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*Research article*

## Markov-switching threshold stochastic volatility models with regime changes

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**Abstract:** This paper introduces a comprehensive class of models known as Markov-Switching Threshold Stochastic Volatility (MS-TSV) models, specifically designed to address asymmetry and the leverage effect observed in the volatility of financial time series. Extending the classical threshold stochastic volatility model, our approach expresses the parameters governing log-volatility as a function of a homogeneous Markov chain with a finite state space. The primary goal of our proposed model is to capture the dynamic behavior of volatility driven by a Markov chain, enabling the accommodation of both gradual shifts due to economic forces and sudden changes caused by abnormal events. Following the model's definition, we derive several probabilistic properties of the MS-TSV models, including strict (or second-order) stationarity, causality, ergodicity, and the computation of higher-order moments. Additionally, we provide the expression for the covariance function of the squared (or powered) process. Furthermore, we establish the limit theory for the Quasi-Maximum Likelihood Estimator (QMLE) and demonstrate the strong consistency of this estimator. Finally, a simulation study is presented to assess the performance of the proposed estimation method.

**Keywords:** Markov chain; threshold stochastic volatility; stationarity; QMLE

**Mathematics Subject Classification:** 60G10, 62F12

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### 1. Introduction

Over recent years, Markov-switching models (MSMs) have garnered considerable scholarly attention, emerging as potent tools for modeling and characterizing asymmetric business cycles within the realm of econometrics. The selection of these models is grounded in their notable flexibility to capture stability and/or asymmetric effects in volatility shocks, as well as their efficacy in

modeling time series data. Initially highlighted by Hamilton [1, 2], these models have been actively employed in statistical applications, addressing various time series phenomena. Several authors have extensively explored aspects such as stationarity, the existence of moments, geometric ergodicity, statistical inference, and asymptotic theory for both linear and nonlinear Markov-switching models, including MS-ARMA models [3–5], nonlinear MS-ARMA models [6], MS-GARCH models [7, 8], MS-BL models [9–13], MS-BLGARCH models [14, 15], doubly MS-AR models [16], MSAR-SV models [17], and MS-AlogGARCH models [18], while also encompassing a distinct case known as the periodic model [19, 20]. In our study, we introduce an alternative perspective by presenting a Markov-switching threshold stochastic volatility process. This process incorporates a standard threshold stochastic volatility [21, 22] representation within each local regime. Notably, the log-volatility process in this model follows an  $r^{\text{th}}$ -order Markov-switching threshold autoregression (TAR), with coefficients contingent on a Markov chain. This approach is recognized in the literature as a compelling substitute for MS-ARCH-type models, which rely on exogenous innovations to drive volatility. Our presented model can be viewed as a logical expansion of the MSAR-SV model initially proposed by So et al. [23], thereby incorporating heavy-tailed innovations to describe the observed process. For further nuanced insights, a more qualitative discussion on this approach can be found in the works of Casarin [24]. The primary rationale behind opting for the MS-TSV model is its remarkable enhancement of predictive capabilities compared to the standard TSV model. This model effectively captures pivotal events that impact the oil market, demonstrating superior performance. Additionally, it adeptly accommodates the typical fluctuating behavior of volatility attributable to economic dynamics, while simultaneously addressing abrupt, discrete shifts in volatility resulting from unexpected extraordinary events. The goals of this paper can be summarized as follows: (1) delving into the probabilistic properties of the MS-TSV model. In doing so, we establish the necessary and sufficient assumptions required to ensure the existence of a stationary solution. It's noteworthy that the MS-TSV coefficients linked to the Markov chain can diverge from the conventional stationary assumptions associated with standard TSV models; (2) centering on analyzing the strong consistency of the QMLE for MS-TSV models. Prior to delving into the analysis, we introduce a set of symbols to facilitate the forthcoming discussion.

Throughout the paper, the following symbols are employed:

- The symbol  $I_{(\cdot)}$  represents a square matrix in which each main diagonal entry is 1, while all other entries are set to 0. Additionally,  $O_{(n,m)}$  signifies a  $n \times m$  matrix in which all entries are zeros. Meanwhile,  $\underline{F}' := (I_{(1)}, \underline{O}_{(1,r-1)}, I_{(1)}, \underline{O}_{(1,r-1)})$ . The function  $I_{\{\cdot\}}$  refers to an indicator function.
- The notation  $\rho(\Gamma)$  denotes the spectral radius of a square matrix  $\Gamma$ .
- The symbol  $\|\cdot\|$  represents any norm applicable to  $m \times n$  matrices (or  $m \times 1$  vectors). Meanwhile, the symbol  $\otimes$  signifies the Kronecker product operation.
- The sequence  $(\Delta_t, t \in \mathbb{Z})$  represents a stationary Markov chain that is both irreducible and aperiodic.
- The matrix  $\mathbb{Q}^{(n)} = (q_{ij}^{(n)}, (i, j) \in \mathbb{E} \times \mathbb{E})$  represents the  $n$ -step transition probability matrix, where  $q_{ij}^{(n)} = P(\Delta_t = j | \Delta_{t-n} = i)$  with one-step transition probability matrix  $\mathbb{Q} := (q_{ij}, (i, j) \in \mathbb{E} \times \mathbb{E})$  where  $q_{ij} := q_{ij}^{(1)} = P(\Delta_t = j | \Delta_{t-1} = i)$  for  $i, j \in \mathbb{E} = \{1, \dots, e\}$ .
- The vector  $\underline{\Pi}' = (\pi(1), \dots, \pi(d))$  represents the initial stationary distribution, where  $\pi(i) = P(\Delta_0 = i)$ ,  $i = 1, \dots, e$ , such that  $\underline{\Pi}' = \underline{\Pi}'\mathbb{Q}$ .

- When considering a collection of deterministic matrices denoted as  $\Gamma := \{\Gamma(i), i \in \mathbb{E}\}$ , it is important to observe that:

$$\mathbb{Q}^{(n)}(\Gamma) = \begin{pmatrix} q_{11}^{(n)}\Gamma(1) & \dots & q_{e1}^{(n)}\Gamma(1) \\ \vdots & \dots & \vdots \\ q_{1e}^{(n)}\Gamma(e) & \dots & q_{ee}^{(n)}\Gamma(e) \end{pmatrix}, \quad \underline{\Pi}(\Gamma) = \begin{pmatrix} \pi(1)\Gamma(1) \\ \vdots \\ \pi(e)\Gamma(e) \end{pmatrix}$$

with  $\mathbb{Q}^{(1)}(\Gamma) = \mathbb{Q}(\Gamma)$ .

The remaining content of the paper is structured in the following manner. In Section 2, we introduce the MS-TSV model, shedding light on its distinctive probabilistic characteristics. Emphasis is placed on the existence of a strictly or second (or higher)-order stationary solution for the MS-TSV model. Additionally, we establish autocovariance functions corresponding to the squared and powered processes. Section 3 unveils our proposition: a meticulously tailored QMLE for the MS-TSV model. This section not only elucidates the essence of QMLE but also establishes its strong consistency within the MS-TSV framework. Dedicated to presenting the outcomes of our simulations, Section 4 provides a comprehensive analysis of the performance of the proposed QMLE within the MS-TSV model framework. Section 5 serves as the conclusion of this paper.

## 2. MS-TSV model

The univariate Markov-switching threshold stochastic volatility model, denoted as *MS – TSV*( $r$ ), is defined by the following equation:

$$\begin{cases} X_t = \sigma_t^{1/2} \varepsilon_t \\ \log \sigma_t = \alpha_0(\Delta_t) + \sum_{i=1}^r (\alpha_i(\Delta_t) I_{\{X_{t-i} > 0\}} + \beta_i(\Delta_t) I_{\{X_{t-i} < 0\}}) \log \sigma_{t-i} + \beta_0(\Delta_t) e_t. \end{cases} \quad (2.1)$$

In Eq (2.1), the two processes  $\{\varepsilon_t, t \in \mathbb{Z}\}$  and  $\{e_t, t \in \mathbb{Z}\}$  represent two independent and identically distributed (**i.i.d.**) sequences of random variables with zero mean and unit variance. The functions  $\alpha_i(\cdot)$  and  $\beta_i(\cdot)$ ,  $i = 0, \dots, r$  are related to the unobserved Markov chain  $(\Delta_t, t \in \mathbb{Z})$ . Additionally, we assume that  $(\varepsilon_t, e_t)$  and  $\{(X_{u-1}, \Delta_u), u \leq t\}$  are independent. The objective of this section is to demonstrate some important probabilistic properties of the MS-TSV model. To facilitate the analysis, it is often useful to express Eq (2.1) in an equivalent state-space representation. In this context, we can rewrite Eq (2.1) in the form of a multivariate autoregressions model with Markov-switching dynamics:

$$\underline{\Lambda}_t = \Psi(\Delta_t) \underline{\Lambda}_{t-1} + \underline{\Upsilon}_t(\Delta_t) \quad (2.2)$$

and

$$X_t = \varepsilon_t \exp\left(\frac{1}{2} \underline{F}' \underline{\Lambda}_t\right)$$

where

$$\underline{\Lambda}_t' := (I_{\{e_t > 0\}} \log \sigma_t, \dots, I_{\{e_{t-r+1} > 0\}} \log \sigma_{t-r+1}, I_{\{e_t < 0\}} \log \sigma_t, \dots, I_{\{e_{t-r+1} < 0\}} \log \sigma_{t-r+1}), \\ \underline{\Upsilon}_t(\Delta_t) := (\alpha_0(\Delta_t) + \beta_0(\Delta_t) e_t) (I_{\{e_t > 0\}}, O_{(1,r-1)}, I_{\{e_t < 0\}}, O_{(1,r-1)})'$$

and

$$\Psi(\Delta_t) := \begin{pmatrix} \underline{A}'(\Delta_t) I_{\{e_t > 0\}} & \alpha_r(\Delta_t) I_{\{e_t > 0\}} & \underline{B}'(\Delta_t) I_{\{e_t > 0\}} & \beta_r(\Delta_t) I_{\{e_t > 0\}} \\ I_{(r-1)} & O_{(r-1,1)} & O_{(r-1,r-1)} & O_{(r-1,1)} \\ \underline{A}'(\Delta_t) I_{\{e_t < 0\}} & \alpha_r(\Delta_t) I_{\{e_t < 0\}} & \underline{B}'(\Delta_t) I_{\{e_t < 0\}} & \beta_r(\Delta_t) I_{\{e_t < 0\}} \\ O_{(r-1,r-1)} & O_{(r-1,1)} & I_{(r-1)} & O_{(r-1,1)} \end{pmatrix}$$

$$\underline{A}'(\Delta_t) = (\alpha_1(\Delta_t), \dots, \alpha_{r-1}(\Delta_t)), \quad \underline{B}'(\Delta_t) = (\beta_1(\Delta_t), \dots, \beta_{r-1}(\Delta_t)).$$

The process  $((\underline{A}', \Delta_t)', t \in \mathbb{Z})$  represents a Markov chain on  $\mathbb{R}^{2r} \times \mathbb{E}$ . However, when investigating the probabilistic properties of the model described in Eq (2.1), it is more convenient and advantageous to utilize the model presented in Eq (2.2). Equation (2.2) is identical to the definition used for the recently studied D-MSAR model by Ghezal [16]. Firstly, we establish the following significant result, implying strict stationarity.

**Theorem 2.1.** *The multivariate model with Markov-switching (2.2) is under consideration. Here, we present the following:*

i. *Sufficient condition: If*

$$\gamma(\Psi) := \lim_{n \rightarrow \infty} E \left\{ \frac{1}{n} \log \left\| \prod_{j=0}^{n-1} \Psi(\Delta_{n-j}) \right\| \right\} \stackrel{a.s.}{=} \lim_{n \rightarrow \infty} \left\{ \frac{1}{n} \log \left\| \prod_{j=0}^{n-1} \Psi(\Delta_{n-j}) \right\| \right\} < 0$$

then Eq (2.2) admits a unique, strictly stationary, causal and ergodic solution given by the following series

$$X_t = \varepsilon_t \prod_{k=0}^{\infty} \exp \left( \frac{1}{2} \underline{F}' \left\{ \prod_{j=0}^{k-1} \Psi(\Delta_{t-j}) \right\} \underline{\Upsilon}_{t-k}(\Delta_{t-k}) \right) \quad (2.3)$$

which converges absolutely almost surely for all  $t \in \mathbb{Z}$ .

ii. *Necessary condition: If  $\{\underline{\Upsilon}_t(\Delta_t), \Psi(\Delta_t)\}$  is controllable [17] and the multivariate stochastic volatility model with Markov-switching (2.2) has a strictly stationary solution, then it follows that  $\gamma(\Psi) < 0$ .*

*Proof.* i. **Sufficient Condition:** A sufficient condition is provided by the subadditive ergodic theorem. Almost surely, we have

$$\limsup_k \left\| \prod_{j=0}^{k-1} \Psi(\Delta_{t-j}) \right\|^{1/k} \leq \exp \{ \gamma(\Psi) \} < 1.$$

Conversely, utilizing the Borel-Cantelli lemma, it follows that

$$P \left( \limsup_{k \rightarrow +\infty} |e_{t-k}|^{1/k} > \lambda \right) = 0 \text{ for all } \lambda > 1.$$

Consequently,

$$\limsup_{k \rightarrow +\infty} \left\| \left\{ \prod_{j=0}^{k-1} \Psi(\Delta_{t-j}) \right\} \underline{\Upsilon}_{t-k}(\Delta_{t-k}) \right\|^{1/k} \leq \exp \{ \gamma(\Psi) \} < 1$$

and by Cauchy's root test, the series (2.3) converges absolutely almost surely.

ii. Necessary Condition: As for the second assertion, we establish a necessary condition. If there exists a strictly stationary solution for Eq (2.2), thus

$$\left\| \left\{ \prod_{j=0}^{k-1} \Psi(\Delta_{t-j}) \right\} \beta_0(\Delta_t) \right\| \xrightarrow[k \rightarrow \infty]{} 0 \text{ in probability.}$$

By controllability, we consequently derive  $\left\| \left\{ \prod_{j=0}^{k-1} \Psi(\Delta_{t-j}) \right\} \right\| \xrightarrow[k \rightarrow \infty]{} 0$  in probability. Through a straightforward modification of Lemma 3.4 in Picard [25], we deduce that  $\gamma(\Psi) < 0$ .

□

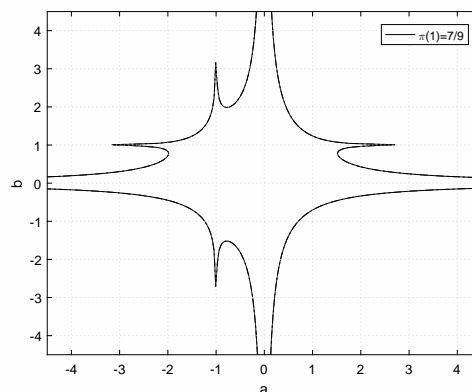
**Remark 2.1.** *If any of the following conditions is satisfied, then it implies that  $\gamma(\Psi) < 0$ :*

- $E \left\{ \log \left\| \prod_{j=0}^{n-1} \Psi(\Delta_{n-j}) \right\| \right\} < 0$ ,
- $E \left\{ \left\| \prod_{j=0}^{n-1} \Psi(\Delta_{n-j}) \right\| \right\} < 1$ ,
- $\rho(|\Psi|) < 1$ , where  $|\Psi| = E\{|\Psi(\Delta_n)|\}$ .

**Example 2.1.** *The MS-TSV(1) model satisfies the following sufficient condition:  $\prod_{k=1}^e |\alpha_1(k)|^{\kappa\pi(k)} |\beta_1(k)|^{(1-\kappa)\pi(k)} < 1$ , where  $\kappa = P(\varepsilon_0 > 0) > 0$ . Consequently, in this state, there exists a unique, strictly stationary, causal, and ergodic solution for the model. Hence, the requirement for local strict stationarity is not essential. In other words, the presence of burst regimes (i.e.,  $|\alpha_1(k_0)|^{\kappa\pi(k_0)} |\beta_1(k_0)|^{(1-\kappa)\pi(k_0)} > 1$ ) does not preclude the possibility of global strict stationarity. For the specific case of MS-TSV(1) with two-regimes,  $X_t = \sigma_t^{1/2} \varepsilon_t$  and*

$$\log \sigma_t = \begin{cases} 1 + (aI_{\{X_{t-1}>0\}} + bI_{\{X_{t-1}<0\}}) \log \sigma_{t-1} + e_t & \text{if } \Delta_t = 1 \\ ((a+1)I_{\{X_{t-1}>0\}} + (b-1)I_{\{X_{t-1}<0\}}) \log \sigma_{t-1} + e_t & \text{if } \Delta_t = 2 \end{cases}$$

$\pi(1) = 7/9$  with  $e_t \sim \mathcal{N}(0, 1)$ , the zone of strict stationarity is illustrated in Figure 1 below.



**Figure 1.** The zone of strict stationarity for MS-TSV(1) model.

The graphical representation in Figure 1 offers a comprehensive insight into the strict stationarity region of the MS-TSV(1) process under the assumption of  $e_t \sim \mathcal{N}(0, 1)$ . The illustration delineates two clearly defined zones:

- The inner zone signifies strict stationarity.
- The outer zone denotes nonstationarity.

This visualization not only facilitates a qualitative assessment of the model's validity but also provides valuable insights into the sensitivity of the model to various inputs.

The distinct delineation of these zones aids in understanding the behavior of the process and contributes to a deeper comprehension of its dynamics.

It's great to hear that other properties of the MS-TSV model, such as second-order stationarity and the existence of moments, are clear and easily obtainable. These properties are essential in understanding the behavior and statistical characteristics of the model. Second-order stationarity ensures that the model's statistical properties remain consistent over time, and the existence of moments indicates that the model's random variables have well-defined statistical properties, such as mean, variance, and higher-order moments.

**Theorem 2.2.** Consider the MS-TSV( $r$ ) model (2.1) with its state-space representation (2.2). If

$$\rho(\mathbb{Q}(\Psi^{(2)})) < 1 \quad (2.4)$$

for  $\Psi^{(2)} := \{\Psi^{\otimes 2}(i), i \in \mathbb{E}\}$ , hence, Eq (2.2) possesses a unique second-order stationary solution represented by the Series (2.3). This solution demonstrates absolute almost sure convergence and convergence in  $\mathbb{L}_2$ . Moreover, it is both strictly stationary and ergodic.

*Proof.* The result is derived from the second-order stationarity of the  $(\underline{\Lambda}_t, t \in \mathbb{Z})$  defined by Eq (2.2). This conclusion is obtained using the findings of Ghezal et al. [10].  $\square$

To demonstrate this, we provide the explicit expressions of the moments up to the second-order in the following result:

**Proposition 2.1.** Consider the MS-TSV( $r$ ) model (2.1), if  $X_t \in \mathbb{L}_2$ , then

i.  $E\{X_t\} = 0$ .

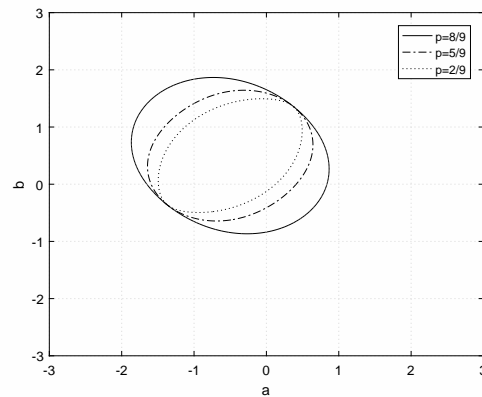
ii.  $\gamma_X(h) = E\{X_t X_{t-h}\} = \sum_{y_t, y_{t-1}, \dots \in \mathbb{E}} \prod_{k \geq 0} q_{y_{t-k-1} y_{t-k}} E\left\{ \prod_{k=0}^{\infty} \exp\left( F' \left\{ \prod_{j=0}^{k-1} \Psi(y_{t-j}) \right\} \underline{\Upsilon}_{t-k}(y_{t-k}) \right) \right\} I_{\{h=0\}}$ .

*Proof.* Given the last condition, obtaining the second-order moments becomes straightforward. For brevity, specific details are omitted.  $\square$

**Example 2.2.** In the case of the MS-TSV(1) model, the Condition (2.4) simplifies to  $\rho(\mathbb{Q}(\underline{\zeta}^{(2)})) < 1$ , where  $\underline{\zeta}^{(2)} := (\zeta^{(2)}(k) = \kappa \alpha_1^2(k) + (1 - \kappa) \beta_1^2(k), k \in \mathbb{S})'$ . Specifically, for two regimes with  $\alpha_1(1) = \alpha_1(2) - 1 = a$ ,  $\beta_1(1) = \beta_1(2) + 1 = b$ ,  $q_{11} = q_{22} = 1 - p$ ,  $q_{12} = q_{21} = p$  and  $e_t \sim \mathcal{N}(0, 1)$ , the Condition (2.4) can be expressed as the following two equivalent conditions:

$$\begin{cases} (2p - 1)(a^2 + b^2)((a + 1)^2 + (b - 1)^2) + 2(1 - p)\left(2\left(a + \frac{1}{2}\right)^2 + 2\left(b - \frac{1}{2}\right)^2 + 1\right) < 4 \\ (1 - p)\left(2\left(a + \frac{1}{2}\right)^2 + 2\left(b - \frac{1}{2}\right)^2 + 1\right) \leq 4. \end{cases}$$

The zone of second-order stationarity is illustrated in Figure 2.



**Figure 2.** Plots of the frontier curves  $\rho(\mathbb{Q}(\underline{\zeta}^{(2)})) = 1$  for MS-TSV(1) model.

Figure 2 provides a nuanced understanding of the second-order stationarity region within the MS-TSV(1) process, assuming  $e_t \sim \mathcal{N}(0, 1)$ . The graphical representation delineates three crucial zones:

- The inner zone signifies second-order stationarity.
- The boundary curve represents the integrated MS-TSV(1), when  $\rho(\mathbb{Q}(\underline{\zeta}^{(2)})) = 1$ .
- The outer zone indicates non-second-order stationarity.

This figure prominently highlights that the second-order stationary zone of MS-TSV(1) is more significant when considering a smaller value for  $p$ . Additionally, the visual representation of second-order stationary zones serves as a valuable tool to observe the model's behavior under different conditions, enhancing our understanding of its dynamics.

Certainly, for the MS-TSV( $r$ ) model with a multivariate representation (2.2), certain assumptions are required to ensure the existence of higher-order moments. These assumptions play a crucial role in understanding the statistical properties and stability of the model.

**Remark 2.2.** When the odd-order moments of  $(X_t, t \in \mathbb{Z})$  exist, they are null. On the other hand, the existence of even-order moments of  $(X_t, t \in \mathbb{Z})$  is succinctly summarized in the following theorem.

**Theorem 2.3.** Consider the MS-TSV( $r$ ) model (2.1) with its state-space representation (2.2). For all integer  $l \geq 1$ , assume that  $E\{(\max(\varepsilon_t, e_t))^l\} < +\infty$  and

$$\rho(\mathbb{Q}(\Psi^{(l)})) < 1 \quad (2.5)$$

where  $\Psi^{(l)} := \{\Psi^{\otimes l}(j), j \in \mathbb{E}\}$ . As a result, the MS-TSV model defined by the state-space representation (2.2) possesses a unique, causal, ergodic, and strictly stationary solution given by (2.3). This solution encompasses moments up to the  $l$ -order. Moreover, the closed form expression for the  $l$ -th moment of  $X_t$  is as follows:

$$E\{X_t^l\} = E\{\varepsilon_t^l\} \sum_{y_t, y_{t-1}, \dots \in \mathbb{E}} \prod_{k \geq 0} q_{y_{t-k-1}, y_{t-k}} E\left\{ \exp\left\{ \frac{l}{2} F' \left\{ \prod_{j=0}^{k-1} \Psi(y_{t-j}) \right\} \underline{\Upsilon}_{t-k}(y_{t-k}) \right\} \right\}.$$

*Proof.* The proof presented in the previous theorem remains applicable, and the results obtained can be extended accordingly. Therefore, we have decided to omit the details.  $\square$

The autocovariance function of the  $(X_t^2, t \in \mathbb{Z})$  process is concisely presented in the following theorem

**Theorem 2.4.** *Given the assumptions stated in the previous theorem, we can deduce the following result:*

(1) *If  $(X_t, t \in \mathbb{Z})$  follows the MS-TSV model (2.1) and  $X_t \in \mathbb{L}_4$ , then*

$$\gamma_{X^2}(0) = E\{\varepsilon_t^4\} \sum_{y_t, y_{t-1}, \dots \in \mathbb{B}} \prod_{k \geq 0} q_{y_{t-k-1} y_{t-k}} E\left\{ \exp\left\{ \underline{F}' \left\{ \prod_{j=0}^{k-1} \Psi(y_{t-j}) \right\} \underline{\Upsilon}_{t-k}(y_{t-k}) \right\} \right\} - \gamma_X^2(0)$$

and  $\gamma_{X^2}(h) = 0$  otherwise.

(2) *If  $(X_t, t \in \mathbb{Z})$  follows the MS-TSV model (2.1) and  $X_t \in \mathbb{L}_{2l}$ , then*

$$\gamma_{X^l}(0) = E\{\varepsilon_t^{2l}\} \sum_{y_t, y_{t-1}, \dots \in \mathbb{B}} \prod_{k \geq 0} q_{y_{t-k-1} y_{t-k}} E\left\{ \exp\left\{ l \underline{F}' \left\{ \prod_{j=0}^{k-1} \Psi(y_{t-j}) \right\} \underline{\Upsilon}_{t-k}(y_{t-k}) \right\} \right\} - (E\{X_t^l\})^2$$

and  $\gamma_{X^l}(h) = 0$  otherwise.

*Proof.* Indeed, it suffices to note that both processes  $(X_t^2)$  and  $(X_t^{2l})$  are white noise processes.  $\square$

### 3. Estimation

Estimating Markov-switching models is a complex task, and the literature has considered specific models to address this challenge [15–18, 26]. Various established Markov Chain Monte Carlo procedures exist for estimating certain states of Eq (2.1), as discussed in [23, 27], and other works. In our study, we focus on a given realization  $(X_1, X_2, \dots, X_n)$  generated from a unique, causal, and strictly stationary MS-TSV model. We assume that  $r$  and  $e$  are known, and  $(\varepsilon_t)$  follows a standard Gaussian distribution. The unknown parameters  $\alpha_i(\cdot)$  and  $\beta_i(\cdot)$ ,  $i = 0, \dots, r$  and  $(q_{i,j}, i, j = 1, \dots, e, i \neq j)$  are combined in a vector  $\underline{\theta}$  belonging to the parameter space  $\Theta$ , with  $\underline{\theta}_0$  representing the true values. Xie [28] has advocated using QMLE and proved its strong consistency for MS-GARCH models. Additionally, Ghezal et al. [16, 17]) introduced certain assumptions ensuring the strong consistency of QMLE for the doubly MS-AR model and symmetric MSAR-SV. The Gaussian likelihood function can be expressed as

$$L_n(\underline{\theta}) = \sum_{\Delta_1, \dots, \Delta_n \in \mathbb{S}} \pi(\Delta_1) \left\{ \prod_{i=2}^n q_{\Delta_{i-1}, \Delta_i} \right\} \left\{ \prod_{i=1}^n h_{\Delta_i}(X_1, \dots, X_i) \right\} \quad (3.1)$$

where

$$h_{\Delta_i}(X_1, \dots, X_i) = \frac{1}{(2\pi\sigma_{\Delta_i}(X_1, \dots, X_{i-1}))^{1/2}} \exp\left\{ -\frac{X_i^2}{2\sigma_{\Delta_i}(X_1, \dots, X_{i-1})} \right\}$$

with the log –transformed conditional stochastic variance process, denoted by  $\log \sigma_{\Delta_i}(X_1, \dots, X_{i-1})$ ,

$$\log \sigma_{\Delta_i}(X_1, \dots, X_{i-1}) = \alpha_0(\Delta_i) + \sum_{i=1}^r (\alpha_i(\Delta_i) I_{\{X_{t-i} > 0\}} + \beta_i(\Delta_i) I_{\{X_{t-i} < 0\}}) \log \sigma_{t-i} + \beta_0(\Delta_i) e_t$$



is defined by the second equation in Eq (2.1). Furthermore, the expression for this likelihood function can be articulated in the subsequent manner:

$$L_n(\underline{\theta}) = \mathbf{1}'_{(e)} \left\{ \prod_{i=1}^n \mathbb{P}_{\underline{\theta}}(h(X_1, \dots, X_i)) \right\} \underline{\Pi}(h(X_1)). \quad (3.2)$$

A quasi-maximum likelihood estimation of  $\underline{\theta}_0$  is determined as any discernible solution,  $\widehat{\underline{\theta}}_n$ , in the context of:

$$\widehat{\underline{\theta}}_n = \arg \max_{\underline{\theta} \in \Theta} L_n(\underline{\theta}). \quad (3.3)$$

In this section, consider  $h_{\Delta_t}(X_t | \underline{X}_{t-1})$  (resp.  $h_{\Delta_t}(X_t | \underline{X}_1)$ ) to be the density function characterizing  $X_t$  given the all past observations (resp. past observations up to  $\varepsilon_1$ ). Similarly, let  $k_{\underline{\theta}}(X_t | \underline{X}_{t-1})$  (resp.  $k_{\underline{\theta}}(X_t | \underline{X}_1)$ ) denote the corresponding logarithmic conditional density of  $X_t$  given  $\{X_{t-1}, X_{t-2}, \dots\}$  (resp.  $\{X_{t-1}, X_{t-2}, \dots, X_1\}$ ). Now, we proceed to establish the likelihood function  $\widetilde{L}_n(\underline{\theta})$  based on all past observations. This function, referred to as  $L_n(\underline{\theta})$  in Eq (3.1), is fashioned by substituting the density  $h_{\Delta_t}(X_1, \dots, X_t)$  with  $h_{\Delta_t}(X_t | \underline{X}_{t-1})$ . Elaborating further,  $\widetilde{L}_n(\underline{\theta})$  can be represented as:

$$\widetilde{L}_n(\underline{\theta}) = \mathbf{1}'_{(e)} \left\{ \prod_{t=2}^n \mathbb{P}_{\underline{\theta}}(h(X_t | \underline{X}_{t-1})) \right\} \underline{\Pi}(h(X_1 | \underline{X}_0)). \quad (3.4)$$

Here, the matrix  $\mathbb{P}_{\underline{\theta}}(h(X_i | \underline{X}_{i-1}))$  (resp. the vector  $\underline{\Pi}(h(X_1 | \underline{X}_0))$ ) takes the place of  $h_{\Delta_t}(X_1, \dots, X_i)$  by  $h_{\Delta_t}(X_i | \underline{X}_{i-1})$  in  $\mathbb{P}_{\underline{\theta}}(h(X_1, \dots, X_i))$  (resp.  $\underline{\Pi}(h(X_1))$ ), for  $i = 1, \dots, n$ .

### 3.1. Demonstration of robust convergence for QMLE

To establish the robust convergence of the QMLE, we rely on the following assumptions:

- A1.**  $\Theta$  constitutes a compact subset of  $\mathbb{R}^{2e(r+1)}$ , encompassing the true value  $\underline{\theta}_0$  within its bounds.
- A2.** For any  $\underline{\theta} \in \Theta$ , the sequence  $\Psi^0$  (derived by modifying the parameters  $\underline{\theta}_0$ ) satisfies  $\gamma_L(\Psi^0) < 0$ .
- A3.** Given any  $\underline{\theta}$  and  $\underline{\theta}^*$  within  $\Theta$ , if  $k_{\underline{\theta}}(X_t | \underline{X}_{t-1})$  equals  $k_{\underline{\theta}^*}(X_t | \underline{X}_{t-1})$  almost surely, then it logically follows that  $\underline{\theta}$  equals  $\underline{\theta}^*$ .

While the first assumption, **A1**, is a familiar cornerstone adopted extensively in various real analysis results, the second assumption, **A2**, secures the principle of strict stationarity for the process  $(X_t, t \in \mathbb{Z})$ . Moreover, **A3**, our third assumption, guarantees the distinguishability of the parameter  $\underline{\theta}$ . To forge ahead, we lay down the foundation of our discourse through the presentation of pivotal lemmas.

**Lemma 3.1.** *Given the robust underpinnings of Assumptions **A2** and **A3**, almost surely, we have*

$$\lim_{n \rightarrow \infty} \log L_n^{1/n}(\underline{\theta}) = \lim_{n \rightarrow \infty} \log \widetilde{L}_n^{1/n}(\underline{\theta}) = E_{\underline{\theta}_0} \left\{ k_{\underline{\theta}}(X_t | \underline{X}_{t-1}) \right\}.$$

*Proof.* Harnessing the potency of the logarithmic function, we attain:

$$\log \tilde{L}_n(\underline{\theta}) = \sum_{t=1}^n k_{\underline{\theta}}(X_t | \underline{X}_{t-1}) \text{ and } \log L_n(\underline{\theta}) = \sum_{t=1}^n k_{\underline{\theta}}(X_t | \underline{X}_1).$$

Hence,

$$\frac{1}{n} \sum_{t=1}^n k_{\underline{\theta}}(X_t | \underline{X}_1) = \frac{1}{n} \sum_{t=1}^n k_{\underline{\theta}}(X_t | \underline{X}_{t-1}) + \frac{1}{n} \sum_{t=1}^n (k_{\underline{\theta}}(X_t | \underline{X}_1) - k_{\underline{\theta}}(X_t | \underline{X}_{t-1})).$$

Presently, for all  $\kappa \in \mathbb{R}$ , the process  $(U_t(s), t \in \mathbb{Z})$  is defined as  $U_t(s) = \sup_{\kappa \geq s} |k_{\underline{\theta}}(X_t | \underline{X}_{t-\kappa}) - k_{\underline{\theta}}(X_t | \underline{X}_{t-1})|$ . For a fixed value of  $s$ , the sequence  $(U_t(s), t \in \mathbb{Z})$  represents a strictly stationary and ergodic process, with  $E_{\underline{\theta}_0} \{U_t(s)\} < +\infty$ . We have

$$\begin{aligned} & \limsup_n \left| \frac{1}{n} \sum_{t=1}^n (k_{\underline{\theta}}(X_t | \underline{X}_1) - k_{\underline{\theta}}(X_t | \underline{X}_{t-1})) \right| \\ & \leq \limsup_n \frac{1}{n} \sum_{t=1}^n |k_{\underline{\theta}}(X_t | \underline{X}_1) - k_{\underline{\theta}}(X_t | \underline{X}_{t-1})| \\ & \leq \limsup_n \frac{1}{n} \sum_{t=s+1}^n U_t(s) = E_{\underline{\theta}_0} \{U_1(s)\} \end{aligned}$$

the result is established.  $\square$

The following lemma provides a comparison between the ratios  $\frac{L_n(\underline{\theta})}{L_n(\underline{\theta}_0)}$  and  $\frac{\tilde{L}_n(\underline{\theta})}{\tilde{L}_n(\underline{\theta}_0)}$ . Define  $T_n(\underline{\theta}) = \log(L_n^{1/n}(\underline{\theta})/L_n^{1/n}(\underline{\theta}_0))$ . With this definition, we can observe that:

**Lemma 3.2.** *Given the robust underpinnings of Assumptions A1–A3, we have*

$$\lim_n (\tilde{L}_n^{1/n}(\underline{\theta})/\tilde{L}_n^{1/n}(\underline{\theta}_0)) = \lim_n T_n(\underline{\theta}) \leq 0$$

with  $\lim_n T_n(\underline{\theta}) = 0$  iff  $\underline{\theta} = \underline{\theta}_0$  for all  $\underline{\theta} \in \Theta$ .

*Proof.* Under assumptions A1–A3, the function  $T_n(\underline{\theta})$  is well-defined. Additionally, leveraging Lemma 3.1 and Jensen's inequality, we obtain:

$$\begin{aligned} \lim_n T_n(\underline{\theta}) &= E_{\underline{\theta}_0} \left\{ \log \left( k_{\underline{\theta}}(X_t | \underline{X}_{t-1}) / k_{\underline{\theta}_0}(X_t | \underline{X}_{t-1}) \right) \right\} \\ &\leq \log E_{\underline{\theta}_0} \left\{ k_{\underline{\theta}}(X_t | \underline{X}_{t-1}) / k_{\underline{\theta}_0}(X_t | \underline{X}_{t-1}) \right\} \\ &= 0. \end{aligned}$$

Given Assumption A3, it follows that  $T_n(\underline{\theta})$  converges to the Kullback-Leinbler information, which attains the value of zero only when  $\underline{\theta} = \underline{\theta}_0$ .  $\square$

**Lemma 3.3.** *Under assumptions A1–A3, for all  $\tilde{\underline{\theta}} \neq \underline{\theta}_0$ , there exists a neighborhood  $\mathcal{V}(\tilde{\underline{\theta}})$  of  $\tilde{\underline{\theta}}$  such that*

$$\limsup_n \sup_{\underline{\theta} \in \mathcal{V}(\tilde{\underline{\theta}})} T_n(\underline{\theta}) < 0 \text{ almost surely.}$$

*Proof.* In Eq (3.4), we derive

$$\min_j \pi(j) h_j \left( X_1 | \underline{X}_0 \right) \left\| \left\{ \prod_{t=2}^n \mathbb{P}_{\underline{\theta}} \left( h \left( X_t | \underline{X}_{t-1} \right) \right) \right\} \right\| \leq \tilde{L}_n(\underline{\theta}) \leq \max_j \pi(j) h_j \left( X_1 | \underline{X}_0 \right) \left\| \left\{ \prod_{t=2}^n \mathbb{P}_{\underline{\theta}} \left( f \left( X_t | \underline{X}_{t-1} \right) \right) \right\} \right\|.$$

Hence, we obtain,

$$\lim_n \log \tilde{L}_n^{1/n}(\underline{\theta}) = \lim_n \log \left\| \left\{ \prod_{t=2}^n \mathbb{P}_{\underline{\theta}} \left( h \left( X_t | \underline{X}_{t-1} \right) \right) \right\} \right\|^{1/n} = E_{\underline{\theta}_0} \left\{ k_{\underline{\theta}} \left( X_t | \underline{X}_{t-1} \right) \right\}.$$

Consider the set  $\mathcal{V}_s(\underline{\theta}) = \{ \underline{\theta} : \|\underline{\theta} - \bar{\underline{\theta}}\| \leq s^{-1} \}$  and define  $\Omega_{2:n}(s) = \sup_{\underline{\theta} \in \mathcal{V}_s(\bar{\underline{\theta}})} \left\| \prod_{t=2}^n \mathbb{P}_{\underline{\theta}} \left( h \left( X_t | \underline{X}_{t-1} \right) \right) \right\|$ . Due to the multiplicativity of the norm, we derive the following result on  $\mathcal{V}_s(\bar{\underline{\theta}})$ ,

$$\sup_{\underline{\theta}} \left\| \prod_{t=2}^{n+k} \mathbb{P}_{\underline{\theta}} \left( h \left( X_t | \underline{X}_{t-1} \right) \right) \right\| \leq \sup_{\underline{\theta}} \left\| \prod_{t=2}^n \mathbb{P}_{\underline{\theta}} \left( h \left( X_t | \underline{X}_{t-1} \right) \right) \right\| \cdot \sup_{\underline{\theta}} \left\| \prod_{t=n+1}^{n+k} \mathbb{P}_{\underline{\theta}} \left( h \left( X_t | \underline{X}_{t-1} \right) \right) \right\|$$

implying:

$$\log \Omega_{2:n+k}(s) \leq \log \Omega_{2:n}(s) + \log \Omega_{n+1:n+k}(s), \text{ for all } n, k.$$

Now, the process  $(\log \Omega_{2:n}(s))$  is both strictly stationary and ergodic, with  $E_{\underline{\theta}_0} \{ \log \Omega_{2:n}(s) \}$  being finite. Consequently, we obtain:

$$\kappa_s(\bar{\underline{\theta}}) = \lim_n \log \Omega_{2:n}^{1/n}(s) = \inf_{n>1} E_{\underline{\theta}_0} \{ \log \Omega_{2:n}^{1/n}(s) \} \text{ almost surely}$$

where  $\gamma_{\underline{\theta}}(H)$  represents the Lyapunov exponent of the sequence  $H = \left( \mathbb{P}_{\underline{\theta}_0} \left( h \left( X_t | \underline{X}_{t-1} \right) \right), t \in \mathbb{Z} \right)$ , that is:

$$\gamma_{\underline{\theta}}(H) = \inf_{n>1} E_{\underline{\theta}_0} \left\{ \log \left\| \prod_{t=2}^n \mathbb{P}_{\underline{\theta}_0} \left( h \left( X_t | \underline{X}_{t-1} \right) \right) \right\|^{1/n} \right\} \stackrel{a.s.}{=} \lim_n \log \left\| \prod_{t=2}^n \mathbb{P}_{\underline{\theta}_0} \left( h \left( X_t | \underline{X}_{t-1} \right) \right) \right\|^{1/n}.$$

Therefore, by utilizing Lemma 3.2, we can establish the existence of  $\delta > 0$  and  $n_\delta \in \mathbb{N}$  such that

$$\frac{1}{n_\delta} E_{\underline{\theta}_0} \left\{ \log \left\| \prod_{t=2}^{n_\delta} \mathbb{P}_{\underline{\theta}} \left( h \left( X_t | \underline{X}_{t-1} \right) \right) \right\| \right\} < \gamma_{\underline{\theta}_0}(H) - \delta.$$

Applying the dominated convergence theorem, we deduce that for sufficiently large  $s$ :

$$\gamma_{\underline{\theta},s}(H) \leq E_{\underline{\theta}_0} \left\{ \log \left\| \prod_{t=2}^{n_\delta} \mathbb{P}_{\underline{\theta}} \left( h \left( X_t | \underline{X}_{t-1} \right) \right) \right\|^{1/n_\delta} \right\} + \frac{\delta}{2} < \gamma_{\underline{\theta}_0}(H) - \frac{\delta}{2}.$$

The final result follows from Lemma 3.1. □

Additionally, we present the following main theorem.

**Theorem 3.1.** *Under Assumptions A1–A3, the sequence of QML estimators  $(\widehat{\underline{\theta}}_n)_n$  satisfying (3.3) exhibits strong consistency, meaning that:*

$$\widehat{\underline{\theta}}_n \rightarrow \underline{\theta}_0 \text{ almost surely when } n \rightarrow +\infty.$$

*Proof.* Let's assume that  $\widehat{\underline{\theta}}_n$  does not converge to  $\underline{\theta}_0$  almost surely, i.e.,

$$\forall n, \exists \delta > 0, N > n, \text{ such that } \|\widehat{\underline{\theta}}_N - \underline{\theta}_0\| \geq \delta.$$

Using the Lemma 3.3, we establish that  $L_n(\widehat{\underline{\theta}}_n) < L_n(\underline{\theta}_0)$ . However, according to the QMLE given in (3.3), we have:

$$L_n(\widehat{\underline{\theta}}_n) = \sup_{\underline{\theta} \in \widetilde{\Theta}} L_n(\underline{\theta}) \geq L_n(\underline{\theta}_0)$$

for any compact subset  $\widetilde{\Theta}$  of  $\Theta$  containing  $\underline{\theta}_0$ . This inconsistency contradicts the result we aim to prove.  $\square$

In the following remark, we delve into the consideration of an open problem

**Remark 3.1.** *Multifractal processes have emerged as a novel formalism for modeling the time series of returns in finance. The notable appeal of these processes lies in their capacity to generate varying degrees of long memory across different powers of returns, a characteristic prevalent in virtually all financial data. In contrast to MS-TSV-type models, multifractal models, as recently developed, are distinguished by a multiplicative structure inherent in the volatility process. Within the multifractal framework, instantaneous volatility is conceptualized as a product of  $m$  volatility components or multipliers and a positive scale factor  $\sigma^2$ ,*

$$X_t = \sigma^2 \left( \sigma_t^{(1)} \sigma_t^{(2)} \cdots \sigma_t^{(m)} \right)^{1/2} \varepsilon_t.$$

*The random multipliers or volatility components  $\sigma_t^{(l)}$  are non-negative. For simplicity, we assume that the multipliers  $\sigma_t^{(1)}, \sigma_t^{(2)}, \dots, \sigma_t^{(m)}$  at a given time  $t$  are statistically independent. This model structure, as outlined by Calvet et al. in [29] and [30], as well as Lux in [31], introduces a new perspective in the representation of financial volatility. To address initial challenges stemming from non-stationarity and the combinatorial nature of the original model, Calvet et al. [29] proposed an iterative multifractal model. This iteration not only overcomes the challenges but also facilitates the estimation of model parameters through methods such as maximum likelihood, providing a robust framework for Bayesian forecasting of volatility in financial time series data.*

#### 4. Simulation study

We conducted a simulation study to assess the performance of the QML method for parameter estimation. The study was based on the Gaussian MS-TSV(r) model with  $e = 2$ . We generated 500 data samples with varying lengths. The sample sizes considered in this simulation study were  $n \in \{750, 1500, 3000\}$ . The chosen parameter values were designed to satisfy the stationarity condition  $\gamma_L(\Psi) < 0$ . For each data trajectory, we estimated the vector  $\underline{\theta}$  of the parameters of interest using the QMLE, denoted as  $\widehat{\underline{\theta}}$ . The QMLE algorithm was executed using the “fminsearch.m” minimizer

function in MATLAB8. In the tables presented below, the root mean square errors (RMSE) of  $\widehat{\theta}$  are displayed in parentheses. Additionally, the true values (TV) of the parameters for each of the considered data-generating processes are reported.

The primary emphasis of this study centers on analyzing the root mean square errors. The results obtained provide initial insights into the finite sample properties of the QMLE within the framework of the MS-TSV model. It is evident from the analysis that the QMLE method delivers effective parameter estimates. Upon examining Table 1, a noteworthy observation is the strong consistency of the QMLE for the MS-TSV model. The corresponding root mean square errors demonstrate a significant reduction as the sample size increases. This suggests that the estimation method becomes more robust with larger datasets. The outcomes presented in Table 2 further reinforce the strong consistency of the QMLE for the MS-TSV model. Notably, even with a relatively small sample size, the estimation procedure produces favorable and reliable results.

**Table 1.** Average and RMSE of QMLE for Gaussian MS-TSV(1) models with varying sample sizes.

	Tv/n	750	1500	3000
$p_{11}$	0.25	0.2471 (0.0053)	0.2529 (0.0032)	0.2479 (0.0014)
$p_{22}$	0.85	0.8534 (0.0072)	0.8521 (0.0056)	0.8509 (0.0023)
$\alpha_0(1)$	1.00	0.9925 (0.0355)	1.0051 (0.0169)	0.9973 (0.0087)
$\alpha_0(2)$	-1.50	-1.5060 (0.0175)	-1.5071 (0.0083)	-1.5031 (0.0038)
$\alpha_1(1)$	0.45	0.4457 (0.0017)	0.4493 (0.0007)	0.4487 (0.0004)
$\alpha_1(2)$	0.25	0.2504 (0.0012)	0.2504 (0.0006)	0.2499 (0.0003)
$\beta_1(1)$	-0.55	-0.5524 (0.0018)	-0.5523 (0.0009)	-0.5511 (0.0004)
$\beta_1(2)$	-0.25	-0.2480 (0.0013)	-0.2496 (0.0005)	-0.2506 (0.0003)
$\beta_0(1)$	0.85	0.8555 (0.0076)	0.8570 (0.0034)	0.8513 (0.0015)
$\beta_0(2)$	-0.50	-0.5000 (0.0023)	-0.4969 (0.0011)	-0.4995 (0.0006)

**Table 2.** Average and RMSE of QMLE for Gaussian MS-TSV(1) models with varying sample sizes.

	Tv/n	750	1500	3000
$p_{11}$	0.95	0.9483 (0.0114)	0.9494 (0.0082)	0.9506 (0.0028)
$p_{22}$	0.15	0.1476 (0.0089)	0.1489 (0.0043)	0.1496 (0.0017)
$\alpha_0(1)$	1.00	0.9835 (0.0290)	0.9950 (0.0158)	1.0011 (0.0073)
$\alpha_0(2)$	1.50	1.4887 (0.0360)	1.4910 (0.0167)	1.4978 (0.0085)
$\alpha_1(1)$	-0.45	-0.4532 (0.0029)	-0.4493 (0.0014)	-0.4510 (0.0007)
$\alpha_1(2)$	0.25	0.2520 (0.0028)	0.2507 (0.0017)	0.2503 (0.0007)
$\beta_1(1)$	0.15	0.1499 (0.0025)	0.1487 (0.0013)	0.1497 (0.0006)
$\beta_1(2)$	0.55	0.5479 (0.0033)	0.5492 (0.0015)	0.5480 (0.0007)
$\beta_0(1)$	0.00	0.0038 (0.0065)	-0.0004 (0.0030)	-0.0023 (0.0015)
$\beta_0(2)$	0.50	0.5053 (0.0168)	0.5032 (0.0078)	0.4998 (0.0038)

## 5. Conclusions

In conclusion, this paper has introduced and thoroughly explored the MS-TSV model, a versatile class specifically designed to address asymmetry and the leverage effect in financial time series volatility. Building upon the classical threshold stochastic volatility model, the MS-TSV model incorporates a homogeneous Markov chain to parameterize log-volatility dynamics. The paper derived essential probabilistic properties of MS-TSV models, including strict stationarity, causality, ergodicity, and higher-order moments, along with providing the covariance function of the squared process. The QMLE for the MS-TSV model was introduced and its strong consistency was demonstrated through a simulation study. The MS-TSV model stands out as a robust alternative to traditional models, particularly in capturing nuanced volatility dynamics influenced by economic factors and unexpected events. This research significantly contributes to the broader understanding of modeling time series data and holds practical applications in financial analysis. However, it's crucial to recognize that ongoing research in this field, especially the exploration of multifractal processes, opens avenues for further investigation. The integration of MS-TSV models with multifractal processes represents a promising direction, offering a more nuanced perspective on volatility modeling in finance. Future work could delve into this intersection, advancing our understanding and refining tools for modeling complex financial time series data.

### Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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### Conflict of interest

All authors declare no conflicts of interest in this paper.

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