x \neq 0,$$

where

$$H(x,y) = \begin{cases} H^+(x,y), x > 0, \\ H^-(x,y), x < 0, \end{cases}$$

and $H^{\pm}(x,y) \in C^{\omega}$ with $H^{\pm}(0,0) = 0$. They studied the number of limit cycles appearing in Hopf bifurcations of piecewise planar Hamiltonian system.

Motivated by the works mentioned above, in this paper, we consider a piecewise Hamiltonian system of the form

$$\begin{cases} \dot{x} = H_y(x,y,\epsilon), \\ \dot{y} = -H_x(x,y,\epsilon), \end{cases} \tag{1.1}$$

where

$$H(x,y,\epsilon) = \begin{cases} H^+(x,y,\epsilon), (x,y) \in \Sigma_1, \\ H^-(x,y,\epsilon), (x,y) \in \Sigma_2, - \end{cases}$$

$$H^{\pm}(x,y,\epsilon) = H_0^{\pm}(x,y) + \epsilon H_1^{\pm}(x,y) + \epsilon^2 H_2^{\pm}(x,y) + \cdots, \tag{1.2}$$

with $H_i^{\pm}(x,y) \in C^{\infty},\ i = 0,1,2,\cdots,\ \epsilon \geq 0$ is a small real parameter, Σ_1 and Σ_2 are the regions with a common boundary consisting of two semi-straight lines. Let $\Sigma_1 \cup \Sigma_2 \subset \mathbb{R}^2$ with $l_j \subset \Sigma_j$, $\bar{l}_1 \cup \bar{l}_2 = \overline{\Sigma}_1 \cap \overline{\Sigma}_2$, and $\Sigma_1 \cap \Sigma_2 = \varnothing$. Suppose that the two lines satisfy

$$l_j \subset \{(x, k_j x) | \mu_j x > 0\},\ j = 1, 2,$$

where $\mu_1,\ \mu_2 = \pm 1$ with $(k_1, \mu_1) \neq (k_2, \mu_2)$, see Figure 1. It is easy to see that the condition $(k_1, \mu_1) \neq (k_2, \mu_2)$ means $l_1 \neq l_2$. When $k_1 = k_2$ and $\mu_1 = -\mu_2$, the two semi-straight lines degenerate into a regular single line. The authors in [23] pointed that when the separation line is nonregular, there will be more limit cycles in general.

The rest of this paper is organized as follows. In Section 2, we establish a bifurcation function of system (1.1) and present expressions of any order Melnikov functions. In Section 3, we give an application to illustrate our results and estimate the number of limit cycles bifurcated from a piecewise polynomial Hamiltonian system.

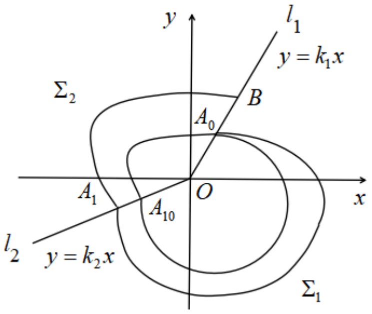


Figure 1. Phase portrait of system 1.1.

2. Expressions of any order Melnikov functions

Consider system (1.1). We make the following basic assumptions for the unperturbed system $(1.1)|_{\epsilon=0}$ as in [11]:

(***A1***) There exists an interval $J = (\alpha, \beta)$ and two points $A_0(h) = (a_0(h), k_1 a_0(h)) \in l_1$ and $A_{10}(h) = (a_{10}(h), k_2 a_{10}(h)) \in l_2$ such that for $h \in J$

$$H_0^+(A_0(h)) = H_0^+(A_{10}(h)) = h,\ H_0^-(A_0(h)) = H_0^-(A_{10}(h)). \tag{2.1}$$

(***A2***) There is a family of closed orbits denoted by $L_h = L_h^1 \cup L_h^2, h \in J$ with clockwise orientation, where L_h^1 is defined by $H_0^+(x, y) = h$, $(x, y) \in \Sigma_1$, starting from $A_0(h)$ and ending at $A_{10}(h)$, and L_h^2 is defined by $H_0^-(x, y) = H^-(A_0(h))$, $(x, y) \in \Sigma_2$, starting from $A_{10}(h)$ and ending at $A_0(h)$.

(***A3***) The arcs L_h^1 and L_h^2 are not tangent to the switching lines l_1 and l_2 at points $A_0(h)$ and $A_{10}(h)$ for $h \in J$. In other words, for each $h \in J$,

$$\begin{aligned} &\frac{\partial H_0^\pm}{\partial x}\Big(A_{10}(h)\Big)a_{10}'(h) + k_2 \frac{\partial H_0^\pm}{\partial y}\Big(A_{10}(h)\Big)a_{10}'(h) \neq 0, \\ &\frac{\partial H_0^\pm}{\partial x}\Big(A_0(h)\Big)a_0'(h) + k_1 \frac{\partial H_0^\pm}{\partial y}\Big(A_0(h)\Big)a_0'(h) \neq 0. \end{aligned} \tag{2.2}$$

Our main goal is to study the number of limit cycles bifurcated from the period annulus $\{L_h\}_{h\in J}$. First of all, we introduce a bifurcation function of system (1.1). Consider the orbit of system (1.1) starting from $A_0(h) \in l_1$. For sufficiently small $|\epsilon| > 0$, it has a first intersection point with the line l_2, denoted by

$$A_1(\epsilon, h) = (a(\epsilon, h), k_2 a(\epsilon, h)), \tag{2.3}$$

such that $\widehat{A_0A_1} \subset \overline{\Sigma}_1$.

The orbit of system (1.1) starting from $A_1(\epsilon, h) \in l_2$ has its first intersection point with the line l_1, denoted by

$$B(\epsilon, h) = (b(\epsilon, h), k_1 b(\epsilon, h)). \tag{2.4}$$

Then, $\widehat{A_1B} \subset \overline{\Sigma}_2$. See Figure 1 for illustration. From [11] we know that if assumptions (***A1***)–(***A3***) hold, then the functions $A_0(h), A_{10}(h)$, $A_1(\epsilon, h)$, and $B(\epsilon, h)$ are all C^∞ smooth with respect to (h, ϵ) on their domain.

Following [11, 12], for any integer $k \geq 1$, we can write for $h \in J$ and $|\epsilon| > 0$ sufficiently small

$$H_0^+(B(\epsilon, h)) - H_0^+(A_0(h)) = \epsilon F(h, \epsilon) = \sum_{j=1}^{k} \epsilon^j M_j(h) + O(\epsilon^{k+1}). \tag{2.5}$$

Here, the functions $F(h, \epsilon)$ and $M_j(h)$ are called a bifurcation function and the jth order Melnikov function of system (1.1), respectively. The orbit from $A_0(h)$ to $B(\epsilon, h)$ defines a Poincaré map or return map of system (1.1). From [7], we have the following bifurcation theorem.

Lemma 1. *[7] Let the assumptions (**A1**)–(**A3**) be satisfied. Suppose that for all* $1 \leq j \leq k-1$, $M_j(h) \equiv 0$ *in (2.5) and that* $M_k(h)$ *has at most* l *zeros in* $h \in J$*, multiplicity taken into account. Then, for small* $|\epsilon| > 0$*, system (1.1) has at most* l *limit cycles bifurcating from the period annulus* $\{L_h\}_{h \in J}$*, multiplicity taken into account.*

The aim of this paper is to develop formulas of the Melnikov functions up to higher order. From (2.5), it is obvious that

$$M_j(h) = \frac{1}{j!} \frac{\partial^j V_0^+}{\partial \epsilon^j}(0, h),\ 1 \leq j \leq k, \tag{2.6}$$

where $V_0^+(\epsilon, h) = H_0^+(B(\epsilon, h))$.

Before presenting our main results, we first give four preliminary lemmas which will be used in deducing expressions of $M_j(h)$ in (2.6). For convenience, we introduce the following notations and functions.

Define the following functions of (ϵ, h)

$$\begin{aligned}
&S_i^\pm(\epsilon, h) = H_i^\pm(A_1(\epsilon, h)),\ V_i^\pm(\epsilon, h) = H_i^\pm(B(\epsilon, h)),\\
&f_{ir}^\pm(\epsilon, h) = \frac{\partial^r S_i^\pm}{\partial \epsilon^r}(\epsilon, h),\ g_{ir}^\pm(\epsilon, h) = \frac{\partial^r V_i^\pm}{\partial \epsilon^r}(\epsilon, h),\\
&\mathbb{K}_{ij}^\pm(\epsilon, h) = \sum_{p=0}^{j} C_j^p k_2^p \frac{\partial^j H_i^\pm}{\partial x^{j-p} \partial y^p}(A_1(\epsilon, h)),\\
&\mathbb{W}_{ij}^\pm(\epsilon, h) = \sum_{q=0}^{j} C_j^q k_1^q \frac{\partial^j H_i^\pm}{\partial x^{j-q} \partial y^q}(B(\epsilon, h)), i = 0, 1, 2, \cdots, j = 1, 2, 3, \cdots.
\end{aligned} \tag{2.7}$$

Define the following functions of h

$$\begin{aligned}
&v_i^\pm(h) = H_i^\pm(A_0(h)) - H_i^\pm(A_{10}(h)),\\
&f_{ir}^\pm(h) = f_{ir}^\pm(0, h) = \frac{\partial^r S_i^\pm}{\partial \epsilon^r}(0, h),\ g_{ir}^\pm(h) = g_{ir}^\pm(0, h) = \frac{\partial^r V_i^\pm}{\partial \epsilon^r}(0, h),\\
&K_{ij}^\pm(h) = \mathbb{K}_{ij}^\pm(0, h) = \sum_{p=0}^{j} C_j^p k_2^p \frac{\partial^j H_i^\pm}{\partial x^{j-p} \partial y^p}(A_{10}(h)),\\
&W_{ij}^\pm(h) = \mathbb{W}_{ij}^\pm(0, h) = \sum_{q=0}^{j} C_j^q k_1^q \frac{\partial^j H_i^\pm}{\partial x^{j-q} \partial y^q}(A_0(h)), i = 0, 1, 2, \cdots, j = 1, 2, 3, \cdots.
\end{aligned} \tag{2.8}$$

For integers r and l satisfying $1 \leq l \leq r$, we define

$$\begin{aligned} T_{rl} &= \{(b_1, b_2, \cdots, b_r) | b_1 + 2b_2 + \cdots + rb_r = r, l = b_1 + b_2 + \cdots + b_r, b_1, \cdots, b_r \in \mathbb{N}\}, \\ N(b_1, b_2, \cdots, b_r) &= \frac{r!}{b_1! b_2! 2!^{b_2} \cdots b_r! r!^{b_r}}, \end{aligned} \tag{2.9}$$

where $\mathbb{N}$ represents the set of non-negative integers. Taking $r = 4$ as an example, we have

$$T_{41} = \{(0, 0, 0, 1)\}, \ T_{42} = \{(0, 2, 0, 0), (1, 0, 1, 0)\}, \ T_{43} = \{(2, 1, 0, 0)\}, \ T_{44} = \{(4, 0, 0, 0)\}.$$

Lemma 2. *Under the notations in (2.8), we have*

$$\frac{\partial \mathbb{K}_{ij}^{\pm}}{\partial \epsilon}(\epsilon, h) = \mathbb{K}_{i,j+1}^{\pm}(\epsilon, h) \frac{\partial a}{\partial \epsilon}(\epsilon, h), \ \frac{\partial \mathbb{W}_{ij}^{\pm}}{\partial \epsilon}(\epsilon, h) = \mathbb{W}_{i,j+1}^{\pm}(\epsilon, h) \frac{\partial b}{\partial \epsilon}(\epsilon, h). \tag{2.10}$$

Proof. Taking the partial derivative with respect to ϵ on both sides of $\mathbb{K}_{ij}^{\pm}(\epsilon, h)$ in (2.7) gives

$$\frac{\partial \mathbb{K}_{ij}^{\pm}}{\partial \epsilon}(\epsilon, h) = (\omega_1(\epsilon, h) + \omega_2(\epsilon, h)) \frac{\partial a}{\partial \epsilon}(\epsilon, h), \tag{2.11}$$

where

$$\begin{aligned} \omega_1(\epsilon, h) &= \sum_{p=0}^{j} C_j^p k_2^p \frac{\partial^{j+1} H_i^{\pm}}{\partial x^{j+1-p} \partial y^p}(A_1(\epsilon, h)), \\ \omega_2(\epsilon, h) &= \sum_{p=0}^{j} C_j^p k_2^{p+1} \frac{\partial^{j+1} H_i^{\pm}}{\partial x^{j-p} \partial y^{p+1}}(A_1(\epsilon, h)). \end{aligned}$$

By direct calculation, we have

$$\begin{aligned} \omega_1(\epsilon, h) &= C_j^0 k_2^0 \frac{\partial^{j+1} H_i^{\pm}}{\partial x^{j+1}}(A_1(\epsilon, h)) + C_j^1 k_2^1 \frac{\partial^{j+1} H_i^{\pm}}{\partial x^j \partial y}(A_1(\epsilon, h)) + C_j^2 k_2^2 \frac{\partial^{j+1} H_i^{\pm}}{\partial x^{j-1} \partial y^2}(A_1(\epsilon, h)) \\ &\quad + \cdots + C_j^j k_2^j \frac{\partial^{j+1} H_i^{\pm}}{\partial x \partial y^j}(A_1(\epsilon, h)), \\ \omega_2(\epsilon, h) &= C_j^0 k_2^1 \frac{\partial^{j+1} H_i^{\pm}}{\partial x^j \partial y}(A_1(\epsilon, h)) + C_j^1 k_2^2 \frac{\partial^{j+1} H_i^{\pm}}{\partial x^{j-1} \partial y^2}(A_1(\epsilon, h)) + C_j^2 k_2^3 \frac{\partial^{j+1} H_i^{\pm}}{\partial x^{j-2} \partial y^3}(A_1(\epsilon, h)) \\ &\quad + \cdots + C_j^j k_2^{j+1} \frac{\partial^{j+1} H_i^{\pm}}{\partial y^{j+1}}(A_1(\epsilon, h)). \end{aligned}$$

Adding the above two equalities together, we have

$$\begin{aligned} \omega_1(\epsilon, h) + \omega_2(\epsilon, h) &= C_{j+1}^0 k_2^0 \frac{\partial^{j+1} H_i^{\pm}}{\partial x^{j+1}}(A_1(\epsilon, h)) + \sum_{p=0}^{j-1} (C_j^{p+1} + C_j^p) k_2^{p+1} \frac{\partial^{j+1} H_i^{\pm}}{\partial x^{j-p} \partial y^{p+1}}(A_1(\epsilon, h)) \\ &\quad + C_{j+1}^{j+1} k_2^{j+1} \frac{\partial^{j+1} H_i^{\pm}}{\partial y^{j+1}}(A_1(\epsilon, h)). \end{aligned} \tag{2.12}$$

Note that $C_j^{p+1} + C_j^p = C_{j+1}^{p+1}$ and

$$\sum_{p=0}^{j-1} C_{j+1}^{p+1} k_2^{p+1} \frac{\partial^{j+1} H_i^{\pm}}{\partial x^{j-p} \partial y^{p+1}} (A_1(\epsilon, h)) = \sum_{p=1}^{j} C_{j+1}^{p} k_2^{p} \frac{\partial^{j+1} H_i^{\pm}}{\partial x^{j-p+1} \partial y^{p}} (A_1(\epsilon, h)).$$

Substituting the above equality into (2.12) yields

$$\omega_1(\epsilon, h) + \omega_2(\epsilon, h) = \sum_{p=0}^{j+1} C_{j+1}^{p} k_2^{p} \frac{\partial^{j+1} H_i^{\pm}}{\partial x^{j-p+1} \partial y^{p}} (A_1(\epsilon, h)) \frac{\partial a}{\partial \epsilon}(\epsilon, h).$$

Hence, by (2.11), we have

$$\frac{\partial \mathbb{K}_{i,j}^{\pm}}{\partial \epsilon}(\epsilon, h) = \mathbb{K}_{i,j+1}^{\pm}(\epsilon, h) \frac{\partial a}{\partial \epsilon}(\epsilon, h).$$

$\mathbb{W}_{ij}^{\pm}(\epsilon, h)$ in (2.10) can be obtained in a manner similar to the previous steps. This ends the proof. □

Lemma 3. *Under the notations in (2.8), we have*

$$\begin{aligned} f_{ir}^{\pm}(\epsilon, h) &= \sum_{l=1}^{r} \sum_{(b_1, \cdots, b_r) \in T_{rl}} N(b_1, \cdots, b_r) \mathbb{K}_{il}^{\pm}(\epsilon, h) \prod_{j=1}^{r} \left(\frac{\partial^j a}{\partial \epsilon^j}(\epsilon, h) \right)^{b_j}, \\ g_{ir}^{\pm}(\epsilon, h) &= \sum_{l=1}^{r} \sum_{(b_1, \cdots, b_r) \in T_{rl}} N(b_1, \cdots, b_r) \mathbb{W}_{il}^{\pm}(\epsilon, h) \prod_{j=1}^{r} \left(\frac{\partial^j b}{\partial \epsilon^j}(\epsilon, h) \right)^{b_j}. \end{aligned} \tag{2.13}$$

Proof. We continue our proof of the formula of $f_{ir}^{\pm}(\epsilon, h)$ in (2.13) by induction on r. For $r = 1$, from the definition of $f_{ir}^{\pm}(\epsilon, h)$ in (2.7), we can obtain

$$\begin{aligned} f_{i1}^{\pm}(\epsilon, h) &= \frac{\partial H_i^{\pm}}{\partial x}(A_1(\epsilon, h)) \frac{\partial a}{\partial \epsilon}(\epsilon, h) + k_2 \frac{\partial H_i^{\pm}}{\partial y}(A_1(\epsilon, h)) \frac{\partial a}{\partial \epsilon}(\epsilon, h) \\ &= \left(\frac{\partial H_i^{\pm}}{\partial x}(A_1(\epsilon, h)) + k_2 \frac{\partial H_i^{\pm}}{\partial y}(A_1(\epsilon, h)) \right) \frac{\partial a}{\partial \epsilon}(\epsilon, h) \\ &= \mathbb{K}_{i1}^{\pm}(\epsilon, h) \frac{\partial a}{\partial \epsilon}(\epsilon, h) = \sum_{(b_1) \in T_{11}} N(b_1) \mathbb{K}_{i1}^{\pm}(\epsilon, h) \prod_{j=1} \left(\frac{\partial^j a}{\partial \epsilon^j}(\epsilon, h) \right)^{b_j}. \end{aligned}$$

Thus, the first formula in (2.13) is true for $r = 1$.

Suppose that it is true for $r = n$. That is,

$$f_{in}^{\pm}(\epsilon, h) = \sum_{l=1}^{n} \sum_{(b_1, \cdots, b_n) \in T_{nl}} N(b_1, \cdots, b_n) \mathbb{K}_{il}^{\pm}(\epsilon, h) \prod_{j=1}^{n} \left(\frac{\partial^j a}{\partial \epsilon^j}(\epsilon, h) \right)^{b_j}.$$

We want to prove that the conclusion on $f_{ir}^{\pm}(\epsilon, h)$ in (2.13) is true for $r = n + 1$. By the definition of T_{nl} and $N(b_1, \cdots, b_n)$ in (2.9), the above equality can be rewritten as

$$f_{in}^{\pm}(\epsilon, h) = \kappa_1(\epsilon, h) + \kappa_2(\epsilon, h) + \kappa_3(\epsilon, h), \tag{2.14}$$

where

$$\kappa_1(\epsilon,h) = \mathbb{K}_{i1}^{\pm}(\epsilon,h)\frac{\partial^n a}{\partial\epsilon^n}(\epsilon,h),$$
$$\kappa_2(\epsilon,h) = \sum_{l=2}^{n-1}\sum_{(b_1,\cdots,b_{n-1},0)\in T_{nl}} N(b_1,\cdots,b_{n-1},0)\mathbb{K}_{il}^{\pm}(\epsilon,h)\prod_{j=1}^{n-1}\left(\frac{\partial^j a}{\partial\epsilon^j}(\epsilon,h)\right)^{b_j},$$
$$\kappa_3(\epsilon,h) = \mathbb{K}_{in}^{\pm}(\epsilon,h)\left(\frac{\partial a}{\partial\epsilon}(\epsilon,h)\right)^n.$$

By the definition in (2.7), we have

$$f_{i,n+1}^{\pm}(\epsilon,h) = \frac{\partial^{n+1}S_i^{\pm}}{\partial\epsilon^{n+1}}(\epsilon,h) = \frac{\partial}{\partial\epsilon}\left(\frac{\partial^n S_i^{\pm}}{\partial\epsilon^n}(\epsilon,h)\right) = \frac{\partial f_{in}^{\pm}}{\partial\epsilon}(\epsilon,h).$$

By (2.14),

$$\frac{\partial f_{in}^{\pm}}{\partial\epsilon}(\epsilon,h) = \frac{\partial\kappa_1}{\partial\epsilon}(\epsilon,h) + \frac{\partial\kappa_2}{\partial\epsilon}(\epsilon,h) + \frac{\partial\kappa_3}{\partial\epsilon}(\epsilon,h), \tag{2.15}$$

where

$$\frac{\partial\kappa_1}{\partial\epsilon}(\epsilon,h) = \frac{\partial\mathbb{K}_{i1}^{\pm}}{\partial\epsilon}(\epsilon,h)\frac{\partial^n a}{\partial\epsilon^n}(\epsilon,h) + \mathbb{K}_{i1}^{\pm}(\epsilon,h)\frac{\partial^{n+1}a}{\partial\epsilon^{n+1}}(\epsilon,h),$$
$$\frac{\partial\kappa_2}{\partial\epsilon}(\epsilon,h) = \sum_{l=2}^{n-1}\sum_{(b_1,\cdots,b_{n-1},0)\in T_{nl}} N(b_1,\cdots,b_{n-1},0)\left[\frac{\partial\mathbb{K}_{il}^{\pm}}{\partial\epsilon}(\epsilon,h)\prod_{j=1}^{n-1}\left(\frac{\partial^j a}{\partial\epsilon^j}(\epsilon,h)\right)^{b_j}\right.$$
$$\left. +\mathbb{K}_{il}^{\pm}(\epsilon,h)\sum_{p=1}^{n-1}\left(b_p\left(\frac{\partial^p a}{\partial\epsilon^p}(\epsilon,h)\right)^{b_p-1}\frac{\partial^{p+1}a}{\partial\epsilon^{p+1}}(\epsilon,h)\prod_{j=1,p\neq j}^{n-1}\left(\frac{\partial^j a}{\partial\epsilon^j}(\epsilon,h)\right)^{b_j}\right)\right],$$
$$\frac{\partial\kappa_3}{\partial\epsilon}(\epsilon,h) = \frac{\partial\mathbb{K}_{in}^{\pm}}{\partial\epsilon}(\epsilon,h)\left(\frac{\partial a}{\partial\epsilon}(\epsilon,h)\right)^n + n\mathbb{K}_{i,n}^{\pm}(\epsilon,h)\left(\frac{\partial a}{\partial\epsilon}(\epsilon,h)\right)^{n-1}\frac{\partial^2 a}{\partial\epsilon^2}(\epsilon,h).$$

By applying Lemma 2 to above equalities, we have

$$\frac{\partial f_{i,n}^{\pm}}{\partial\epsilon}(\epsilon,h) = \mathbb{K}_{i1}^{\pm}(\epsilon,h)\frac{\partial^{n+1}a}{\partial\epsilon^{n+1}}(\epsilon,h) + \sum_{l=2}^{n}\sum_{(b_1,\cdots,b_n,0)\in T_{n+1,l}} N(b_1,\cdots,b_n,0)\mathbb{K}_{il}^{\pm}(\epsilon,h)\prod_{j=1}^{n}\left(\frac{\partial^j a}{\partial\epsilon^j}(\epsilon,h)\right)^{b_j}$$
$$+ \mathbb{K}_{i,n+1}^{\pm}(\epsilon,h)\left(\frac{\partial a}{\partial\epsilon}(\epsilon,h)\right)^{n+1},$$

which means that

$$f_{i,n+1}^{\pm}(\epsilon,h) = \sum_{l=1}^{n+1}\sum_{(b_1,\cdots,b_{n+1})\in T_{n+1,l}} N(b_1,\cdots,b_{n+1})\mathbb{K}_{il}^{\pm}(\epsilon,h)\prod_{j=1}^{n+1}\left(\frac{\partial^j a}{\partial\epsilon^j}(\epsilon,h)\right)^{b_j}.$$

Therefore, the first formula in (2.13) is true for $r = n+1$. The second formula on $g_{i,n+1}^{\pm}(\epsilon,h)$ in (2.13) can also be proved in a similar manner. This ends the proof. □

Lemma 4. *Under the notations in (2.8), for the function $a(\epsilon, h)$ in (2.3), we have*

$$\frac{\partial a}{\partial \epsilon}(0,h) = \frac{v_1^+(h)}{K_{01}^+(h)},$$

$$\frac{\partial^j a}{\partial \epsilon^j}(0,h) = \frac{1}{K_{01}^+(h)}\Bigg[j!v_j^+(h) - \sum_{i=1}^{j-1}\frac{j!}{(j-i)!}\sum_{l=1}^{j-i}\sum_{(b_1,\cdots,b_{j-i})\in T_{j-i,l}} N(b_1,\cdots,b_{j-i})K_{il}^+(h)\prod_{s=1}^{j-i}\phi_s(h)$$
$$-\sum_{l=2}^{j}\sum_{(b_1,\cdots,b_{j-1},0)\in T_{jl}} N(b_1,\cdots,b_{j-1},0)K_{0l}^+(h)\prod_{s=1}^{j-1}\phi_s(h)\Bigg],\ j\geq 2,$$

where $\phi_s(h) = \left(\frac{\partial^s a}{\partial \epsilon^s}(0,h)\right)^{b_s}$.

Proof. From (1.2),

$$H^{\pm}(A_1(\epsilon,h),\epsilon) = S_0^{\pm}(\epsilon,h) + \sum_{j=1}^{n}\epsilon^j S_j^{\pm}(\epsilon,h) + O(\epsilon^{n+1}).$$

For $|\epsilon| > 0$ sufficiently small, $S_i^{\pm}(\epsilon,h)$ has the following Taylor expansion in ϵ

$$S_i^{\pm}(\epsilon,h) = H_i^{\pm}(A_{10}(h)) + \sum_{j=1}^{n}\frac{\epsilon^j}{j!}f_{ij}^{\pm}(h) + O(\epsilon^{n+1}). \tag{2.16}$$

By using Lemma 3 and letting $\epsilon = 0$ in (2.13), we get

$$f_{ir}^{\pm}(h) = \sum_{l=1}^{r}\sum_{(b_1,\cdots,b_r)\in T_{rl}} N(b_1,\cdots,b_r)K_{il}^{\pm}(h)\prod_{j=1}^{r}\phi_j(h), \tag{2.17}$$

where $1 \leq l \leq r$, T_{rl}, and $N(b_1,\cdots,b_r)$ are defined in (2.9).

Substituting (2.16) into the above equation yields

$$H^{\pm}(A_1(\epsilon,h),\epsilon) = H_0^{\pm}(A_{10}(h)) + \sum_{j=1}^{n}\epsilon^j\left[\sum_{i=0}^{j-1}\frac{1}{(j-i)!}f_{i,j-i}^{\pm}(h) + H_j^{\pm}(A_{10}(h))\right] + O(\epsilon^{n+1}). \tag{2.18}$$

Again, from (1.2),

$$H^{\pm}(A_0(h),\epsilon) = H_0^{\pm}(A_0(h)) + \sum_{j=1}^{n}\epsilon^j H_j^{\pm}(A_0(h)) + O(\epsilon^{n+1}). \tag{2.19}$$

Then note that system (1.1) is Hamiltonian and

$$H^+(A_1(\epsilon,h),\epsilon) = H^+(A_0(h),\epsilon).$$

By inserting (2.19) and the expansion of $H^+(A_1(\epsilon,h),\epsilon)$ in (2.18) into the above equality, and comparing the like powers of ϵ, we can obtain

$$\sum_{i=0}^{j-1}\frac{1}{(j-i)!}f_{i,j-i}^{+}(h) + H_j^{+}(A_{10}(h)) = H_j^{+}(A_0(h)),\ j = 1,2,3,\cdots. \tag{2.20}$$

For $j = 1$, (2.20) gives us

$$f_{01}^{+}(h) = H_1^{+}(A_0(h)) - H_1^{+}(A_{10}(h)).$$

Substituting (2.17) into the above equation, we have $\frac{\partial a}{\partial \epsilon}(0,h) = \frac{v_1^{+}(h)}{K_{01}^{+}(h)}$.

In order to get $\frac{\partial^j a}{\partial \epsilon^j}(0,h)$ for $j \geq 2$, substituting (2.17) into (2.20) and multiplying both sides of (2.20) by $j!$ lead to

$$\sum_{i=0}^{j-1} \frac{j!}{(j-i)!} \sum_{l=1}^{j-i} \sum_{(b_1,\cdots,b_{j-i})\in T_{j-i,l}} N(b_1,\cdots,b_{j-i}) K_{il}^{+}(h) \prod_{s=1}^{j-i} \phi_s(h) = j! v_j^{+}(h). \tag{2.21}$$

The left side of (2.21) can be rewritten as

$$\begin{aligned}
&\sum_{l=1}^{j-i} \sum_{(b_1,\cdots,b_j)\in T_{jl}} N(b_1,\cdots,b_j) K_{0l}^{+}(h) \prod_{s=1}^{j} \phi_s(h) \\
&+ \sum_{i=1}^{j-1} \frac{j!}{(j-i)!} \sum_{l=1}^{j-i} \sum_{(b_1,\cdots,b_{j-i})\in T_{j-i,l}} N(b_1,\cdots,b_{j-i}) K_{il}^{+}(h) \prod_{s=1}^{j-i} \phi_s(h) \\
= &\sum_{(b_1,\cdots,b_j)\in T_{j1}} N(b_1,\cdots,b_j) K_{01}^{+}(h) \prod_{s=1}^{j} \phi_s(h) + \sum_{l=2}^{j} \sum_{(b_1,\cdots,b_j)\in T_{jl}} N(b_1,\cdots,b_j) K_{0l}^{+}(h) \prod_{s=1}^{j} \phi_s(h) \\
&+ \sum_{i=1}^{j-1} \frac{j!}{(j-i)!} \sum_{l=1}^{j-i} \sum_{(b_1,\cdots,b_{j-i})\in T_{j-i,l}} N(b_1,\cdots,b_{j-i}) K_{il}^{+}(h) \prod_{s=1}^{j-i} \phi_s(h).
\end{aligned} \tag{2.22}$$

From (2.9), we know that if $l = 1$

$$T_{j1} = \{(0,\cdots,0,1)\},\ N(b_1,\cdots,b_j) = 1, \tag{2.23}$$

and if $l \geq 2$, then

$$\begin{aligned}
T_{jl} = \{(b_1, b_2, \cdots, b_{j-1}, 0) | (b_1 + 2b_2 + \cdots + (j-1)b_{j-1} = j,& \\
l = b_1 + b_2 + \cdots + b_{j-1}, b_1, b_2, \cdots, b_{j-1} \in \mathbb{N}\}.&
\end{aligned} \tag{2.24}$$

By direct calculation from (2.23) and (2.24), if $l = 1$, then

$$\prod_{s=1}^{j} \phi_s(h) = \left(\frac{\partial a}{\partial \epsilon}(0,h)\right)^0 \cdots \left(\frac{\partial^{j-1} a}{\partial \epsilon^{j-1}}(0,h)\right)^0 \left(\frac{\partial^j a}{\partial \epsilon^j}(0,h)\right)^1 = \frac{\partial^j a}{\partial \epsilon^j}(0,h), \tag{2.25}$$

and if $l \geq 2$, then

$$\prod_{s=1}^{j} \phi_s(h) = \left(\frac{\partial a}{\partial \epsilon}(0,h)\right)^{b_1} \cdots \left(\frac{\partial^{j-1} a}{\partial \epsilon^{j-1}}(0,h)\right)^{b_{j-1}} \left(\frac{\partial^j a}{\partial \epsilon^j}(0,h)\right)^0 = \prod_{s=1}^{j-1} \phi_s(h). \tag{2.26}$$

Substituting (2.23)–(2.26) into (2.22) and (2.21), we have

$$K_{01}^{+}(h)\frac{\partial^j a}{\partial\epsilon^j}(0,h)+\sum_{l=2}^{j}\sum_{(b_1,\cdots,b_{j-1},0)\in T_{jl}}N(b_1,\cdots,b_{j-1},0)K_{0l}^{+}(h)\prod_{s=1}^{j-1}\phi_s(h)$$
$$+\sum_{i=1}^{j-1}\frac{j!}{(j-i)!}\sum_{l=1}^{j-i}\sum_{(b_1,\cdots,b_{j-i})\in T_{j-i,l}}N(b_1,\cdots,b_{j-i})K_{il}^{+}(h)\prod_{s=1}^{j-i}\phi_s(h)=j!v_j^{+}(h),$$

which implies that

$$\frac{\partial^j a}{\partial\epsilon^j}(0,h)=\frac{1}{K_{01}^{+}(h)}\Big[j!v_j^{+}(h)-\sum_{i=1}^{j-1}\frac{j!}{(j-i)!}\sum_{l=1}^{j-i}\sum_{(b_1,\cdots,b_{j-i})\in T_{j-i,l}}N(b_1,\cdots,b_{j-i})K_{il}^{+}(h)\prod_{s=1}^{j-i}\phi_s(h)$$
$$-\sum_{l=2}^{j}\sum_{(b_1,\cdots,b_{j-1},0)\in T_{jl}}N(b_1,\cdots,b_{j-1},0)K_{0l}^{+}(h)\prod_{s=1}^{j-1}\phi_s(h)\Big],\ j\geq 2.$$

For example,

$$\frac{\partial^2 a}{\partial\epsilon^2}(0,h)=\frac{1}{K_{01}^{+}(h)}\Big[2!v_2^{+}(h)-\frac{2!}{1!}K_{11}^{+}(h)\frac{\partial a}{\partial\epsilon}(0,h)-K_{02}^{+}(h)\Big(\frac{\partial a}{\partial\epsilon}(0,h)\Big)^2\Big],$$
$$\frac{\partial^3 a}{\partial\epsilon^3}(0,h)=\frac{1}{K_{01}^{+}(h)}\Big[3!v_3^{+}(h)-\frac{3!}{2!}\Big(K_{11}^{+}(h)\frac{\partial^2 a}{\partial\epsilon^2}(0,h)+K_{12}^{+}(h)\Big(\frac{\partial a}{\partial\epsilon}(0,h)\Big)^2\Big)$$
$$-\frac{3!}{1!}K_{21}^{+}(h)\frac{\partial a}{\partial\epsilon}(0,h)-\Big(3K_{02}^{+}(h)\frac{\partial a}{\partial\epsilon}(0,h)\frac{\partial^2 a}{\partial\epsilon^2}(0,h)+K_{03}^{+}(h)\Big(\frac{\partial a}{\partial\epsilon}(0,h)\Big)^3\Big)\Big].$$

This ends the proof. □

Lemma 5. *Under the notations in (2.8), for the function $b(\epsilon,h)$ in (2.4), we have*

$$\frac{\partial b}{\partial\epsilon}(0,h)=\frac{1}{W_{01}^{-}(h)}\Big(-v_1^{-}(h)+K_{01}^{-}(h)\frac{\partial a}{\partial\epsilon}(0,h)\Big),$$
$$\frac{\partial^j b}{\partial\epsilon^j}(0,h)=\frac{1}{W_{01}^{-}(h)}\Big[-j!v_j^{-}(h)+\sum_{i=0}^{j-1}\frac{j!}{(j-i)!}\sum_{l=1}^{j-i}\sum_{(b_1,\cdots,b_{j-i})\in T_{j-i,l}}N(b_1,\cdots,b_{j-i})K_{il}^{-}(h)\prod_{s=1}^{j}\phi_s(h)$$
$$-\sum_{l=2}^{j}\sum_{(b_1,\cdots,b_{j-1},0)\in T_{jl}}N(b_1,\cdots,b_{j-1},0)W_{0l}^{-}(h)\prod_{s=1}^{j-1}\psi_s(h)$$
$$-\sum_{i=1}^{j-1}\frac{j!}{(j-i)!}\sum_{l=1}^{j-i}\sum_{(b_1,\cdots,b_{j-i})\in T_{j-i,l}}N(b_1,\cdots,b_{j-i})W_{il}^{-}(h)\prod_{s=1}^{j-i}\psi_s(h)\Big],\ j\geq 2,$$

where

$$\phi_s(h)=\Big(\frac{\partial^s a}{\partial\epsilon^s}(0,h)\Big)^{b_s},\ \psi_s(h)=\Big(\frac{\partial^s b}{\partial\epsilon^s}(0,h)\Big)^{b_s}.$$

Proof. We obtain from (1.2) that

$$H^-(B(\epsilon,h),\epsilon) = V_0^-(\epsilon,h) + \sum_{j=1}^{n}\epsilon^j V_j^-(\epsilon,h) + O(\epsilon^{n+1}).$$

For $|\epsilon| > 0$ sufficiently small, $V_i^\pm(\epsilon,h)$ has the following Taylor expansion

$$V_i^\pm(\epsilon,h) = H_i^\pm(A_0(h)) + \sum_{j=1}^{n}\frac{\epsilon^j}{j!}g_{ij}^\pm(h) + O(\epsilon^{n+1}),\ j \geq 1. \tag{2.27}$$

By Lemma 3, we have

$$g_{ir}^\pm(h) = \sum_{l=1}^{r}\sum_{(b_1,\cdots,b_r)\in T_{rl}} N(b_1,\cdots,b_r)W_{il}^\pm(h)\prod_{j=1}^{r}\psi_j(h), \tag{2.28}$$

where $1 \leq l \leq r$. T_{rl} and $N(b_1,\cdots,b_r)$ are defined in (2.9). Substituting (2.27) into the above equation, similar to (2.18), we have

$$H^-(B(\epsilon,h),\epsilon) = H_0^-(A_0(h)) + \sum_{j=1}^{n}\epsilon^j\left[\sum_{i=0}^{j-1}\frac{1}{(j-i)!}g_{i,j-i}^-(h) + H_j^-(A_0(h))\right] + O(\epsilon^{n+1}). \tag{2.29}$$

Noting that system (1.1) is Hamiltonian and

$$H^-(A_1(\epsilon,h),\epsilon) = H^-(B(\epsilon,h),\epsilon),$$

substituting $H^-(A_1(\epsilon,h),\epsilon)$ in (2.18) and (2.29) into the equality above gives us

$$\begin{aligned} &H_0^-(A_{10}(h)) + \sum_{j=1}^{n}\epsilon^j\left[\sum_{i=0}^{j-1}\frac{1}{(j-i)!}f_{i,j-i}^-(h) + H_j^-(A_{10}(h))\right] \\ =&H_0^-(A_0(h)) + \sum_{j=1}^{n}\epsilon^j\left[\sum_{i=0}^{j-1}\frac{1}{(j-i)!}g_{i,j-i}^-(h) + H_j^-(A_0(h))\right]. \end{aligned} \tag{2.30}$$

According to the second equation of (2.1) and comparing the like powers of ϵ on the left and right sides of (2.30), we have

$$\sum_{i=0}^{j-1}\frac{1}{(j-i)!}f_{i,j-i}^-(h) + H_j^-(A_{10}(h)) = \sum_{i=0}^{j-1}\frac{1}{(j-i)!}g_{i,j-i}^-(h) + H_j^-(A_0(h)),\ j \geq 1. \tag{2.31}$$

Taking $j = 1$ in (2.31) gives us

$$f_{01}^-(h) + H_1^-(A_{10}(h)) = g_{01}^-(h) + H_1^-(A_0(h)).$$

Substituting (2.17) and (2.28) into the above equation yields

$$K_{01}^-(h)\frac{\partial a}{\partial\epsilon}(0,h) + H_1^-(A_{10}(h)) = W_{01}^-(h)\frac{\partial b}{\partial\epsilon}(0,h) + H_1^-(A_0(h)),$$

which gives us

$$\frac{\partial b}{\partial \epsilon}(0,h)=\frac{1}{W_{01}^{-}(h)}\Big(-v_1^{-}(h)+K_{01}^{-}(h)\frac{\partial a}{\partial \epsilon}(0,h)\Big).$$

For $j\geq 2$, in order to solve $\frac{\partial^j b}{\partial \epsilon^j}(0,h)$ from (2.31), substituting (2.17) and (2.28) into (2.31) and multiplying both sides of (2.31) by $j!$ yields

$$\begin{aligned}&\sum_{i=0}^{j-1}\frac{j!}{(j-i)!}\sum_{l=1}^{j-i}\sum_{(b_1,\cdots,b_{j-i})\in T_{j-i,l}}N(b_1,\cdots,b_{j-i})K_{il}^{-}(h)\prod_{s=1}^{j-i}\phi_s(h)+j!H_j^{-}(A_{10}(h))\\&=\sum_{i=0}^{j-1}\frac{j!}{(j-i)!}\sum_{l=1}^{j-i}\sum_{(b_1,\cdots,b_{j-i})\in T_{j-i,l}}N(b_1,\cdots,b_{j-i})W_{il}^{-}(h)\prod_{s=1}^{j-i}\psi_s(h)+j!H_j^{-}(A_0(h)).\end{aligned}\tag{2.32}$$

Similar to (2.22), we have

$$\begin{aligned}&\sum_{i=0}^{j-1}\frac{j!}{(j-i)!}\sum_{l=1}^{j-i}\sum_{(b_1,\cdots,b_{j-i})\in T_{j-i,l}}N(b_1,\cdots,b_{j-i})W_{il}^{-}(h)\prod_{s=1}^{j-i}\psi_s(h)\\&=\sum_{l=1}^{j-i}\sum_{(b_1,\cdots,b_j)\in T_{jl}}N(b_1,\cdots,b_j)W_{0l}^{-}(h)\prod_{s=1}^{j}\psi_s(h)\\&+\sum_{i=1}^{j-1}\frac{j!}{(j-i)!}\sum_{l=1}^{j-i}\sum_{(b_1,\cdots,b_{j-i})\in T_{j-i,l}}N(b_1,\cdots,b_{j-i})W_{il}^{-}(h)\prod_{s=1}^{j-i}\psi_s(h)\\&=\sum_{(b_1,\cdots,b_j)\in T_{j1}}N(b_1,\cdots,b_j)W_{01}^{-}(h)\prod_{s=1}^{j}\psi_s(h)+\sum_{l=2}^{j}\sum_{(b_1,\cdots,b_j)\in T_{jl}}N(b_1,\cdots,b_j)W_{0l}^{-}(h)\prod_{s=1}^{j}\psi_s(h)\\&+\sum_{i=1}^{j-1}\frac{j!}{(j-i)!}\sum_{l=1}^{j-i}\sum_{(b_1,\cdots,b_{j-i})\in T_{j-i,l}}N(b_1,\cdots,b_{j-i})W_{il}^{-}(h)\prod_{s=1}^{j-i}\psi_s(h).\end{aligned}$$

Similar to (2.23) and (2.24), we can see that the above equation can be simplified to

$$\begin{aligned}&\sum_{i=0}^{j-1}\frac{j!}{(j-i)!}\sum_{l=1}^{j-i}\sum_{(b_1,\cdots,b_{j-i})\in T_{j-i,l}}N(b_1,\cdots,b_{j-i})W_{il}^{-}(h)\prod_{s=1}^{j-i}\psi_s(h)\\&=W_{01}^{-}(h)\frac{\partial^j b}{\partial \epsilon^j}(0,h)+\sum_{l=2}^{j}\sum_{(b_1,\cdots,b_{j-1},0)\in T_{jl}}N(b_1,\cdots,b_{j-1},0)W_{0l}^{-}(h)\prod_{s=1}^{j-1}\psi_s(h)\\&+\sum_{i=1}^{j-1}\frac{j!}{(j-i)!}\sum_{l=1}^{j-i}\sum_{(b_1,\cdots,b_{j-i})\in T_{j-i,l}}N(b_1,\cdots,b_{j-i})W_{il}^{-}(h)\prod_{s=1}^{j-i}\psi_s(h).\end{aligned}\tag{2.33}$$

Substituting (2.33) into (2.32) yields

$$\begin{aligned}
&\sum_{i=0}^{j-1}\frac{j!}{(j-i)!}\sum_{l=1}^{j-i}\sum_{(b_1,\cdots,b_{j-i})\in T_{j-i,l}}N(b_1,\cdots,b_{j-i})K_{il}^-(h)\prod_{s=1}^{j-i}\phi_s(h)+j!H_j^-(A_{10}(h))\\
=&W_{01}^-(h)\frac{\partial^j b}{\partial\epsilon^j}(0,h)+\sum_{l=2}^{j}\sum_{(b_1,\cdots,b_{j-1},0)\in T_{jl}}N(b_1,\cdots,b_{j-1},0)W_{0l}^-(h)\prod_{s=1}^{j-1}\psi_s(h)\\
&+\sum_{i=1}^{j-1}\frac{j!}{(j-i)!}\sum_{l=1}^{j-i}\sum_{(b_1,\cdots,b_{j-i})\in T_{j-i,l}}N(b_1,\cdots,b_{j-i})W_{il}^-(h)\prod_{s=1}^{j-i}\psi_s(h)+j!H_j^-(A_0(h)),
\end{aligned}$$

which implies that

$$\begin{aligned}
&\frac{\partial^j b}{\partial\epsilon^j}(0,h)\\
=&\frac{1}{W_{01}^-(h)}\Bigg[-j!v_j^-(h)+\sum_{i=0}^{j-1}\frac{j!}{(j-i)!}\sum_{l=1}^{j-i}\sum_{(b_1,\cdots,b_{j-i})\in T_{j-i,l}}N(b_1,\cdots,b_{j-i})K_{il}^-(h)\prod_{s=1}^{j-i}\phi_s(h)\\
&-\sum_{l=2}^{j}\sum_{(b_1,\cdots,b_{j-1},0)\in T_{jl}}N(b_1,\cdots,b_{j-1},0)W_{0l}^-(h)\prod_{s=1}^{j-1}\psi_s(h)\\
&-\sum_{i=1}^{j-1}\frac{j!}{(j-i)!}\sum_{l=1}^{j-i}\sum_{(b_1,\cdots,b_{j-i})\in T_{j-i,l}}N(b_1,\cdots,b_{j-i})W_{il}^-(h)\prod_{s=1}^{j-i}\psi_s(h)\Bigg].
\end{aligned}$$

For example,

$$\begin{aligned}
\frac{\partial^2 b}{\partial\epsilon^2}(0,h)=&\frac{1}{W_{01}^-(h)}\Bigg[-2!v_2^-(h)+\frac{2!}{2!}\left(K_{01}^-(h)\frac{\partial^2 a}{\partial\epsilon^2}(0,h)+K_{02}^-(h)\left(\frac{\partial a}{\partial\epsilon}(0,h)\right)^2\right)+\frac{2!}{1!}K_{11}^-(h)\frac{\partial a}{\partial\epsilon}(0,h)\Bigg)\\
&-W_{02}^-(h)\left(\frac{\partial b}{\partial\epsilon}(0,h)\right)^2-\frac{2!}{1!}W_{11}^-(h)\frac{\partial b}{\partial\epsilon}(0,h)\Bigg].
\end{aligned}$$

This ends the proof. □

Based on the previous discussion, we now present explicit formulas of any order Melnikov functions of piecewise Hamiltonian system (1.1).

Theorem 1. *Consider system (1.1) with the assumptions (**A1**)-(**A3**). The Melnikov functions $M_j(h)$ have the following formula for $j\geq 2$*

$$M_j(h)=M_j^+(h)+\frac{W_{01}^+(h)}{W_{01}^-(h)}M_j^-(h),$$

where

$$\begin{aligned}
M_j^+(h)=&v_j^+(h)-\sum_{i=1}^{j-1}\frac{1}{(j-i)!}\sum_{l=1}^{j-i}\sum_{(b_1,\cdots,b_{j-i})\in T_{j-i,l}}N(b_1,\cdots,b_{j-i})K_{il}^+(h)\prod_{s=1}^{j-i}\phi_s(h)\\
&-\frac{1}{j!}\sum_{l=2}^{j}\sum_{(b_1,\cdots,b_{j-1},0)\in T_{jl}}N(b_1,\cdots,b_{j-1},0)K_{0l}^+(h)\prod_{s=1}^{j-1}\phi_s(h)=\frac{K_{01}^+(h)}{j!}\frac{\partial^j a}{\partial\epsilon^j}(0,h),
\end{aligned}$$

$$M_j^-(h) = -v_j^-(h) + \sum_{i=1}^{j-1} \frac{1}{(j-i)!} \sum_{l=1}^{j-i} \sum_{(b_1,\cdots,b_{j-i})\in T_{j-i,l}} N(b_1,\cdots,b_{j-i}) K_{il}^-(h) \prod_{s=1}^{j-i} \phi_s(h)$$
$$+ \frac{1}{j!} \sum_{l=2}^{j} \sum_{(b_1,\cdots,b_{j-1},0)\in T_{jl}} N(b_1,\cdots,b_{j-1},0) K_{0l}^-(h) \prod_{s=1}^{j-1} \phi_s(h).$$

Proof. In order to get the explicit expressions of $M_j(h)$ in (2.6), we need to obtain the Taylor expansion of the function $H_0^+(B(\epsilon,h))$ with respect to ϵ, where $B(\epsilon,h) = (b(\epsilon,h), k_1 b(\epsilon,h))$. To that end, we should first find $\frac{\partial^j b}{\partial \epsilon^j}(0,h)$, $j \geq 1$, which are related to $\frac{\partial^j a}{\partial \epsilon^j}(0,h)$, $j \geq 1$.

To simplify the proof, we have presented formulas of $\frac{\partial^j a}{\partial \epsilon^j}(0,h)$, $j \geq 1$ in Lemma 4, and formulas of $\frac{\partial^j b}{\partial \epsilon^j}(0,h)$, $j \geq 1$ in Lemma 5. Next, we want to find $M_j(h)$. According to (2.28) and (2.6), if $j \geq 2$, then

$$\begin{aligned} M_j(h) &= \frac{1}{j!} g_{0j}^+(h) = \frac{1}{j!}\Bigg[\sum_{l=1}^{j} \sum_{(b_1,\cdots,b_r)\in T_{jl}} N(b_1,\cdots,b_r) W_{0l}^+(h) \prod_{s=1}^{j} \psi_s(h)\Bigg] \\ &= \frac{1}{j!}\Bigg[W_{01}^+(h)\frac{\partial^j b}{\partial \epsilon^j}(0,h) + \sum_{l=2}^{j} \sum_{(b_1,\cdots,b_{j-1},0)\in T_{jl}} N(b_1,\cdots,b_{j-1},0) W_{0l}^+(h) \prod_{s=1}^{j-1} \psi_s(h)\Bigg]. \end{aligned} \tag{2.34}$$

Thus, when $M_1(h) = M_2(h) = \cdots = M_{j-1} \equiv 0$, we have

$$\frac{\partial b}{\partial \epsilon}(0,h) = \frac{\partial^2 b}{\partial \epsilon^2}(0,h) = \cdots = \frac{\partial^{j-1} b}{\partial \epsilon^{j-1}}(0,h) \equiv 0. \tag{2.35}$$

Substituting (2.35) into (2.34) yields

$$M_j(h) = \frac{W_{01}^+(h)}{j!} \frac{\partial^j b}{\partial \epsilon^j}(0,h).$$

By substituting the expression of $\frac{\partial^j b}{\partial \epsilon^j}(0,h)$ in Lemma 5 into the above equation, we have

$$\begin{aligned} M_j(h) &= \frac{W_{01}^+(h)}{j! W_{01}^-(h)}\Bigg[-j! v_j^-(h) + \sum_{i=0}^{j-1} \frac{j!}{(j-i)!} \sum_{l=1}^{j-i} \sum_{(b_1,\cdots,b_{j-i})\in T_{j-i,l}} N(b_1,\cdots,b_{j-i}) K_{il}^-(h) \prod_{s=1}^{j-i} \phi_s(h)\Bigg] \\ &= \frac{W_{01}^+(h)}{W_{01}^-(h)}\Bigg[-v_j^-(h) + \sum_{i=0}^{j-1} \frac{1}{(j-i)!} \sum_{l=1}^{j-i} \sum_{(b_1,\cdots,b_{j-i})\in T_{j-i,l}} N(b_1,\cdots,b_{j-i}) K_{il}^-(h) \prod_{s=1}^{j-i} \phi_s(h)\Bigg]. \end{aligned}$$

According to the derivation of (2.22), (2.23), and (2.24), the above equation can be written as

$$\begin{aligned} M_j(h) = \frac{W_{01}^+(h)}{W_{01}^-(h)}\Bigg[&-v_j^-(h) + \frac{1}{j!} K_{01}^-(h) \frac{\partial^j a}{\partial \epsilon^j}(0,h) + \frac{1}{j!} \sum_{l=2}^{j} \sum_{(b_1,\cdots,b_{j-1},0)\in T_{jl}} N(b_1,\cdots,b_{j-1},0) K_{0l}^-(h) \prod_{s=1}^{j-1} \phi_s(h) \\ &+ \sum_{i=1}^{j-1} \frac{1}{(j-i)!} \sum_{l=1}^{j-i} \sum_{(b_1,\cdots,b_{j-i})\in T_{j-i,l}} N(b_1,\cdots,b_{j-i}) K_{il}^-(h) \prod_{s=1}^{j-i} \phi_s(h)\Bigg]. \end{aligned} \tag{2.36}$$

From Lemma 4, it can be calculated directly that

$$\frac{1}{j!}K_{01}^{-}(h)\frac{\partial^j a}{\partial\epsilon^j}(0,h) = \frac{K_{01}^{-}(h)}{K_{01}^{+}(h)}\Big[v_j^{+}(h) - \sum_{i=1}^{j-1}\frac{1}{(j-i)!}\sum_{l=1}^{j-i}\sum_{(b_1,\cdots,b_{j-i})\in T_{j-i,l}} N(b_1,\cdots,b_{j-i})K_{il}^{+}(h)\prod_{s=1}^{j-i}\phi_s(h) \\ -\frac{1}{j!}\sum_{l=2}^{j}\sum_{(b_1,\cdots,b_{j-1},0)\in T_{jl}} N(b_1,\cdots,b_{j-1},0)K_{0l}^{+}(h)\prod_{s=1}^{j-1}\phi_s(h)\Big].$$

Substituting the above equality into (2.36) gives us

$$\begin{aligned} M_j(h) = {} & \frac{W_{01}^{+}(h)}{W_{01}^{-}(h)}\frac{K_{01}^{-}(h)}{K_{01}^{+}(h)}\Big[v_j^{+}(h) - \sum_{i=1}^{j-1}\frac{1}{(j-i)!}\sum_{l=1}^{j-i}\sum_{(b_1,\cdots,b_{j-i})\in T_{j-i,l}} N(b_1,\cdots,b_{j-i})K_{il}^{+}(h)\prod_{s=1}^{j-i}\phi_s(h) \\ & - \frac{1}{j!}\sum_{l=2}^{j}\sum_{(b_1,\cdots,b_{j-1},0)\in T_{jl}} N(b_1,\cdots,b_{j-1},0)K_{0l}^{+}(h)\prod_{s=1}^{j-1}\phi_s(h)\Big] \\ & + \frac{W_{01}^{+}(h)}{W_{01}^{-}(h)}\Big[-v_j^{-}(h) + \sum_{i=1}^{j-1}\frac{1}{(j-i)!}\sum_{l=1}^{j-i}\sum_{(b_1,\cdots,b_{j-i})\in T_{j-i,l}} N(b_1,\cdots,b_{j-i})K_{il}^{-}(h)\prod_{s=1}^{j-i}\phi_s(h) \\ & + \frac{1}{j!}\sum_{l=2}^{j}\sum_{(b_1,\cdots,b_{j-1},0)\in T_{jl}} N(b_1,\cdots,b_{j-1},0)K_{0l}^{-}(h)\prod_{s=1}^{j-1}\phi_s(h)\Big]. \end{aligned} \tag{2.37}$$

According to Theorem 2.2 in [25], we know

$$\frac{W_{01}^{+}(h)}{W_{01}^{-}(h)}\frac{K_{01}^{-}(h)}{K_{01}^{+}(h)} = 1.$$

Hence, (2.37) becomes

$$M_j(h) = M_j^{+}(h) + \frac{W_{01}^{+}(h)}{W_{01}^{-}(h)}M_j^{-}(h),$$

where

$$M_j^{+}(h) = v_j^{+}(h) - \sum_{i=1}^{j-1}\frac{1}{(j-i)!}\sum_{l=1}^{j-i}\sum_{(b_1,\cdots,b_{j-i})\in T_{j-i,l}} N(b_1,\cdots,b_{j-i})K_{il}^{+}(h)\prod_{s=1}^{j-i}\phi_s(h) \\ - \frac{1}{j!}\sum_{l=2}^{j}\sum_{(b_1,\cdots,b_{j-1},0)\in T_{jl}} N(b_1,\cdots,b_{j-1},0)K_{0l}^{+}(h)\prod_{s=1}^{j-1}\phi_s(h) = \frac{K_{01}^{+}(h)}{j!}\frac{\partial^j a}{\partial\epsilon^j}(0,h),$$

$$M_j^{-}(h) = -v_j^{-}(h) + \sum_{i=1}^{j-1}\frac{1}{(j-i)!}\sum_{l=1}^{j-i}\sum_{(b_1,\cdots,b_{j-i})\in T_{j-i,l}} N(b_1,\cdots,b_{j-i})K_{il}^{-}(h)\prod_{s=1}^{j-i}\phi_s(h) \\ + \frac{1}{j!}\sum_{l=2}^{j}\sum_{(b_1,\cdots,b_{j-1},0)\in T_{jl}} N(b_1,\cdots,b_{j-1},0)K_{0l}^{-}(h)\prod_{s=1}^{j-1}\phi_s(h).$$

For example, when $j = 2, 3$, we have

$$M_2^+(h) = v_2^+(h) - K_{11}^+(h)\frac{\partial a}{\partial \epsilon}(0,h) - \frac{1}{2!}K_{02}^+(h)\left(\frac{\partial a}{\partial \epsilon}(0,h)\right)^2 = \frac{K_{01}^+(h)}{2!}\frac{\partial^2 a}{\partial \epsilon^2}(0,h),$$
$$M_2^-(h) = -v_2^-(h) + K_{11}^-(h)\frac{\partial a}{\partial \epsilon}(0,h) + \frac{1}{2!}K_{02}^-(h)\left(\frac{\partial a}{\partial \epsilon}(0,h)\right)^2,$$
$$M_3^+(h) = v_3^+(h) - \frac{1}{2!}\left[K_{11}^+(h)\frac{\partial^2 a}{\partial \epsilon^2}(0,h) + K_{12}^+(h)\left(\frac{\partial a}{\partial \epsilon}(0,h)\right)^2\right] - \frac{1}{1!}K_{21}^+(h)\frac{\partial a}{\partial \epsilon}(0,h)$$
$$- \frac{1}{3!}\left[3K_{02}^+(h)\frac{\partial a}{\partial \epsilon}(0,h)\frac{\partial^2 a}{\partial \epsilon^2}(0,h) + K_{03}^+(h)\left(\frac{\partial a}{\partial \epsilon}(0,h)\right)^3\right] = \frac{K_{01}^+(h)}{3!}\frac{\partial^3 a}{\partial \epsilon^3}(0,h),$$
$$M_3^-(h) = -v_3^-(h) + \frac{1}{2!}\left[K_{11}^-(h)\frac{\partial^2 a}{\partial \epsilon^2}(0,h) + K_{12}^-(h)\left(\frac{\partial a}{\partial \epsilon}(0,h)\right)^2\right] + \frac{1}{1!}K_{21}^-(h)\frac{\partial a}{\partial \epsilon}(0,h)$$
$$+ \frac{1}{3!}\left[3K_{02}^-(h)\frac{\partial a}{\partial \epsilon}(0,h)\frac{\partial^2 a}{\partial \epsilon^2}(0,h) + K_{03}^-(h)\left(\frac{\partial a}{\partial \epsilon}(0,h)\right)^3\right].$$

The proof is completed. □

For $j = 1$, the expression of $M_1(h)$ is given in [25], which is expressed by

$$M_1(h) = Q(h)\Big(- H_1^-(A_0(h)) + H_1^-(A_{10}(h))\Big) + H_1^+(A_0(h)) - H_1^+(A_{10}(h)), \tag{2.38}$$

where $Q(h) = \frac{\frac{\partial H_0^+}{\partial x}(A_0(h))+k_1\frac{\partial H_0^+}{\partial y}(A_0(h))}{\frac{\partial H_0^-}{\partial x}(A_0(h))+k_1\frac{\partial H_0^-}{\partial y}(A_0(h))}$.

To illustrate the impact of one straight line and two semi-lines on the number of limit cycles in a system, we consider the following piecewise polynomial Hamiltonian perturbation of the linear center

$$\begin{cases}\dot{x} = y + \epsilon H_y^+(x,y), \\ \dot{y} = -x - \epsilon H_x^+(x,y),\end{cases} \quad (x,y)\in\Sigma_1,$$
$$\begin{cases}\dot{x} = y + \epsilon H_y^-(x,y), \\ \dot{y} = -x - \epsilon H_x^-(x,y),\end{cases} \quad (x,y)\in\Sigma_2, \tag{2.39}$$

where $0 < \epsilon \ll 1$,

$$\begin{aligned} H^+(x,y) &= a_{01}^+y + a_{02}^+y^2 + a_{11}^+xy + a_{20}^+x^2, \\ H^-(x,y) &= a_{02}^-y^2 + a_{11}^-xy + a_{20}^-x^2. \end{aligned} \tag{2.40}$$

Case 1: $l = l_1 \cup l_2 = \{(x,x)|x \in \mathbb{R}\}$. See Figure 2(a).
Case 2: $l_1 = \{(x,x)|x > 0\}$ and $l_2 = \{(x,-x)|x > 0\}$. See Figure 2(b).

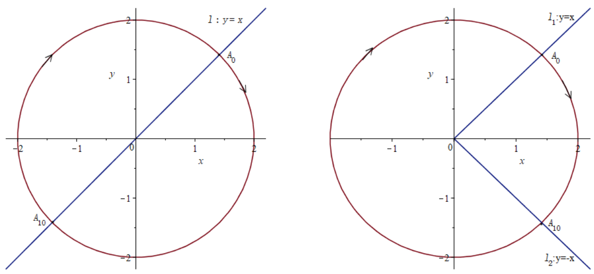
By direct calculation, we have

$$M_1(h) = 2a_{01}^+\sqrt{h} \tag{2.41}$$

for **Case** 1 and

$$M_1(h) = 2a_{01}^{+}\sqrt{h} + 2(a_{11}^{+} - a_{11}^{-})(\sqrt{h})^2 \tag{2.42}$$

for **Case** 2.



(a) (b)

Figure 2. Phase portraits of system $(2.39)|_{\epsilon=0}$ in **Case** 1 and **Case** 2.

From (2.41) and (2.42), one can see that system (2.39) has no limit cycle in **Case** 1 and one limit cycle in **Case** 2 if $a_{01}^{+} < 0$ and $a_{11}^{+} > a_{11}^{-}$.

3. The number of limit cycles of a piecewise polynomial system

In this section, as an application of the main results, we consider a piecewise Hamiltonian polynomial system of the form

$$\begin{cases} \begin{cases} \dot{x} = -y + \epsilon H_{1y}^{+}(x,y) + \epsilon^2 H_{2y}^{+}(x,y) + \epsilon^3 H_{3y}^{+}(x,y) + \epsilon^4 H_{4y}^{+}(x,y), \\ \dot{y} = 1 - x - \epsilon H_{1x}^{+}(x,y) - \epsilon^2 H_{2x}^{+}(x,y) - \epsilon^3 H_{3x}^{+}(x,y) - \epsilon^4 H_{4x}^{+}(x,y), \end{cases} & (x,y) \in \Sigma_1, \\ \begin{cases} \dot{x} = -y + \epsilon H_{1y}^{-}(x,y) + \epsilon^2 H_{2y}^{-}(x,y) + \epsilon^3 H_{3y}^{-}(x,y) + \epsilon^4 H_{4y}^{-}(x,y), \\ \dot{y} = x - \epsilon H_{1x}^{-}(x,y) - \epsilon^2 H_{2x}^{-}(x,y) - \epsilon^3 H_{3x}^{-}(x,y) - \epsilon^4 H_{4x}^{-}(x,y), \end{cases} & (x,y) \in \Sigma_2, \end{cases} \tag{3.1}$$

where $0 < \epsilon \ll 1$, $H_{ix}^{\pm}(x,y)$ and $H_{iy}^{\pm}(x,y)$ represent the partial derivatives of the function $H_i^{\pm}(x,y)$ with respect to x and y, respectively,

$$H_1^{\pm}(x,y) = \sum_{i+j=1}^{m+1} a_{ij}^{\pm}x^iy^j,\ H_2^{\pm}(x,y) = \sum_{i+j=1}^{m+1} b_{ij}^{\pm}x^iy^j, H_3^{\pm}(x,y) = \sum_{i+j=1}^{m+1} c_{ij}^{\pm}x^iy^j,\ H_4^{\pm}(x,y) = \sum_{i+j=1}^{m+1} d_{ij}^{\pm}x^iy^j, \tag{3.2}$$

and Σ_1 and Σ_2 are the regions bounded by the two semi-straight lines $l_1 : y = x,\ x > 0$ and $l_2 : y = -x,\ x > 0$. See Figure 3 for illustration.

Obviously, for (3.1) we have

$$H_0^{+}(x,y) = \frac{1}{2}\left[(x-1)^2 - y^2\right],\ H_0^{-}(x,y) = -\frac{1}{2}(x^2 + y^2).$$

Let

$$H_0(x,y) = \begin{cases} H_0^{+}(x,y),\ (x,y) \in \Sigma_1, \\ H_0^{-}(x,y),\ (x,y) \in \Sigma_2. \end{cases} \tag{3.3}$$

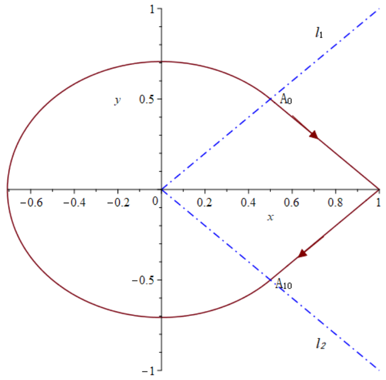


Figure 3. Phase portrait of system $(3.1)|_{\epsilon=0}$.

In this case, we have $A_0(h) = (-\frac{2h-1}{2}, -\frac{2h-1}{2})$ and $A_{10}(h) = (-\frac{2h-1}{2}, \frac{2h-1}{2})$. For any integer m, we further investigate the upper bound of the number of limit cycles by calculating $M_1(h)$, $M_2(h)$, $M_3(h)$, and $M_4(h)$ for piecewise polynomial Hamiltonian system (3.1).

Theorem 2. *If the first order Melnikov function $M_1(h)$ of (3.1) is not zero identically, then for sufficiently small $|\epsilon| > 0$, it has at most $m + 1$ limit cycles bifurcated from the period annulus defined by $\{L_h\}_{h\in J}$, multiplicity taken into account.*

Proof. Below, we give an expression of $M_1(h)$ to estimate the maximum number of limit cycles of system (3.1). According to (2.38), the first order Melnikov function of system (3.1) is

$$M_1(h) = \frac{W_{01}^+(h)}{W_{01}^-(h)}\Big(-v_1^-(h)\Big) + v_1^+(h). \tag{3.4}$$

According to (2.8), we have

$$W_{01}^+(h) = -1,\ W_{01}^-(h) = 2h - 1. \tag{3.5}$$

According to (2.8) and (3.2), we have

$$v_1^\pm(h) = \sum_{i+j=1}^{m+1} a_{ij}^\pm \left(\frac{1}{2}\right)^{i+j} (-1)^i \left[(-1)^j - 1\right](2h-1)^{i+j}. \tag{3.6}$$

Substituting (3.5) and (3.6) into (3.4) yields

$$\begin{aligned} M_1(h) &= \sum_{i+j=1}^{m+1}\left[a_{ij}^-\left(\frac{1}{2}\right)^{i+j}(-1)^i\left[(-1)^j-1\right](2h-1)^{i+j-1} + a_{ij}^+\left(\frac{1}{2}\right)^{i+j}(-1)^i\left[(-1)^j-1\right](2h-1)^{i+j}\right] \\ &= -a_{01}^- + \sum_{k=1}^{m}\left[\frac{(-1)^k}{2^{k-1}}\sum_{\substack{i\ge0,j\ge0,\\ i+2j+1=k}} a_{i,2j+1}^+ + \frac{(-1)^{k+1}}{2^k}\sum_{\substack{i\ge0,j\ge0,\\ i+2j+1=k+1}} a_{i,2j+1}^-\right](2h-1)^k \\ &\quad + \sum_{k=m+1}\left[\frac{(-1)^k}{2^{k-1}}\sum_{\substack{i\ge0,j\ge0,\\ i+2j+1=k}} a_{i,2j+1}^+\right](2h-1)^k. \end{aligned} \tag{3.7}$$

From (3.7) it is easy to observe that $M_1(h)$ has at most $m+1$ zeros in h on $(0,+\infty)$, multiplicity taken into account. Hence, by Lemma 1, system (3.1) has at most n limit cycles bifurcated from the period annulus $\{L_h\}_{h\in J}$, multiplicity taken into account. □

Theorem 3. *Suppose that $M_1(h)\equiv 0$ and $M_2(h)\not\equiv 0$. Then, for sufficiently small $|\epsilon|>0$, system (3.1) has at most $2m$ limit cycles bifurcated from the period annulus defined by $\{L_h\}_{h\in J}$, multiplicity taken into account.*

Proof. When $M_1(h)\equiv 0$, we can obtain from (3.7) that

$$a_{01}^-=0,\ \sum_{\substack{i\ge0,j\ge0,\\ i+2j+1=k}} a_{i,2j+1}^+=\frac{1}{2}\sum_{\substack{i\ge0,j\ge0,\\ i+2j+1=k+1}} a_{i,2j+1}^-,\ k=1,\cdots,m,\ \sum_{\substack{i\ge0,j\ge0,\\ i+2j+1=m+1}} a_{i,2j+1}^+=0.$$

Below, we give an expression of $M_2(h)$ in the case of $M_1(h)\equiv 0$ to estimate the maximum number of limit cycles of system (3.1). According to Theorem 1, the second order Melnikov function of system (3.1) is

$$M_2(h)=M_2^+(h)+\frac{W_{01}^+(h)}{W_{01}^-(h)}M_2^-(h), \tag{3.8}$$

where

$$\begin{aligned} M_2^+(h)&=v_2^+(h)-K_{11}^+(h)\frac{v_1^+(h)}{K_{01}^+(h)}-\frac{K_{02}^+(h)}{2}\Big(\frac{v_1^+(h)}{K_{01}^+(h)}\Big)^2,\\ M_2^-(h)&=-v_2^-(h)+K_{11}^-(h)\frac{v_1^+(h)}{K_{01}^+(h)}+\frac{K_{02}^-(h)}{2}\Big(\frac{v_1^+(h)}{K_{01}^+(h)}\Big)^2. \end{aligned} \tag{3.9}$$

In the following, we calculate the expressions of $K_{01}^+(h)$, $K_{02}^\pm(h)$, $K_{11}^\pm(h)$, and $v_2^\pm(h)$. According to (2.8), (3.2), and (3.3), we have

$$\begin{aligned} K_{01}^+(h)&=\left(\frac{\partial H_0^+}{\partial x}-\frac{\partial H_0^+}{\partial y}\right)\Bigg|_{A_{10}(h)}=-1,\\ K_{02}^+(h)&=\left(\frac{\partial^2 H_0^+}{\partial x^2}-2\frac{\partial^2 H_0^+}{\partial x\partial y}+\frac{\partial^2 H_0^+}{\partial y^2}\right)\Bigg|_{A_{10}(h)}=0,\\ K_{02}^-(h)&=\left(\frac{\partial^2 H_0^-}{\partial x^2}-2\frac{\partial^2 H_0^-}{\partial x\partial y}+\frac{\partial^2 H_0^-}{\partial y^2}\right)\Bigg|_{A_{10}(h)}=-2. \end{aligned} \tag{3.10}$$

According to (2.8) and (3.2), we have

$$v_2^\pm(h)=H_2^\pm(A_0)-H_2^\pm(A_{10})=\sum_{i+j=1}^{m+1} b_{ij}^\pm\left(\frac{1}{2}\right)^{i+j}(-1)^i\left[(-1)^j-1\right](2h-1)^{i+j}. \tag{3.11}$$

Similarly, according to (2.8) and (3.2), we have

$$K_{11}^\pm(h)=\left(\frac{\partial H_1^\pm}{\partial x}-\frac{\partial H_1^\pm}{\partial y}\right)\Bigg|_{A_{10}(h)}=\sum_{i+j=1}^{m+1} a_{ij}^\pm\left(\frac{1}{2}\right)^{i+j-1}(i+j)(-1)^{i-1}(2h-1)^{i+j-1}. \tag{3.12}$$

Substituting (3.10)–(3.12) into (3.8) yields

$$M_2(h) = -\frac{1}{2h-1}\Big[-v_2^-(h) - K_{11}^-(h)v_1^+(h) - (v_1^+(h))^2\Big] + v_2^+(h) + K_{11}^+(h)v_1^+(h) = G_3(h) + G_4(h), \quad (3.13)$$

where

$$G_3(h) = \frac{v_2^-(h)}{2h-1} + v_2^+(h), G_4(h) = \Big(K_{11}^-(h) + (2h-1)K_{11}^+(h) + v_1^+(h)\Big)\frac{v_1^+(h)}{2h-1}.$$

Inserting (3.11) into $G_3(h)$ yields

$$\begin{aligned} G_3(h) = &\sum_{i+j=1}^{m+1}\left[b_{ij}^-\left(\frac{1}{2}\right)^{i+j}(-1)^i\left[(-1)^j-1\right](2h-1)^{i+j-1} + b_{ij}^+\left(\frac{1}{2}\right)^{i+j}(-1)^i\left[(-1)^j-1\right](2h-1)^{i+j}\right] \\ &= -b_{01}^- + \sum_{k=1}^{m}\left[\frac{(-1)^k}{2^{k-1}}\sum_{\substack{i\ge0,j\ge0,\\ i+2j+1=k}} b_{i,2j+1}^+ + \frac{(-1)^{k-1}}{2^k}\sum_{\substack{i\ge0,j\ge0,\\ i+2j+1=k+1}} b_{i,2j+1}^-\right](2h-1)^k \\ &+ \sum_{k=m+1}\left[\frac{(-1)^k}{2^{k-1}}\sum_{\substack{i\ge0,j\ge0,\\ i+2j+1=k}} b_{i,2j+1}^+\right](2h-1)^k \triangleq -b_{01}^- + \sum_{k=1}^{m}\xi_1(k)(2h-1)^k + \sum_{k=m+1}\xi_2(k)(2h-1)^k. \end{aligned} \quad (3.14)$$

Substituting (3.12) and (3.6) into $G_4(h)$ yields

$$\begin{aligned} G_4(h) = &\Big[\sum_{i+j=1}^{m+1} a_{ij}^-(-1)^{i-1}(i+j)\left(\frac{1}{2}\right)^{i+j-1}(2h-1)^{i+j-1} + \sum_{i+j=1}^{m+1} a_{ij}^+\left(\frac{1}{2}\right)^{i+j}(-1)^i\left[(-1)^j-1\right](2h-1)^{i+j} \\ &+ \sum_{i+j=1}^{m+1} a_{ij}^+(-1)^{i-1}(i+j)\left(\frac{1}{2}\right)^{i+j-1}(2h-1)^{i+j}\Big]\sum_{i+j=1}^{m+1} a_{ij}^+\left(\frac{1}{2}\right)^{i+j}(-1)^i\left[(-1)^j-1\right](2h-1)^{i+j-1} \\ &\triangleq G_{41}(h)G_{42}(h), \end{aligned} \quad (3.15)$$

where

$$\begin{aligned} G_{41}(h) = &\sum_{k=0}^{m}\left[\frac{(-1)^{k-1}k}{2^{k-1}}\sum_{i+2j=k} a_{i,2j}^+ + \frac{(-1)^k(k+1)}{2^k}\sum_{i+2j=k+1} a_{i,2j}^-\right](2h-1)^k \\ &+ \sum_{k=m+1}\left[\frac{(-1)^{k-1}k}{2^{k-1}}\sum_{i+2j=k} a_{i,2j}^+\right](2h-1)^k \triangleq \sum_{k=0}^{m}\xi_4(k)(2h-1)^k + \sum_{k=m+1}\xi_5(k)(2h-1)^k \end{aligned} \quad (3.16)$$

and

$$G_{42}(h) = \sum_{k=0}^{m-1}\left[\frac{(-1)^{k+1}}{2^k}\sum_{i+2j+1=k+1} a_{i,2j+1}^+\right](2h-1)^k \triangleq \sum_{k=0}^{m-1}\xi_3(k)(2h-1)^k. \quad (3.17)$$

Substituting (3.16) and (3.17) into (3.15) yields

$$
\begin{aligned}
G_4(h) &= \left\{ \sum_{k=0}^{m} \xi_4(k)(2h-1)^k + \sum_{k=m+1} \xi_5(k)(2h-1)^k \right\} \sum_{k=0}^{m-1} \xi_3(k)(2h-1)^k \\
&= \sum_{k=0}^{2m-1} \left[\sum_{\substack{k_1+k_3=k,\\ 0\le k_1\le m,\\ 0\le k_3\le m-1}} \xi_4(k_1)\xi_3(k_3) \right] (2h-1)^k + \sum_{k=m+1}^{2m} \left[\sum_{\substack{k_2+k_3=k,\\ k_2=m+1,\\ 0\le k_3\le m-1}} \xi_5(k_2)\xi_3(k_3) \right] (2h-1)^k \\
&= -a_{01}^{+}a_{10}^{-} + \sum_{k=1}^{m} \left[\sum_{\substack{k_1+k_3=k,\\ 0\le k_1\le m,\\ 0\le k_3\le m-1}} \xi_4(k_1)\xi_3(k_3) \right] (2h-1)^k + \sum_{k=m+1} \left[\sum_{\substack{k_1+k_3=k,\\ 0\le k_1\le m,\\ 0\le k_3\le m-1}} \xi_4(k_1)\xi_3(k_3) \right. \\
&\quad \left. + \sum_{\substack{k_2+k_3=k,\\ k_2=m+1,\\ 0\le k_3\le m-1}} \xi_5(k_2)\xi_3(k_3) \right] (2h-1)^k + \sum_{k=m+2}^{2m-1} \left[\sum_{\substack{k_1+k_3=k,\\ 0\le k_1\le m,\\ 0\le k_3\le m-1}} \xi_4(k_1)\xi_3(k_3) + \sum_{\substack{k_2+k_3=k,\\ k_2=m+1,\\ 0\le k_3\le m-1}} \xi_5(k_2)\xi_3(k_3) \right] (2h-1)^k \\
&\quad + \sum_{k=2m} \left[\sum_{\substack{k_2+k_3=k,\\ k_2=m+1,\\ 0\le k_3\le m-1}} \xi_5(k_2)\xi_3(k_3) \right] (2h-1)^k.
\end{aligned}
\tag{3.18}
$$

Inserting (3.14) and (3.18) into (3.13) yields

$$
\begin{aligned}
M_2(h) = -b_{01}^{-} - a_{01}^{+}a_{10}^{-} &+ \sum_{k=1}^{m} \left[\xi_1(k) + \sum_{\substack{k_1+k_3=k,\\ 0\le k_1\le m,\\ 0\le k_3\le m-1}} \xi_4(k_1)\xi_3(k_3) \right] (2h-1)^k \\
&+ \sum_{k=m+1} \left[\xi_2(k) + \sum_{\substack{k_1+k_3=k,\\ 0\le k_1\le m,\\ 0\le k_3\le m-1}} \xi_4(k_1)\xi_3(k_3) + \sum_{\substack{k_2+k_3=k,\\ k_2=m+1,\\ 0\le k_3\le m-1}} \xi_5(k_2)\xi_3(k_3) \right] (2h-1)^k \\
&+ \sum_{k=m+2}^{2m-1} \left[\sum_{\substack{k_1+k_3=k,\\ 0\le k_1\le m,\\ 0\le k_3\le m-1}} \xi_4(k_1)\xi_3(k_3) + \sum_{\substack{k_2+k_3=k,\\ k_2=m+1,\\ 0\le k_3\le m-1}} \xi_5(k_2)\xi_3(k_3) \right] (2h-1)^k \\
&+ \sum_{k=2m} \left[\sum_{\substack{k_2+k_3=k,\\ k_2=m+1,\\ k_3=m-1}} \xi_5(k_2)\xi_3(k_3) \right] (2h-1)^k,
\end{aligned}
\tag{3.19}
$$

where

$$\xi_1(k) = \frac{(-1)^k}{2^{k-1}} \sum_{\substack{i\geq 0, j\geq 0,\\ i+2j+1=k}} b^+_{i,2j+1} + \frac{(-1)^{k-1}}{2^k} \sum_{\substack{i\geq 0, j\geq 0,\\ i+2j+1=k+1}} b^-_{i,2j+1},\ 1 \leq k \leq m,$$

$$\xi_2(k) = \frac{(-1)^k}{2^{k-1}} \sum_{\substack{i\geq 0, j\geq 0,\\ i+2j+1=k}} b^+_{i,2j+1},\ k = m+1,$$

$$\xi_3(k) = \frac{(-1)^{k+1}}{2^k} \sum_{i+2j+1=k+1} a^+_{i,2j+1},\ 0 \leq k \leq m-1,$$

$$\xi_4(k) = \frac{(-1)^{k-1}k}{2^{k-1}} \sum_{i+2j=k} a^+_{i,2j} + \frac{(-1)^k(k+1)}{2^k} \sum_{i+2j=k+1} a^-_{i,2j},\ 0 \leq k \leq m,$$

$$\xi_5(k) = \frac{(-1)^{k-1}k}{2^{k-1}} \sum_{i+2j=k} a^+_{i,2j},\ k = m+1.$$

Thus, according to Lemma 1 and (3.19), system (3.1) has at most $2m$ limit cycles bifurcated from the period annulus defined by $\{L_h\}_{h\in J}$ by the second order Melnikov function, multiplicity taken into account. □

Theorem 4. *Suppose that $M_1(h) = M_2(h) \equiv 0$ and $M_3(h) \not\equiv 0$. Then, for sufficiently small $|\epsilon| > 0$, system (3.1) has at most $3m-1$ limit cycles bifurcated from the period annulus defined by $\{L_h\}_{h\in J}$, multiplicity taken into account.*

Proof. When $M_2(h) \equiv 0$, we can obtain from (3.19) that

$$\begin{aligned}
&-b^-_{01} - a^+_{01}a^-_{10} = 0,\\
&\xi_1(k) + \sum_{\substack{k_1+k_3=k,\\ 0\leq k_1\leq m,\\ 0\leq k_3\leq m-1}} \xi_4(k_1)\xi_3(k_3) = 0,\ 1 \leq k \leq m,\\
&\xi_2(k) + \sum_{\substack{k_1+k_3=k,\\ 0\leq k_1\leq m,\\ 0\leq k_3\leq m-1}} \xi_4(k_1)\xi_3(k_3) + \sum_{\substack{k_2+k_3=k,\\ k_2=m+1,\\ 0\leq k_3\leq m-1}} \xi_5(k_2)\xi_3(k_3) = 0,\ k = m+1,\\
&\sum_{\substack{k_1+k_3=k,\\ 0\leq k_1\leq m,\\ 0\leq k_3\leq m-1}} \xi_4(k_1)\xi_3(k_3) + \sum_{\substack{k_2+k_3=k,\\ k_2=m+1,\\ 0\leq k_3\leq m-1}} \xi_5(k_2)\xi_3(k_3),\ m+2 \leq k \leq 2m-1,\\
&\xi_5(m+1)\xi_3(m-1) = 0.
\end{aligned}$$

Below, we give an expression of $M_3(h)$ in the case of $M_1(h) = M_2(h) \equiv 0$ to estimate the maximum number of limit cycles of system (3.1). According to Theorem 1, the third order Melnikov function of system (3.1) is

$$M_3(h) = M_3^+(h) + \frac{W^+_{01}(h)}{W^-_{01}(h)} M_3^-(h), \tag{3.20}$$

where

$$M_3^+(h) = v_3^+(h) - K_{21}^+(h)\frac{v_1^+(h)}{K_{01}^+(h)} - \frac{K_{12}^+(h)}{2}\Big(\frac{v_1^+(h)}{K_{01}^+(h)}\Big)^2 - \frac{K_{03}^+(h)}{6}\Big(\frac{v_1^+(h)}{K_{01}^+(h)}\Big)^3 - \frac{K_{11}^+(h)}{K_{01}^+(h)}M_2^+(h) - \frac{K_{02}^+(h)}{K_{01}^+(h)}\frac{v_1^+(h)}{K_{01}^+(h)}M_2^+(h),$$

$$M_3^-(h) = -v_3^-(h) + K_{21}^-(h)\frac{v_1^+(h)}{K_{01}^+(h)} + \frac{K_{12}^-(h)}{2}\Big(\frac{v_1^+(h)}{K_{01}^+(h)}\Big)^2 + \frac{K_{03}^-(h)}{6}\Big(\frac{v_1^+(h)}{K_{01}^+(h)}\Big)^3 + \frac{K_{11}^-(h)}{K_{01}^+(h)}M_2^+(h) + \frac{K_{02}^-(h)}{K_{01}^+(h)}\frac{v_1^+(h)}{K_{01}^+(h)}M_2^+(h).$$

In the following, we calculate the expressions of $K_{12}^+(h)$, $K_{21}^{\pm}(h)$, $v_3^{\pm}(h)$, and $M_2^+(h)$ based on (2.8) and (3.3).

According to (2.8) and (3.2), we have

$$v_3^{\pm}(h) = H_3^{\pm}(A_0) - H_3^{\pm}(A_{10}) = \sum_{i+j=1}^{m+1} c_{ij}^{\pm}\left(\frac{1}{2}\right)^{i+j}(-1)^i\left[(-1)^j - 1\right](2h-1)^{i+j}. \tag{3.21}$$

We can directly obtain from (2.8) and (3.2) that

$$K_{21}^{\pm}(h) = \left(\frac{\partial H_2^{\pm}}{\partial x} - \frac{\partial H_2^{\pm}}{\partial y}\right)\bigg|_{A_{10}(h)} = \sum_{i+j=1}^{m+1} b_{ij}^{\pm}(-1)^{i-1}(i+j)\left(\frac{1}{2}\right)^{i+j-1}(2h-1)^{i+j-1}. \tag{3.22}$$

According to (2.8) and (3.2), we have

$$\begin{aligned} K_{12}^{\pm}(h) &= \left(\frac{\partial^2 H_1^{\pm}}{\partial x^2} - 2\frac{\partial^2 H_1^{\pm}}{\partial x\partial y} + \frac{\partial^2 H_1^{\pm}}{\partial y^2}\right)\bigg|_{A_{10}(h)} \\ &= \sum_{i+j=2}^{m+1}\left(a_{ij}^{\pm}\left(\frac{1}{2}\right)^{i+j-2}(-1)^i(i+j-1)(i+j)\right)(2h-1)^{i+j-2}. \end{aligned} \tag{3.23}$$

According to (3.3), we have

$$K_{03}^{\pm}(h) = \left(\frac{\partial^3 H_0^{\pm}}{\partial x^3} - 3\frac{\partial^3 H_0^{\pm}}{\partial x^2\partial y} + 3\frac{\partial^3 H_0^{\pm}}{\partial x\partial y^2} - \frac{\partial^3 H_0^{\pm}}{\partial y^3}\right)\bigg|_{A_{10}(h)} = 0. \tag{3.24}$$

Substituting $K_{01}^+(h)$ and $K_{02}^+(h)$ in (3.10) into $M_2^+(h)$ in (3.9) yields

$$\begin{aligned} M_2^+(h) =& v_2^+(h) + K_{11}^+(h)v_1^+(h) \\ =& \sum_{i+j=1}^{m+1} b_{ij}^+\left(\frac{1}{2}\right)^{i+j}(-1)^i\left[(-1)^j - 1\right](2h-1)^{i+j} + \sum_{i+j=1}^{m+1} a_{ij}^+(-1)^{i-1}(i+j)\left(\frac{1}{2}\right)^{i+j-1}(2h-1)^{i+j-1} \\ &\times \sum_{i+j=1}^{m+1} a_{ij}^+\left(\frac{1}{2}\right)^{i+j}(-1)^i\left[(-1)^j - 1\right](2h-1)^{i+j}. \end{aligned} \tag{3.25}$$

Substituting (3.10) and (3.24) into (3.20) yields

$$\begin{aligned}M_3(h) =&\frac{W_{01}^+(h)}{W_{01}^-(h)}\Big[-v_3^-(h) - K_{21}^-(h)v_1^+(h) + \frac{1}{2}K_{12}^-(h)(v_1^+(h))^2 - K_{11}^-(h)M_2^+(h) - 2v_1^+(h)M_2^+(h)\Big]\\&+ v_3^+(h) + K_{21}^+(h)v_1^+(h) - \frac{1}{2}K_{12}^+(h)(v_1^+(h))^2 + K_{11}^+(h)M_2^+(h)\\\triangleq&\widetilde{H}_1(h) + \widetilde{H}_2(h) + \widetilde{H}_3(h) + \widetilde{H}_4(h),\end{aligned} \tag{3.26}$$

where

$$\begin{aligned}\widetilde{H}_1(h) &= -\frac{W_{01}^+(h)}{W_{01}^-(h)}v_3^-(h) + v_3^+(h),\\\widetilde{H}_2(h) &= \left(-\frac{W_{01}^+(h)}{W_{01}^-(h)}K_{21}^-(h) + K_{21}^+(h)\right)v_1^+(h),\\\widetilde{H}_3(h) &= \frac{1}{2}\left(\frac{W_{01}^+(h)}{W_{01}^-(h)}K_{12}^-(h) - K_{12}^+(h)\right)(v_1^+(h))^2,\\\widetilde{H}_4(h) &= \left[-\frac{W_{01}^+(h)}{W_{01}^-(h)}\Big(K_{11}^-(h) + 2v_1^+(h)\Big) + K_{11}^+(h)\right]M_2^+(h).\end{aligned} \tag{3.27}$$

In the following, we will give an expression of $M_3(h)$ by evaluating $\widetilde{H}_1(h)$, $\widetilde{H}_2(h)$, $\widetilde{H}_3(h)$, and $\widetilde{H}_4(h)$ one by one.

Inserting (3.21) and (3.51) into the first equation in (3.27) yields

$$\begin{aligned}\widetilde{H}_1(h) &= \frac{1}{2h-1}v_3^-(h) + v_3^+(h)\\&= \sum_{i+j=1}^{m+1}\left[c_{ij}^-\left(\frac{1}{2}\right)^{i+j}(-1)^i\left[(-1)^j - 1\right](2h-1)^{i+j-1} + c_{ij}^+\left(\frac{1}{2}\right)^{i+j}(-1)^i\left[(-1)^j - 1\right](2h-1)^{i+j}\right]\\&= -c_{01}^- + \sum_{k=1}^{m}\left[\frac{(-1)^k}{2^{k-1}}\sum_{\substack{i\ge0,j\ge0,\\i+2j+1=k}}c_{i,2j+1}^+ + \frac{(-1)^{k+1}}{2^k}\sum_{\substack{i\ge0,j\ge0,\\i+2j+1=k+1}}c_{i,2j+1}^-\right](2h-1)^k\\&\quad+ \sum_{k=m+1}\left[\frac{(-1)^k}{2^{k-1}}\sum_{\substack{i\ge0,j\ge0,\\i+2j+1=k}}c_{i,2j+1}^+\right](2h-1)^k\\&\triangleq -c_{01}^- + \sum_{k=1}^{m}\mathcal{I}_1(k)(2h-1)^k + \sum_{k=m+1}\mathcal{I}_2(k)(2h-1)^k.\end{aligned} \tag{3.28}$$

We can obtain from (3.28) that

$$\deg\left(\widetilde{H}_1(h)\right) = m + 1. \tag{3.29}$$

Substituting (3.5) into the second equation in (3.27) gives

$$\widetilde{H}_2(h) = \Big(K_{21}^-(h) + (2h-1)K_{21}^+(h)\Big)\frac{v_1^+(h)}{2h-1} = \widetilde{H}_{21}(h)\widetilde{H}_{22}(h). \tag{3.30}$$

For $\widetilde{H}_{21}(h)$, substituting (3.22) into (3.30) yields

$$\begin{aligned}\widetilde{H}_{21}(h) &= \sum_{i+j=1}^{m+1}\left[b_{ij}^-(-1)^{i-1}(i+j)\left(\frac{1}{2}\right)^{i+j-1}(2h-1)^{i+j-1} + b_{ij}^+(-1)^{i-1}(i+j)\left(\frac{1}{2}\right)^{i+j-1}(2h-1)^{i+j}\right]\\ &= \sum_{k=0}^{m}\left[\frac{k}{2^{k-1}}\sum_{i+j=k}(-1)^{i-1}b_{ij}^+ + \frac{k+1}{2^k}\sum_{i+j=k+1}(-1)^{i-1}b_{ij}^-\right](2h-1)^k\\ &+ \sum_{k=m+1}\left[\frac{k}{2^{k-1}}\sum_{i+j=k}(-1)^{i-1}b_{ij}^+\right](2h-1)^k\\ &\triangleq \sum_{k=0}^{m}\widetilde{L}(k)(2h-1)^k + \sum_{k=m+1}\widetilde{N}(k)(2h-1)^k.\end{aligned} \tag{3.31}$$

We can obtain from (3.31) that

$$\deg\Big(\widetilde{H}_{21}(h)\Big) = m+1. \tag{3.32}$$

For $\widetilde{H}_{22}(h)$, similar to $G_{42}(h)$, we have

$$\widetilde{H}_{22}(h) = G_{42}(h) = \sum_{k=0}^{m-1}\xi_3(k)(2h-1)^k. \tag{3.33}$$

We can obtain from (3.33) that

$$\deg\Big(\widetilde{H}_{22}(h)\Big) = m-1. \tag{3.34}$$

Substituting (3.31) and (3.33) into (3.30) gives

$$\deg\Big(\widetilde{H}_2(h)\Big) = 2m. \tag{3.35}$$

For $\widetilde{H}_3(h)$, inserting (3.10) into $\widetilde{H}_3(h)$ in (3.27) yields

$$\begin{aligned}\widetilde{H}_3(h) &= \frac{1}{2}\left(-\frac{1}{2h-1}K_{12}^-(h) - K_{12}^+(h)\right)(v_1^+(h))^2\\ &= \left(-\frac{1}{2}K_{12}^-(h) - \frac{2h-1}{2}K_{12}^+(h)\right)\frac{v_1^+(h)}{2h-1}v_1^+(h)\\ &= \widetilde{H}_{31}(h)\widetilde{H}_{32}(h)\widetilde{H}_{33}(h).\end{aligned} \tag{3.36}$$

In the following, we give $\widetilde{H}_{31}(h)$, $\widetilde{H}_{32}(h)$, and $\widetilde{H}_{33}(h)$ one by one. Substituting (3.23) into $\widetilde{H}_{31}(h)$ yields

$$
\begin{aligned}
\widetilde{H}_{31}(h) = & \sum_{i+j=2}^{m+1} \left[(-1)^{i-1} \left(\frac{1}{2}\right)^{i+j-1} (i+j-1)(i+j) a_{ij}^{-} (2h-1)^{i+j-2} \right. \\
& \left. + (-1)^{i-1} \left(\frac{1}{2}\right)^{i+j-1} (i+j-1)(i+j) a_{ij}^{+} (2h-1)^{i+j-1} \right] \\
= & \sum_{k=0}^{m-1} \left[\frac{(-1)^k k(k+1)}{2^k} \sum_{i+2j=k+1} a_{i,2j}^{+} + \frac{(-1)^k (k+1)}{2^k} \sum_{i+2j+1=k+2} a_{i,2j+1}^{-} \right. \\
& \left. + \frac{(-1)^{k+1}(k+1)(k+2)}{2^{k+1}} \sum_{i+2j=k+2} a_{i,2j}^{-} \right] (2h-1)^k \\
& + \sum_{k=m} \left[\frac{(-1)^k k(k+1)}{2^k} \sum_{i+2j=k+1} a_{i,2j}^{+} \right] (2h-1)^k \\
\triangleq & \sum_{k=0}^{m-1} \widetilde{P}(k)(2h-1)^k + \sum_{k=m} \widetilde{Q}(k)(2h-1)^k.
\end{aligned}
\tag{3.37}
$$

We can obtain from (3.37) that

$$
\deg\left(\widetilde{H}_{31}(h)\right) = m. \tag{3.38}
$$

For $\widetilde{H}_{32}(h)$, similar to $G_{42}(h)$, we have

$$
\widetilde{H}_{32}(h) = G_{42}(h) = \sum_{k=0}^{m-1} \xi_3(k)(2h-1)^k.
$$

From the above equality, we can obtain

$$
\deg\left(\widetilde{H}_{32}(h)\right) = m-1. \tag{3.39}
$$

Inserting (3.23) into $\widetilde{H}_{33}(h)$ gives us

$$
\begin{aligned}
\widetilde{H}_{33}(h) = & \sum_{i+j=1}^{m+1} a_{ij}^{+} \left(\frac{1}{2}\right)^{i+j} (-1)^i \left[(-1)^j - 1\right] (2h-1)^{i+j} \\
= & \sum_{k=1}^{m} \left[\frac{(-1)^k}{2^{k-1}} \sum_{i+2j+1=k} a_{i,2j+1}^{+} \right] (2h-1)^k = \sum_{k=1}^{m} R(k)(2h-1)^k.
\end{aligned}
\tag{3.40}
$$

From (3.40), we can obtain

$$
\deg\left(\widetilde{H}_{33}(h)\right) = m. \tag{3.41}
$$

Substituting (3.37) and (3.40) into (3.36) yields

$$
\deg\left(\widetilde{H}_3(h)\right) = \deg\left(\widetilde{H}_{31}(h)\right) + \deg\left(\widetilde{H}_{32}(h)\right) + \deg\left(\widetilde{H}_{33}(h)\right) = 3m-1. \tag{3.42}
$$

For $\widetilde{H}_4(h)$, inserting (3.10) into $H_4(h)$ in (3.27) yields

$$\begin{aligned}\widetilde{H}_4(h) &= \left[\frac{1}{2h-1}\Big(K_{11}^-(h) + 2v_1^+(h)\Big) + K_{11}^+(h)\right] M_2^+(h)\\ &= \left[\Big(K_{11}^-(h) + 2v_1^+(h)\Big) + (2h-1)K_{11}^+(h)\right] \frac{M_2^+(h)}{2h-1}\\ &= \widetilde{H}_{41}(h)\widetilde{H}_{42}(h).\end{aligned} \tag{3.43}$$

Plugging (3.12) into $\widetilde{H}_{41}(h)$ yields

$$\begin{aligned}\widetilde{H}_{41}(h) = &\sum_{i+j=1}^{m+1}\Bigg[a_{ij}^-(-1)^{i-1}(i+j)\left(\frac{1}{2}\right)^{i+j-1}(2h-1)^{i+j-1} + a_{ij}^+\left(\frac{1}{2}\right)^{i+j-1}(-1)^i\left[(-1)^j - 1\right](2h-1)^{i+j}\\ &+ a_{ij}^+(-1)^{i-1}(i+j)\left(\frac{1}{2}\right)^{i+j-1}(2h-1)^{i+j}\Bigg]\\ &= a_{10}^- + \sum_{k=1}^{m}\left[\frac{(-1)^{k-1}k}{2^{k-1}}\sum_{i+2j=k} a_{i,2j}^+ + \frac{(-1)^k}{2^k}\sum_{i+2j+1=k+1} a_{i,2j+1}^- + \frac{(-1)^k(k+1)}{2^k}\sum_{i+2j=k+1} a_{i,2j}^-\right](2h-1)^k\\ &+ \sum_{k=m+1}\left[\frac{(-1)^{k-1}k}{2^{k-1}}\sum_{i+2j=k} a_{i,2j}^+\right](2h-1)^k \triangleq a_{10}^- + \sum_{k=1}^{m}\widetilde{S}(k)(2h-1)^k + \sum_{k=m+1}\xi_5(k)(2h-1)^k.\end{aligned} \tag{3.44}$$

From (3.44), we can obtain

$$\deg\Big(\widetilde{H}_{41}(h)\Big) = m+1. \tag{3.45}$$

For $\widetilde{H}_{42}(h)$, we have

$$\begin{aligned}\widetilde{H}_{42}(h) = &\frac{v_2^+(h)}{2h-1} + K_{11}^+\cdot\frac{v_1^+(h)}{2h-1} = \sum_{k=0}^{m}\left[\frac{(-1)^{k+1}}{2^k}\sum_{i+2j+1=k+1} b_{i,2j+1}^+\right](2h-1)^k\\ &+ \Bigg(\sum_{k=0}^{m-1}\left[\frac{(-1)^k(k+1)}{2^k}\sum_{i+2j=k+1} a_{i,2j}^+ + \frac{(-1)^{k+1}(k+1)}{2^{k+1}}\sum_{i+2j+1=k+2} a_{i,2j+1}^-\right](2h-1)^k\\ &+ \sum_{k=m}\left[\frac{(-1)^k(k+1)}{2^k}\sum_{i+2j=k+1} a_{i,2j}^+\right](2h-1)^k\Bigg)\\ &\times\sum_{k=0}^{m-1}\left[\frac{(-1)^{k+1}}{2^k}\sum_{i+2j+1=k+1} a_{i,2j+1}^+\right](2h-1)^k\\ &\triangleq \sum_{k=0}^{m} U(k)(2h-1)^k + \left(\sum_{k=0}^{m-1} V_1(k)(2h-1)^k + \sum_{k=m} V_2(k)(2h-1)^k\right)\cdot\sum_{k=0}^{m-1}\xi_3(k)(2h-1)^k.\end{aligned} \tag{3.46}$$

From (3.46), we can obtain

$$\deg\Big(\widetilde{H}_{42}(h)\Big) = 2m-1. \tag{3.47}$$

Due to $\xi_5(m+1)\xi_3(m-1)=0$, substituting (3.44) and (3.46) into (3.43) gives us

$$\deg\left(\widetilde{H}_4(h)\right)=\deg\left(\widetilde{H}_{41}(h)\right)+\deg\left(\widetilde{H}_{42}(h)\right)=3m-1. \tag{3.48}$$

Substituting (3.28), (3.35), (3.42), and (3.48) into (3.26) yields

$$\begin{aligned}\deg\left(M_3(h)\right)&=\max\left\{\deg\left(\widetilde{H}_1(h)\right),\deg\left(\widetilde{H}_2(h)\right),\deg\left(\widetilde{H}_3(h)\right),\deg\left(\widetilde{H}_4(h)\right)\right\}\\&=\max\{m+1,2m,3m-1,3m-1\}\\&=3m-1.\end{aligned} \tag{3.49}$$

Thus, according to Lemma 1 and (3.49), system (3.1) has at most $3m-1$ limit cycles bifurcated from the period annulus defined by $\{L_h\}_{h\in J}$ by the third order Melnikov function, multiplicity taken into account. □

Theorem 5. *Suppose that $M_1(h)=M_2(h)=M_3(h)\equiv 0$ and $M_4(h)\not\equiv 0$. Then, for sufficiently small $|\epsilon|>0$, system (3.1) has at most $4m$ limit cycles bifurcated from the period annulus defined by $\{L_h\}_{h\in J}$, multiplicity taken into account.*

Proof. Below, we give an expression of $M_4(h)$ in the case of $M_1(h)=M_2(h)=M_3(h)\equiv 0$ to estimate the maximum number of limit cycles of system (3.1). According to Theorem 1, the fourth order Melnikov function of system (3.1) is

$$M_4(h)=M_4^+(h)+\frac{W_{01}^+(h)}{W_{01}^-(h)}M_4^-(h), \tag{3.50}$$

where

$$\begin{aligned}M_4^+(h)=&\,v_4^+(h)-K_{31}^+(h)\frac{v_1^+(h)}{K_{01}^+(h)}-\frac{K_{22}^+(h)}{2}\Big(\frac{v_1^+(h)}{K_{01}^+(h)}\Big)^2-\frac{K_{13}^+(h)}{6}\Big(\frac{v_1^+(h)}{K_{01}^+(h)}\Big)^3-\frac{K_{04}^+(h)}{24}\Big(\frac{v_1^+(h)}{K_{01}^+(h)}\Big)^4\\&-\frac{K_{21}^+(h)}{K_{01}^+(h)}M_2^+(h)-\frac{K_{12}^+(h)}{K_{01}^+(h)}\frac{v_1^+(h)}{K_{01}^+(h)}M_2^+(h)-\frac{1}{2}\frac{K_{03}^+(h)}{K_{01}^+(h)}\Big(\frac{v_1^+(h)}{K_{01}^+(h)}\Big)^2M_2^+(h)\\&-\frac{K_{11}^+(h)}{K_{01}^+(h)}M_3^+(h)-\frac{K_{02}^+(h)}{2}\Big(\frac{M_2^+(h)}{K_{01}^+(h)}\Big)^2-\frac{K_{02}^+(h)}{K_{01}^+(h)}\frac{v_1^+(h)}{K_{01}^+(h)}M_3^+(h),\end{aligned}$$

$$\begin{aligned}M_4^-(h)=&-v_4^-(h)+K_{31}^-(h)\frac{v_1^+(h)}{K_{01}^+(h)}+\frac{K_{22}^-(h)}{2}\Big(\frac{v_1^+(h)}{K_{01}^+(h)}\Big)^2+\frac{K_{13}^-(h)}{6}\Big(\frac{v_1^+(h)}{K_{01}^+(h)}\Big)^3+\frac{K_{04}^-(h)}{24}\Big(\frac{v_1^+(h)}{K_{01}^+(h)}\Big)^4\\&+\frac{K_{21}^-(h)}{K_{01}^+(h)}M_2^+(h)+\frac{K_{12}^-(h)}{K_{01}^+(h)}\frac{v_1^+(h)}{K_{01}^+(h)}M_2^+(h)+\frac{1}{2}\frac{K_{03}^-(h)}{K_{01}^+(h)}\Big(\frac{v_1^+(h)}{K_{01}^+(h)}\Big)^2M_2^+(h)\\&+\frac{K_{11}^-(h)}{K_{01}^+(h)}M_3^+(h)+\frac{K_{02}^-(h)}{2}\Big(\frac{M_2^+(h)}{K_{01}^+(h)}\Big)^2+\frac{K_{02}^-(h)}{K_{01}^+(h)}\frac{v_1^+(h)}{K_{01}^+(h)}M_3^+(h).\end{aligned}$$

In the following, we calculate $v_4^\pm(h)$, $K_{13}^\pm(h)$, $K_{22}^\pm(h)$, $K_{31}^\pm(h)$, and $M_3^+(h)$ based on (2.8) and (3.3). According to (2.8) and (3.2), we have

$$v_4^\pm(h)=H_4^\pm(A_0)-H_4^\pm(A_{10})=\sum_{i+j=1}^{m+1}d_{ij}^\pm\left(\frac{1}{2}\right)^{i+j}(-1)^i\left[(-1)^j-1\right](2h-1)^{i+j}. \tag{3.51}$$

According to (2.8) and (3.2), we have

$$\begin{aligned} K_{13}^{\pm}(h) &= \left(\frac{\partial^3 H_1^{\pm}}{\partial x^3} - 3\frac{\partial^3 H_1^{\pm}}{\partial x^2 \partial y} + 3\frac{\partial^3 H_1^{\pm}}{\partial x \partial y^2} - \frac{\partial^3 H_1^{\pm}}{\partial y^3}\right)\Bigg|_{A_{10}(h)} \\ &= \sum_{i+j=3}^{m+1}\left[a_{ij}^{\pm}\left(\frac{1}{2}\right)^{i+j-3}(-1)^{i-1}\Big(i(i-1)(i-2)+3ij(i+j-2)+j(j-1)(j-2)\Big)\right](2h-1)^{i+j-3}. \end{aligned} \tag{3.52}$$

According to (2.8) and (3.2), similar to the derivation of (3.23), we have

$$\begin{aligned} K_{22}^{\pm}(h) &= \left(\frac{\partial^2 H_2^{\pm}}{\partial x^2} - 2\frac{\partial^2 H_2^{\pm}}{\partial x \partial y} + \frac{\partial^2 H_2^{\pm}}{\partial y^2}\right)\Bigg|_{A_{10}(h)} \\ &= \sum_{i+j=2}^{m+1}\left(b_{ij}^{\pm}\left(\frac{1}{2}\right)^{i+j-2}(-1)^{i}(i+j-1)(i+j)\right)(2h-1)^{i+j-2}. \end{aligned} \tag{3.53}$$

Similar to (3.12) and (3.22), from (2.8) and (3.2), we can directly obtain

$$K_{31}^{\pm}(h) = \left(\frac{\partial H_3^{\pm}}{\partial x} - \frac{\partial H_3^{\pm}}{\partial y}\right)\Bigg|_{A_{10}(h)} = \sum_{i+j=1}^{m+1} c_{ij}^{\pm}(-1)^{i-1}(i+j)\left(\frac{1}{2}\right)^{i+j-1}(2h-1)^{i+j-1}. \tag{3.54}$$

For $M_3^+(h)$, according to (3.20),

$$M_3^+(h) = v_3^+(h) + K_{21}^+(h)v_1^+(h) - \frac{1}{2}K_{12}^+(h)\Big(v_1^+(h)\Big)^2 + K_{11}^+(h)M_2^+(h). \tag{3.55}$$

Returning our attention to (3.50), substituting $K_{01}^+(h)$, $K_{02}^{\pm}(h)$, $K_{03}^{\pm}(h)$, and $K_{04}^{\pm}(h)$ into (3.50) yields

$$M_4(h) = \widetilde{Y}_1(h) + \widetilde{Y}_2(h) + \widetilde{Y}_3(h) + \widetilde{Y}_4(h) + \widetilde{Y}_5(h) + \widetilde{Y}_6(h) + \widetilde{Y}_7(h) + \widetilde{Y}_8(h), \tag{3.56}$$

where

$$\begin{aligned} \widetilde{Y}_1(h) &= -\frac{W_{01}^+(h)}{W_{01}^-(h)}v_4^-(h) + v_4^+(h), \\ \widetilde{Y}_2(h) &= \left(-\frac{W_{01}^+(h)}{W_{01}^-(h)}K_{31}^-(h) + K_{31}^+(h)\right)v_1^+(h), \\ \widetilde{Y}_3(h) &= \frac{1}{2}\left(\frac{W_{01}^+(h)}{W_{01}^-(h)}K_{22}^-(h) - K_{22}^+(h)\right)(v_1^+(h))^2, \\ \widetilde{Y}_4(h) &= \frac{1}{6}\left(-\frac{W_{01}^+(h)}{W_{01}^-(h)}K_{13}^-(h) + K_{13}^+(h)\right)(v_1^+(h))^3, \\ \widetilde{Y}_5(h) &= \left(-\frac{W_{01}^+(h)}{W_{01}^-(h)}K_{21}^-(h) + K_{21}^+(h)\right)M_2^+(h), \\ \widetilde{Y}_6(h) &= \left(\frac{W_{01}^+(h)}{W_{01}^-(h)}K_{12}^-(h) - K_{12}^+(h)\right)v_1^+(h)M_2^+(h), \\ \widetilde{Y}_7(h) &= -\frac{W_{01}^+(h)}{W_{01}^-(h)}(M_2^+(h))^2, \\ \widetilde{Y}_8(h) &= \left[-\frac{W_{01}^+(h)}{W_{01}^-(h)}\Big(K_{11}^-(h) + 2v_1^+(h)\Big) + K_{11}^+(h)\right]M_3^+(h). \end{aligned}$$

In the following, we will give an expression for $M_4(h)$ by evaluating $Y_i(h)$, $(i = 1, \cdots, 8)$.

For $\widetilde{Y}_1(h)$, substituting (3.5) and (3.51) into $\widetilde{Y}_1(h)$ yields

$$\begin{aligned}
\widetilde{Y}_1(h) &= \frac{1}{2h-1} v_4^-(h) + v_4^+(h) \\
&= \sum_{i+j=1}^{m+1} \left[d_{ij}^- \left(\frac{1}{2}\right)^{i+j} (-1)^i \left[(-1)^j - 1\right] (2h-1)^{i+j-1} + d_{ij}^+ \left(\frac{1}{2}\right)^{i+j} (-1)^i \left[(-1)^j - 1\right] (2h-1)^{i+j} \right] \\
&= -d_{01}^- + \sum_{k=1}^{m} \left[\frac{(-1)^k}{2^{k-1}} \sum_{\substack{i\geq 0, j\geq 0,\\ i+2j+1=k}} d_{i,2j+1}^+ + \frac{(-1)^{k+1}}{2^k} \sum_{\substack{i\geq 0, j\geq 0,\\ i+2j+1=k+1}} d_{i,2j+1}^- \right] (2h-1)^k \\
&+ \sum_{k=m+1} \left[\frac{(-1)^k}{2^{k-1}} \sum_{\substack{i\geq 0, j\geq 0,\\ i+2j+1=k}} d_{i,2j+1}^+ \right] (2h-1)^k \\
&\triangleq -d_{01}^- + \sum_{k=1}^{m} \widetilde{Y}_{1,1}(k)(2h-1)^k + \sum_{k=m+1} \widetilde{Y}_{1,2}(k)(2h-1)^k.
\end{aligned} \tag{3.57}$$

From (3.57), we can obtain

$$\deg\left(\widetilde{Y}_1(h)\right) = m + 1. \tag{3.58}$$

For $\widetilde{Y}_2(h)$, substituting (3.54) and (3.6) into $\widetilde{Y}_2(h)$ yields

$$\begin{aligned}
\widetilde{Y}_2(h) &= \left(K_{31}^-(h) + (2h-1)K_{31}^+(h)\right) \frac{v_1^+(h)}{2h-1} \\
&= \widetilde{Y}_{21}(h)\widetilde{Y}_{22}(h).
\end{aligned} \tag{3.59}$$

Inserting (3.54) into $\widetilde{Y}_{21}(h)$ yields

$$\begin{aligned}
\widetilde{Y}_{21}(h) &= \sum_{i+j=1}^{m+1} \left[c_{ij}^-(-1)^{i-1}(i+j) \left(\frac{1}{2}\right)^{i+j-1} (2h-1)^{i+j-1} + c_{ij}^+(-1)^{i-1}(i+j) \left(\frac{1}{2}\right)^{i+j-1} (2h-1)^{i+j} \right] \\
&= \sum_{k=0}^{m} \left[\frac{k}{2^{k-1}} \sum_{i+j=k} (-1)^{i-1} c_{ij}^+ + \frac{k+1}{2^k} \sum_{i+j=k+1} (-1)^{i-1} c_{ij}^- \right] (2h-1)^k \\
&+ \sum_{k=m+1} \left[\frac{k}{2^{k-1}} \sum_{i+j=k} (-1)^{i-1} c_{ij}^+ \right] (2h-1)^k \\
&\triangleq \sum_{k=0}^{m} \widetilde{Y}_{21,1}(k)(2h-1)^k + \sum_{k=m+1} \widetilde{Y}_{21,2}(k)(2h-1)^k.
\end{aligned} \tag{3.60}$$

From (3.60), we can obtain

$$\deg\left(\widetilde{Y}_{21}(h)\right) = m + 1. \tag{3.61}$$

Similarly, we have

$$\deg\left(\widetilde{Y}_{22}(h)\right) = \deg\left(G_{42}(h)\right) = m - 1. \tag{3.62}$$

Plugging (3.60) and (3.62) into (3.59) yields

$$\deg\left(\widetilde{Y}_{2}(h)\right) = \deg\left(\widetilde{Y}_{21}(h)\right) + \deg\left(\widetilde{Y}_{22}(h)\right) = 2m. \tag{3.63}$$

Substituting (3.53) and (3.6) into $\widetilde{Y}_3(h)$ yields

$$\begin{aligned}\widetilde{Y}_3(h) &= \left(-\frac{1}{2}K_{22}^{-}(h) - \frac{2h-1}{2}K_{22}^{+}(h)\right)\frac{v_1^{+}(h)}{2h-1}v_1^{+}(h)\\ &= \widetilde{Y}_{31}(h)\widetilde{Y}_{32}(h)\widetilde{Y}_{33}(h).\end{aligned} \tag{3.64}$$

Inserting (3.53) into $\widetilde{Y}_{31}(h)$ yields

$$\begin{aligned}\widetilde{Y}_{31}(h) = &\sum_{k=0}^{m-1}\left[\frac{k(k+1)}{2^k}\sum_{i+j=k+1}(-1)^{i-1}b_{ij}^{+} + \frac{(k+1)(k+2)}{2^{k+1}}\sum_{i+j=k+2}(-1)^{i-1}b_{ij}^{-}\right](2h-1)^k\\ &+ \sum_{k=m}\left[\frac{k(k+1)}{2^k}\sum_{i+j=k+1}(-1)^{i-1}b_{ij}^{+}\right](2h-1)^k\\ &\triangleq \sum_{k=0}^{m-1}\widetilde{Y}_{31,1}(k)(2h-1)^k + \sum_{k=m}\widetilde{Y}_{31,2}(k)(2h-1)^k.\end{aligned} \tag{3.65}$$

From (3.65), we can obtain

$$\deg\left(\widetilde{Y}_{31}(h)\right) = m. \tag{3.66}$$

Similarly, we have

$$\begin{aligned}&\deg\left(\widetilde{Y}_{32}(h)\right) = \deg\left(G_{42}(h)\right) = m - 1,\\ &\deg\left(\widetilde{Y}_{33}(h)\right) = \deg\left(\widetilde{H}_{33}(h)\right) = m.\end{aligned} \tag{3.67}$$

Substituting (3.65) and (3.67) into (3.64) yields

$$\begin{aligned}\deg\left(\widetilde{Y}_{3}(h)\right) &= \deg\left(\widetilde{Y}_{31}(h)\right) + \deg\left(\widetilde{Y}_{32}(h)\right) + \deg\left(\widetilde{Y}_{33}(h)\right)\\ &= 3m - 1.\end{aligned} \tag{3.68}$$

For $\widetilde{Y}_4(h)$, substituting (3.52) and (3.6)into $\widetilde{Y}_4(h)$ yields

$$\begin{aligned}\widetilde{Y}_4(h) &= \frac{1}{6}\left(K_{13}^{-}(h) + (2h-1)K_{13}^{+}(h)\right)\frac{(v_1^{+}(h))^2}{2h-1}v_1^{+}(h)\\ &= \widetilde{Y}_{41}(h)\widetilde{Y}_{42}(h)\widetilde{Y}_{43}(h).\end{aligned} \tag{3.69}$$

Inserting (3.52) into $\widetilde{Y}_{41}(h)$ yields

$$\begin{aligned}\widetilde{Y}_{41}(h) &= \frac{1}{6}\Big(K_{13}^{-}(h) + (2h-1)K_{13}^{+}(h)\Big)\\ &= \sum_{k=0}^{m-2}\Bigg[\frac{(-1)^{k-1}}{2^{k-1}}\frac{k(k+1)(k+2)}{6}\sum_{i+2j=k+2} a_{i,2j}^{+} + \frac{(-1)^{k+1}}{2^{k}}\frac{3(k+1)(k+2)}{6}\sum_{i+2j+1=k+3} a_{i,2j+1}^{-}\\ &+ \frac{(-1)^{k}}{2^{k}}\frac{(k+1)(k+2)(k+3)}{6}\sum_{i+2j=k+3} a_{i,2j}^{-}\Bigg](2h-1)^{k}\\ &+ \sum_{k=m-1}\Bigg[\frac{(-1)^{k-1}}{2^{k-1}}\frac{k(k+1)(k+2)}{6}\sum_{i+2j=k+2} a_{i,2j}^{+}\Bigg](2h-1)^{k}\\ &\triangleq \sum_{k=0}^{m-2} Y_{41,1}(k)(2h-1)^{k} + \sum_{k=m-1} Y_{41,2}(k)(2h-1)^{k}.\end{aligned} \tag{3.70}$$

From (3.70), we can obtain

$$\deg\Big(\widetilde{Y}_{41}(h)\Big) = m-1. \tag{3.71}$$

Similarly, we have

$$\begin{aligned}&\deg\Big(\widetilde{Y}_{42}(h)\Big) = \deg\Big(\widetilde{Y}_{32}(h)\Big) + \deg\Big(\widetilde{Y}_{33}(h)\Big) = 2m-1,\\ &\deg\Big(\widetilde{Y}_{43}(h)\Big) = \deg\Big(\widetilde{Y}_{33}(h)\Big) = m.\end{aligned} \tag{3.72}$$

Substituting (3.70) and (3.72) into (3.69) yields

$$\deg\Big(\widetilde{Y}_{4}(h)\Big) = \deg\Big(\widetilde{Y}_{41}(h)\Big) + \deg\Big(\widetilde{Y}_{42}(h)\Big) + \deg\Big(\widetilde{Y}_{43}(h)\Big) = 4m-2. \tag{3.73}$$

Inserting (3.22) and (3.25) into $\widetilde{Y}_{5}(h)$ yields

$$\begin{aligned}\widetilde{Y}_{5}(h) &= \Big(K_{21}^{-}(h) + (2h-1)K_{21}^{+}(h)\Big)\frac{M_{2}^{+}(h)}{2h-1}\\ &= \widetilde{H}_{21}(h)\widetilde{H}_{42}(h).\end{aligned} \tag{3.74}$$

From (3.61), (3.47), and (3.74), we can obtain

$$\deg\Big(\widetilde{Y}_{5}(h)\Big) = \deg\Big(\widetilde{H}_{21}(h)\Big) + \deg\Big(\widetilde{H}_{42}(h)\Big) = m+1+2m-1 = 3m. \tag{3.75}$$

Plugging (3.23) and (3.25) into $\widetilde{Y}_{6}(h)$ yileds

$$\begin{aligned}\widetilde{Y}_{6}(h) &= \Big(-K_{12}^{-}(h) - (2h-1)K_{12}^{+}(h)\Big)\frac{M_{2}^{+}(h)}{2h-1}v_{1}^{+}(h)\\ &= \widetilde{Y}_{61}(h)\widetilde{Y}_{62}(h)\widetilde{Y}_{63}(h).\end{aligned} \tag{3.76}$$

For $\widetilde{Y}_{61}(h)$, we have

$$\begin{aligned}\widetilde{Y}_{61}(h) = 2\widetilde{H}_{31} = &\sum_{k=0}^{m-1}\left[\frac{(-1)^k k(k+1)}{2^{k-1}}\sum_{i+2j=k+1} a^+_{i,2j} + \frac{(-1)^k(k+1)}{2^{k-1}}\sum_{i+2j+1=k+2} a^-_{i,2j+1}\right.\\ &\left.+\frac{(-1)^{k+1}(k+1)(k+2)}{2^k}\sum_{i+2j=k+2} a^-_{i,2j}\right](2h-1)^k + \sum_{k=m}\left[\frac{(-1)^k k(k+1)}{2^{k-1}}\sum_{i+2j=k+1} a^+_{i,2j}\right](2h-1)^k\\ &\triangleq \sum_{k=0}^{m-1}\widetilde{Y}_{61,1}(k)(2h-1)^k + \sum_{k=m}\widetilde{Y}_{61,2}(k)(2h-1)^k.\end{aligned} \tag{3.77}$$

From (3.77), we can obtain

$$\deg\left(\widetilde{Y}_{61}(h)\right) = m. \tag{3.78}$$

Similarly, we have

$$\begin{aligned}&\deg\left(\widetilde{Y}_{62}(h)\right) = \deg\left(\widetilde{H}_{42}(h)\right) = 2m-1,\\ &\deg\left(\widetilde{Y}_{63}(h)\right) = \deg\left(\widetilde{H}_{33}(h)\right) = m.\end{aligned} \tag{3.79}$$

Substituting (3.77) and (3.79) into (3.76) yields

$$\deg\left(\widetilde{Y}_{6}(h)\right) = \deg\left(\widetilde{Y}_{61}(h)\right) + \deg\left(\widetilde{Y}_{62}(h)\right) + \deg\left(\widetilde{Y}_{63}(h)\right) = 4m-1. \tag{3.80}$$

Inserting (3.25) into $\widetilde{Y}_7(h)$ yields

$$\widetilde{Y}_7(h) = \frac{M_2^+(h)}{2h-1}M_2^+(h) = \widetilde{H}_{42}(h)M_2^+(h). \tag{3.81}$$

By direct calculation, we have

$$\begin{aligned}M_2^+(h) = &\sum_{k=1}^{m+1}\frac{(-1)^k}{2^{k-1}}\sum_{i+2j+1=k} b^+_{i,2j+1}(2h-1)^k + \left(\sum_{k=0}^{m-1}V_1(k)(2h-1)^k + \sum_{k=m}V_2(k)(2h-1)^k\right)\\ &\times\sum_{k=1}^{m}R(k)(2h-1)^k.\end{aligned} \tag{3.82}$$

From (3.82), we can obtain

$$\deg\left(M_2^+(h)\right) = 2m. \tag{3.83}$$

Hence, we have

$$\deg\left(\widetilde{Y}_{7}(h)\right) = \deg\left(\widetilde{H}_{42}(h)\right) + \deg\left(M_2^+(h)\right) = 2m-1+2m = 4m-1. \tag{3.84}$$

For $Y_8(h)$, substituting (3.12) and (3.6) into $\widetilde{Y}_8(h)$ yields

$$\begin{aligned}\widetilde{Y}_8(h) &= \left[\frac{1}{2h-1}(K_{11}^-(h)+2v_1^+(h))+K_{11}^+(h)\right]M_3^+(h)\\ &= \left[(K_{11}^-(h)+2v_1^+(h))+(2h-1)K_{11}^+(h)\right]\frac{M_3^+(h)}{2h-1}\\ &= \widetilde{Y}_{81}(h)\widetilde{Y}_{82}(h).\end{aligned} \tag{3.85}$$

Plugging (3.12) and (3.6) into $\widetilde{Y}_{81}(h)$ yields

$$\widetilde{Y}_{81}(h) = \widetilde{H}_{41}(h) = a_{10}^- + \sum_{k=1}^{m}\widetilde{S}(k)(2h-1)^k + \sum_{k=m+1}\widetilde{T}(k)(2h-1)^k. \tag{3.86}$$

From (3.86), we can obtain

$$\deg\left(\widetilde{Y}_{81}(h)\right) = m+1. \tag{3.87}$$

Substituting (3.55) into $\widetilde{Y}_{82}(h)$ yields

$$\begin{aligned}\widetilde{Y}_{82}(h) &= \frac{1}{2h-1}\left[v_3^+(h)+K_{21}^+(h)v_1^+(h)-\frac{1}{2}K_{12}^+(h)(v_1^+(h))^2+K_{11}^+(h)M_2^+(h)\right]\\ &= \frac{v_3^+(h)}{2h-1}+K_{21}^+(h)\cdot\frac{v_1^+(h)}{2h-1}-\frac{1}{2}K_{12}^+(h)\cdot\frac{(v_1^+(h))^2}{2h-1}+K_{11}^+(h)\cdot\frac{M_2^+(h)}{2h-1}\\ &= Y_{82,1}(h)+Y_{82,2}(h)+Y_{82,3}(h)+Y_{82,4}(h).\end{aligned} \tag{3.88}$$

By direct calculation, we have

$$Y_{82,1}(h) = \frac{v_3^+(h)}{2h-1} = \sum_{k=1}^{m+1}\left[\frac{(-1)^k}{2^k}\sum_{i+2j+1=k}c_{i,2j+1}^+\right](2h-1)^k.$$

Thus,

$$\deg\left(Y_{82,1}(h)\right) = m+1. \tag{3.89}$$

Similarly, we have

$$Y_{82,2}(h) = K_{21}^+(h)\cdot\frac{v_1^+(h)}{2h-1} = \sum_{k=0}^{m}\left[\frac{k+1}{2^k}\sum_{i+j=k+1}(-1)^{i-1}b_{ij}^+\right](2h-1)^k\cdot\sum_{k=0}^{m-1}\xi_3(k)(2h-1)^k.$$

Thus,

$$\deg\left(Y_{82,2}(h)\right) = 2m-1. \tag{3.90}$$

Similarly, we have

$$
\begin{aligned}
Y_{82,3}(h) &= -\frac{1}{2}K_{12}^{+}(h)\cdot\frac{v_1^{+}(h)}{2h-1}v_1^{+}(h) = -\frac{1}{2}K_{12}^{+}(h)G_{42}(h)\widetilde{H}_{33}(h)\\
&= \Bigg(\sum_{k=0}^{m-2}\left[\frac{(-1)^{k+1}(k+1)(k+2)}{2^{k+1}}\sum_{i+2j=k+2}a_{i,2j}^{+} + \frac{(-1)^{k+1}(k+1)(k+2)}{2^{k+2}}\sum_{i+2j+1=k+3}a_{i,2j+1}^{-}\right](2h-1)^k\\
&\quad + \sum_{k=m-1}\left[\frac{(-1)^{k+1}(k+1)(k+2)}{2^{k+1}}\sum_{i+2j=k+2}a_{i,2j}^{+}\right](2h-1)^k\Bigg)\times\sum_{k=0}^{m-1}\xi_3(k)(2h-1)^k\sum_{k=1}^{m}R(k)(2h-1)^k.
\end{aligned}
$$

Thus,

$$
\deg\left(Y_{82,3}(h)\right) = 3m-2. \tag{3.91}
$$

Similarly, we have

$$
\begin{aligned}
Y_{82,4}(h) &= K_{11}^{+}(h)\cdot\frac{M_2^{+}(h)}{2h-1} = K_{11}^{+}(h)\widetilde{H}_{42}(h)\\
&= \Bigg(\sum_{k=0}^{m-1}V_1(k)(2h-1)^k + \sum_{k=m}V_2(k)(2h-1)^k\Bigg)\cdot\Bigg[\sum_{k=0}^{m}U(k)(2h-1)^k + \Bigg(\sum_{k=0}^{m-1}V_1(k)(2h-1)^k\\
&\quad + \sum_{k=m}V_2(k)(2h-1)^k\Bigg)\cdot\sum_{k=0}^{m-1}\xi_3(k)(2h-1)^k\Bigg].
\end{aligned}
$$

Thus,

$$
\deg\left(Y_{82,4}(h)\right) = 3m-1. \tag{3.92}
$$

Substituting (3.89)–(3.91) and (3.93) into (3.88) yields

$$
\begin{aligned}
\deg\left(\widetilde{Y}_{82}(h)\right) &= \max\{\deg\left(Y_{82,1}(h)\right),\deg\left(Y_{82,2}(h)\right),\deg\left(Y_{82,3}(h)\right),\\
\deg\left(Y_{82,4}(h)\right)\} &= \max\{m+1, 2m-1, 3m-2, 3m-1\} = 3m-1.
\end{aligned} \tag{3.93}
$$

Inserting (3.87) and (3.93) into (3.85) yields

$$
\deg\left(\widetilde{Y}_8(h)\right) = \deg\left(\widetilde{Y}_{81}(h)\right) + \deg\left(\widetilde{Y}_{82}(h)\right) = m+1+3m-1 = 4m. \tag{3.94}
$$

Plugging (3.58), (3.63), (3.68), (3.73), (3.75), (3.80), (3.84), and (3.94) into (3.56) yields

$$
\begin{aligned}
\deg\left(M_4(h)\right) = \max\Big\{&\deg\left(\widetilde{Y}_1(h)\right),\deg\left(\widetilde{Y}_2(h)\right),\deg\left(\widetilde{Y}_3(h)\right),\deg\left(\widetilde{Y}_4(h)\right),\deg\left(\widetilde{Y}_5(h)\right),\deg\left(\widetilde{Y}_6(h)\right),\\
&\deg\left(\widetilde{Y}_7(h)\right),\deg\left(\widetilde{Y}_8(h)\right)\Big\}\\
=\ &\max\{m+1, 2m, 3m-1, 4m-2, 3m, 4m-1, 4m-1, 4m\}\\
=\ &4m.
\end{aligned} \tag{3.95}
$$

Thus, according to Lemma 1 together with (3.95), system (3.1) has at most $4m$ limit cycles bifurcated from the period annulus defined by $\{L_h\}_{h\in J}$ by the fourth order Melnikov function, multiplicity taken into account. □

In this part, we consider the piecewise polynomial Hamiltonian system (3.1) with $m = 1, 2$. Before giving the main results, we first introduce the following two lemmas, which are calculated directly from Theorems 2-5.

Lemma 6. *When $m = 1$, we have*

$$M_1(h) = -a_{01}^- + \left(-a_{01}^+ + \frac{1}{2}a_{11}^-\right)(2h-1) + \frac{1}{2}a_{11}^+(2h-1)^2.$$

$$\begin{aligned} M_2(h) = & -b_{01}^- - a_{10}^- a_{01}^+ + \left[-b_{01}^+ + \frac{1}{2}b_{11}^- - a_{01}^+ \left(a_{10}^+ - a_{20}^- - a_{02}^-\right)\right](2h-1) \\ & + \left[\frac{1}{2}b_{11}^+ + a_{01}^+ \left(a_{02}^+ + a_{20}^+\right)\right](2h-1)^2. \end{aligned}$$

$$M_3(h) = \sum_{k=0}^{2} A_k(2h-1)^k,$$

where

$$\begin{aligned} A_0 = & -c_{01}^- - a_{01}^+(-b_{01}^- + b_{10}^-) - a_{10}^- \left[b_{01}^+ + a_{01}^+ \left(a_{10}^+ - \frac{1}{2}a_{11}^-\right)\right], \\ A_1 = & -c_{01}^+ + \frac{1}{2}c_{11}^- + a_{01}^+(b_{01}^+ - b_{10}^+ + b_{02}^- - b_{11}^- + b_{20}^-) + (a_{01}^+)^2(a_{11}^- - a_{02}^- - a_{20}^-) \\ & + \left(a_{10}^+ - \frac{1}{2}a_{11}^- - a_{02}^- - a_{20}^-\right)\left[-b_{01}^+ - a_{01}^+ \left(a_{10}^+ - \frac{1}{2}a_{11}^-\right)\right], \\ A_2 = & \frac{1}{2}c_{11}^+ + a_{01}^+(b_{02}^+ - b_{11}^+ + b_{20}^+) - (a_{01}^+)^2(a_{02}^+ + a_{20}^+) + (a_{02}^+ + a_{20}^+)\left[b_{01}^+ + a_{01}^+ \left(a_{10}^+ - \frac{1}{2}a_{11}^-\right)\right]. \end{aligned}$$

$$M_4(h) = \sum_{k=0}^{4} A_k(2h-1)^k,$$

where

$$\begin{aligned} A_0 &= A_{01} + A_{02}, \\ A_1 &= A_{11} + A_{12} + A_{13} + A_{14} + A_{15}, \\ A_2 &= A_{21} + A_{22} + A_{23} + A_{24} + A_{25}, \\ A_3 &= A_{31} + A_{32}, \\ A_4 &= -\frac{1}{4}c_{11}^+(a_{02}^+ + a_{20}^+), \end{aligned}$$

and

$$\begin{aligned} A_{01} &= -d_{01}^- + a_{01}^+(c_{01}^- - c_{10}^-) + (-b_{01}^- + b_{10}^-)\left[-b_{01}^+ - a_{01}^+\left(a_{10}^+ - \frac{1}{2}a_{11}^-\right)\right], \\ A_{02} &= a_{10}^-\left\{-a_{01}^+(-b_{01}^+ + b_{10}^+) + \left(a_{10}^+ - \frac{1}{2}a_{11}^-\right)\left[-b_{01}^+ - a_{01}^+\left(a_{10}^+ - \frac{1}{2}a_{11}^-\right)\right]\right\}, \\ A_{11} &= -d_{01}^+ + \frac{1}{2}d_{11}^- + a_{01}^+(c_{01}^+ - c_{11}^+ + c_{02}^- - c_{11}^- + c_{20}^-) + (a_{01}^+)^2(-b_{02}^- + b_{11}^- - b_{20}^-), \end{aligned}$$

$$A_{12} = (-b_{01}^{+} + b_{10}^{+} - b_{02}^{-} + b_{11}^{-} - b_{20}^{-})\left[-b_{01}^{+} - a_{01}^{+}\left(a_{10}^{+} - \frac{1}{2}a_{11}^{-}\right)\right],$$

$$A_{13} = +2a_{01}^{+}(a_{11}^{-} - a_{02}^{-} - a_{20}^{-})\left[b_{01}^{+} + a_{01}^{+}\left(a_{10}^{+} - \frac{1}{2}a_{11}^{-}\right)\right] + \left[-b_{01}^{+} - a_{01}^{+}\left(a_{10}^{+} - \frac{1}{2}a_{11}^{-}\right)\right]^{2},$$

$$A_{14} = a_{10}^{-}\left\{-\frac{1}{2}c_{01}^{+} - a_{01}^{+}(-b_{02}^{+} + b_{11}^{+} - b_{20}^{+}) - (a_{01}^{+})^{2}(a_{02}^{+} + a_{20}^{+}) + (a_{02}^{+} + a_{20}^{+})\left[b_{01}^{+} + a_{01}^{+}\left(a_{10}^{+} - \frac{1}{2}a_{11}^{-}\right)\right]\right\},$$

$$A_{15} = \left(a_{10}^{+} - \frac{1}{2}a_{11}^{-} - a_{02}^{-} - a_{20}^{-}\right)\left\{-a_{01}^{+}(-b_{01}^{+} + b_{10}^{+}) + \left(a_{10}^{+} - \frac{1}{2}a_{11}^{-}\right)\left[-b_{01}^{+} - a_{01}^{+}\left(a_{10}^{+} - \frac{1}{2}a_{11}^{-}\right)\right]\right\},$$

$$A_{21} = \frac{1}{2}d_{11}^{+} + a_{01}^{+}(c_{02}^{+} - c_{11}^{+} + c_{20}^{+}) + (a_{01}^{+})^{2}(-b_{02}^{+} + b_{11}^{+} - b_{20}^{+}),$$

$$A_{22} = (-b_{02}^{+} + b_{11}^{+} - b_{20}^{+})\left[-b_{01}^{+} - a_{01}^{+}\left(a_{10}^{+} - \frac{1}{2}a_{11}^{-}\right)\right],$$

$$A_{23} = -2a_{01}^{+}(a_{02}^{+} + a_{20}^{+})\left[b_{01}^{+} + a_{01}^{+}\left(a_{10}^{+} - \frac{1}{2}a_{11}^{-}\right)\right] + \frac{1}{4}c_{11}^{+}a_{10}^{-},$$

$$A_{24} = \left(a_{10}^{+} - \frac{1}{2}a_{11}^{-} - a_{02}^{-} - a_{20}^{-}\right)\left\{-\frac{1}{2}c_{01}^{+} - a_{01}^{+}(-b_{02}^{+} + b_{11}^{+} - b_{20}^{+}) - (a_{01}^{+})^{2}(a_{02}^{+} + a_{20}^{+}) + (a_{02}^{+} + a_{20}^{+})\left[b_{01}^{+}\right.\right.$$
$$\left.\left.+a_{01}^{+}\left(a_{10}^{+} - \frac{1}{2}a_{11}^{-}\right)\right]\right\},$$

$$A_{25} = (-a_{02}^{+} - a_{20}^{+})\left\{-a_{01}^{+}(-b_{01}^{+} + b_{10}^{+}) + \left(a_{10}^{+} - \frac{1}{2}a_{11}^{-}\right)\left[-b_{01}^{+} - a_{01}^{+}\left(a_{10}^{+} - \frac{1}{2}a_{11}^{-}\right)\right]\right\},$$

$$A_{31} = \frac{1}{4}c_{11}^{+}\left(a_{10}^{+} - \frac{1}{2}a_{11}^{-} - a_{02}^{-} - a_{20}^{-}\right),$$

$$A_{32} = (-a_{02}^{+} - a_{20}^{+})\left\{-\frac{1}{2}c_{01}^{+} - a_{01}^{+}(-b_{02}^{+} + b_{11}^{+} - b_{20}^{+}) - (a_{01}^{+})^{2}(a_{02}^{+} + a_{20}^{+}) + (a_{02}^{+} + a_{20}^{+})\left[b_{01}^{+} + a_{01}^{+}\left(a_{10}^{+}\right.\right.\right.$$
$$\left.\left.\left.-\frac{1}{2}a_{11}^{-}\right)\right]\right\}.$$

Lemma 7. *When $m = 2$, we have*

$$M_1(h) = \sum_{k=0}^{3} A_k(2h-1)^k,$$

where

$$A_0 = -a_{01}^{-},$$
$$A_1 = -a_{01}^{+} + \frac{1}{2}a_{11}^{-},$$
$$A_2 = \frac{1}{2}a_{11}^{+} - \frac{1}{4}a_{21}^{-} - \frac{1}{4}a_{03}^{-},$$
$$A_3 = -\frac{1}{4}a_{21}^{+} - \frac{1}{4}a_{03}^{+}.$$

$$M_2(h) = \sum_{k=0}^{4} A_k (2h-1)^k,$$

where

$$\begin{aligned}
A_0 &= -b_{01}^- - a_{01}^+ a_{10}^-,\\
A_1 &= -b_{01}^+ + \frac{1}{2} b_{11}^- - a_{01}^+ (a_{10}^+ - a_{20}^- - a_{02}^-) + \frac{1}{2} a_{10}^- a_{11}^+,\\
A_2 &= \frac{1}{2} b_{11}^+ - \frac{1}{4}(b_{21}^- + b_{03}^-) - a_{01}^+ \left[-a_{02}^+ - a_{20}^+ + \frac{3}{4}(a_{30}^- + a_{12}^-) \right] + \frac{1}{2} a_{11}^+ (a_{10}^+ - a_{20}^- - a_{02}^-),\\
A_3 &= -\frac{1}{4}(b_{21}^+ + b_{03}^+) + \frac{1}{2} a_{11}^+ \left[-a_{02}^+ - a_{20}^+ + \frac{3}{4}(a_{30}^- + a_{12}^-) \right] - \frac{3}{4} a_{01}^+ (a_{30}^+ + a_{12}^+),\\
A_4 &= \frac{3}{8} a_{11}^+ (a_{30}^+ + a_{12}^+).
\end{aligned}$$

$M_3(h)$ and $M_4(h)$ are expressed in the following three cases.
Case 1: *When $a_{11}^+ = 0,\ a_{30}^+ + a_{12}^+ \neq 0$, we have*

$$M_3(h) = \sum_{k=0}^{4} A_k (2h-1)^k,$$

where

$$\begin{aligned}
A_0 &= -c_{01}^- - a_{01}^+(-b_{01}^- + b_{10}^-) + a_{10}^- \left[-b_{01}^+ - a_{01}^+ \left(a_{10}^+ - \frac{1}{2} a_{11}^- \right) \right],\\
A_1 &= A_{11} + A_{12} + A_{13},\\
A_2 &= A_{21} + A_{22} + A_{23} + A_{24},\\
A_3 &= A_{31} + A_{32} + A_{33},\\
A_4 &= \frac{3}{4}(a_{30}^+ + a_{12}^+) \left\{ \frac{1}{2} b_{11}^+ - a_{01}^+ \left[-a_{02}^+ - a_{20}^+ + \frac{1}{2}(a_{21}^- + a_{03}^-) \right] \right\},
\end{aligned}$$

and

$$\begin{aligned}
A_{11} &= -c_{01}^+ + \frac{1}{2} c_{11}^- - a_{01}^+(-b_{01}^+ + b_{10}^+ - b_{02}^- + b_{11}^- - b_{20}^-) + (a_{01}^+)^2 (a_{11}^- - a_{20}^- - a_{02}^-),\\
A_{12} &= a_{10}^- \left\{ \frac{1}{2} b_{11}^+ - a_{01}^+ \left[-a_{20}^+ - a_{02}^+ + \frac{1}{2}(a_{21}^- + a_{03}^-) \right] \right\},\\
A_{13} &= (a_{10}^+ - \frac{1}{2} a_{11}^- - a_{20}^- - a_{02}^-) \left[-b_{01}^+ - a_{01}^+ \left(a_{10}^+ - \frac{1}{2} a_{11}^- \right) \right],\\
A_{21} &= \frac{1}{2} c_{11}^+ - \frac{1}{4}(c_{21}^- + c_{03}^-) - a_{01}^+ \left[-b_{02}^+ + b_{11}^+ - b_{20}^+ + \frac{3}{4}(b_{03}^- + b_{12}^- - b_{21}^- + b_{30}^-) \right],\\
A_{22} &= (a_{01}^+)^2 \left[-a_{20}^+ - a_{02}^+ - a_{21}^- - a_{03}^- + \frac{3}{2}(a_{30}^- + a_{12}^-) \right],\\
A_{23} &= \left(a_{10}^+ - \frac{1}{2} a_{11}^- - a_{20}^- - a_{02}^- \right) \left\{ \frac{1}{2} b_{11}^+ - a_{01}^+ \left[-a_{02}^+ - a_{20}^+ + \frac{1}{2}(a_{21}^- + a_{03}^-) \right] \right\},\\
A_{24} &= \left[-a_{02}^+ - a_{20}^+ + \frac{1}{4}(a_{21}^- + a_{03}^-) + \frac{3}{4}(a_{30}^- + a_{12}^-) \right] \left[-b_{01}^+ - a_{01}^+ \left(a_{10}^+ - \frac{1}{2} a_{11}^- \right) \right],
\end{aligned}$$

$$A_{31} = -\frac{1}{4}\left(c_{21}^{+} + c_{03}^{+}\right) - \frac{3}{4}a_{01}^{+}(-b_{03}^{+} + b_{12}^{+} - b_{21}^{+} + b_{30}^{+}) + \frac{3}{2}(a_{01}^{+})^{2}(a_{30}^{+} + a_{12}^{+}),$$

$$A_{32} = \left[-a_{02}^{+} - a_{20}^{+} + \frac{1}{4}(a_{21}^{-} + a_{03}^{-}) + \frac{3}{4}(a_{30}^{-} + a_{12}^{-})\right]\left\{\frac{1}{2}b_{11}^{+} - a_{01}^{+}\left[-a_{02}^{+} - a_{20}^{+} + \frac{1}{2}(a_{21}^{-} + a_{03}^{-})\right]\right\},$$

$$A_{33} = \frac{3}{4}(a_{30}^{+} + a_{12}^{+})\left[-b_{01}^{+} - a_{01}^{+}\left(a_{10}^{+} - \frac{1}{2}a_{11}^{-}\right)\right].$$

Case 2: *When* $a_{30}^{+} + a_{12}^{+} = 0,\ a_{11}^{+} \neq 0$, *we have*

$$M_3(h) = \sum_{k=0}^{4} A_k(2h-1)^k,$$

where

$$A_0 = -c_{01}^{-} - a_{01}^{+}(-b_{01}^{-} + b_{10}^{-}) + a_{10}^{-}\left[-b_{01}^{+} - a_{01}^{+}\left(a_{10}^{+} - \frac{1}{2}a_{11}^{-}\right)\right],$$
$$A_1 = A_{11} + A_{12} + A_{13} + A_{14},$$
$$A_2 = A_{21} + A_{22} + A_{23} + A_{24} + A_{25} + A_{26},$$
$$A_3 = A_{31} + A_{32} + A_{33} + A_{34} + A_{35} + A_{36},$$
$$A_4 = A_{41} + A_{42},$$

and

$$A_{11} = -c_{01}^{+} + \frac{1}{2}c_{11}^{-} - a_{01}^{+}(-b_{01}^{+} + b_{10}^{+} - b_{02}^{-} + b_{11}^{-} - b_{20}^{-}),$$

$$A_{12} = \frac{1}{2}a_{11}^{+}(-b_{01}^{-} + b_{10}^{-}) + (a_{01}^{+})^{2}(a_{11}^{-} - a_{20}^{-} - a_{02}^{-}),$$

$$A_{13} = a_{10}^{-}\left\{\frac{1}{2}b_{11}^{+} + \frac{1}{2}a_{11}^{+}\left(a_{10}^{+} - \frac{1}{2}a_{11}^{-}\right) - a_{01}^{+}\left[-a_{20}^{+} - a_{02}^{+} + \frac{1}{2}(a_{21}^{-} + a_{03}^{-})\right]\right\},$$

$$A_{14} = (a_{10}^{+} - \frac{1}{2}a_{11}^{-} - a_{20}^{-} - a_{02}^{-})\left[-b_{01}^{+} - a_{01}^{+}\left(a_{10}^{+} - \frac{1}{2}a_{11}^{-}\right)\right],$$

$$A_{21} = \frac{1}{2}c_{11}^{+} - \frac{1}{4}(c_{21}^{-} + c_{03}^{-}) - a_{01}^{+}\left[-b_{02}^{+} + b_{11}^{+} - b_{20}^{+} + \frac{3}{4}(b_{03}^{-} + b_{12}^{-} - b_{21}^{-} + b_{30}^{-})\right],$$

$$A_{22} = \frac{1}{2}a_{11}^{+}(-b_{01}^{+} + b_{10}^{+} - b_{02}^{-} + b_{11}^{-} - b_{20}^{-}) - a_{01}^{+}a_{11}^{+}(a_{11}^{-} - a_{20}^{-} - a_{02}^{-}),$$

$$A_{23} = (a_{01}^{+})^{2}\left[-a_{20}^{+} - a_{02}^{+} - a_{21}^{-} - a_{03}^{-} + \frac{3}{2}(a_{30}^{-} + a_{12}^{-})\right],$$

$$A_{24} = a_{10}^{-}\left\{-\frac{1}{4}(b_{21}^{+} + b_{03}^{+}) + \frac{1}{2}a_{11}^{+}\left[-a_{02}^{+} - a_{20}^{+} + \frac{1}{2}(a_{21}^{-} + a_{03}^{-})\right]\right\},$$

$$A_{25} = \left(a_{10}^{+} - \frac{1}{2}a_{11}^{-} - a_{20}^{-} - a_{02}^{-}\right)\left\{\frac{1}{2}b_{11}^{+} + \frac{1}{2}a_{11}^{+}\left(a_{10}^{+} - \frac{1}{2}a_{11}^{-}\right) - a_{01}^{+}\left[-a_{02}^{+} - a_{20}^{+} + \frac{1}{2}(a_{21}^{-} + a_{03}^{-})\right]\right\},$$

$$A_{26} = \left[-a_{02}^{+} - a_{20}^{+} + \frac{1}{4}(a_{21}^{-} + a_{03}^{-}) + \frac{3}{4}(a_{30}^{-} + a_{12}^{-})\right]\left[-b_{01}^{+} - a_{01}^{+}\left(a_{10}^{+} - \frac{1}{2}a_{11}^{-}\right)\right],$$

$$
\begin{aligned}
A_{31} &= -\frac{1}{4}\left(c_{21}^{+} + c_{03}^{+}\right) - \frac{3}{4}a_{01}^{+}(-b_{03}^{+} + b_{12}^{+} - b_{21}^{+} + b_{30}^{+}),\\
A_{32} &= \frac{1}{2}a_{11}^{+}\left[-b_{02}^{+} + b_{11}^{+} - b_{20}^{+} + \frac{3}{4}(-b_{03}^{+} + b_{12}^{+} - b_{21}^{-} + b_{30}^{-})\right],\\
A_{33} &= -a_{01}^{+}a_{11}^{+}\left[-a_{20}^{+} - a_{02}^{+} - a_{21}^{-} - a_{03}^{-} + \frac{3}{2}(a_{30}^{-} + a_{12}^{-})\right],\\
A_{34} &= \frac{1}{4}(a_{11}^{+})^2(a_{11}^{-} - a_{20}^{-} - a_{02}^{-}),\\
A_{35} &= \left(a_{10}^{+} - \frac{1}{2}a_{11}^{-} - a_{20}^{-} - a_{02}^{-}\right)\left\{-\frac{1}{4}(b_{21}^{+} + b_{03}^{+}) + \frac{1}{2}a_{11}^{+}\left[-a_{02}^{+} - a_{20}^{+} + \frac{1}{2}(a_{21}^{-} + a_{03}^{-})\right]\right\},\\
A_{36} &= \left[-a_{02}^{+} - a_{20}^{+} + \frac{1}{4}(a_{21}^{-} + a_{03}^{-}) + \frac{3}{4}(a_{30}^{-} + a_{12}^{-})\right]\left\{\frac{1}{2}b_{11}^{+} + \frac{1}{2}a_{11}^{+}\left(a_{10}^{+} - \frac{1}{2}a_{11}^{-}\right) - a_{01}^{+}\left[-a_{02}^{+} - a_{20}^{+}\right.\right.\\
&\quad \left.\left.+\frac{1}{2}(a_{21}^{-} + a_{03}^{-})\right]\right\},\\
A_{41} &= \frac{3}{8}a_{11}^{+}(-b_{03}^{+} + b_{12}^{+} - b_{21}^{+} + b_{30}^{+}) + \frac{1}{4}(a_{11}^{+})^2\left[-a_{20}^{+} - a_{02}^{+} - a_{21}^{-} - a_{03}^{-} + \frac{3}{2}(a_{30}^{-} + a_{12}^{-})\right],\\
A_{42} &= \left[-a_{02}^{+} - a_{20}^{+} + \frac{1}{4}(a_{21}^{-} + a_{03}^{-}) + \frac{3}{4}(a_{30}^{-} + a_{12}^{-})\right]\left\{-\frac{1}{4}(b_{21}^{+} + b_{03}^{+}) + \frac{1}{2}a_{11}^{+}\left[-a_{02}^{+} - a_{20}^{+} + \frac{1}{2}(a_{21}^{-} + a_{03}^{-})\right]\right\}.
\end{aligned}
$$

Case 3: *When $a_{30}^{+} + a_{12}^{+} = 0$ and $a_{11}^{+} = 0$, we have*

$$M_3(h) = \sum_{k=0}^{3} A_k(2h-1)^k,$$

where

$$
\begin{aligned}
A_0 &= -c_{01}^{-} - a_{01}^{+}(-b_{01}^{-} + b_{10}^{-}) + a_{10}^{-}\left[-b_{01}^{+} - a_{01}^{+}\left(a_{10}^{+} - \frac{1}{2}a_{11}^{-}\right)\right],\\
A_1 &= A_{11} + A_{12} + A_{13},\\
A_2 &= A_{21} + A_{22} + A_{23} + A_{24},\\
A_3 &= A_{31} + A_{32},
\end{aligned}
$$

and

$$
\begin{aligned}
A_{11} &= -c_{01}^{+} + \frac{1}{2}c_{11}^{-} - a_{01}^{+}(-b_{01}^{+} + b_{10}^{+} - b_{02}^{-} + b_{11}^{-} - b_{20}^{-}) + (a_{01}^{+})^2(a_{11}^{-} - a_{20}^{-} - a_{02}^{-}),\\
A_{12} &= a_{10}^{-}\left\{\frac{1}{2}b_{11}^{+} - a_{01}^{+}\left[-a_{20}^{+} - a_{02}^{+} + \frac{1}{2}(a_{21}^{-} + a_{03}^{-})\right]\right\},\\
A_{13} &= (a_{10}^{+} - \frac{1}{2}a_{11}^{-} - a_{20}^{-} - a_{02}^{-})\left[-b_{01}^{+} - a_{01}^{+}\left(a_{10}^{+} - \frac{1}{2}a_{11}^{-}\right)\right],\\
A_{21} &= \frac{1}{2}c_{11}^{+} - \frac{1}{4}(c_{21}^{-} + c_{03}^{-}) - a_{01}^{+}\left[-b_{02}^{+} + b_{11}^{+} - b_{20}^{+} + \frac{3}{4}(b_{03}^{-} + b_{12}^{-} - b_{21}^{-} + b_{30}^{-})\right],\\
A_{22} &= (a_{01}^{+})^2\left[-a_{20}^{+} - a_{02}^{+} - a_{21}^{-} - a_{03}^{-} + \frac{3}{2}(a_{30}^{-} + a_{12}^{-})\right],
\end{aligned}
$$

$$A_{23} = \left(a_{10}^{+} - \frac{1}{2}a_{11}^{-} - a_{20}^{-} - a_{02}^{-}\right)\left\{\frac{1}{2}b_{11}^{+} - a_{01}^{+}\left[-a_{02}^{+} - a_{20}^{+} + \frac{1}{2}(a_{21}^{-} + a_{03}^{-})\right]\right\},$$
$$A_{24} = \left[-a_{02}^{+} - a_{20}^{+} + \frac{1}{4}(a_{21}^{-} + a_{03}^{-}) + \frac{3}{4}(a_{30}^{-} + a_{12}^{-})\right]\left[-b_{01}^{+} - a_{01}^{+}\left(a_{10}^{+} - \frac{1}{2}a_{11}^{-}\right)\right],$$
$$A_{31} = -\frac{1}{4}\left(c_{21}^{+} + c_{03}^{+}\right) - \frac{3}{4}a_{01}^{+}(b_{12}^{+} + b_{30}^{+}),$$
$$A_{32} = \left[-a_{02}^{+} - a_{20}^{+} + \frac{1}{4}(a_{21}^{-} + a_{03}^{-}) + \frac{3}{4}(a_{30}^{-} + a_{12}^{-})\right]\left\{\frac{1}{2}b_{11}^{+} - a_{01}^{+}\left[-a_{02}^{+} - a_{20}^{+} + \frac{1}{2}(a_{21}^{-} + a_{03}^{-})\right]\right\}.$$

Case 1: *When* $a_{11}^{+} = 0,\ a_{30}^{+} + a_{12}^{+} \neq 0$*, we have*

$$M_4(h) = \sum_{k=0}^{6} A_k(2h-1)^k,$$

where

$$A_0 = A_{01} + A_{02},$$
$$A_1 = A_{11} + A_{12} + A_{13} + A_{14} + A_{15} + A_{16},$$
$$A_2 = A_{21} + A_{22} + A_{23} + A_{24} + A_{25} + A_{26} + A_{27},$$
$$A_3 = A_{31} + A_{32} + A_{33} + A_{34} + A_{35} + A_{36} + A_{37},$$
$$A_4 = A_{41} + A_{42} + A_{43} + A_{44},$$
$$A_5 = A_{51} + A_{52},$$
$$A_6 = \frac{3}{4}(a_{30}^{+} + a_{12}^{+})\left[-\frac{1}{8}(c_{21}^{+} + c_{03}^{+})\right],$$

and

$$A_{01} = -d_{01}^{-} - \frac{1}{2}a_{01}^{+}(-c_{01}^{-} + c_{10}^{-}) + (-b_{01}^{-} + b_{10}^{-})\left[-b_{01}^{+} - a_{01}^{+}\left(a_{10}^{+} - \frac{1}{2}a_{11}^{-}\right)\right],$$
$$A_{02} = a_{10}^{-}\left\{-a_{01}^{+}(-b_{01}^{+} + b_{10}^{+}) + \left(a_{10}^{+} - \frac{1}{2}a_{11}^{-}\right)\left[-b_{01}^{+} - a_{01}^{+}\left(a_{10}^{+} - \frac{1}{2}a_{11}^{-}\right)\right]\right\},$$
$$A_{11} = \left(-d_{01}^{+} + \frac{1}{2}d_{11}^{-}\right) - a_{01}^{+}(-c_{01}^{+} + c_{10}^{+} - c_{02}^{-} + c_{11}^{-} - c_{20}^{-}) + (a_{01}^{+})^2(-b_{02}^{-} + b_{11}^{-} - b_{20}^{-}),$$
$$A_{12} = \left(-b_{01}^{+} + a_{01}^{+}a_{10}^{+} - \frac{1}{2}a_{01}^{+}a_{11}^{-}\right)(-b_{01}^{+} + b_{10}^{+} - b_{02}^{-} + b_{11}^{-} - b_{20}^{-}),$$
$$A_{13} = (2a_{11}^{-} - 2a_{20}^{-} - 2a_{02}^{-})\left[a_{01}^{+}b_{01}^{+} + (a_{01}^{+})^2\left(a_{10}^{+} - \frac{1}{2}a_{11}^{-}\right)\right],$$
$$A_{14} = \left[-b_{01}^{+} - a_{01}^{+}\left(a_{10}^{+} - \frac{1}{2}a_{11}^{-}\right)\right]^2,$$
$$A_{15} = a_{10}^{-}\left\{-\frac{1}{2}c_{01}^{+} + (-a_{01}^{+}(-b_{02}^{+} + b_{11}^{+} - b_{20}^{+})) + (a_{01}^{+})^2\left[-(a_{02}^{+} + a_{20}^{+}) - \frac{1}{2}(a_{21}^{-} + a_{03}^{-})\right]\right.$$
$$\left. + \left[-b_{01}^{+} - a_{01}^{+}\left(a_{10}^{+} - \frac{1}{2}a_{11}^{-}\right)\right]\left[-(a_{02}^{+} + a_{20}^{+}) + \frac{1}{2}(a_{21}^{-} + a_{03}^{-})\right]\right\},$$

$$A_{16}=\left\{-a_{01}^{+}(-b_{01}^{+}+b_{10}^{+})+\left(a_{10}^{+}-\frac{1}{2}a_{11}^{-}\right)\left[-b_{01}^{+}-a_{01}^{+}\left(a_{10}^{+}-\frac{1}{2}a_{11}^{-}\right)\right]\right\}\left[a_{10}^{+}-\frac{1}{2}a_{11}^{-}-(a_{20}^{-}+a_{02}^{-})\right],$$

$$A_{21}=\left(\frac{1}{2}d_{11}^{+}-\frac{1}{4}d_{03}^{-}-\frac{1}{4}d_{21}^{-}\right)+\left[-a_{01}^{+}(-c_{02}^{+}+c_{11}^{+}-c_{20}^{+}-\frac{3}{4}c_{03}^{-}+\frac{3}{4}c_{12}^{-}-\frac{3}{4}c_{21}^{-}+\frac{3}{4}c_{30}^{-})\right],$$

$$A_{22}=(a_{01}^{+})^{2}(-b_{02}^{+}+b_{11}^{+}-b_{20}^{+}-\frac{3}{2}b_{03}^{-}+\frac{3}{2}b_{12}^{-}-\frac{3}{2}b_{21}^{-}+\frac{3}{2}b_{30}^{-})+(a_{01}^{+})^{3}(a_{21}^{-}+a_{03}^{-}-a_{30}^{-}-a_{12}^{-}),$$

$$A_{23}=\left(b_{01}^{+}-a_{01}^{+}a_{10}^{+}+\frac{1}{2}a_{01}^{+}a_{11}^{-}\right)\left(-b_{02}^{+}+b_{11}^{+}-b_{20}^{+}-\frac{3}{4}b_{03}^{-}+\frac{3}{4}b_{12}^{-}-\frac{3}{4}b_{21}^{-}+\frac{3}{4}b_{30}^{-}\right),$$

$$A_{24}=\left[a_{01}^{+}b_{01}^{+}+(a_{01}^{+})^{2}\left(a_{10}^{+}-\frac{1}{2}a_{11}^{-}\right)\right]\left[-2(a_{20}^{+}+a_{02}^{+})-2(a_{21}^{-}+a_{03}^{-})+3(a_{30}^{-}+a_{12}^{-})\right],$$

$$A_{25}=a_{10}^{-}\left\{\frac{1}{4}c_{11}^{+}-\frac{3}{4}a_{01}^{+}(-b_{03}^{+}+b_{12}^{+}-b_{21}^{+}+b_{30}^{+})+\frac{3}{2}(a_{01}^{+})^{2}(a_{30}^{+}+a_{12}^{+})+\frac{3}{4}(a_{30}^{+}+a_{12}^{+})\left[-b_{01}^{+}\right.\right.$$
$$\left.\left.-a_{01}^{+}\left(a_{10}^{+}-\frac{1}{2}a_{11}^{-}\right)\right]\right\},$$

$$A_{26}=\left[a_{10}^{+}-\frac{1}{2}a_{11}^{-}-(a_{20}^{-}+a_{02}^{-})\right]\left\{-\frac{1}{2}c_{01}^{+}+(-a_{01}^{+}(-b_{02}^{+}+b_{11}^{+}-b_{20}^{+}))+(a_{01}^{+})^{2}\right.$$
$$\left.\times\left[-(a_{02}^{+}+a_{20}^{+})-\frac{1}{2}(a_{21}^{-}+a_{03}^{-})\right]+\left[-b_{01}^{+}-a_{01}^{+}\left(a_{10}^{+}-\frac{1}{2}a_{11}^{-}\right)\right]\left[-(a_{02}^{+}+a_{20}^{+})+\frac{1}{2}(a_{21}^{-}+a_{03}^{-})\right]\right\},$$

$$A_{27}=\left[-(a_{20}^{+}+a_{02}^{+})+\frac{1}{4}(a_{21}^{-}+a_{03}^{-})+\frac{3}{4}(a_{30}^{-}+a_{12}^{-})\right]\left\{-a_{01}^{+}(-b_{01}^{+}+b_{10}^{+})+\left(a_{10}^{+}-\frac{1}{2}a_{11}^{-}\right)\left[-b_{01}^{+}\right.\right.$$
$$\left.\left.-a_{01}^{+}\left(a_{10}^{+}-\frac{1}{2}a_{11}^{-}\right)\right]\right\},$$

$$A_{31}=\left(-\frac{1}{4}d_{03}^{+}-\frac{1}{4}d_{21}^{+}\right)-a_{01}^{+}(-\frac{3}{4}c_{03}^{+}+\frac{3}{4}c_{12}^{+}-\frac{3}{4}c_{21}^{+}+\frac{3}{4}c_{30}^{+})+\left[\frac{3}{2}(a_{01}^{+})^{2}(-b_{03}^{+}+b_{12}^{+}-b_{21}^{+}+b_{30}^{+})\right],$$

$$A_{32}=-(a_{01}^{+})^{3}(a_{30}^{+}+a_{12}^{+})+\frac{3}{4}(-b_{03}^{+}+b_{12}^{+}-b_{21}^{+}+b_{30}^{+})(-b_{01}^{+}-a_{01}^{+}a_{10}^{+}+\frac{1}{2}a_{01}^{+}a_{11}^{-}),$$

$$A_{33}=(2a_{11}^{-}-2a_{20}^{-}-2a_{02}^{-})\left[\frac{1}{4}a_{01}^{+}(b_{21}^{+}+b_{03}^{+})+\frac{3}{4}(a_{01}^{+})^{2}(a_{30}^{+}+a_{12}^{+})\right],$$

$$A_{34}=3(a_{30}^{+}+a_{12}^{+})\left[a_{01}^{+}b_{01}^{+}+(a_{01}^{+})^{2}\left(a_{10}^{+}-\frac{1}{2}a_{11}^{-}\right)\right],A_{35}=a_{10}^{-}\left[-\frac{1}{8}(c_{21}^{+}+c_{03}^{+})\right],$$

$$A_{36}=\left[a_{10}^{+}-\frac{1}{2}a_{11}^{-}-(a_{20}^{-}+a_{02}^{-})\right]\left\{\frac{1}{4}c_{11}^{+}-\frac{3}{4}a_{01}^{+}(-b_{03}^{+}+b_{12}^{+}-b_{21}^{+}+b_{30}^{+})+\frac{3}{2}(a_{01}^{+})^{2}(a_{30}^{+}+a_{12}^{+})\right.$$
$$\left.+\frac{3}{4}(a_{30}^{+}+a_{12}^{+})\times\left[-b_{01}^{+}-a_{01}^{+}\left(a_{10}^{+}-\frac{1}{2}a_{11}^{-}\right)\right]\right\},$$

$$A_{37}=\left[-(a_{20}^{+}+a_{02}^{+})+\frac{1}{4}(a_{21}^{-}+a_{03}^{-})+\frac{3}{4}(a_{30}^{-}+a_{12}^{-})\right]\left\{-\frac{1}{2}c_{01}^{+}-a_{01}^{+}(-b_{02}^{+}+b_{11}^{+}-b_{20}^{+})\right.$$
$$+(a_{01}^{+})^{2}\left[-(a_{02}^{+}+a_{20}^{+})\ -\frac{1}{2}(a_{21}^{-}+a_{03}^{-})\right]+\left[-b_{01}^{+}-a_{01}^{+}\left(a_{10}^{+}-\frac{1}{2}a_{11}^{-}\right)\right]\left[-(a_{02}^{+}+a_{20}^{+})+\frac{1}{2}(a_{21}^{-}+a_{03}^{-})\right]$$
$$\left.+\left[\frac{3}{4}(a_{30}^{+}+a_{12}^{+})\right]\times\left\{-a_{01}^{+}(-b_{01}^{+}+b_{10}^{+})+\left(a_{10}^{+}-\frac{1}{2}a_{11}^{-}\right)\left[-b_{01}^{+}-a_{01}^{+}\left(a_{10}^{+}-\frac{1}{2}a_{11}^{-}\right)\right]\right\}\right\},$$

$$A_{41} = [-2(a_{20}^+ + a_{02}^+) - 2(a_{21}^- + a_{03}^-) + 3(a_{30}^- + a_{12}^-)]\left[\frac{1}{4}a_{01}^+(b_{21}^+ + b_{03}^+) + \frac{3}{4}(a_{01}^+)^2(a_{30}^+ + a_{12}^+)\right],$$

$$A_{42} = \left[a_{10}^+ - \frac{1}{2}a_{11}^- - (a_{20}^- + a_{02}^-)\right]\left[-\frac{1}{8}(c_{21}^+ + c_{03}^+)\right],$$

$$A_{43} = \left[-(a_{20}^+ + a_{02}^+) + \frac{1}{4}(a_{21}^- + a_{03}^-) + \frac{3}{4}(a_{30}^- + a_{12}^-)\right]\left\{\frac{1}{4}c_{11}^+ - \frac{3}{4}a_{01}^+(-b_{03}^+ + b_{12}^+ - b_{21}^+ + b_{30}^+)\right.$$
$$\left.+\frac{3}{2}(a_{01}^+)^2(a_{30}^+ + a_{12}^+) + \frac{3}{4}(a_{30}^+ + a_{12}^+)\left[-b_{01}^+ - a_{01}^+\left(a_{10}^+ - \frac{1}{2}a_{11}^-\right)\right]\right\},$$

$$A_{44} = \frac{3}{4}(a_{30}^+ + a_{12}^+)\left\{-\frac{1}{2}c_{01}^+ - a_{01}^+(-b_{02}^+ + b_{11}^+ - b_{20}^+) + (a_{01}^+)^2\left[-(a_{02}^+ + a_{20}^+) - \frac{1}{2}(a_{21}^- + a_{03}^-)\right]\right.$$
$$\left.+\left[-b_{01}^+ - a_{01}^+\left(a_{10}^+ - \frac{1}{2}a_{11}^-\right)\right]\left[-(a_{02}^+ + a_{20}^+) + \frac{1}{2}(a_{21}^- + a_{03}^-)\right]\right\}\Bigg\},$$

$$A_{51} = \left[-(a_{20}^+ + a_{02}^+) + \frac{1}{4}(a_{21}^- + a_{03}^-) + \frac{3}{4}(a_{30}^- + a_{12}^-)\right]\left[-\frac{1}{8}(c_{21}^+ + c_{03}^+)\right],$$

$$A_{52} = \frac{3}{4}(a_{30}^+ + a_{12}^+)\left\{\frac{1}{4}c_{11}^+ - \frac{3}{4}a_{01}^+(-b_{03}^+ + b_{12}^+ - b_{21}^+ + b_{30}^+) + \frac{3}{2}(a_{01}^+)^2(a_{30}^+ + a_{12}^+) + \frac{3}{4}(a_{30}^+ + a_{12}^+)\right.$$
$$\left.\times\left[-b_{01}^+ - a_{01}^+\left(a_{10}^+ - \frac{1}{2}a_{11}^-\right)\right]\right\}.$$

Case 2: *When* $a_{30}^+ + a_{12}^+ = 0,\ a_{11}^+ \neq 0$, *we have*

$$M_4(h) = \sum_{k=0}^{5} A_k(2h-1)^k,$$

where

$$A_0 = A_{01} + A_{02}, A_1 = \sum_{j=1}^{7} A_{1j}, A_2 = \sum_{j=1}^{13} A_{2j}, A_3 = \sum_{j=1}^{14} A_{3j}, A_4 = \sum_{j=1}^{10} A_{4j}, A_5 = \sum_{j=1}^{5} A_{5j},$$

and

$$A_{01} = -d_{01}^- - \frac{1}{2}a_{01}^+(-c_{01}^- + c_{10}^-) + (-b_{01}^- + b_{10}^-)\left[-b_{01}^+ - a_{01}^+\left(a_{10}^+ - \frac{1}{2}a_{11}^-\right)\right],$$

$$A_{02} = a_{10}^-\left\{-a_{01}^+(-b_{01}^+ + b_{10}^+) + \left(a_{10}^+ - \frac{1}{2}a_{11}^-\right)\left[-b_{01}^+ - a_{01}^+\left(a_{10}^+ - \frac{1}{2}a_{11}^-\right)\right]\right\},$$

$$A_{11} = -d_{01}^+ + \frac{1}{2}d_{11}^- + \frac{1}{2}a_{11}^+(-c_{01}^- + c_{10}^-) - a_{01}^+(-c_{01}^+ + c_{10}^+ - c_{02}^- + c_{11}^- - c_{20}^-) + (a_{01}^+)^2(-b_{02}^- + b_{11}^- - b_{20}^-),$$

$$A_{12} = (-b_{01}^- + b_{10}^-)\left(\frac{1}{2}b_{11}^+ + \frac{1}{2}a_{11}^+\left(a_{10}^+ - \frac{1}{2}a_{11}^-\right) + a_{01}^+\left(a_{20}^+ + a_{02}^+ - \frac{1}{2}a_{21}^- - \frac{1}{2}a_{03}^-\right)\right),$$

$$A_{13} = (-b_{01}^+ + a_{01}^+a_{10}^+ - \frac{1}{2}a_{01}^+a_{11}^-)(-b_{01}^+ + b_{10}^+ - b_{02}^- + b_{11}^- - b_{20}^-),$$

$$A_{14} = (2a_{11}^- - 2a_{20}^- - 2a_{02}^-)\left[a_{01}^+b_{01}^+ + (a_{01}^+)^2\left(a_{10}^+ - \frac{1}{2}a_{11}^-\right)\right],$$

$$A_{15} = \left[-b_{01}^+ - a_{01}^+\left(a_{10}^+ - \frac{1}{2}a_{11}^-\right)\right]^2,$$

$$
\begin{aligned}
A_{16} &= a_{10}^{-}\left\{-\frac{1}{2}c_{01}^{+}+\left(\frac{1}{2}a_{11}^{+}(-b_{01}^{+}+b_{10}^{+})-a_{01}^{+}(-b_{02}^{+}+b_{11}^{+}-b_{20}^{+})\right)+(a_{01}^{+})^{2}\left[-(a_{02}^{+}+a_{20}^{+})-\frac{1}{2}(a_{21}^{-}+a_{03}^{-})\right]\right.\\
&\quad+\left(a_{10}^{+}-\frac{1}{2}a_{11}^{-}\right)\left[\frac{1}{2}b_{11}^{+}+\frac{1}{2}a_{11}^{+}\left(a_{10}^{+}-\frac{1}{2}a_{11}^{-}\right)-a_{01}^{+}\left(-(a_{02}^{+}+a_{20}^{+})+\frac{1}{2}(a_{21}^{-}+a_{03}^{-})\right)\right]\\
&\quad\left.+\left[-b_{01}^{+}-a_{01}^{+}\left(a_{10}^{+}-\frac{1}{2}a_{11}^{-}\right)\right]\left[-(a_{02}^{+}+a_{20}^{+})+\frac{1}{2}(a_{21}^{-}+a_{03}^{-})\right]\right\},\\
A_{17} &= \left\{-a_{01}^{+}(-b_{01}^{+}+b_{10}^{+})+\left(a_{10}^{+}-\frac{1}{2}a_{11}^{-}\right)\left[-b_{01}^{+}-a_{01}^{+}\left(a_{10}^{+}-\frac{1}{2}a_{11}^{-}\right)\right]\right\}\left[a_{10}^{+}-\frac{1}{2}a_{11}^{-}-(a_{20}^{-}+a_{02}^{-})\right],\\
A_{21} &= \frac{1}{2}d_{11}^{+}-\frac{1}{4}d_{03}^{-}-\frac{1}{4}d_{21}^{-},\\
A_{22} &= \frac{1}{2}a_{11}^{+}(-c_{01}^{+}+c_{10}^{+}-c_{02}^{-}+c_{11}^{-}-c_{20}^{-})-a_{01}^{+}(-c_{02}^{+}+c_{11}^{+}-c_{20}^{+}-\frac{3}{4}c_{03}^{-}+\frac{3}{4}c_{12}^{-}-\frac{3}{4}c_{21}^{-}+\frac{3}{4}c_{30}^{-}),\\
A_{23} &= -a_{01}^{+}a_{11}^{+}(-b_{02}^{-}+b_{11}^{-}-b_{20}^{-})+(a_{01}^{+})^{2}(-b_{02}^{+}+b_{11}^{+}-b_{20}^{+}-\frac{3}{2}b_{03}^{-}+\frac{3}{2}b_{12}^{-}-\frac{3}{2}b_{21}^{-}+\frac{3}{2}b_{30}^{-}),\\
A_{24} &= (a_{01}^{+})^{3}(a_{21}^{-}+a_{03}^{-}-a_{30}^{-}-a_{12}^{-}),\\
A_{25} &= (-b_{01}^{-}+b_{10}^{-})\left[-\frac{1}{4}(b_{21}^{+}+b_{03}^{+})+\frac{1}{2}a_{11}^{+}\left(-a_{20}^{+}-a_{02}^{+}+\frac{1}{2}a_{21}^{-}+\frac{1}{2}a_{03}^{-}\right)\right],\\
A_{26} &= (-b_{01}^{+}+b_{10}^{+}-b_{02}^{-}+b_{11}^{-}-b_{20}^{-})\left[\frac{1}{2}b_{11}^{+}+\frac{1}{2}a_{11}^{+}(a_{10}^{+}-\frac{1}{2}a_{11}^{-})+a_{01}^{+}(a_{20}^{+}+a_{02}^{+}-\frac{1}{2}a_{21}^{-}-\frac{1}{2}a_{03}^{-})\right],\\
A_{27} &= \left(b_{01}^{+}-a_{01}^{+}a_{10}^{+}+\frac{1}{2}a_{01}^{+}a_{11}^{-}\right)\left(-b_{02}^{+}+b_{11}^{+}-b_{20}^{+}-\frac{3}{4}b_{03}^{-}+\frac{3}{4}b_{12}^{-}-\frac{3}{4}b_{21}^{-}+\frac{3}{4}b_{30}^{-}\right),\\
A_{28} &= (2a_{11}^{-}-2a_{20}^{-}-2a_{02}^{-})\left[-\frac{1}{2}a_{11}^{+}b_{01}^{+}-\frac{1}{2}a_{11}^{+}a_{01}^{+}\left(a_{10}^{+}-\frac{1}{2}a_{11}^{-}\right)-\frac{1}{2}a_{01}^{+}b_{11}^{+}-\frac{1}{2}a_{11}^{+}a_{01}^{+}\left(a_{10}^{-}-\frac{1}{2}a_{11}^{-}\right)\right.\\
&\quad\left.+(a_{01}^{+})^{2}\left(-a_{20}^{+}-a_{02}^{+}+\frac{1}{2}a_{21}^{-}+\frac{1}{2}a_{03}^{-}\right)\right],\\
A_{29} &= \left[a_{01}^{+}b_{01}^{+}+(a_{01}^{+})^{2}\left(a_{10}^{+}-\frac{1}{2}a_{11}^{-}\right)\right][-2(a_{20}^{+}+a_{02}^{+})-2(a_{21}^{-}+a_{03}^{-})+3(a_{30}^{-}+a_{12}^{-})],\\
A_{2,10} &= 2\left[-b_{01}^{+}-a_{01}^{+}\left(a_{10}^{+}-\frac{1}{2}a_{11}^{-}\right)\right]\left[\frac{1}{2}b_{11}^{+}+\frac{1}{2}a_{11}^{+}\left(a_{10}^{+}-\frac{1}{2}a_{11}^{-}\right)+a_{01}^{+}\left(a_{02}^{+}+a_{20}^{+}-\frac{1}{2}a_{21}^{-}-\frac{1}{2}a_{03}^{-}\right)\right],\\
A_{2,11} &= a_{10}^{-}\left\{\frac{1}{4}c_{11}^{+}+\frac{1}{2}a_{11}^{+}(-b_{02}^{+}+b_{11}^{+}-b_{20}^{+})-\frac{3}{4}a_{01}^{+}(-b_{03}^{+}+b_{12}^{+}-b_{21}^{+}+b_{30}^{+})-a_{01}^{+}a_{11}^{+}\right.\\
&\quad\times\left[-(a_{02}^{+}+a_{20}^{+})-\frac{1}{2}(a_{21}^{-}+a_{03}^{-})\right]+\left(a_{10}^{+}-\frac{1}{2}a_{11}^{-}\right)\left[-\frac{1}{4}(b_{21}^{+}+b_{03}^{+})+\frac{1}{2}a_{11}^{+}\left(-(a_{02}^{+}+a_{20}^{+})\right.\right.\\
&\quad\left.\left.+\frac{1}{2}(a_{21}^{-}+a_{03}^{-})\right)\right]+\left[\frac{1}{2}b_{11}^{+}+\frac{1}{2}a_{11}^{+}\left(a_{10}^{+}-\frac{1}{2}a_{11}^{-}\right)-a_{01}^{+}\left(-(a_{02}^{+}+a_{20}^{+})+\frac{1}{2}(a_{21}^{-}+a_{03}^{-})\right)\right]\\
&\quad\left.\times\left[-(a_{02}^{+}+a_{20}^{+})+\frac{1}{2}(a_{21}^{-}+a_{03}^{-})\right]\right\},
\end{aligned}
$$

$$A_{2,12}=\left[a_{10}^{+}-\frac{1}{2}a_{11}^{-}-(a_{20}^{-}+a_{02}^{-})\right]\left\{-\frac{1}{2}c_{01}^{+}+\left(\frac{1}{2}a_{11}^{+}(-b_{01}^{+}+b_{10}^{+})-a_{01}^{+}(-b_{02}^{+}+b_{11}^{+}-b_{20}^{+})\right)\right.$$
$$+(a_{01}^{+})^{2}\left[-(a_{02}^{+}+a_{20}^{+})-\frac{1}{2}(a_{21}^{-}+a_{03}^{-})\right]+\left(a_{10}^{+}-\frac{1}{2}a_{11}^{-}\right)\left[\frac{1}{2}b_{11}^{+}+\frac{1}{2}a_{11}^{+}\left(a_{10}^{+}-\frac{1}{2}a_{11}^{-}\right)\right.$$
$$\left.\left.-a_{01}^{+}\left(-(a_{02}^{+}+a_{20}^{+})+\frac{1}{2}(a_{21}^{-}+a_{03}^{-})\right)\right]+\left[-b_{01}^{+}-a_{01}^{+}\left(a_{10}^{+}-\frac{1}{2}a_{11}^{-}\right)\right]\left[-(a_{02}^{+}+a_{20}^{+})+\frac{1}{2}(a_{21}^{-}+a_{03}^{-})\right]\right\},$$
$$A_{2,13}=\left[-(a_{20}^{+}+a_{02}^{+})+\frac{1}{4}(a_{21}^{-}+a_{03}^{-})+\frac{3}{4}(a_{30}^{-}+a_{12}^{-})\right]\left\{-a_{01}^{+}(-b_{01}^{+}+b_{10}^{+})+\left(a_{10}^{+}-\frac{1}{2}a_{11}^{-}\right)\left[-b_{01}^{+}\right.\right.$$
$$\left.\left.-a_{01}^{+}\left(a_{10}^{+}-\frac{1}{2}a_{11}^{-}\right)\right]\right\},$$
$$A_{31}=-\frac{1}{4}d_{03}^{+}-\frac{1}{4}d_{21}^{+},$$
$$A_{32}=\frac{1}{2}a_{11}^{+}(-c_{02}^{+}+c_{11}^{+}-c_{20}^{+}-\frac{3}{4}c_{03}^{-}+\frac{3}{4}c_{12}^{-}-\frac{3}{4}c_{21}^{-}+\frac{3}{4}c_{30}^{-})-a_{01}^{+}(-\frac{3}{4}c_{03}^{+}+\frac{3}{4}c_{12}^{+}-\frac{3}{4}c_{21}^{+}+\frac{3}{4}c_{30}^{+}),$$
$$A_{33}=\left[\frac{3}{2}(a_{01}^{+})^{2}(-b_{03}^{+}+b_{12}^{+}-b_{21}^{+}+b_{30}^{+})-a_{01}^{+}a_{11}^{+}(-b_{02}^{+}+b_{11}^{+}-b_{20}^{+}-\frac{3}{2}b_{03}^{-}+\frac{3}{2}b_{12}^{-}-\frac{3}{2}b_{21}^{-}+\frac{3}{2}b_{30}^{-})\right.$$
$$\left.+\frac{1}{4}(a_{11}^{+})^{2}(-b_{02}^{-}+b_{11}^{-}-b_{20}^{-})\right],$$
$$A_{34}=-\frac{3}{2}a_{11}^{+}(a_{01}^{+})^{2}(a_{21}^{-}+a_{03}^{-}-a_{30}^{-}-a_{12}^{-}),$$
$$A_{35}=(-b_{01}^{+}+b_{10}^{+}-b_{02}^{-}+b_{11}^{-}-b_{20}^{-})\left[-\frac{1}{4}(b_{21}^{+}+b_{03}^{+})+\frac{1}{2}a_{11}^{+}\left(-a_{20}^{+}-a_{02}^{+}+\frac{1}{2}a_{21}^{-}+\frac{1}{2}a_{03}^{-}\right)\right],$$
$$A_{36}=(-b_{02}^{+}+b_{11}^{+}-b_{20}^{+}-\frac{3}{4}b_{03}^{-}+\frac{3}{4}b_{12}^{-}-\frac{3}{4}b_{21}^{-}+\frac{3}{4}b_{30}^{-})\left[\frac{1}{2}b_{11}^{+}+\frac{1}{2}a_{11}^{+}(a_{10}^{+}-\frac{1}{2}a_{11}^{-})\right.$$
$$\left.+a_{01}^{+}(a_{20}^{+}+a_{02}^{+}-\frac{1}{2}a_{21}^{-}-\frac{1}{2}a_{03}^{-})\right],$$
$$A_{37}=\frac{3}{4}(-b_{03}^{+}+b_{12}^{+}-b_{21}^{+}+b_{30}^{+})(-b_{01}^{+}-a_{01}^{+}a_{10}^{+}+\frac{1}{2}a_{01}^{+}a_{11}^{-}),$$
$$A_{38}=(2a_{11}^{-}-2a_{20}^{-}-2a_{02}^{-})\left[\frac{1}{4}a_{11}^{+}b_{11}^{+}+\frac{1}{4}(a_{11}^{+})^{2}\left(a_{10}^{+}-\frac{1}{2}a_{11}^{-}\right)-\frac{1}{2}a_{11}^{+}a_{01}^{+}\left(-a_{20}^{+}-a_{02}^{+}+\frac{1}{2}a_{21}^{-}+\frac{1}{2}a_{03}^{-}\right)\right.$$
$$\left.+\frac{1}{4}a_{01}^{+}(b_{21}^{+}+b_{03}^{+})-\frac{1}{2}a_{11}^{+}a_{01}^{+}\left(-a_{20}^{+}-a_{02}^{+}+\frac{1}{2}a_{21}^{-}+\frac{1}{2}a_{03}^{-}\right)\right],$$
$$A_{39}=[-2(a_{20}^{+}+a_{02}^{+})-2(a_{21}^{-}+a_{03}^{-})+3(a_{30}^{-}+a_{12}^{-})]\left[-\frac{1}{2}a_{11}^{+}b_{01}^{+}-\frac{1}{2}a_{11}^{+}a_{01}^{+}\left(a_{10}^{+}-\frac{1}{2}a_{11}^{-}\right)-\frac{1}{2}a_{01}^{+}b_{11}^{+}\right.$$
$$\left.-\frac{1}{2}a_{11}^{+}a_{01}^{+}\left(a_{10}^{-}-\frac{1}{2}a_{11}^{-}\right)+(a_{01}^{+})^{2}\left(-a_{20}^{+}-a_{02}^{+}+\frac{1}{2}a_{21}^{-}+\frac{1}{2}a_{03}^{-}\right)\right],$$
$$A_{3,10}=2\left[-b_{01}^{+}-a_{01}^{+}\left(a_{10}^{+}-\frac{1}{2}a_{11}^{-}\right)\right]\left[-\frac{1}{4}(b_{21}^{+}+b_{03}^{+})+\frac{1}{2}a_{11}^{+}\left(-a_{02}^{+}-a_{20}^{+}+\frac{1}{2}a_{21}^{-}+\frac{1}{2}a_{03}^{-}\right)\right],$$
$$A_{3,11}=\left[\frac{1}{2}b_{11}^{+}+\frac{1}{2}a_{11}^{+}\left(a_{10}^{+}-\frac{1}{2}a_{11}^{-}\right)+a_{01}^{+}\left(a_{02}^{+}+a_{20}^{+}-\frac{1}{2}a_{21}^{-}-\frac{1}{2}a_{03}^{-}\right)\right]^{2},$$

$$
\begin{aligned}
A_{3,12} &= a_{10}^{-}\left\{-\frac{1}{8}(c_{21}^{+}+c_{03}^{+})+\frac{3}{8}a_{11}^{+}(-b_{03}^{+}+b_{12}^{+}-b_{21}^{+}+b_{30}^{+})+\frac{1}{4}(a_{11}^{+})^{2}\right.\\
&\quad\times\left[-(a_{02}^{+}+a_{20}^{+})-\frac{1}{2}(a_{21}^{-}+a_{03}^{-})\right]+\left[-(a_{02}^{+}+a_{20}^{+})+\frac{1}{2}(a_{21}^{-}+a_{03}^{-})\right]\\
&\quad\left.\times\left[-\frac{1}{4}(b_{21}^{+}+b_{03}^{+})+\frac{1}{2}a_{11}^{+}\left(-(a_{02}^{+}+a_{20}^{+})+\frac{1}{2}(a_{21}^{-}+a_{03}^{-})\right)\right]\right\},\\
A_{3,13} &= \left[a_{10}^{+}-\frac{1}{2}a_{11}^{-}-(a_{20}^{-}+a_{02}^{-})\right]\left\{\frac{1}{4}c_{11}^{+}+\frac{1}{2}a_{11}^{+}(-b_{02}^{+}+b_{11}^{+}-b_{20}^{+})-\frac{3}{4}a_{01}^{+}(-b_{03}^{+}+b_{12}^{+}-b_{21}^{+}+b_{30}^{+})\right.\\
&\quad -a_{01}^{+}a_{11}^{+}\left[-(a_{02}^{+}+a_{20}^{+})\frac{1}{2}(a_{21}^{-}+a_{03}^{-})\right]+\left(a_{10}^{+}-\frac{1}{2}a_{11}^{-}\right)\left[-\frac{1}{4}(b_{21}^{+}+b_{03}^{+})+\frac{1}{2}a_{11}^{+}\left(-(a_{02}^{+}+a_{20}^{+})\right.\right.\\
&\quad\left.\left.+\frac{1}{2}(a_{21}^{-}+a_{03}^{-})\right)\right]+\left[\frac{1}{2}b_{11}^{+}+\frac{1}{2}a_{11}^{+}\left(a_{10}^{+}-\frac{1}{2}a_{11}^{-}\right)-a_{01}^{+}\left(-(a_{02}^{+}+a_{20}^{+})+\frac{1}{2}(a_{21}^{-}+a_{03}^{-})\right)\right]\\
&\quad\left.\times\left[-(a_{02}^{+}+a_{20}^{+})+\frac{1}{2}(a_{21}^{-}+a_{03}^{-})\right]\right\},\\
A_{3,14} &= \left[-(a_{20}^{+}+a_{02}^{+})+\frac{1}{4}(a_{21}^{-}+a_{03}^{-})+\frac{3}{4}(a_{30}^{-}+a_{12}^{-})\right]\left\{-\frac{1}{2}c_{01}^{+}+\left(\frac{1}{2}a_{11}^{+}(-b_{01}^{+}+b_{10}^{+})-a_{01}^{+}(-b_{02}^{+}+b_{11}^{+}-b_{20}^{+})\right)\right.\\
&\quad +(a_{01}^{+})^{2}\left[-(a_{02}^{+}+a_{20}^{+})-\frac{1}{2}(a_{21}^{-}+a_{03}^{-})\right]+\left(a_{10}^{+}-\frac{1}{2}a_{11}^{-}\right)\left[\frac{1}{2}b_{11}^{+}+\frac{1}{2}a_{11}^{+}\left(a_{10}^{+}-\frac{1}{2}a_{11}^{-}\right)\right.\\
&\quad\left.\left.-a_{01}^{+}\left(-(a_{02}^{+}+a_{20}^{+})+\frac{1}{2}(a_{21}^{-}+a_{03}^{-})\right)\right]+\left[-b_{01}^{+}-a_{01}^{+}\left(a_{10}^{+}-\frac{1}{2}a_{11}^{-}\right)\right]\left[-(a_{02}^{+}+a_{20}^{+})+\frac{1}{2}(a_{21}^{-}+a_{03}^{-})\right]\right\},\\
A_{41} &= \frac{1}{2}a_{11}^{+}(-\frac{3}{4}c_{03}^{+}+\frac{3}{4}c_{12}^{+}-\frac{3}{4}c_{21}^{+}+\frac{3}{4}c_{30}^{+}),\\
A_{42} &= -\frac{3}{2}a_{01}^{+}a_{11}^{+}(-b_{03}^{+}+b_{12}^{+}-b_{21}^{+}+b_{30}^{+})+\frac{1}{4}(a_{11}^{+})^{2}(-b_{02}^{+}+b_{11}^{+}-b_{20}^{+}-\frac{3}{2}b_{03}^{-}+\frac{3}{2}b_{12}^{-}-\frac{3}{2}b_{21}^{-}+\frac{3}{2}b_{30}^{-}),\\
A_{43} &= \frac{3}{4}(a_{11}^{+})^{2}a_{01}^{+}(a_{21}^{-}+a_{03}^{-}-a_{30}^{-}-a_{12}^{-}),\\
A_{44} &= (-b_{02}^{+}+b_{11}^{+}-b_{20}^{+}-\frac{3}{4}b_{03}^{-}+\frac{3}{4}b_{12}^{-}-\frac{3}{4}b_{21}^{-}+\frac{3}{4}b_{30}^{-})\left[-\frac{1}{4}(b_{21}^{+}+b_{03}^{+})+\frac{1}{2}a_{11}^{+}(-a_{20}^{+}-a_{02}^{+}+\frac{1}{2}a_{21}^{-}+\frac{1}{2}a_{03}^{-})\right],\\
A_{45} &= \frac{3}{4}(-b_{03}^{+}+b_{12}^{+}-b_{21}^{+}+b_{30}^{+})\left[\frac{1}{2}b_{11}^{+}+\frac{1}{2}a_{11}^{+}(a_{10}^{+}-\frac{1}{2}a_{11}^{-})+a_{10}^{+}(a_{20}^{+}+a_{02}^{+}-\frac{1}{2}a_{21}^{-}-\frac{1}{2}a_{03}^{-})\right],\\
A_{46} &= (2a_{11}^{-}-2a_{20}^{-}-2a_{02}^{-})\left[-\frac{1}{8}a_{11}^{+}(b_{21}^{+}+b_{03}^{+})+\frac{1}{4}(a_{11}^{+})^{2}\left(-a_{20}^{+}-a_{02}^{+}+\frac{1}{2}a_{21}^{-}+\frac{1}{2}a_{03}^{-}\right)\right],\\
A_{47} &= [-2(a_{20}^{+}+a_{02}^{+})-2(a_{21}^{-}+a_{03}^{-})+3(a_{30}^{-}+a_{12}^{-})]\left[\frac{1}{4}a_{11}^{+}b_{11}^{+}+\frac{1}{4}(a_{11}^{+})^{2}\left(a_{10}^{+}-\frac{1}{2}a_{11}^{-}\right)\right.\\
&\quad\left.-\frac{1}{2}a_{11}^{+}a_{01}^{+}\left(-a_{20}^{+}-a_{02}^{+}+\frac{1}{2}a_{21}^{-}+\frac{1}{2}a_{03}^{-}\right)+\frac{1}{4}a_{01}^{+}(b_{21}^{+}+b_{03}^{+})-\frac{1}{2}a_{11}^{+}a_{01}^{+}\left(-a_{20}^{+}-a_{02}^{+}+\frac{1}{2}a_{21}^{-}+\frac{1}{2}a_{03}^{-}\right)\right],\\
A_{48} &= 2\left[\frac{1}{2}b_{11}^{+}+\frac{1}{2}a_{11}^{+}\left(a_{10}^{+}-\frac{1}{2}a_{11}^{-}\right)+a_{01}^{+}\left(a_{02}^{+}+a_{20}^{+}-\frac{1}{2}a_{21}^{-}-\frac{1}{2}a_{03}^{-}\right)\right]\left[-\frac{1}{4}(b_{21}^{+}+b_{03}^{+})\right.\\
&\quad\left.+\frac{1}{2}a_{11}^{+}\left(-a_{02}^{+}-a_{20}^{+}+\frac{1}{2}a_{21}^{-}+\frac{1}{2}a_{03}^{-}\right)\right],
\end{aligned}
$$

$$A_{49} = \left[a_{10}^{+} - \frac{1}{2}a_{11}^{-} - (a_{20}^{-} + a_{02}^{-})\right]\left\{-\frac{1}{8}(c_{21}^{+} + c_{03}^{+}) + \frac{3}{8}a_{11}^{+}(-b_{03}^{+} + b_{12}^{+} - b_{21}^{+} + b_{30}^{+})\right.$$
$$+\frac{1}{4}(a_{11}^{+})^{2}\left[-(a_{02}^{+} + a_{20}^{+}) - \frac{1}{2}(a_{21}^{-} + a_{03}^{-})\right] + \left[-(a_{02}^{+} + a_{20}^{+}) + \frac{1}{2}(a_{21}^{-} + a_{03}^{-})\right]\left[-\frac{1}{4}(b_{21}^{+} + b_{03}^{+})\right.$$
$$\left.\left.+\frac{1}{2}a_{11}^{+}\left(-(a_{02}^{+} + a_{20}^{+}) + \frac{1}{2}(a_{21}^{-} + a_{03}^{-})\right)\right]\right\},$$
$$A_{4,10} = \left[-(a_{20}^{+} + a_{02}^{+}) + \frac{1}{4}(a_{21}^{-} + a_{03}^{-}) + \frac{3}{4}(a_{30}^{-} + a_{12}^{-})\right]\left\{\frac{1}{4}c_{11}^{+} + \frac{1}{2}a_{11}^{+}(-b_{02}^{+} + b_{11}^{+} - b_{20}^{+})\right.$$
$$-\frac{3}{4}a_{01}^{+}(-b_{03}^{+} + b_{12}^{+} - b_{21}^{+} + b_{30}^{+}) - a_{01}^{+}a_{11}^{+}\left[-(a_{02}^{+} + a_{20}^{+}) - \frac{1}{2}(a_{21}^{-} + a_{03}^{-})\right]$$
$$+\left(a_{10}^{+} - \frac{1}{2}a_{11}^{-}\right)\left[-\frac{1}{4}(b_{21}^{+} + b_{03}^{+}) + \frac{1}{2}a_{11}^{+}\left(-(a_{02}^{+} + a_{20}^{+}) + \frac{1}{2}(a_{21}^{-} + a_{03}^{-})\right)\right]$$
$$\left.+\left[\frac{1}{2}b_{11}^{+} + \frac{1}{2}a_{11}^{+}\left(a_{10}^{+} - \frac{1}{2}a_{11}^{-}\right) - a_{01}^{+}\left(-(a_{02}^{+} + a_{20}^{+}) + \frac{1}{2}(a_{21}^{-} + a_{03}^{-})\right)\right]\left[-(a_{02}^{+} + a_{20}^{+}) + \frac{1}{2}(a_{21}^{-} + a_{03}^{-})\right]\right\},$$
$$A_{51} = \frac{3}{8}(a_{11}^{+})^{2}(-b_{03}^{+} + b_{12}^{+} - b_{21}^{+} + b_{30}^{+}) + \frac{1}{8}(a_{11}^{+})^{3}(a_{21}^{-} - a_{03}^{-} + a_{30}^{-} + a_{12}^{-}),$$
$$A_{52} = \frac{3}{4}(-b_{03}^{+} + b_{12}^{+} - b_{21}^{+} + b_{30}^{+})\left[-\frac{1}{4}(b_{21}^{+} + b_{03}^{+}) + \frac{1}{2}a_{11}^{+}\left(-a_{20}^{+} - a_{02}^{+} + \frac{1}{2}a_{21}^{-} + \frac{1}{2}a_{03}^{-}\right)\right],$$
$$A_{53} = [-2(a_{20}^{+} + a_{02}^{+}) - 2(a_{21}^{-} + a_{03}^{-}) + 3(a_{30}^{-} + a_{12}^{-})]\left[-\frac{1}{8}a_{11}^{+}(b_{21}^{+} + b_{03}^{+})\right.$$
$$\left.+\frac{1}{4}(a_{11}^{+})^{2}\left(-a_{20}^{+} - a_{02}^{+} + \frac{1}{2}a_{21}^{-} + \frac{1}{2}a_{03}^{-}\right)\right],$$

$$A_{54} = \left[-\frac{1}{4}(b_{21}^{+} + b_{03}^{+}) + \frac{1}{2}a_{11}^{+}\left(-a_{02}^{+} - a_{20}^{+} + \frac{1}{2}a_{21}^{-} + \frac{1}{2}a_{03}^{-}\right)\right]^{2},$$
$$A_{55} = \left[-(a_{20}^{+} + a_{02}^{+}) + \frac{1}{4}(a_{21}^{-} + a_{03}^{-}) + \frac{3}{4}(a_{30}^{-} + a_{12}^{-})\right]\left\{-\frac{1}{8}(c_{21}^{+} + c_{03}^{+}) + \frac{3}{8}a_{11}^{+}(-b_{03}^{+} + b_{12}^{+} - b_{21}^{+} + b_{30}^{+})\right.$$
$$+\frac{1}{4}(a_{11}^{+})^{2}\left[-(a_{02}^{+} + a_{20}^{+}) - \frac{1}{2}(a_{21}^{-} + a_{03}^{-})\right] + \left[-(a_{02}^{+} + a_{20}^{+}) + \frac{1}{2}(a_{21}^{-} + a_{03}^{-})\right]\left[-\frac{1}{4}(b_{21}^{+} + b_{03}^{+})\right.$$
$$\left.\left.+\frac{1}{2}a_{11}^{+}\left(-(a_{02}^{+} + a_{20}^{+}) + \frac{1}{2}(a_{21}^{-} + a_{03}^{-})\right)\right]\right\}.$$

Case 3: *When* $a_{30}^{+} + a_{12}^{+} = 0$ *and* $a_{11}^{+} = 0$, *we have*

$$M_4(h) = \sum_{k=0}^{4} A_k(2h-1)^k,$$

where

$$A_0 = A_{01} + A_{02}, A_1 = \sum_{j=1}^{5} A_{1j}, A_2 = \sum_{j=1}^{9} A_{2j}, A_3 = \sum_{j=1}^{8} A_{3j}, A_4 = A_{41} + A_{42},$$

and

$$
\begin{aligned}
A_{01} &= -d_{01}^{-} - \frac{1}{2}a_{01}^{+}(-c_{01}^{-} + c_{10}^{-}) + (-b_{01}^{-} + b_{10}^{-})\left[-b_{01}^{+} - a_{01}^{+}\left(a_{10}^{+} - \frac{1}{2}a_{11}^{-}\right)\right],\\
A_{02} &= a_{10}^{-}\left\{-a_{01}^{+}(-b_{01}^{+} + b_{10}^{+}) + \left(a_{10}^{+} - \frac{1}{2}a_{11}^{-}\right)\left[-b_{01}^{+} - a_{01}^{+}\left(a_{10}^{+} - \frac{1}{2}a_{11}^{-}\right)\right]\right\},\\
A_{11} &= \left(-d_{01}^{+} + \frac{1}{2}d_{11}^{-}\right) + -a_{01}^{+}(-c_{01}^{+} + c_{10}^{+} - c_{02}^{-} + c_{11}^{-} - c_{20}^{-}) + (a_{01}^{+})^{2}(-b_{02}^{-} + b_{11}^{-} - b_{20}^{-}),\\
A_{12} &= (-b_{01}^{-} + b_{10}^{-})\left(\frac{1}{2}b_{11}^{+} + a_{01}^{+}\left(a_{20}^{+} + a_{02}^{+} - \frac{1}{2}a_{21}^{-} - \frac{1}{2}a_{03}^{-}\right)\right) + (-b_{01}^{+} + a_{01}^{+}a_{10}^{+} - \frac{1}{2}a_{01}^{+}a_{11}^{-})\\
&\quad \times (-b_{01}^{+} + b_{10}^{+} - b_{02}^{-} + b_{11}^{-} - b_{20}^{-}),\\
A_{13} &= (2a_{11}^{-} - 2a_{20}^{-} - 2a_{02}^{-})\left[a_{01}^{+}b_{01}^{+} + (a_{01}^{+})^{2}\left(a_{10}^{+} - \frac{1}{2}a_{11}^{-}\right)\right] + \left[-b_{01}^{+} - a_{01}^{+}\left(a_{10}^{+} - \frac{1}{2}a_{11}^{-}\right)\right]^{2},\\
A_{14} &= a_{10}^{-}\left\{-\frac{1}{2}c_{01}^{+} - a_{01}^{+}(-b_{02}^{+} + b_{11}^{+} - b_{20}^{+}) + (a_{01}^{+})^{2}\left[-(a_{02}^{+} + a_{20}^{+}) - \frac{1}{2}(a_{21}^{-} + a_{03}^{-})\right] + \left(a_{10}^{+} - \frac{1}{2}a_{11}^{-}\right)\right.\\
&\quad \times\left[\frac{1}{2}b_{11}^{+} - a_{01}^{+}\left(-(a_{02}^{+} + a_{20}^{+}) + \frac{1}{2}(a_{21}^{-} + a_{03}^{-})\right)\right] + \left[-b_{01}^{+} - a_{01}^{+}\left(a_{10}^{+} - \frac{1}{2}a_{11}^{-}\right)\right][-(a_{02}^{+} + a_{20}^{+})\\
&\quad \left. + \frac{1}{2}(a_{21}^{-} + a_{03}^{-})]\right\},\\
A_{15} &= \left\{-a_{01}^{+}(-b_{01}^{+} + b_{10}^{+}) + \left(a_{10}^{+} - \frac{1}{2}a_{11}^{-}\right)\left[-b_{01}^{+} - a_{01}^{+}\left(a_{10}^{+} - \frac{1}{2}a_{11}^{-}\right)\right]\right\}\left[a_{10}^{+} - \frac{1}{2}a_{11}^{-} - (a_{20}^{-} + a_{02}^{-})\right],\\
A_{21} &= \frac{1}{2}d_{11}^{+} - \frac{1}{4}d_{03}^{-} - \frac{1}{4}d_{21}^{-} - a_{01}^{+}(-c_{02}^{+} + c_{11}^{+} - c_{20}^{+} - \frac{3}{4}c_{03}^{-} + \frac{3}{4}c_{12}^{-} - \frac{3}{4}c_{21}^{-} + \frac{3}{4}c_{30}^{-}),\\
A_{22} &= (a_{01}^{+})^{2}(-b_{02}^{+} + b_{11}^{+} - b_{20}^{+} - \frac{3}{2}b_{03}^{-} + \frac{3}{2}b_{12}^{-} - \frac{3}{2}b_{21}^{-} + \frac{3}{2}b_{30}^{-}) + (a_{01}^{+})^{3}(a_{21}^{-} + a_{03}^{-} - a_{30}^{-} - a_{12}^{-}),\\
A_{23} &= (-b_{01}^{+} + b_{10}^{+} - b_{02}^{-} + b_{11}^{-} - b_{20}^{-})\left[\frac{1}{2}b_{11}^{+} + a_{01}^{+}(a_{20}^{+} + a_{02}^{+} - \frac{1}{2}a_{21}^{-} - \frac{1}{2}a_{03}^{-})\right],\\
A_{24} &= \left(b_{01}^{+} - a_{01}^{+}a_{10}^{+} + \frac{1}{2}a_{01}^{+}a_{11}^{-}\right)\left(-b_{02}^{+} + b_{11}^{+} - b_{20}^{+} - \frac{3}{4}b_{03}^{-} + \frac{3}{4}b_{12}^{-} - \frac{3}{4}b_{21}^{-} + \frac{3}{4}b_{30}^{-}\right),\\
A_{25} &= (2a_{11}^{-} - 2a_{20}^{-} - 2a_{02}^{-})\left[-\frac{1}{2}a_{01}^{+}b_{11}^{+} + (a_{01}^{+})^{2}\left(-a_{20}^{+} - a_{02}^{+} + \frac{1}{2}a_{21}^{-} + \frac{1}{2}a_{03}^{-}\right)\right],\\
A_{26} &= \left[a_{01}^{+}b_{01}^{+} + (a_{01}^{+})^{2}\left(a_{10}^{+} - \frac{1}{2}a_{11}^{-}\right)\right][-2(a_{20}^{+} + a_{02}^{+}) - 2(a_{21}^{-} + a_{03}^{-}) + 3(a_{30}^{-} + a_{12}^{-})],\\
A_{27} &= 2\left[-b_{01}^{+} - a_{01}^{+}\left(a_{10}^{+} - \frac{1}{2}a_{11}^{-}\right)\right]\left[\frac{1}{2}b_{11}^{+} + a_{01}^{+}\left(a_{02}^{+} + a_{20}^{+} - \frac{1}{2}a_{21}^{-} - \frac{1}{2}a_{03}^{-}\right)\right],\\
A_{28} &= a_{10}^{-}\left\{\frac{1}{4}c_{11}^{+} - \frac{3}{4}a_{01}^{+}(-b_{03}^{+} + b_{12}^{+} - b_{21}^{+} + b_{30}^{+}) + \left[\frac{1}{2}b_{11}^{+} - a_{01}^{+}\left(-(a_{02}^{+} + a_{20}^{+}) + \frac{1}{2}(a_{21}^{-} + a_{03}^{-})\right)\right]\right.\\
&\quad \left.\times\left[-(a_{02}^{+} + a_{20}^{+}) + \frac{1}{2}(a_{21}^{-} + a_{03}^{-})\right]\right\},
\end{aligned}
$$

$$
\begin{aligned}
A_{29} &= \left[a_{10}^{+} - \frac{1}{2}a_{11}^{-} - (a_{20}^{-} + a_{02}^{-})\right]\left\{-\frac{1}{2}c_{01}^{+} + -a_{01}^{+}(-b_{02}^{+} + b_{11}^{+} - b_{20}^{+})\right.\\
&\quad +(a_{01}^{+})^2\left[-(a_{02}^{+} + a_{20}^{+}) - \frac{1}{2}(a_{21}^{-} + a_{03}^{-})\right] + \left(a_{10}^{+} - \frac{1}{2}a_{11}^{-}\right)\left[\frac{1}{2}b_{11}^{+} - a_{01}^{+}\left(-(a_{02}^{+} + a_{20}^{+}) + \frac{1}{2}(a_{21}^{-} + a_{03}^{-})\right)\right]\\
&\quad \left.+\left[-b_{01}^{+} - a_{01}^{+}\left(a_{10}^{+} - \frac{1}{2}a_{11}^{-}\right)\right]\left[-(a_{02}^{+} + a_{20}^{+}) + \frac{1}{2}(a_{21}^{-} + a_{03}^{-})\right]\right\} + \left[-(a_{20}^{+} + a_{02}^{+}) + \frac{1}{4}(a_{21}^{-} + a_{03}^{-})\right.\\
&\quad \left.+\frac{3}{4}(a_{30}^{-} + a_{12}^{-})\right]\left\{-a_{01}^{+}(-b_{01}^{+} + b_{10}^{+}) + \left(a_{10}^{+} - \frac{1}{2}a_{11}^{-}\right)\left[-b_{01}^{+} - a_{01}^{+}\left(a_{10}^{+} - \frac{1}{2}a_{11}^{-}\right)\right]\right\},\\
A_{31} &= -\frac{1}{4}d_{03}^{+} - \frac{1}{4}d_{21}^{+} - a_{01}^{+}(-\frac{3}{4}c_{03}^{+} + \frac{3}{4}c_{12}^{+} - \frac{3}{4}c_{21}^{+} + \frac{3}{4}c_{30}^{+}) + \frac{3}{2}(a_{01}^{+})^2(-b_{03}^{+} + b_{12}^{+} - b_{21}^{+} + b_{30}^{+}),\\
A_{32} &= (-b_{02}^{+} + b_{11}^{+} - b_{20}^{+} - \frac{3}{4}b_{03}^{-} + \frac{3}{4}b_{12}^{-} - \frac{3}{4}b_{21}^{-} + \frac{3}{4}b_{30}^{-})\left[\frac{1}{2}b_{11}^{+} + a_{01}^{+}(a_{20}^{+} + a_{02}^{+} - \frac{1}{2}a_{21}^{-} - \frac{1}{2}a_{03}^{-})\right],\\
A_{33} &= \frac{3}{4}(-b_{03}^{+} + b_{12}^{+} - b_{21}^{+} + b_{30}^{+})(-b_{01}^{+} - a_{01}^{+}a_{10}^{+} + \frac{1}{2}a_{01}^{+}a_{11}^{-}),\\
A_{34} &= [-2(a_{20}^{+} + a_{02}^{+}) - 2(a_{21}^{-} + a_{03}^{-}) + 3(a_{30}^{-} + a_{12}^{-})]\left[-\frac{1}{2}a_{01}^{+}b_{11}^{+} + (a_{01}^{+})^2\left(-a_{20}^{+} - a_{02}^{+} + \frac{1}{2}a_{21}^{-} + \frac{1}{2}a_{03}^{-}\right)\right],\\
A_{35} &= \left[\frac{1}{2}b_{11}^{+} + a_{01}^{+}\left(a_{02}^{+} + a_{20}^{+} - \frac{1}{2}a_{21}^{-} - \frac{1}{2}a_{03}^{-}\right)\right]^2,\\
A_{36} &= a_{10}^{-}\left[-\frac{1}{8}(c_{21}^{+} + c_{03}^{+})\right],\\
A_{37} &= \left[a_{10}^{+} - \frac{1}{2}a_{11}^{-} - (a_{20}^{-} + a_{02}^{-})\right]\left\{\frac{1}{4}c_{11}^{+} - \frac{3}{4}a_{01}^{+}(-b_{03}^{+} + b_{12}^{+} - b_{21}^{+} + b_{30}^{+})\right.\\
&\quad \left.+\left[\frac{1}{2}b_{11}^{+} - a_{01}^{+}\left(-(a_{02}^{+} + a_{20}^{+}) + \frac{1}{2}(a_{21}^{-} + a_{03}^{-})\right)\right]\left[-(a_{02}^{+} + a_{20}^{+}) + \frac{1}{2}(a_{21}^{-} + a_{03}^{-})\right]\right\},
\end{aligned}
$$

$$
\begin{aligned}
A_{38} &= \left[-(a_{20}^{+} + a_{02}^{+}) + \frac{1}{4}(a_{21}^{-} + a_{03}^{-}) + \frac{3}{4}(a_{30}^{-} + a_{12}^{-})\right]\left\{-\frac{1}{2}c_{01}^{+} + (-a_{01}^{+}(-b_{02}^{+} + b_{11}^{+} - b_{20}^{+}))\right.\\
&\quad +(a_{01}^{+})^2\left[-(a_{02}^{+} + a_{20}^{+}) - \frac{1}{2}(a_{21}^{-} + a_{03}^{-})\right] + \left(a_{10}^{+} - \frac{1}{2}a_{11}^{-}\right)\left[\frac{1}{2}b_{11}^{+} - a_{01}^{+}\left(-(a_{02}^{+} + a_{20}^{+}) + \frac{1}{2}(a_{21}^{-} + a_{03}^{-})\right)\right]\\
&\quad \left.+\left[-b_{01}^{+} - a_{01}^{+}\left(a_{10}^{+} - \frac{1}{2}a_{11}^{-}\right)\right]\left[-(a_{02}^{+} + a_{20}^{+}) + \frac{1}{2}(a_{21}^{-} + a_{03}^{-})\right]\right\},\\
A_{41} &= \left[a_{10}^{+} - \frac{1}{2}a_{11}^{-} - (a_{20}^{-} + a_{02}^{-})\right]\left[-\frac{1}{8}(c_{21}^{+} + c_{03}^{+})\right],\\
A_{42} &= \left[-(a_{20}^{+} + a_{02}^{+}) + \frac{1}{4}(a_{21}^{-} + a_{03}^{-}) + \frac{3}{4}(a_{30}^{-} + a_{12}^{-})\right]\left\{\frac{1}{4}c_{11}^{+} - \frac{3}{4}a_{01}^{+}(-b_{03}^{+} + b_{12}^{+} - b_{21}^{+} + b_{30}^{+})\right.\\
&\quad \left.+\left[\frac{1}{2}b_{11}^{+} - a_{01}^{+}\left(-(a_{02}^{+} + a_{20}^{+}) + \frac{1}{2}(a_{21}^{-} + a_{03}^{-})\right)\right]\left[-(a_{02}^{+} + a_{20}^{+}) + \frac{1}{2}(a_{21}^{-} + a_{03}^{-})\right]\right\}.
\end{aligned}
$$

Theorem 6. *Consider the piecewise polynomial Hamiltonian system (3.1) with m=1. By using the up to the fourth order Melnikov function, the following statements hold.*

(i) *If the first order Melnikov function $M_1(h)$ is not zero identically, then for sufficiently small $|\epsilon| > 0$, system (3.1) has at most 2 limit cycles bifurcated from the period annulus $\{L_h\}_{h\in J}$, multiplicity taken into account. Moreover, this maximum is achievable.*
(ii) *If $M_1(h) \equiv 0$ and $M_2(h) \not\equiv 0$, for sufficiently small $|\epsilon| > 0$, system (3.1) has at most 2 limit cycles bifurcated from the period annulus $\{L_h\}_{h\in J}$, multiplicity taken into account. Moreover, this maximum is achievable.*
(iii) *If $M_1(h) = M_2(h) \equiv 0$ and $M_3(h) \not\equiv 0$, for sufficiently small $|\epsilon| > 0$, system (3.1) has at most 2 limit cycles bifurcated from the period annulus $\{L_h\}_{h\in J}$, multiplicity taken into account. Moreover, this maximum is achievable.*
(iv) *If $M_1(h) = M_2(h) = M_3(h) \equiv 0$ and $M_4(h) \not\equiv 0$, for sufficiently small $|\epsilon| > 0$, system (3.1) has at most 4 limit cycles bifurcated from the period annulus $\{L_h\}_{h\in J}$, multiplicity taken into account. Moreover, this maximum is achievable.*

Proof. According to the expression for $M_1(h)$ in Lemma 6, we know that

$$M_1(h) = \sum_{k=0}^{2} A_k(2h-1)^k,$$

where

$$A_0 = -a_{01}^-,\ A_1 = -a_{01}^+ + \frac{1}{2}a_{11}^-,\ A_2 = \frac{1}{2}a_{11}^+.$$

Let $a_{01}^- = 0,\ a_{11}^+ = 2,\ a_{11}^- = 2$. Denote

$$\delta = (a_{01}^-, a_{01}^+),\ \delta_0 = (0, 1).$$

Then, through direct calculation, we have

$$A_0(\delta_0) = 0,\ A_1(\delta_0) = 0.\ A_2(\delta_0) = 1 \neq 0.$$

Further more,

$$\det \frac{\partial(A_0, A_1)}{\partial(a_{01}^-, a_{01}^+)}(\delta_0) = \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} = 1 \neq 0.$$

It follows that A_0, A_1 can be taken as free parameters. We can obtain that system (3.1) has 2 limit cycles for $m = 1$. The proof of (**i**) is completed.

According to the expression for $M_2(h)$ in Lemma 6, we know that

$$M_2(h) = \sum_{k=0}^{2} A_k(2h-1)^k,$$

where

$$\begin{aligned} A_0 &= -b_{01}^- - a_{10}^- a_{01}^+, \\ A_1 &= -b_{01}^+ + \frac{1}{2}b_{11}^- - a_{01}^+ \left(a_{10}^+ - a_{20}^- - a_{02}^-\right), \\ A_2 &= \frac{1}{2}b_{11}^+ + a_{01}^+ \left(a_{02}^+ + a_{20}^+\right). \end{aligned}$$

Let $a_{01}^{+}=0,\ b_{01}^{+}=1,\ b_{11}^{+}=2$. Denote

$$\delta=(b_{01}^{-},b_{11}^{-}),\ \delta_0=(0,2).$$

Then, through direct calculation, we have

$$A_0(\delta_0)=0,\ A_1(\delta_0)=0.\ A_2(\delta_0)=\frac{1}{2}\neq 0.$$

Further more,

$$\det\frac{\partial(A_0,A_1)}{\partial(b_{01}^{-},b_{11}^{-})}(\delta_0)=\begin{vmatrix}-1 & 0\\ 0 & \frac{1}{2}\end{vmatrix}=-\frac{1}{2}\neq 0.$$

It follows that A_0,A_1 can be taken as free parameters. We can obtain that system (3.1) has 2 limit cycles for $m=1$. The proof of (**ii**) is completed.

According to the expression for $M_3(h)$ in Lemma 6, let $a_{01}^{+}=0,\ b_{01}^{+}=0,\ c_{11}^{-}=2,\ c_{11}^{+}=1$. Denote

$$\delta=(c_{01}^{-},c_{01}^{+}),\ \delta_0=(0,1).$$

Then, through direct calculation, we have

$$A_0(\delta_0)=0,\ A_1(\delta_0)=0,\ A_2(\delta_0)=\frac{1}{2}\neq 0.$$

Further more,

$$\det\frac{\partial(A_0,A_1)}{\partial(c_{01}^{-},c_{01}^{+})}(\delta_0)=\begin{vmatrix}-1 & 0\\ 0 & -1\end{vmatrix}=1\neq 0.$$

It follows that A_0,A_1,A_2 can be taken as free parameters. We can obtain that system (3.1) has 2 limit cycles for $m=1$. The proof of (**iii**) is completed.

In the expression of $M_4(h)$ in Lemma 6, let $a_{01}^{+}=0,\ b_{01}^{+}=0,\ a_{02}^{+}=\frac{1}{2},\ a_{20}^{+}=\frac{1}{2},\ c_{01}^{+}=-2,\ a_{10}^{+}=1,\ a_{11}^{-}=0,\ a_{02}^{-}=0,\ a_{20}^{-}=0,\ a_{10}^{-}=-2,\ d_{11}^{-}=2$. Denote

$$\delta=(d_{01}^{-},d_{01}^{+},d_{11}^{+},c_{11}^{+}),\ \delta_0=(0,-1,2,4).$$

Then, through direct calculation, we have

$$A_0(\delta_0)=0,\ A_1(\delta_0)=0,\ A_2(\delta_0)=0,\ A_3(\delta_0)=0,\ A_4(\delta_0)=-1\neq 0.$$

Further more,

$$\det\frac{\partial(A_0,A_1,A_2,A_3)}{\partial(d_{01}^{-},d_{01}^{+},d_{11}^{+},c_{11}^{+})}(\delta_0)=\begin{vmatrix}-1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & \frac{1}{2} & 0\\ 0 & 0 & -\frac{1}{2} & \frac{1}{4}\end{vmatrix}=\frac{1}{8}\neq 0.$$

It follows that A_0,A_1,A_2,A_3 can be taken as free parameters. We can obtain that system (3.1) has 4 limit cycles for $m=1$. The proof of (**iv**) is completed. □

Theorem 7. *Consider the piecewise polynomial Hamiltonian system (3.1) with m=2. By using the up to the fourth order Melnikov function, the following statements hold.*
(i) *If the first order Melnikov function $M_1(h)$ is not zero identically, then for sufficiently small $|\epsilon| > 0$, system (3.1) has at most 3 limit cycles bifurcated from the period annulus $\{L_h\}_{h\in J}$, multiplicity taken into account.*
(ii) *If $M_1(h) \equiv 0$ and $M_2(h) \not\equiv 0$, for sufficiently small $|\epsilon| > 0$, system (3.1) has at most 4 limit cycles bifurcated from the period annulus $\{L_h\}_{h\in J}$, multiplicity taken into account.*
(iii) *If $M_1(h) = M_2(h) \equiv 0$ and $M_3(h) \not\equiv 0$, for sufficiently small $|\epsilon| > 0$, system (3.1) has at most 4 limit cycles bifurcated from the period annulus $\{L_h\}_{h\in J}$, multiplicity taken into account.*
(iv) *If $M_1(h) = M_2(h) = M_3(h) \equiv 0$ and $M_4(h) \not\equiv 0$, for sufficiently small $|\epsilon| > 0$, system (3.1) has at most 6 limit cycles bifurcated from the period annulus $\{L_h\}_{h\in J}$, multiplicity taken into account.*

Proof. According to $M_1(h)$ in Lemma 7, let $a_{01}^{+} = 1$, $a_{11}^{-} = 2$, $a_{21}^{-} = 4$, $a_{03}^{-} = 4$, $a_{03}^{+} = 4$, $a_{21}^{+} = 4$. Denote

$$\delta = (a_{01}^{-}, a_{01}^{+}, a_{11}^{+}),\ \delta_0 = (0, 1, 4).$$

Then, through direct calculation, we have

$$A_0(\delta_0) = 0,\ A_1(\delta_0) = 0,\ A_2(\delta_0) = 0,\ A_3(\delta_0) = -2 \neq 0.$$

Further more,

$$\det \frac{\partial(A_0, A_1, A_2)}{\partial(a_{01}^{-}, a_{01}^{+}, a_{11}^{+})}(\delta_0) = \begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \frac{1}{2} \end{vmatrix} = \frac{1}{2} \neq 0.$$

It follows that A_0, A_1, A_2 can be taken as free parameters. We can obtain that system (3.1) has 3 limit cycles for $m = 2$. The proof of (**i**) is completed.

According to $M_2(h)$ in the Lemma 7, let $a_{01}^{+} = 0$, $a_{11}^{+} = 1$, $a_{30}^{+} = \frac{1}{2}$, $a_{12}^{+} = \frac{1}{2}$, $b_{11}^{-} = 1$, $a_{10}^{-} = 1$, $a_{10}^{+} = 0$, $a_{20}^{-} = 0$, $a_{02}^{-} = 0$, $b_{21}^{-} = 2$, $b_{03}^{-} = 2$, $b_{03}^{+} = 2$, $a_{30}^{-} = 2$, $a_{12}^{-} = 2$, $a_{20}^{+} = \frac{1}{2}$, $a_{02}^{+} = \frac{1}{2}$. Denote

$$\delta = (b_{01}^{-}, b_{01}^{+}, b_{11}^{+}, b_{21}^{+}),\ \delta_0 = (0, 1, 2, 2).$$

Then, through direct calculation, we have

$$A_0(\delta_0) = 0,\ A_1(\delta_0) = 0,\ A_2(\delta_0) = 0,\ A_3(\delta_0) = 0,\ A_4(\delta_0) = \frac{3}{8} \neq 0.$$

Further more,

$$\det \frac{\partial(A_0, A_1, A_2, A_3)}{\partial(b_{01}^{-}, b_{01}^{+}, b_{11}^{+}, b_{21}^{+})}(\delta_0) = \begin{vmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{4} \end{vmatrix} = -\frac{1}{8} \neq 0.$$

It follows that A_0, A_1, A_2, A_3 can be taken as free parameters. We can obtain that system (3.1) has 4 limit cycles for $m = 2$. The proof of (**ii**) is completed.

According to the $M_3(h)$ in the **Case 1**, let $a_{01}^+ = 0,\ b_{01}^+ = 0,\ b_{11}^+ = 2,\ a_{10}^- = 1,\ c_{11}^- = 2,\ c_{21}^- = 2,\ c_{03}^- = 2,\ a_{10}^+ = 1,\ c_{03}^+ = 14,\ a_{11}^- = 2,\ a_{30}^+ = 2,\ a_{12}^+ = 2,\ a_{20}^- = 1,\ a_{02}^- = -1,\ a_{02}^+ = 0,\ a_{20}^+ = 0,\ a_{21}^- = 2,\ a_{03}^- = 2,\ a_{30}^- = 2,\ a_{12}^- = 2$. Denote

$$\delta = (c_{01}^-, c_{01}^+, c_{11}^+, c_{21}^+),\ \delta_0 = (0, 2, 2, 2).$$

Then, through direct calculation, we have

$$A_0(\delta_0) = 0,\ A_1(\delta_0) = 0,\ A_2(\delta_0) = 0,\ A_3(\delta_0) = 0,\ A_4(\delta_0) = 3 \neq 0.$$

Further more,

$$\det \frac{\partial(A_0, A_1, A_2, A_3)}{\partial(c_{01}^-, c_{01}^+, c_{11}^+, c_{21}^+)}(\delta_0) = \begin{vmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{4} \end{vmatrix} = -\frac{1}{8} \neq 0.$$

It follows that A_0, A_1, A_2, A_3 can be taken as free parameters. We can obtain that system (3.1) has 4 limit cycles for $m = 2$ in **Case 1**.

According to $M_3(h)$ in **Case 2**, Let $a_{01}^+ = 0,\ b_{01}^+ = 0,\ a_{10}^- = 0,\ c_{11}^- = 2,\ a_{11}^+ = 2,\ b_{01}^- = 1,\ b_{10}^- = 1,\ a_{02}^+ = 0,\ a_{20}^+ = 0,\ a_{21}^- = 0,\ a_{03}^- = 0,\ a_{30}^- = 0,\ a_{12}^- = 0,\ b_{03}^+ = \frac{1}{2},\ b_{12}^+ = 1,\ b_{21}^+ = \frac{1}{2},\ b_{30}^+ = 1,\ a_{10}^+ = 0,\ a_{11}^- = 0,\ a_{20}^- = 0,\ a_{02}^- = 0,\ b_{21}^- = 1,\ b_{30}^- = \frac{1}{2},\ c_{03}^+ = 2,\ c_{21}^- = 2,\ c_{03}^- = 2,\ b_{10}^+ = 0,\ b_{02}^- = 0,\ b_{11}^- = 0,\ b_{20}^- = 0,\ b_{02}^+ = 0,\ b_{11}^+ = \frac{3}{4},\ b_{20}^+ = 0$. Denote

$$\delta = (c_{01}^-, c_{01}^+, c_{11}^+, c_{21}^+),\ \delta_0 = (0, 1, 2, 1).$$

Then, through direct calculation, we have

$$A_0(\delta_0) = 0,\ A_1(\delta_0) = 0,\ A_2(\delta_0) = 0,\ A_3(\delta_0) = 0,\ A_4(\delta_0) = \frac{3}{4} \neq 0.$$

Further more,

$$\det \frac{\partial(A_0, A_1, A_2, A_3)}{\partial(c_{01}^-, c_{01}^+, c_{11}^+, c_{21}^+)}(\delta_0) = \begin{vmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{4} \end{vmatrix} = -\frac{1}{8} \neq 0.$$

It follows that A_0, A_1, A_2, A_3 can be taken as free parameters. We can obtain that system (3.1) has 4 limit cycles for $m = 2$ in **Case 2**.

According to $M_3(h)$ in **Case 3**, Let $a_{01}^+ = 0,\ b_{01}^+ = 0,\ a_{10}^- = 0,\ c_{11}^- = 2,\ b_{11}^+ = 0,\ c_{21}^- = 2,\ c_{03}^- = 2,\ c_{21}^+ = 2,\ c_{03}^+ = 2$. Denote

$$\delta = (c_{01}^-, c_{01}^+, c_{11}^+),\ \delta_0 = (0, 1, 2).$$

Then, through direct calculation, we have

$$A_0(\delta_0) = 0,\ A_1(\delta_0) = 0,\ A_2(\delta_0) = 0,\ A_3(\delta_0) = -1 \neq 0.$$

Further more,

$$\det \frac{\partial(A_0, A_1, A_2)}{\partial(c_{01}^-, c_{01}^+, c_{11}^+)}(\delta_0) = \begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \frac{1}{2} \end{vmatrix} = \frac{1}{2} \neq 0.$$

It follows that A_0, A_1, A_2 can be taken as free parameters. We can obtain that system (3.1) has 3 limit cycles for $m = 2$ in **Case 3**. The proof of (**iii**) is completed.

According to $M_4(h)$ in **Case 1**,let $a_{01}^+ = 0$, $b_{01}^+ = 0$, $a_{10}^- = 0$, $d_{11}^- = 2$, $c_{01}^+ = 0$, $d_{03}^- = 4$, $d_{21}^- = 4$, $d_{21}^+ = 4$, $a_{20}^- = 1$, $a_{02}^- = 1$, $a_{11}^- = 2$, $b_{03}^+ = 0$, $b_{12}^+ = 0$, $b_{21}^+ = 0$, $b_{30}^+ = 0$, $a_{20}^+ = 0$, $a_{02}^+ = 0$, $a_{21}^- = 0$, $a_{03}^- = 0$, $a_{30}^- = 0$, $a_{12}^- = 0$, $a_{30}^+ = 2$, $a_{12}^+ = 2$, $c_{21}^+ = 4$, $c_{03}^+ = 4$. Denote

$$\delta = (d_{01}^-, d_{01}^+, d_{11}^+, d_{03}^+, a_{10}^+, c_{11}^+),\ \delta_0 = (0, 1, 4, -4, 3, 0).$$

Then, through direct calculation, we have

$$A_0(\delta_0) = 0,\ A_1(\delta_0) = 0,\ A_2(\delta_0) = 0,\ A_3(\delta_0) = 0,\ A_4(\delta_0) = 0,\ A_5(\delta_0) = 0,\ A_6(\delta_0) = -3 \neq 0.$$

Further more,

$$\det \frac{\partial(A_0, A_1, A_2, A_3, A_4, A_5)}{\partial(d_{01}^-, d_{01}^+, d_{11}^+, d_{03}^+, a_{10}^+, c_{11}^+)}(\delta_0) = \begin{vmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{3}{4} \end{vmatrix} = -\frac{3}{32} \neq 0.$$

It follows that $A_0, A_1, A_2, A_3, A_4, A_5$ can be taken as free parameters. We can obtain that system (3.1) has 6 limit cycles for $m = 2$ in **Case 1**.

According to $M_4(h)$ in **Case 2**, Let $a_{01}^+ = 0$, $b_{01}^+ = 0$, $a_{10}^- = 0$, $a_{11}^+ = 1$, $b_{11}^+ = 0$, $c_{01}^- = 0$, $c_{10}^- = 0$, $b_{01}^- = 1$, $b_{10}^- = 1$, $d_{11}^- = 2$, $c_{01}^+ = 1$, $c_{10}^+ = 1$, $c_{02}^- = 1$, $c_{11}^- = 1$, $c_{20}^- = 1$, $b_{10}^+ = 1$, $b_{02}^- = 1$, $b_{11}^- = 1$, $b_{20}^- = 1$, $a_{10}^+ = 1$, $a_{11}^- = 2$, $a_{20}^- = 1$, $a_{02}^- = -1$, $d_{03}^- = -4$, $d_{21}^- = 4$, $c_{03}^- = 1$, $c_{12}^- = 1$, $c_{21}^- = 1$, $c_{30}^- = 1$, $c_{02}^+ = 1$, $c_{11}^+ = 0$, $c_{20}^+ = 1$, $d_{21}^+ = -3$, $a_{30}^- = 2$, $a_{12}^- = 2$, $c_{21}^+ = -1$, $a_{20}^+ = 0$, $a_{02}^+ = 0$, $a_{21}^- = 0$, $a_{03}^- = 0$, $b_{03}^+ = 0$, $b_{12}^+ = 0$, $b_{21}^+ = 0$, $b_{30}^+ = 0$, $b_{02}^+ = 0$, $b_{20}^+ = 0$, $b_{03}^- = 0$, $b_{12}^- = 0$, $b_{21}^- = 0$, $b_{30}^- = 0$, $c_{12}^+ = 0$, $c_{30}^+ = 0$. Denote

$$\delta = (d_{01}^-, d_{01}^+, d_{11}^+, d_{03}^+, c_{03}^+),\ \delta_0 = (0, 1, 1, -2, 1).$$

Then, through direct calculation, we have

$$A_0(\delta_0) = 0,\ A_1(\delta_0) = 0,\ A_2(\delta_0) = 0,\ A_3(\delta_0) = 0,\ A_4(\delta_0) = 0,\ A_5(\delta_0) = \frac{1}{2} \neq 0.$$

Further more,

$$\det \frac{\partial(A_0, A_1, A_2, A_3, A_4)}{\partial(d_{01}^-, d_{01}^+, d_{11}^+, d_{03}^+, c_{03}^+)}(\delta_0) = \begin{vmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & -\frac{3}{8} \end{vmatrix} = \frac{3}{64} \neq 0.$$

It follows that A_0, A_1, A_2, A_3, A_4 can be taken as free parameters. We can obtain that system (3.1) has 5 limit cycles for $m = 2$ in **Case 2**.

According to $M_4(h)$ in **Case 3**, let $a^+_{01} = 0$, $b^+_{01} = 0$, $a^-_{10} = 0$, $b^+_{11} = 0$, $d^-_{11} = 2$, $c^+_{01} = 0$, $a^-_{02} = 0$, $a^+_{10} = -1$, $d^-_{03} = 0$, $d^-_{21} = 0$, $d^+_{21} = 4$, $a^-_{11} = 0$, $a^-_{20} = 0$, $c^+_{21} = 4$, $c^+_{03} = 4$, $c^+_{11} = 0$. Denote

$$\delta = (d^-_{01}, d^+_{01}, d^+_{11}, d^+_{03}),\ \delta_0 = (0, 1, 0, -4).$$

Then, through direct calculation, we have

$$A_0(\delta_0) = 0,\ A_1(\delta_0) = 0,\ A_2(\delta_0) = 0,\ A_3(\delta_0) = 0,\ A_4(\delta_0) = 1 \neq 0.$$

Further more,

$$\det \frac{\partial(A_0, A_1, A_2, A_3)}{\partial(d^-_{01}, d^+_{01}, d^+_{11}, d^+_{03})}(\delta_0) = \begin{vmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{4} \end{vmatrix} = -\frac{1}{8} \neq 0.$$

It follows that A_0, A_1, A_2, A_3 can be taken as free parameters. We can obtain that system (3.1) has 4 limit cycles for $m = 2$ in **Case 3**. The proof of (**iv**) is completed. □

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The corresponding author is supported by the National Natural Science Foundation of China (11931016).

Conflict of interest

The authors declare that they have no conflict of interest.

References

1. X. Chen, M. Han, Number of limit cycles from a class of perturbed piecewise polynomial systems, *Internat. J. Bifur. Chaos*, **31** (2021), 2150123. https://doi.org/10.1142/S0218127421501236
2. X. Chen, T. Li, J. Llibre, Melnikov functions of arbitrary order for piecewise smooth differential systems in $\mathbb{R}^n$ and applications, *J. Differ. Equ.*, **314** (2022), 340–369. https://doi.org/10.1016/j.jde.2022.01.019
3. J. Giné, J. Llibre, K. Wu, X. Zhang, Averaging methods of arbitrary order, periodic solutions and integrability, *J. Differ. Equ.*, **260** (2016), 4130–4156. https://doi.org/10.1016/j.jde.2015.11.005
4. M. Han, On the maximum number of periodic solutions of piecewise smooth periodic equations by average method, *J. Differ. Equ.*, **7** (2017), 788–794. https://doi.org/10.11948/2017049

5. M. Han, V. G. Romanovski, X. Zhang, Equivalence of the Melnikov function method and the averaging method, *Qual. Theory Dyn. Syst.*, **15** (2016), 471–479. https://doi.org/10.1007/s12346-015-0179-3
6. M. Han, L. Sheng, Bifurcation of limit cycles in piecewise smooth systems via Melnikov function, *J. Appl. Anal. Comput.*, **5** (2015), 809–815.
7. M. Han, J. Yang, The maximum number of zeros of functions with parameters and application to differential equations, *J. Nonlinear Model. Anal.*, **3** (2021), 13–34. https://doi.org/10.12150/jnma.2021.13
8. J. Huang, Y. Jin, Bifurcation of a kind of 1D piecewise differential equation and its application to piecewise planar polynomial systems, *Int. J. Bifur. Chaos Appl. Sci. Engrg.*, **29** (2019), 1950072. https://doi.org/10.1142/S021812741950072X
9. J. Itikawa, J. Llibre, D. D. Novaes, A new result on averaging theory for a class of discontinuous planar differential systems with applications, *Rev. Mat. Iberoam.*, **33** (2017), 1247–1265. https://doi.org/10.4171/RMI/970
10. G. Ji, Y. Sun, Bifurcation for a class of piecewise cubic systems with two centers, *Electron. J. Qual. Theory Differ. Equ.*, **46** (2022), 1–12. https://doi.org/10.14232/ejqtde.2022.1.46
11. W. Liu, M. Han, Limit cycle bifurcations of near-Hamiltonian systems with multiple switching curves and applications, *Discrete Contin. Dyn. Syst. Ser. S*, **16** (2022), 498–532. https://doi.org/10.3934/dcdss.2022053
12. X. Liu, M. Han, Bifurcation of limit cycles by perturbing piecewise Hamiltonian systems, *Int. J. Bifur. Chaos Appl. Sci. Engrg.*, **20** (2010), 1379–1390. https://doi.org/10.1142/S021812741002654X
13. S. Liu, M. Han, Bifurcation of limit cycles in a family of piecewise smooth systems via averaging theory, *Discrete Contin. Dyn. Syst., Ser. B*, **13** (2020), 3115–3124. https://doi.org/10.3934/dcdss.2020133
14. S. Liu, M. Han, J. Li, Bifurcation methods of periodic orbits for piecewise smooth systems, *J. Differ. Equ.*, **275** (2021), 204–233. https://doi.org/10.1016/j.jde.2020.11.040
15. S. Liu, X. Jin, Y. Xiong, The number of limit cycles in a class of piecewise polynomial systems, *J. Nonlinear Model. Anal.*, **4** (2022), 352–370. https://doi.org/10.12150/jnma.2022.352
16. J. Llibre, A. C. Mereu, D. D. Novaes, Averaging theory for discontinuous piecewise differential systems, *J. Differ. Equ.*, **258** (2015), 4007–4032. https://doi.org/10.1016/j.jde.2015.01.022
17. L. Sheng, S. Wang, X. Li, M. Han, Bifurcation of periodic orbits of periodic equations with multiple parameters by averaging method, *J. Math. Anal. Appl.*, **490** (2020), 124311. https://doi.org/10.1016/j.jmaa.2020.124311
18. H. Tian, M. Han, Limit cycle bifurcations of piecewise smooth near-Hamiltonian systems with a switching curve, *Discrete Contin. Dyn. Syst. Ser. B*, **26** (2021), 5581–5599. https://doi.org/10.3934/dcdsb.2020368
19. Y. Xiong, M. Han, Limit cycle bifurcations in discontinuous planar systems with multiple lines, *J. Appl. Anal. Comput.*, **10** (2020), 361–377. https://doi.org/10.11948/20190274
20. J. Yang, Limit cycles appearing from the perturbation of differential systems with multiple switching curves, *Chaos Solitons Fract.*, **135** (2020), 109764. https://doi.org/10.1016/j.chaos.2020.109764

21. J. Yang, L. Zhao, Limit cycle bifurcations for piecewise smooth Hamiltonian systems with a generalized eye-figure loop, *Int. J. Bifur. Chaos Appl. Sci. Engrg.*, **26** (2016), 1650204. https://doi.org/10.1142/S0218127416502047
22. J. Yang, M. Han, W. Huang, On Hopf bifurcations of piecewise planar Hamiltonian systems, *J. Differ. Equ.*, **250** (2011), 1026–1051. https://doi.org/10.1016/j.jde.2010.06.012
23. P. Yang, Y. Yang, J. Yu, Up to second order Melnikov functions for general piecewise Hamiltonian systems with nonregular separation line, *J. Differ. Equ.*, **285** (2021), 583–606. https://doi.org/10.1016/j.jde.2021.03.020
24. F. Li, P. Yu, Y. Liu, Y. Liu, Centers and isochronous centers of a class of quasi-analytic switching systems, *Sci. China Math.*, **61** (2018), 1201–1218. https://doi.org/10.1007/s11425-016-9158-2
25. W. Hou, S. Liu, Melnikov functions for a class of piecewise Hamiltonian systems, *J. Nonlinear Model. Anal.*, **5** (2023), 123–145. https://doi.org/10.12150/jnma.2023.123