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Research article

Neumann gradient estimate for nonlinear heat equation under integral Ricci curvature bounds

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Abstract: In this paper, we consider a Li-Yau gradient estimate on the positive solution to the following nonlinear parabolic equation

$$\frac{\partial}{\partial t}f = \Delta f + af(\ln f)^p$$

with Neumann boundary conditions on a compact Riemannian manifold satisfying the integral Ricci curvature assumption, where $p \geq 0$ is a real constant. This contrasts Olivé's gradient estimate, which works mainly for the heat equation rather than nonlinear parabolic equations and the result can be regarded as a generalization of the Li-Yau [P. Li, S. T. Yau, On the parabolic kernel of the Schrödinger operator, *Acta Math.*, **156** (1986), 153–201] and Olivé [X. R. Olivé, Neumann Li-Yau gradient estimate under integral Ricci curvature bounds, *Proc. Amer. Math. Soc.*, **147** (2019), 411–426] gradient estimates.

Keywords: nonlinear parabolic equation; Li-Yau gradient estimate; integral Ricci curvature; Neumann boundary condition
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1. Introduction

We investigate the Li-Yau gradient estimate for positive solutions to the following nonlinear parabolic equation:

$$\frac{\partial}{\partial t}f = \Delta f + af(\ln f)^p.$$

This equation is subject to Neumann boundary conditions on a compact Riemannian manifold that fulfills the integral Ricci curvature assumption. Here, $p \geq 0$ is a real constant. It is noteworthy that, when $a = 0$, the aforementioned equation aligns with the universally recognized heat equation. Empirical studies have underscored the extensive applicability of scrutinizing solutions to parabolic equations on a Riemannian manifold in various fields, encompassing biology, physics, dynamical systems, and image processing. Furthermore, it serves as a crucial instrument in the domain of geometric analysis. In the pursuit of understanding such correlated issues, it becomes essential to contemplate initial estimates of the solution. In this regard, the Li-Yau type gradient estimation emerges as a significant subject.

The exploration of Li-Yau gradient estimates for the heat equation was initiated in the seminal paper by Li and Yau [1]. In this work, they derived gradient estimates for positive solutions of the heat equation on a complete Riemannian manifold with bounded Ricci curvature, employing the maximum principle. Subsequently, Hamilton utilized a similar technique to establish Harnack inequalities for the Ricci flow. Moreover, Hamilton extended the Li-Yau gradient estimates of the heat equation to a full matrix version in [2]. From this point forward, differential Harnack inequalities have played a pivotal role in the research of geometric flows. In 2002, Perelman [3] applied this method to formulate the Harnack inequality for the fundamental solution of the conjugate heat equation. Leveraging the Harnack inequality, Perelman demonstrated the non-collapse theorem for the Ricci flow. This ground breaking achievement laid the essential foundation for proving the feasibility of the Poincaré conjecture. Numerous papers have been published on the study of parabolic equations based on the Ricci curvature condition, including works by Hamilton [2], Li-Xu [4], and others [5–10]. Additionally, the authors [11, 12] examined a heat equation problem within the framework of control theory.

It is a logical approach to investigate the gradient estimation of positive solutions to equations under integral curvature conditions. It is worth mentioning that this research direction has seen substantial contributions. Prior to presenting the results, we establish some notations. For each $x \in M^n$, let $g(x)$ represent the smallest eigenvalue for the Ricci tensor Ric: $T_xM \to T_xM$. For any given number K,

$$Ric^-(x) = \max\{0, (n-1)K - g(x)\}$$

the amount of n-dimensional Ricci curvature lying below $(n-1)K$. Now consider, for any constants $q, r > 0$,

$$k(q, r) = \sup_{x \in M} r^2 \Big(\oint_{B(x,r)} |Ric^-|^q dV \Big)^{\frac{1}{q}}.$$

Note that $k(q, r) = 0$ if and only if $Ric \geq (n-1)K$. It is crucial to recognize that the integral curvature bound is a natural condition, which is substantially weaker than a lower bound Ricci curvature condition. Consequently, the relationship between the integral Ricci curvature condition and the topology and geometry of manifolds necessitates comprehensive examination. Specifically, by formulating a global version of gradient estimates, Wang [13] achieved the Li-Yau type gradient estimates for the positive solution of the heat equation with Neumann boundary condition. In this scenario, the boundary of M is not convex, but it satisfies the interior rolling R-ball condition, with constants

$$C_1 = \frac{6nb(b-1)(1+H)^7 K}{(b-(1+H)^2)^2} + \frac{309n^2 b^3 (b-1)(1+H)^{10} H}{(b-(1+H)^2)^4 R^2 d},$$

$$C_2 = \frac{nb^2(b-1)^2(1+H)^4}{(2-d)(1-d)(b-(1+H)^2)^2}.$$

It should be noted that our research also encompasses the second fundamental form of ∂M and the interior rolling R-ball condition. The interior rolling ball condition was employed by Chen in [14] to derive an estimate for the first nonzero Neumann eigenvalue of M.

The central focus in gradient estimation under integral curvature conditions, tracing back to the paper by Petersen and Wei [15], revolves around an extension of the Bishop-Gromov relative volume comparison estimate to scenarios where there is only an integral bound for the portion of the Ricci curvature that falls below a specified number. Utilizing Yang's estimates on the Sobolev constants from [16] and capitalizing on the relative volume comparison for integral Ricci curvature, Petersen and Wei [17] broadened several geometric results for manifolds with lower Ricci curvature bounds to integral lower bounds. Building on this groundwork, Zhang and Zhu [18] proposed a Li-Yau gradient bound for positive solutions of the heat equation on compact manifolds. This result was predicated on the assumption that either $|Ric^-| \in L_p(M)$ for some $p > \frac{n}{2}$ and the manifold is noncollapsed, or a certain Kato type of norm of $|Ric^-|$ is finite and the heat kernel possesses a Gaussian upper bound. Subsequently, Zhang and Zhu [19] expanded the result in [18] with the aid of the Sobolev inequality demonstrated by Dai, Wei, and Zhang [20].

In a similar vein, the application of analogous methods to the study of general equations has yielded significant results and findings. The utilization of such methodologies facilitates valuable insights and advancements in our comprehension of diverse techniques. Wang [21] extended the gradient estimates for the heat equation $(\partial_t - \Delta)f = 0$ to $(\partial_t - \Delta)f = af \ln f$, where a is a constant. These outcomes represented substantial generalizations of the Li-Yau [1], Hamilton [2], and Li-Xu [4] type gradient estimates under the integral Ricci curvature bounds. Wang [22] and Wang [23] conducted research on the elliptic gradient estimates on a Riemannian manifold with control on integral Ricci curvature. Wang and Wang [24] concentrated on investigating a class of nonlinear elliptic equations on a collapsed complete Riemannian manifold and its parabolic counterpart under integral curvature conditions.

In this context, for the scenario where the boundary of M is not convex, Olivé [25] formulated a Li-Yau type gradient estimate for positive solutions to the heat equation with Neumann boundary conditions under the integral Ricci curvature assumption. To be more precise, the principal theorem proposed by Olivé was:

Theorem A. Given $H > 0$, $n > 0$, $p > \frac{n}{2}$, and $R > 0$ small enough, there exists $K(n,p) > 0$ such that if M^n is a compact Riemannian submanifold with boundary of a Riemannian manifold N^n with the properties:

(1) $D^2 \sup_{x\in N}(\fint_{B(x,D)} |Ric^-|^q dy)^{\frac{1}{q}} < K$, where $diam(M) \leq D$;

(2) $II \geq -H$, where II is the second fundamental form of ∂M;

(3) M satisfies the interior rolling R-ball condition,

then any positive solution $u(x,t)$ to:

$$\begin{cases} \dfrac{\partial}{\partial t} f = \Delta f, & in \overset{\circ}{M} \times (0,\infty), \\ \partial_\nu f = 0, & on\ \partial M \times (0,\infty), \end{cases}$$

satisfies the Li-Yau type gradient estimate:

$$\alpha \underline{J} \frac{|\nabla f|^2}{f^2} - \frac{\partial_t f}{f} \leq \frac{C_2}{\underline{J}} \frac{1}{t} + C_1. \tag{1.1}$$

where, given any $0 < \xi < 1$, we can choose any $0 < \alpha < \frac{1-\xi}{(1+H)^2}$ and any $0 < \beta \leq \frac{\xi^2(1-\xi)}{2\xi^2+n^2(1+H)^2}$, and where

$$\begin{aligned} C_1 =& \frac{n^2}{\alpha\sqrt{2\xi^3(1-2\beta)}}\Big(\frac{32n^2\alpha H^2(1+H)^2}{\xi^3 R^2} + 2\alpha(1+H)[\frac{H}{R^2} + \frac{2(n-1)H(3H+1)}{R}] \\ &+ (\beta + 4\alpha^{-1})[\frac{4\alpha H(1+H)}{R}]^2\Big), \end{aligned}$$

$$C_2 = \frac{n^2}{\alpha(1-2\beta)}.$$

Our objective is to derive a gradient estimate for the positive solutions to the nonlinear heat equation with Neumann conditions, under the integral Ricci curvature assumption. The methodology utilized to establish the primary results fundamentally draws upon Olivé [25]. However, in contrast to the previously mentioned points, our preliminary task entails identifying a methodology that can proficiently manage the nonlinear terms. Moreover, the function $Q(t)$ can be inferred through analysis, leveraging the Sobolev inequality [20] and volume doubling results [17].

The remainder of this paper is structured as follows. In Section 2, we revisit some definitions and prior works that will serve a fundamental role. Our principal results are presented in Section 3.

2. Preliminaries

In this section, we will effectively derive and present the definitions and lemmas that were instrumental in shaping the arguments and conclusions of this paper. Initially, let $\tau(s)$ denote a nonnegative function on $[0, \infty)$ and the properties of $\tau(s)$ as

$$\begin{cases} \tau(s) \leq H, & s \in [0, \frac{1}{2}), \\ \tau(s) = H, & s \in [1, \infty) \end{cases}$$

with $\tau(0) = 0$, $0 \leq \tau'(s) \leq 2H$, $\tau'(0) = H$, $\tau''(s) \geq -H$.

For convenience, $\tau(\frac{D(s)}{R})$ is written as $\sigma(s)$, where $D(s)$ denotes the distance from a point $s \in M$ to the boundary of M. Then, we introduce two functions, namely $\omega(s)$ and $\tilde{\omega}(s)$, as proposed in references [13, 14]. The specific expressions of two functions are defined as

$$\omega(s) := (1 + \sigma(s))^2, \quad \tilde{\omega}(s) := b\sigma(s)$$

where b satisfies the conditions outlined in the following lemma.

Lemma 2.1. ([13, 14]) The function $\tilde{\omega}$ satisfies

$$b \leq \tilde{\omega} \leq b(1+H)^2, \tag{2.1}$$

$$|\nabla \tilde{\omega}| \leq \frac{4bH(1+H)}{R}, \tag{2.2}$$

$$\Delta\tilde{\omega} \geq -2b(1+H)[\frac{H}{R^2} + \frac{2(n-1)H(3H+1)}{R}]. \tag{2.3}$$

Let $\frac{\partial}{\partial\nu}$ be the outward pointing unit normal vector to ∂M. To estimate a unique smooth solution $Q(x,t)$, we need the following lemma provided in [25] and the key tools are referenced in [26–28].

Lemma 2.2. ([25]) There exists a unique smooth solution $Q(x,t)$ to the problem

$$\begin{cases} \Delta Q - \frac{\partial}{\partial t}Q - c\frac{|\nabla Q|^2}{Q} - 2Q|Ric^-| = 0, & in\ \overset{\circ}{M}\times(0,\infty), \\ \frac{\partial}{\partial \nu}Q = 0, & on\ \partial M\times(0,\infty), \\ Q = 1, & on\ M\times\{0\}, \end{cases}$$

where $c > 1$ is a constant, and satisfies

$$0 < \underline{Q} \leq Q \leq 1$$

with

$$\underline{Q} = \underline{Q}(t) := 2^{-\frac{1}{c-1}}e^{-\frac{\widetilde{C}_3}{c-1}t}$$

and

$$\widetilde{C}_3 = C(c,n,p)[\frac{K}{D^{2-\frac{n}{p}}R^{\frac{n}{p}}} + \frac{K^{\frac{2p}{2p-n}}}{D^{\frac{4p-6n}{2p-n}}R^{\frac{4n}{2p-n}}}] > 0.$$

Remark 2.1. ([25]) The function $\underline{Q}(t)$ is decreasing in t.

Definition 2.1. ([14]) Let $O(x,r)$ be a geodesic ball centered at x with raduis r. For any $y \in \partial N$, there exists $x \in N$ such that $O(x,r) \subseteq N$, and $O(x,r)\cap\partial N = y$. Then, this condition is commonly referred to as the interior rolling r-ball condition on a Riemannian manifold N with boundary.

Based on the above important definitions and lemmas, for $q > \frac{n}{2}$ and $n > 0$, we need to point out that this paper investigates manifold M^n in light of the scaling invariant curvature condition

$$D^2\sup_{x\in N}(\oint_{B(x,D)}|Ric^-|^q dy)^{\frac{1}{q}} < K$$

where $K = K(n,q) > 0$ small enough satisfies the volume doubling proprty [17] and $diam(M) \leq D$.

3. Results

The objective of this section is to establish estimates on the positive solution of the equation

$$\begin{cases} \frac{\partial}{\partial t}f = \Delta f + au(\ln f)^p & in\ \overset{\circ}{M}\times(0,\infty), \\ \frac{\partial}{\partial \nu}f = 0 & on\ \partial M\times(0,\infty) \end{cases} \tag{3.1}$$

and we will give the proof of the main theorem. In order to use the maximum principle, by substituting $u = \ln f$ into (3.1), we can transform equation (3.1).

Theorem 3.1. Let M^n be a compact Riemannian submanifold with boundary of a Riemannian manifold N^n and K, H, and r small enough be nonnegative constants. Given $H > 0$, $n > 0$, $p > \frac{n}{2}$, there exists $K(n, p) > 0$. The second fundamental form denoted as II and the boundary of the manifold ∂M is bounded from below by $-H$. Suppose that M satisfies the interior rolling r-ball conditon. If $u(x, t)$ is a positive solution of equation (3.1) on $M \times (0, \infty)$, then the Li-Yau type gradient estimate is given by

$$b\underline{Q}\frac{|\nabla f|^2}{f^2} - \frac{\partial_t f}{f} + af(\ln f)^p \leq \frac{C_1}{\underline{Q}}\frac{1}{t} + C_2 \tag{3.2}$$

for all constants $0 < b < \frac{1-\xi}{(1+H)^2}$, $0 < d \leq \frac{\xi^2(1-\xi)}{2\xi^2+n^2(1+H)^2}$, $0 < \xi < 1$, where

$$\begin{aligned} C_2 =& \frac{n^2}{b\sqrt{2\xi^3(1-2d)}}\Big(\frac{32n^2bH^2(1+H)^2}{\xi^3R^2} + 2b(1+H)[\frac{H}{R^2} + \frac{2(n-1)H(3H+1)}{R}] \\ &+ (d + 4b^{-1})[\frac{4bH(1+H)}{R}]^2 - 2t_0ap\xi|u^{p-1}|_\infty \\ &- t_0ap|u^{p-1}|_\infty(\tilde{\omega}\underline{Q} - 1) - t_0ap(p-1)|u^{p-2}|_\infty\Big), \end{aligned}$$

$$C_3 = 1 + t_0ap|u^{p-1}|_\infty, \qquad C_1 = \frac{C_3}{E},$$

and $\underline{Q}$ satisfies Lemma 2.2 corresponding to $c = (3 + \frac{1}{b})\frac{1}{d}$.

Proof. Before we give our proof of main results in this section, for simplicity of notation, we define

$$P(x, t) := t[\tilde{\omega}\underline{Q}(|\nabla u|^2 + \varepsilon) - \partial_t u + au^p] \tag{3.3}$$

where $u = \ln f$, $\varepsilon > 0$, and $\underline{Q}$ satisfies the lemma corresponding to $c = (3 + \frac{1}{b})\frac{1}{d}$. For $T > 0$, we suppose that P takes the maximum at point (x, t_0) in $M \times [0, T]$, thus $\Delta P - \partial_t P \leq 0$ at point (x, t_0). The difference from [25] is that we need to deal with the terms involving the coefficient of au^p.

Now we deal with the case that, for $x \in \mathring{M} = M\backslash\partial M$, P takes the maximum at point (x, t_0). Without loss of generality, we can assume that $P(x, t_0) > 0$, then

$$\tilde{P} = \tilde{\omega}\underline{Q}h - \partial_t u + au^p \tag{3.4}$$

where $h = |\nabla u|^2 + \varepsilon$.

Obviously, the first step of this section should calculate the evolution equation. Computing the evolution equation of $\tilde{P}$ and using Bochner's formula, the following result is straightforward:

$$\begin{aligned} &\Delta(\tilde{\omega}\underline{Q}h - \partial_t u + au^p) \\ =&\Delta(\tilde{\omega}\underline{Q})h + 2\nabla(\tilde{\omega}\underline{Q})\nabla h + 2\tilde{\omega}\underline{Q}[|\nabla_i\nabla_j u|^2 + \nabla u\nabla(\Delta u) \\ &+ Ric(\nabla u, \nabla u)] - \partial_t(\Delta u) + ap(p-1)u^{p-2}|\nabla u|^2 + apu^{p-1}\Delta u \\ =&\Delta(\tilde{\omega}\underline{Q})h + 2\nabla(\tilde{\omega}\underline{Q})\nabla h + 2\tilde{\omega}\nabla\underline{Q}\nabla h + 2\tilde{\omega}\underline{Q}[|\nabla_i\nabla_j u|^2 + \nabla u\nabla(\Delta u) \\ &+ Ric(\nabla u, \nabla u)] - \partial_t(\Delta u) + ap(p-1)u^{p-2}|\nabla u|^2 + apu^{p-1}\Delta u \end{aligned} \tag{3.5}$$

and

$$\partial_t(\tilde{\omega}\underline{Q}h - \partial_t u + au^p) = (\tilde{\omega}\underline{Q})_t h + \tilde{\omega}\underline{Q}\partial_t h - \partial_t(\Delta u) - \partial_t|\nabla u|^2. \tag{3.6}$$

Combine (3.5) and (3.6), leading to

$$(\Delta-\partial_t)\tilde{P}=(\Delta-\partial_t)(\tilde{\omega}Q)h+2\nabla\tilde{\omega}\nabla hQ+2\tilde{\omega}\nabla Q\nabla h+2\tilde{\omega}Q[|\nabla_i\nabla_j f|^2+Ric(\nabla u,\nabla u)] + 2\tilde{\omega}Q\nabla u\nabla(\Delta u)-\tilde{\omega}Q\partial_t h+\partial_t|\nabla u|^2+ap(p-1)f^{p-2}|\nabla u|^2+apu^{p-1}\Delta u. \tag{3.7}$$

We see that

$$(\Delta-\partial_t)\tilde{P}\geq(\Delta-\partial_t)(\tilde{\omega}Q)h+2Q\nabla\tilde{\varphi}\nabla h-\frac{2\tilde{\omega}}{dQ}|\nabla Q|^2|\nabla u|^2-2\tilde{\omega}dQ|\nabla_i\nabla_j u|^2+2\tilde{\omega}Q[|\nabla_i\nabla_j u|^2 + Ric(\nabla u,\nabla u)]+2\tilde{\omega}Q\nabla u\nabla(\Delta u)-\tilde{\omega}Q\partial_t h+\partial_t|\nabla u|^2+ap(p-1)u^{p-2}|\nabla u|^2+apu^{p-1}\Delta u \tag{3.8}$$

and here use the upper bound of the terms, for $d>0$,

$$\nabla Q\nabla(|\nabla u|^2)\geq-\frac{1}{dQ}|\nabla Q|^2|\nabla u|^2-dQ|\partial_i\partial_j u|^2.$$

Now, based on the boundedness of $|\nabla u|^2$ and $Ric(\nabla u,\nabla u)$,

$$|\nabla u|^2\leq h,\qquad Ric(\nabla u,\nabla u)\geq-|Ric^-||\nabla u|^2$$

and it is obvious that

$$(\Delta-\partial_t)\tilde{P}\geq(\Delta-\partial_t)(\tilde{\omega}Q)h+2(1-d)\tilde{\omega}Q|\partial_i\nabla_j u|^2-2\tilde{\omega}Q|Ric^-||\nabla u|^2-\frac{2\tilde{\omega}}{dQ}|\nabla Q|^2|\nabla u|^2 + 2Q\nabla\tilde{\omega}\nabla h+2Q\nabla\tilde{\omega}\nabla u+2\tilde{\omega}h\nabla Q\nabla u-2\nabla u\nabla\tilde{P}+2ap(1-\tilde{\omega}Q)u^{p-1}|\nabla u|^2 + \partial_t|\nabla u|^2+ap(p-1)u^{p-2}|\nabla u|^2+apu^{p-1}\Delta u. \tag{3.9}$$

Towards the same direction, for the term of $-2|\nabla u||\nabla Q|$ on the right-hand side, we need to find the upper bound. Taking $-2|\nabla u||\nabla Q|\geq-\frac{|\nabla Q|^2}{dQ}-dQ|\nabla u|^2$ and the Cauchy-Schwarz inequality into consideration, the derivation of (3.9) is as follows:

$$(\Delta-\partial_t)\tilde{P}\geq(\Delta-\partial_t)(\tilde{\omega}Q)h+2(1-d)\tilde{\omega}Q|\partial_i\nabla_j u|^2-2\tilde{\omega}Qh|Ric^-|-\frac{2\tilde{\omega}}{dQ}|\nabla Q|^2h+2hQ\nabla\tilde{\omega}\nabla u - \frac{|\nabla Q|^2}{dQ}\tilde{\omega}h-dQ\tilde{\omega}h|\nabla u|^2+2Q\nabla\tilde{\omega}\nabla h-2\nabla u\nabla Q+2ap(1-\tilde{\omega}Q)u^{p-1}|\nabla u|^2 + \partial_t|\nabla u|^2+ap(p-1)u^{p-2}|\nabla u|^2+apu^{p-1}\Delta u. \tag{3.10}$$

By employing a similar calculation as previously described, we can determine the upper bound of $2\nabla\tilde{\omega}\nabla Q$, where $2\nabla\tilde{\omega}\nabla Q\geq-\frac{1}{bd}\tilde{\omega}\frac{|\nabla Q|^2}{Q}-dQ|\nabla\tilde{\omega}|^2$. We arrive at

$$\begin{aligned}(\Delta-\partial_t)\tilde{P}\geq&h[\Delta\tilde{\omega}Q+2\nabla\tilde{\omega}\nabla Q]+2(1-d)\tilde{\omega}Q|\nabla_i\nabla_j u|^2-dQ\tilde{\omega}h|\nabla u|^2+2Q\nabla\tilde{\omega}\nabla(|\nabla u|^2)\\&+2hQ\nabla\tilde{\omega}\nabla u-2\nabla u\nabla\tilde{P}+\tilde{\omega}h[\Delta Q-\partial_t Q-\frac{3}{d}\frac{|\nabla Q|^2}{Q}-2Q|Ric^-|]\\&+2ap(1-\tilde{\omega}Q)u^{p-1}|\nabla u|^2+ap(p-1)u^{p-2}|\nabla u|^2+apu^{p-1}\Delta u\\\geq&hQ\Delta\tilde{\omega}+2(1-d)\tilde{\omega}Q|\nabla_i\nabla_j u|^2-dQ\tilde{\omega}h|\nabla u|^2+2Q\nabla\tilde{\omega}\nabla(|\nabla u|^2)+2hQ\nabla\tilde{\omega}\nabla u\\&-2\nabla u\nabla\tilde{P}-dhQ|\nabla\tilde{\omega}|^2+[\Delta Q-\partial_t Q-(3+\frac{1}{b})\frac{1}{d}\frac{|\nabla Q|^2}{Q}-2Q|Ric^-|]\tilde{\omega}h\\&+2ap(1-\tilde{\omega}Q)u^{p-1}|\nabla u|^2+ap(p-1)f^{p-2}|\nabla u|^2+apu^{p-1}\Delta u.\end{aligned} \tag{3.11}$$

Subsequently, from (3.11) we obtain

$$\begin{aligned}(\Delta-\partial_t)\tilde{P} \geq & Q[\Delta\tilde{\omega}h+2(1-d)\tilde{\omega}Q|\nabla_i\nabla_j u|^2-dh\tilde{\omega}|\nabla u|^2+2\nabla\tilde{\omega}\nabla(|\nabla u|^2)+2h\nabla\tilde{\omega}\nabla u-dh|\nabla\tilde{\omega}|^2]\\ &-2\nabla u\nabla\tilde{P}+2ap(1-\tilde{\omega}Q)u^{p-1}|\nabla u|^2+ap(p-1)u^{p-2}|\nabla u|^2+apu^{p-1}\Delta u,\end{aligned} \tag{3.12}$$

where Q satisfies Lemma 2.2 corresponding to $c=(3+\frac{1}{b})\frac{1}{d}$.

Since G achieves its maximum value at the point (x,t_0) in $M^n\times(0,T]$, then

$$\begin{aligned}0\geq & t_0Q[\Delta\tilde{\omega}h+2(1-d)\tilde{\omega}Q|\nabla_i\nabla_j u|^2-dh\tilde{\omega}|\nabla u|^2+2\nabla\tilde{\omega}\nabla(|\nabla u|^2)+2h\nabla\tilde{\omega}\nabla u-dh|\nabla\tilde{\omega}|^2]\\ &-\tilde{P}+2t_0ap(1-\tilde{\omega}Q)u^{p-1}|\nabla u|^2+t_0ap(p-1)u^{p-2}|\nabla u|^2+t_0apu^{p-1}\Delta u.\end{aligned} \tag{3.13}$$

The following identities are widely recognized and will be utilized in (3.13):

$$2\nabla\tilde{\omega}\nabla(|\nabla u|^2)\geq 4b^{-1}|\nabla\tilde{\omega}|^2|\nabla u|^2-b|\nabla_i\nabla_j u|^2$$

and

$$\sum|\nabla_i\nabla_j u|^2\geq\frac{1}{n^2}(\sum_{i,j=1}^{n}|\nabla_i\nabla_j u|)^2\geq\frac{1}{n^2}(\Delta u)^2=\frac{1}{n^2}(|\nabla u|^2+au^p-\partial_t u)^2.$$

Therefore, this can be demonstrated by rearranging the inequality as follows:

$$\begin{aligned}0\geq & t_0Q[h\Delta\tilde{\omega}+(2(1-d)\tilde{\omega}-b)|\nabla_i\nabla_j u|^2-dh\tilde{\omega}|\nabla u|^2-4b^{-1}|\nabla\tilde{\omega}|^2|\nabla u|^2+2h\nabla\tilde{\omega}\nabla u\\ &-dh|\nabla\tilde{\omega}|^2]-\tilde{P}+2t_0ap(1-\tilde{\omega}Q)u^{p-1}|\nabla u|^2+t_0ap(p-1)u^{p-2}|\nabla u|^2+t_0apu^{p-1}\Delta u\\ \geq & t_0Q[h\Delta\tilde{\omega}+\frac{2(1-d)\tilde{\omega}-b}{n^2}(|\nabla u|^2+au^p-\partial_t u)^2-dh\tilde{\omega}|\nabla u|^2-4b^{-1}|\nabla\tilde{\omega}|^2|\nabla u|^2+2h\nabla\tilde{\omega}\nabla u\\ &-dh|\nabla\tilde{\omega}|^2]-\tilde{P}+2t_0ap(1-\tilde{\omega}Q)u^{p-1}|\nabla u|^2+t_0ap(p-1)u^{p-2}|\nabla u|^2+t_0apu^{p-1}\Delta u.\end{aligned} \tag{3.14}$$

Here, we expand the terms containing $h=|\nabla u|^2+\varepsilon$ in (3.14) and note the merging of the terms involving the coefficient of af^p. Since we have the inequality $2\nabla\tilde{\omega}\nabla u\geq-(|\nabla\tilde{\omega}|^2+|\nabla u|^2)$, we check

$$\begin{aligned}0\geq & t_0Q[(\Delta\tilde{\omega}-(d+4b^{-1})|\nabla\tilde{\omega}|^2+\theta(\varepsilon))|\nabla u|^2-2|\nabla\tilde{\omega}||\nabla u|^3-d\tilde{\omega}|\nabla u|^4\\ &+\frac{2(1-d)\tilde{\omega}-b}{n^2}(|\nabla u|^2+au^p-\partial_t u)^2]-\tilde{P}+\theta(\varepsilon)\\ &+2t_0ap(1-\tilde{\omega}Q)u^{p-1}|\nabla u|^2+t_0ap(p-1)u^{p-2}|\nabla u|^2+t_0apu^{p-1}\Delta u\end{aligned} \tag{3.15}$$

where $\theta(\varepsilon)$ denotes a function that goes to zero as ε goes to zero.

To apply the maximum principle, we further simplify the right-hand side of the inequality. It is necessary to compute the relationship between the term $(|\nabla u|^2-\partial_t u)^2$ and $\tilde{P}^2$. That is to say that

$$\begin{aligned}\tilde{P}^2=&(|\nabla u|^2+au^p-\partial_t u)^2+(\tilde{\omega}^2Q^2-1)|\nabla u|^4+2(1-\tilde{\omega}Q)|\nabla u|^2\partial_t u\\ &+\theta(\varepsilon)|\nabla u|^2+\theta(\varepsilon)\partial_t u+\theta(\varepsilon)-2a(1-\tilde{\omega}Q)|\nabla u|^2u^p.\end{aligned} \tag{3.16}$$

By substituting $\partial_t u=\tilde{\omega}Q(|\nabla u|+\varepsilon)+au^p-\tilde{P}$, we derive the transformation of (3.16) to be

$$\tilde{P}^2=(|\nabla u|^2+au^p-\partial_t u)^2-(1-\tilde{\omega}Q)|\nabla u|^4-2(1-\tilde{\omega}Q)\tilde{P}|\nabla u|^2+\theta(\varepsilon)|\nabla u|^2+\theta(\varepsilon)\tilde{P}+\theta(\varepsilon). \tag{3.17}$$

The negativity of $2(1-\tilde{\omega}Q)|\nabla u|^2$ can lead to the inequality

$$(|\nabla u|^2 + au^p - \partial_t u)^2 \geq \tilde{P}^2 + (1-\tilde{\omega}Q)^2|\nabla u|^4 + \theta(\varepsilon)|\nabla u|^2 + \theta(\varepsilon)\tilde{P} + \theta(\varepsilon). \tag{3.18}$$

Combining the above two identities, (3.17) and (3.18), we deduce, at point (x, t_0), that

$$\begin{aligned} 0 \geq & t_0 Q[(\Delta\tilde{\omega} - (d+4b^{-1})|\nabla\tilde{\omega}|^2 + \theta(\varepsilon))|\nabla u|^2 - 2|\nabla\tilde{\omega}||\nabla u|^3 \\ & + (\frac{2(1-d)\tilde{\omega} - b}{n^2}(1-\tilde{\omega}Q)^2 - d\tilde{\omega})|\nabla u|^4 + \frac{2(1-d)\tilde{\omega} - b}{n^2}\tilde{P}^2] - (1+\theta(\varepsilon))\tilde{P} + \theta(\varepsilon) \\ & + 2t_0 ap(1-\tilde{\omega}Q)u^{p-1}|\nabla u|^2 + t_0 ap(p-1)u^{p-2}|\nabla u|^2 + t_0 apu^{p-1}\Delta u. \end{aligned} \tag{3.19}$$

In the following step, we appropriately select values for b, d and estimate each of the coefficient terms. We choose $b \leq \frac{1-\xi}{(1-H)^2}$ for some $0 < \xi < 1$ and choose $d > 0$ so that $\frac{2(1-d)\tilde{\omega}-b}{n^2} > 0$.

The coefficient of $|\nabla u|^4$ in the inequality derived from (3.19) can be further refined. For some $G > 0$, we have

$$\frac{2(1-d)\tilde{\omega} - b}{n^2}(1-\tilde{\omega}Q)^2 - d\tilde{\omega} \geq G. \tag{3.20}$$

Based on Lemma 2.1, it is established that

$$\frac{2(1-d)\tilde{\omega} - b}{n^2}(1-\tilde{\omega}Q)^2 - d\tilde{\omega} \geq \frac{2(1-d)\tilde{\omega} - b}{n^2}\xi^2 - db(1+H)^2. \tag{3.21}$$

Given the evolution of (3.21), the bound of d is given by

$$d \leq \frac{a\xi^2 - Gn^2}{2a\xi^2 + an^2(1+H)^2}. \tag{3.22}$$

In (3.22), we assume that d is a positive and consider choosing $G = \frac{b\xi^2}{n^2}$. Then,

$$d \leq \frac{\xi^2(1-\xi)}{2\xi^2 + n^2(1+H)^2}. \tag{3.23}$$

As Lemma 2.1 holds for $\tilde{\omega}$, it allows that the coefficients of $|\nabla u|^2$ satisfies

$$\Delta\tilde{\omega} - (d+4b^{-1})|\nabla\tilde{\omega}|^2 + \theta(\varepsilon) \geq -C(b,d,n,H,R) + \theta(\varepsilon) \tag{3.24}$$

where $C := 2b(1+H)[\frac{H}{R^2} + \frac{2(n-1)H(3H+1)}{R}] + (d+4b^{-1})[\frac{4bH(1+H)}{R}]^2$.

The scaling forms for the coefficients of $|\nabla u|^3$ and $\tilde{P}^2$, respectively, are

$$-2|\nabla\tilde{\omega}| \geq -\frac{8bH(1+H)}{R} = -B_1(b,H,R) \tag{3.25}$$

and

$$\frac{2(1-d)\tilde{\omega} - b}{n^2} \geq \frac{b(1-2d)}{n^2} = E(b,d,n). \tag{3.26}$$

For convenience, we assume that

$$\begin{aligned}&\Delta\tilde{\omega} - (d + 4b^{-1})|\nabla\tilde{\omega}|^2 - 2t_0ap\xi|u^{p-1}|_\infty|\nabla u|^2 - t_0ap|u^{p-1}|_\infty(\tilde{\omega}Q - 1)|\nabla u|^2 \\ &\quad - t_0ap(p-1)|u^{p-2}|_\infty|\nabla u|^2 + \theta(\varepsilon) \geq -D(b, d, n, H, R, |u|_\infty) + \theta(\varepsilon).\end{aligned} \tag{3.27}$$

Combining the above inequalities, (3.24), (3.25), (3.26), and (3.27), we can construct, after elementary estimates,

$$\begin{aligned}0 \geq & t_0Q[-C|\nabla u|^2 - B_1|\nabla u|^3 + G|\nabla u|^4 + E\tilde{P}^2] - (1 + \theta(\varepsilon))\tilde{P} + \theta(\varepsilon) \\ & + 2t_0ap\xi u^{p-1}|\nabla u|^2 + t_0ap(p-1)u^{p-2}|\nabla u|^2 + t_0apu^{p-1}\Delta u \\ \geq & t_0Q[-C|\nabla u|^2 - B_1|\nabla u|^3 + G|\nabla u|^4 + E\tilde{P}^2] - (1 + t_0apu^{p-1} + \theta(\varepsilon))\tilde{P} + \theta(\varepsilon) \\ & + 2t_0ap\xi u^{p-1}|\nabla u|^2 + t_0apu^{p-1}(\tilde{\omega}Q - 1)|\nabla u|^2 + t_0ap(p-1)u^{p-2}|\nabla u|^2.\end{aligned} \tag{3.28}$$

It follows from Lemma 2.2 that (3.28) attains

$$\begin{aligned}0 \geq & t_0Q[-D|\nabla u|^2 - B_1|\nabla u|^3 + G|\nabla u|^4 + E\tilde{P}^2] - (1 + t_0apu^{p-1} \\ & + \theta(\varepsilon))\tilde{P} + \theta(\varepsilon),\end{aligned} \tag{3.29}$$

where $|u|_\infty = \max_M |u|$.

Since the calculations are similar to that of Wang [13] and Olivé [25], we will only sketch the key steps. Calling $x = |\nabla u|^2$, the coefficient of t_0J is transformed such that

$$\begin{aligned}Ax^2 - Bx^{\frac{3}{2}} - Dx \geq & (\sqrt{\frac{A}{2}}x - \frac{1}{\sqrt{2A}}(\frac{B^2}{2A} + D))^2 - \frac{1}{2A}(\frac{B^2}{2A} + D)^2 \\ \geq & -\frac{1}{2A}(\frac{B^2}{2A} + D)^2 + \theta(\varepsilon) := -M + \theta(\varepsilon).\end{aligned} \tag{3.30}$$

By substituting the result of (3.30) into (3.29), we derive that (3.29) becomes

$$\begin{aligned}0 \geq & Et_0Q\tilde{P}^2 - (1 + t_0apu^{p-1} + \theta(\varepsilon))\tilde{P} - Mt_0Q + \theta(\varepsilon) \\ \geq & Et_0Q\tilde{P}^2 - (1 + t_0ap|u^{p-1}|_\infty + \theta(\varepsilon))\tilde{P} - Mt_0Q + \theta(\varepsilon).\end{aligned} \tag{3.31}$$

Multiplying both sides of (3.31) by t_0 gives rise to

$$0 \geq EQP^2 - (1 + t_0ap|u^{p-1}|_\infty + \theta(\varepsilon))P - Mt_0^2Q + \theta(\varepsilon). \tag{3.32}$$

Obviously, the right side of (3.32) is quadratic in P. So, if it is nonpositive, we need, at point (p, t_0),

$$P(p, t_0) \leq \frac{N}{2EQ(t_0)} + \sqrt{\frac{N^2}{4E^2Q^2(t_0)} + \frac{Mt_0^2}{E} + \theta(\varepsilon)} + \theta(\varepsilon) \tag{3.33}$$

where P takes the maximum at point (p, t_0) in $M \times [0, T]$, $N = 1 + t_0ap|u^{p-1}|_\infty$.

For any point $x \in M$, under the conditions satisfied by Q outlined in Lemma 2.2, (3.33) can be transformed into

$$P(x, T) \leq \frac{N}{2E\underline{Q}(T)} + \sqrt{\frac{N^2}{4E^2\underline{Q}^2(T)} + \frac{MT^2}{E} + \theta(\varepsilon)} + \theta(\varepsilon). \tag{3.34}$$

Expanding the definition of P, we observe

$$T[\tilde{\omega}\underline{Q}(|\nabla u|^2+\varepsilon)-\partial_t u+au^p]\le P\le \frac{N}{2E\underline{Q}}+\sqrt{\frac{N^2}{4E^2\underline{Q}^2}+\frac{MT^2}{E}}+\theta(\varepsilon)+\theta(\varepsilon)$$

where the inequality is being evaluated at (x,T). At this point, the inequality does not depend on the (x,t_0), so we have

$$T[\tilde{\omega}\underline{Q}|\nabla u|^2-\partial_t u+au^p]\le \frac{N}{2E\underline{Q}}+\sqrt{\frac{N^2}{4E^2\underline{Q}^2}+\frac{MT^2}{E}}. \tag{3.35}$$

Then, at any point $(x,t)\in M\times(0,\infty)$, we derive

$$t[\tilde{\omega}Q|\nabla u|^2-\partial_t u+au^p]\le \frac{N}{2E\underline{Q}}+\sqrt{\frac{N^2}{4E^2\underline{Q}^2}+\frac{Mt^2}{E}}\le\frac{N}{E\underline{Q}}+\sqrt{\frac{Mt^2}{E}}. \tag{3.36}$$

Hence, dividing by t on both sides of (3.36) leads to

$$b\underline{Q}\frac{|\nabla f|^2}{f^2}-\frac{\partial_t f}{f}+af(\ln f)^p\le\frac{C_1}{\underline{Q}}\frac{1}{t}+C_2 \tag{3.37}$$

where $0<\xi<1$, $0<b<\frac{1-\xi}{(1+H)^2}$, $0<d\le\frac{\xi^2(1-\xi)}{2\xi^2+n^2(1+H)^2}$, $C_1=\frac{C_3}{E}$, $C_3=1+t_0ap|u^{p-1}|_\infty$,

$$\begin{aligned}C_2=&\frac{n^2}{b\sqrt{2\xi^3(1-2d)}}\Big(\frac{32n^2bH^2(1+H)^2}{\xi^3R^2}+2b(1+H)[\frac{H}{R^2}+\frac{2(n-1)H(3H+1)}{R}]\\&+(d+4b^{-1})[\frac{4bH(1+H)}{R}]^2-2t_0ap\xi|u^{p-1}|_\infty\\&-t_0ap|u^{p-1}|_\infty(\tilde{\varphi}Q-1)-t_0ap(p-1)|u^{p-2}|_\infty\Big).\end{aligned}$$

Finally, we aim to bound the case of $x\in\partial M$. Following [13], here we employ an idea that has been developed in [19] and [18] to deal with manifolds. We will only sketch the key steps. Choosing an orthonormal frame $\{e_1,e_2......e_n\}$ at x so that $e_n=\nu$ and $\partial_\nu P(x,t_0)=0$, we obtain

$$t_0[\partial_\nu\tilde{\varphi}Q(|\nabla u|^2+\varepsilon)+\tilde{\omega}(\partial_\nu Q(|\nabla u|^2+\varepsilon)+2Q\partial_i u\partial_\nu\partial_i u)-\partial_\nu\partial_t f+apu^{p-1}\partial_\nu u]\ge 0.$$

Using the fact that $\partial_\nu u=0$ and $\partial_\nu J=0$ on $\partial M\times(0,\infty)$, and dividing by $t_0\tilde{\omega}(|\nabla u|+\varepsilon)$, at point (x,t_0), one can show that

$$\frac{1}{\widetilde{\omega}}\partial_\nu\widetilde{\omega}+\frac{2\partial_i u\partial_\nu\partial_i u}{|\nabla u|^2+\varepsilon}\ge 0.$$

This leads to a contradiction. Then, this case can not occur, which completes the proof.

□

Remark 3.1. In Table 1, there have been observed inconsistencies in the assignment of constants among the theorems. However, in the case where $a=0$, equation (3.1) can indeed be simplified to equation (1) using appropriate transformations. The constant C_1 in Theorem A is equivalent to the constant C_2 in Theorem 3.1, and the remaining constants in the theorems exhibit a similar relationship. The equivalence of these constants underscores the close relationship between the two theorems, further reinforcing the interconnectedness and consistency of the results.

Table 1. Constants in theorems.

	$\mathbf{C_1}$	$\mathbf{C_2}$
Theorem A	$\frac{n^2}{\alpha\sqrt{2\xi^3(1-2\beta)}}(\frac{32n^2\alpha H^2(1+H)^2}{\xi^3R^2} + 2\alpha(1+H)[\frac{H}{R^2}+\frac{2(n-1)H(3H+1)}{R}] + (\beta+4\alpha^{-1})[\frac{4\alpha H(1+H)}{R}]^2)$	$\frac{n^2}{\alpha(1-2\beta)}$
Theorem 3.1	$\frac{1+t_0ap\lvert u^{p-1}\rvert_\infty}{E}, E=\frac{b(1-2d)}{n^2}$	$\frac{n^2}{b\sqrt{2\xi^3(1-2d)}}(\frac{32n^2bH^2(1+H)^2}{\xi^3R^2} + 2b(1+H)[\frac{H}{R^2}+\frac{2(n-1)H(3H+1)}{R}] + (d+4b^{-1})[\frac{4bH(1+H)}{R}]^2 - 2t_0ap\xi\lvert u^{p-1}\rvert_\infty - t_0ap\lvert u^{p-1}\rvert_\infty(\tilde{\varphi}Q-1) - t_0ap(p-1)\lvert u^{p-2}\rvert_\infty)$

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest in this paper.

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