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Research article

Compactness for commutators of Calderón-Zygmund singular integral on weighted Morrey spaces

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Abstract: We prove boundedness and compactness for the iterated commutators of the θ-type Calderón-Zygmund singular integral and its fractional variant on the weighed Morrey spaces.

Keywords: boundedness and compactness; commutator; singular integral; fractional integral; weighted Morrey spaces
Mathematics Subject Classification: 42B20, 42B25

1. Introduction

The aim of this paper is to establish some new results that focus on boundedness and compactness for the iterated commutators of the θ-type Calderón-Zygmund singular integral and its fractional variant on the weighed Morrey spaces. Let us recall some definitions and the background. For $0 \le \alpha < n$, the θ-type Calderón-Zygmund integral operator T_{K_α} is defined by

$$T_{K_\alpha}(f)(x) = \int_{\mathbb{R}^n} K_\alpha(x,y) f(y) dy \ \text{ for } \ x \notin \operatorname{supp} f \tag{1.1}$$

with kernel K_α satisfying the size condition

$$|K_\alpha(x,y)| \le \frac{C_{K_\alpha}}{|x-y|^{n-\alpha}} \tag{1.2}$$

and a smoothness condition

$$|K_\alpha(x,y) - K_\alpha(z,y)| + |K_\alpha(y,x) - K_\alpha(y,z)| \le \theta\Big(\frac{|x-z|}{|x-y|}\Big)\frac{1}{|x-y|^{n-\alpha}}, \tag{1.3}$$

for all

$$|x-y| > 2|x-z|,$$

where θ: $[0,1] \to [0,\infty)$ is a modulus of continuity, that is, θ is a continuous, increasing, subadditive function with $\theta(0) = 0$ that satisfies the Dini condition

$$\int_0^1 \theta(t)\frac{dt}{t} < \infty.$$

When $\alpha = 0$, we denote $T_{K_\alpha} = T_K$. If T_K is bounded on $L^2(\mathbb{R}^n)$, then T_K is just the θ-type Calderón-Zygmund operator. When $\alpha \in (0,1)$, the operator T_{K_α} is the θ-type fractional integral operator. Particularly, when $\theta(t) = t^\delta$ for some $\delta > 0$, the operator T_K is the classical Calderón-Zygmund singular integral operator. It was shown in [1, 2] that T_K is bounded on $L^p(w)$ for $1 < p < \infty$ and $w \in A_p(\mathbb{R}^n)$. When $\alpha \in (0,1)$, we get from (1.2) that

$$T_{K_\alpha} f \le C_{K_\alpha} I_\alpha |f|,$$

where I_α is the classical fractional integral operator defined by

$$I_\alpha(f)(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$

As an immediate consequence of the boundedness for I_α, we have that T_{K_α} is bounded from $L^p(w^p)$ to $L^q(w^q)$ for $1 < p < q < \infty$, $1/q = 1/p - \alpha/n$ and $w \in A_{p,q}(\mathbb{R}^n)$. Here we recall two definitions.

Definition 1.1. ($A_p(\mathbb{R}^n)$ weight) ([3]) A weight is a nonnegative, locally integrable function on $\mathbb{R}^n$ that takes values in $(0,\infty)$ almost everywhere. For $1 < p < \infty$, a weight w is said to be in the Muckenhoupt weight class $A_p(\mathbb{R}^n)$ if there exists a positive constant C such that

$$\sup_{Q \text{ cubes in } \mathbb{R}^n} \Big(\frac{1}{|Q|}\int_Q w(x)dx\Big)\Big(\frac{1}{|Q|}\int_Q w(x)^{1-p'}dx\Big)^{p-1} \le C. \tag{1.4}$$

The smallest constant C in the inequality (1.4) is the corresponding A_p constant of w, which is denoted by $[w]_{A_p}$.

Definition 1.2. ($A_{p,q}(\mathbb{R}^n)$ weight) ([4]) Let $0 < \alpha < n$, $1 < p, q < \infty$ and $1/q = 1/p - \alpha/n$. A weight w is said to be in the Muckenhoupt weight class $A_{p,q}(\mathbb{R}^n)$ if there exists a positive constant C such that

$$\sup_{Q \text{ cubes in } \mathbb{R}^n} \Big(\frac{1}{|Q|}\int_Q w^q(x)dx\Big)\Big(\frac{1}{|Q|}\int_Q w^{-p'}(x)dx\Big)^{q/p'} \le C. \tag{1.5}$$

The smallest constant C in the inequality (1.5) is the corresponding $A_{p,q}$ constant of w, which is denoted by $[w]_{A_{p,q}}$.

On the other hand, investigation into the boundedness and compactness of the commutators has been the subject of many recent papers in the field of harmonic analysis. In 1976, Coifman et al. [5] first introduced the following commutator

$$[b,T](f)(x) = bTf(x) - T(bf)(x)$$

with the suitable operator T and function b. More precisely, they established the L^p boundedness for $[b,T]$ with T denoting the Riesz transform for $1 < p < \infty$ if and only if $b \in \mathrm{BMO}(\mathbb{R}^n)$. Later on, Uchiyama [6] improved the above result by showing that $[b, T_\Omega]$ with T_Ω being the rough singular integral operator with rough kernel $\Omega \in \mathrm{Lip}_1(S^{n-1})$ is bounded (resp., compact) on $L^p(\mathbb{R}^n)$ for all $p \in (1, \infty)$ if and only if the symbol $b \in \mathrm{BMO}(\mathbb{R}^n)$ (resp., $b \in \mathrm{CMO}(\mathbb{R}^n)$). Here $\mathrm{CMO}(\mathbb{R}^n)$ is the closure of $C_c^\infty(\mathbb{R}^n)$ in the $\mathrm{BMO}(\mathbb{R}^n)$ topology, which coincides with the space of functions of vanishing mean oscillation. Since then, a considerable amount of attention has been focused on studying boundedness and compactness for the commutators of various operators. For examples, see [7–9] for the L^p boundedness of the commutators of the rough singular integral and [10–17] for the L^p compactness of the commutators of various integral operators. Other interesting works related to this topic are [18–24]. Let us recall the definition of $\mathrm{BMO}(\mathbb{R}^n)$.

Definition 1.3. ($\mathrm{BMO}(\mathbb{R}^n)$ space) ([25]) The $\mathrm{BMO}(\mathbb{R}^n)$ space is given by

$$\mathrm{BMO}(\mathbb{R}^n) := \{f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \ \ \|f\|_{\mathrm{BMO}(\mathbb{R}^n)} := \|M^\sharp f\|_{L^\infty(\mathbb{R}^n)} < \infty\},$$

where $M^\sharp f$ is the sharp maximal function, i.e.,

$$M^\sharp f(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

where the supremum is taken over all cubes Q in $\mathbb{R}^n$ that contain the given point x.

In this paper we focus on the commutators of the θ-type integral operators. More precisely, let T_{K_α} be defined in (1.1). For a locally integrable function b defined on $\mathbb{R}^n$, the commutator $[b, T_{K_\alpha}]$ is given by

$$[b, T_{K_\alpha}](f)(x) := b(x)T_{K_\alpha}(f)(x) - T_{K_\alpha}(bf)(x)$$

for suitable functions devoted by f. Let $\mathbb{N} = \{0, 1, \cdots\}$ and $m \in \mathbb{N} \setminus \{0\}$. The m-th iterated commutator $(T_{K_\alpha})^m_b$ is defined by

$$(T_{K_\alpha})^m_b(f) := [b, (T_{K_\alpha})^{m-1}_b](f), \quad (T_{K_\alpha})^1_b(f) := [b, T_{K_\alpha}](f).$$

For convenience, we denote

$$(T_{K_\alpha})^m_b = T_{K_\alpha}$$

when $m = 0$.

Very recently, Guo et al. [14] showed that

Theorem A. ([14]) *Let* $0 \le \alpha < n$, $m \in \mathbb{N} \setminus \{0\}$, $1 < p < q < \infty$, $1/q = 1/p - \alpha/n$ *and* $w \in A_{p,q}(\mathbb{R}^n)$.

(i) *If* $b \in \mathrm{BMO}(\mathbb{R}^n)$*, then*

$$\|(T_{K_\alpha})^m_b(f)\|_{L^q(w^q)} \le C\|b\|^m_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_{L^p(w^p)}, \ \forall f \in L^p(w^p).$$

(ii) *If* $b \in \mathrm{BMO}(\mathbb{R}^n)$*, then* $(T_{K_\alpha})^m_b$ *is a compact operator from* $L^p(w^p)$ *to* $L^q(w^q)$.

The primary motivation of this note is to establish the corresponding results for $(T_{K_\alpha})^m_b$ on weighted Morrey spaces. Let us recall one definition.

Definition 1.4. (Weighted Morrey spaces) ([26]) Let w, v be two weights on $\mathbb{R}^n$. For $1 \le p < \infty$ and $0 \le \beta < 1$, the weighted Morrey space $M^{p,\beta}(w, v)$ is defined as

$$M^{p,\beta}(w, v) := \{f \in L^p_{\mathrm{loc}}(w) : \|f\|_{M^{p,\beta}(w,v)} < \infty\},$$

where

$$\|f\|_{M^{p,\beta}(w,v)} := \sup_{B \text{ balls in } \mathbb{R}^n} \Big(\frac{1}{v(B)^\beta}\int_B |f(x)|^p w(x)dx\Big)^{1/p},$$

where the supremum is taken over all balls in $\mathbb{R}^n$.

This type of Morrey space was originally introduced by Komori and Shirai [26] who established that the fractional maximal operator M_α with $0 < \alpha < n$ is bounded from $M^{p,\beta}(w^p, w^q)$ to $M^{q,q\beta/p}(w^q)$, provided that $1 < p < q < \infty$, $1/q = 1/p - \alpha/n$ and $w \in A_{p,q}(\mathbb{R}^n)$. When $w = v$, then $M^{p,\beta}(w, v)$ reduces to the classical weighted Morrey space $M^{p,\beta}(w)$, which was also introduced by Komori and Shirai [26] who established the boundedness for the Hardy-Littlewood maximal operator and the Calderón-Zygmund singular integral operator on $M^{p,\beta}(w)$. When $w \equiv 1$, the space $M^{p,\beta}(w)$ reduces to the classical Morrey space $M^{p,\beta}(\mathbb{R}^n)$, which was first introduced by Morrey [27] to study the local behavior of solutions to second order elliptic partial differential equations. In 1991, Di Fazio and Ragusa [28] presented a characterization of $M^{p,\beta}(\mathbb{R}^n)$ boundedness for $[b, T_\Omega]$. Since then, the characterizations of boundedness and compactness of $[b, T]$ on Morrey spaces $M^{p,\beta}(\mathbb{R}^n)$ have been studied by many authors (see [29–31]).

In this paper we establish the following results.

Theorem 1.1. *Let $m \in \mathbb{N}$, $0 \le \alpha < n$, $1 < p < q < \infty$, $1/q = 1/p - \alpha/n$, $0 \le \beta < p/q$ and $w \in A_{p,q}(\mathbb{R}^n)$. Let T_m be a linear or sublinear operator satisfying*

$$|T_m(f)(x)| \le C_1 \int_{\mathbb{R}^n} \prod_{j=1}^m |b_j(x) - b_j(y)| \frac{|f(y)|}{|x-y|^{n-\alpha}} dy, \tag{1.6}$$

where $\vec{b} = (b_1, \cdots, b_m)$ with each $b_j \in \mathrm{BMO}(\mathbb{R}^n)$. When $m = 0$, we denote $T_0 = T$. If T_m satisfies

$$\|T_m(f)\|_{L^q(w^q)} \le C_2 \prod_{j=1}^m \|b_j\|_{\mathrm{BMO}(\mathbb{R}^n)} \|f\|_{L^p(w^p)}, \quad \forall f \in L^p(w^p), \tag{1.7}$$

then for any $f \in M^{p,\beta}(w^p, w^q)$,

$$\|T_m(f)\|_{M^{q,q\beta/p}(w^q)} \le C(C_1, C_2, \beta) \prod_{j=1}^m \|b_j\|_{\mathrm{BMO}(\mathbb{R}^n)} \|f\|_{M^{p,\beta}(w^p,w^q)}. \tag{1.8}$$

Theorem 1.2. *Let $m \in \mathbb{N}$, $0 \le \alpha < n$, $1 < p < q < \infty$, $1/q = 1/p - \alpha/n$, $0 \le \beta < p/q$ and $w \in A_{p,q}(\mathbb{R}^n)$.*

(i) *If $b \in \mathrm{BMO}(\mathbb{R}^n)$, then*

$$\|(T_{K_\alpha})_b^m(f)\|_{M^{q,q\beta/p}(w^q)} \le C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}^m \|f\|_{M^{p,\beta}(w^p,w^q)}, \quad \forall f \in M^{p,\beta}(w^p, w^q).$$

(ii) *If $m \in \mathbb{N} \setminus \{0\}$ and $b \in \mathrm{BMO}(\mathbb{R}^n)$, then $(T_{K_\alpha})_b^m$ is a compact operator from $M^{p,\beta}(w^p, w^q)$ to $M^{q,q\beta/p}(w^q)$.*

We would like to remark that Theorem 1.1 provides a boundedness criterion for a class of sublinear operators on weighted Morrey spaces. In fact, Theorem 1.1 can apply to the multilinear commutator. Let $m \in \mathbb{N} \setminus \{0\}$ and T_{K_α} be defined as (1.1). For a vector function $\vec{b} = (b_1, \cdots, b_m)$ with each $b_j \in$

BMO($\mathbb{R}^n$), the multilinear commutator $(T_{K_\alpha})^m_{\vec{b}}$ is defined as

$$\begin{aligned}(T_{K_\alpha})^m_{\vec{b}}(f)(x) &:= [b_m, \cdots [b_2, [b_1, T_{K_\alpha}]] \cdots](x) \\ &= \int_{\mathbb{R}^n} \prod_{j=1}^m (b_j(x) - b_j(y)) \frac{f(y)}{|x-y|^{n-\alpha}} f(y)dy.\end{aligned}$$

Clearly,

$$(T_{K_\alpha})^m_{\vec{b}} = (T_{K_\alpha})^m_b$$

if $\vec{b} = (b_1, \cdots, b_m)$ with $b_j = b$ for $1 \le j \le m$. Recently, Guo et al. [14] proved that $(T_{K_\alpha})^m_{\vec{b}}$ is bounded from $L^p(w^p)$ to $L^q(w^q)$ for $0 \le \alpha < n$, $1 < p < q < \infty$, $1/q = 1/p - \alpha/n$ and $w \in A_{p,q}(\mathbb{R}^n)$, provided that each $b_j \in$ BMO($\mathbb{R}^n$) for all $1 \le j \le m$ (see [14, Theorem 5.3]). It is clear that $(T_{K_\alpha})^m_{\vec{b}}$ satisfies the condition (1.6). These facts, together with Theorem 1.1 and a slight modification of the proof of the compactness part in Theorem 1.2, directly imply the following result.

Corollary 1.1. *Let $m \in \mathbb{N}$, $0 \le \alpha < n$, $1 < p < q < \infty$, $1/q = 1/p - \alpha/n$ and $w \in A_{p,q}(\mathbb{R}^n)$.*

(i) *If $\vec{b} = (b_1, \cdots, b_m)$ with each $b_j \in$ BMO($\mathbb{R}^n$), then*

$$\|(T_{K_\alpha})^m_{\vec{b}}(f)\|_{M^{q,q\beta/p}(w^q)} \le C \prod_{j=1}^m \|b_j\|_{\mathrm{BMO}(\mathbb{R}^n)} \|f\|_{M^{p,\beta}(w^p, w^q)}$$

holds for all $f \in M^{p,\beta}(w^p, w^q)$.

(ii) *If $\vec{b} = (b_1, \cdots, b_m)$ with each $b_j \in$ CMO($\mathbb{R}^n$), then $(T_{K_\alpha})^m_{\vec{b}}$ is a compact operator from $M^{p,\beta}(w^p, w^q)$ to $M^{q,q\beta/p}(w^q)$.*

Remark 1.1. When $\beta = 0$, Theorem 1.2 implies Theorem A. There are some examples that satisfy the conditions of Theorem 1.2, such as $w = |x|^\gamma$ with $\gamma \in (\alpha - \frac{n}{p}, n - \frac{n}{p})$. By Lemma 2.2, it is not difficult to verify that $|x|^\gamma \in A_{p,q}(\mathbb{R}^n)$ for $0 \le \alpha < n$, $1 < p < q < \infty$, $1/q = 1/p - \alpha/n$ and $\gamma \in (\alpha - \frac{n}{p}, n - \frac{n}{p})$.

As an application of Theorem 1.2, we have the corresponding results for the θ-type Calderóna-Zygmund operator and its commutators.

Corollary 1.2. *Let $m \in \mathbb{N}$, $1 < p < \infty$, $0 \le \beta < 1$ and $w \in A_p(\mathbb{R}^n)$.*

(i) *If $b \in$ BMO($\mathbb{R}^n$), then*

$$\|(T_K)^m_b(f)\|_{M^{p,\beta}(w)} \le C\|b\|^m_{\mathrm{BMO}(\mathbb{R}^n)} \|f\|_{M^{p,\beta}(w)}, \ \forall f \in M^{p,\beta}(w).$$

(ii) *If $m \in \mathbb{N} \setminus \{0\}$ and $b \in$ BMO($\mathbb{R}^n$), then $(T_K)^m_b$ is a compact operator from $M^{p,\beta}(w)$ to $M^{p,\beta}(w)$.*

The paper is organized as follows. In Section 2 we present some definitions and lemmas, which are the main ingredients we used to prove our main results. The proofs of Theorems 1.1 and 1.2 will be given in Section 3. We remark that some ideas of our methods are taken from [14, 15, 32], but our methods and techniques are more delicate and complex than that of [14, 15, 32].

Throughout the paper, for any $p \in (1, \infty]$ we let p' denote the conjugate index of p which satisfies that $1/p + 1/p' = 1$ (here we set $\infty' = 1$). The letter C will denote a positive constant that is not necessarily the same at each occurrence but is independent of the essential variables. For $x = (x_1, \cdots, x_n)$ we set

$$|x|_\infty = \max_{1 \le i \le n} |x_i|.$$

2. Materials and methods

In order to prove Theorem 1.2, we need the following properties for $A_p(\mathbb{R}^n)$ and $A_{p,q}(\mathbb{R}^n)$ weights.

Lemma 2.1. ([32]) *Let $1 < p < \infty$ and $w \in A_p(\mathbb{R}^n)$. Then, the following holds*

(i) *There exists a constant $\theta \in (0,1)$ such that $w^{1+\theta} \in A_p(\mathbb{R}^n)$. Both θ and $[w^{1+\theta}]_{A_p}$ depend only on n, p and the A_p constant of w.*

(ii) *There exists a constant $\epsilon \in (0,1)$ such that $w \in A_{p-\epsilon}(\mathbb{R}^n)$.*

(iii) *The measure $w(x)dx$ is doubling, i.e., for all $\lambda > 1$ we have*

$$\sup_{\text{Q cubes in } \mathbb{R}^n} \frac{w(\lambda Q)}{w(Q)} \leq [w]_{A_p} \lambda^{np}.$$

(iv) *There exists a constant $\gamma_w > 1$ such that*

$$\inf_{\text{Q cubes in } \mathbb{R}^n} \frac{w(2Q)}{w(Q)} \geq \gamma_w.$$

(v) *Let $b \in \mathrm{BMO}(\mathbb{R}^n)$. Then we have*

$$\sup_{Q \text{ cubes in } \mathbb{R}^n} \Big(\frac{1}{w(Q)} \int_Q |b(x) - b_Q|^p w(x)dx\Big)^{1/p} \simeq_{p,[w]_{A_p}} \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}.$$

Lemma 2.2. ([4]) *Let $0 < \alpha < n$, $1 < p, q < \infty$, $1/q = 1/p - \alpha/n$ and $w \in A_{p,q}(\mathbb{R}^n)$. Then, the following holds true:*

(i) $w^p \in A_p(\mathbb{R}^n)$, $w^q \in A_q(\mathbb{R}^n)$ *and* $w^{-p'} \in A_{p'}(\mathbb{R}^n)$.

(ii)

$$\begin{aligned} w \in A_{p,q}(\mathbb{R}^n) &\Leftrightarrow w^q \in A_{q(n-\alpha)/n}(\mathbb{R}^n) \\ &\Leftrightarrow w^q \in A_{1+q/p'}(\mathbb{R}^n) \Leftrightarrow w^{-p'} \in A_{1+p'/q}(\mathbb{R}^n). \end{aligned}$$

For convenience, we always use the weighted Morrey spaces associated with cubes. Let $1 \leq p < \infty$ and $0 \leq \beta < 1$. For two weights w and v defined on $\mathbb{R}^n$, the weighted Morrey space associated with cubes is defined by

$$\widetilde{M}^{p,\beta}(w,v) := \{f \in L^p_{\mathrm{loc}}(v) : \|f\|_{\widetilde{M}^{p,\beta}(w,v)} < \infty\},$$

where

$$\|f\|_{\widetilde{M}^{p,\beta}(w,v)} := \sup_{\text{Q cubes in } \mathbb{R}^n} \Big(\frac{1}{v(Q)^\beta} \int_Q |f(x)|^p w(x)dx\Big)^{1/p},$$

where the supremum is taken over all cubes in $\mathbb{R}^n$.

Remark 2.1. If the weight w is doubling, then we have that $\widetilde{M}^{p,\beta}(w,v) = M^{p,\beta}(w,v)$, i.e.,

$$\|f\|_{\widetilde{M}^{p,\beta}(w,v)} \simeq \|f\|_{M^{p,\beta}(w,v)}, \tag{2.1}$$

which can be seen by the doubling property for w and the following observation

$$Q(x_0, r) \subset B(x_0, \sqrt{n}/2r) \subset Q(x_0, \sqrt{n}r), \quad \forall x_0 \in \mathbb{R}^n,\ r > 0.$$

To end this section, we shall present the characterization that a subset in $M^{p,\beta}(w)$ is a strongly pre-compact set, which plays a key role in the proof of compactness part of Theorem 1.2.

Proposition 2.1. ([32]) *Let $1 < p < \infty$, $0 \leq \beta < 1$ and $w \in A_p(\mathbb{R}^n)$. Then a subset $\mathcal{F}$ of $M^{p,\beta}(w)$ is a strongly pre-compact set in $M^{p,\beta}(w)$ if $\mathcal{F}$ satisfies the following conditions:*

(i) $\sup_{f\in\mathcal{F}} \|f\|_{M^{p,\beta}(w)} < \infty$;

(ii) *For any $f \in \mathcal{F}$, we have that*

$$\lim_{N\to+\infty} \|f\chi_{E_N}\|_{M^{p,\beta}(w)} = 0,$$

where $E_N = \{x \in \mathbb{R}^n; |x| > N\}$.

(iii) *For any $f \in \mathcal{F}$, we have that*

$$\lim_{r\to 0} \sup_{h\in B(0,r)} \|f(\cdot + h) - f(\cdot)\|_{M^{p,\beta}(w)} = 0.$$

3. Proofs of main results

In this section we present the proofs of Theorems 1.1 and 1.2. We first prove Theorem 1.1.

Proof of Theorem 1.1. Let $f \in \widetilde{M}^{p,\beta_1}(w^p, w^q)$, $\beta \in (0, p/q)$ and $w \in A_{p,q}(\mathbb{R}^n)$. Fix a cube $Q = Q(x_0, r)$. We divide the proof into two parts:

Step 1. Proof of (1.8) **for** $m = 0$. By Remark 2.1, to prove (1.8), it is enough to show that

$$\Big(\frac{1}{w^q(Q)^{q\beta/p}}\int_Q |T(f)(x)|^q w^q(x)dx\Big)^{1/q} \leq C\|f\|_{\widetilde{M}^{p,\beta}(w^p,w^q)}, \tag{3.1}$$

where $C > 0$ is independent of x_0 and r.

We write f as

$$f = f\chi_{2Q} + f\chi_{(2Q)^c}.$$

Then we have

$$\begin{aligned}
&\Big(\frac{1}{w^q(Q)^{q\beta/p}}\int_Q | T(f)(x)|^q w^q(x)dx\Big)^{1/q}\\
&\leq \Big(\frac{1}{w^q(Q)^{q\beta/p}}\int_Q | T(f\chi_{2Q})(x)|^q w^q(x)dx\Big)^{1/q}\\
&\quad+\Big(\frac{1}{w^q(Q)^{q\beta/p}}\int_Q | T(f\chi_{(2Q)^c})(x)|^q w^q(x)dx\Big)^{1/q}\\
&=: I_1 + I_2.
\end{aligned} \tag{3.2}$$

By Lemma 2.2(i), we have that $w^q \in A_q(\mathbb{R}^n)$. By Lemma 2.1(iii), we see that

$$\frac{w^q(2Q)}{w^q(Q)} \leq [w^q]_{A_q} 2^{nq}.$$

This together with the condition (1.7) with $m = 0$ implies that

$$\begin{aligned} I_1 &= \Big(\frac{1}{w^q(Q)^{q\beta/p}}\int_{2Q}|T(f)(x)|^q w^q(x)dx\Big)^{1/q} \\ &\le C_2\frac{1}{w^q(Q)^{\beta/p}}\Big(\int_{2Q}|f(x)|^p w^p(x)dx\Big)^{1/p} \\ &= C_2\Big(\frac{1}{w^q(Q)^{\beta}}\int_{2Q}|f(x)|^p w^p(x)dx\Big)^{1/p} \\ &\le C_2\Big(\Big(\frac{w^q(2Q)}{w^q(Q)}\Big)^{\beta}\frac{1}{w^q(2Q)^{\beta}}\int_{2Q}|f(x)|^p w^p(x)dx\Big)^{1/p} \\ &\le C(C_2,n,p,q,\beta,[w^q]_{A_q})\|f\|_{\widetilde{M}^{p,\beta}(w^p,w^q)}. \end{aligned} \tag{3.3}$$

We now estimate I_2. Fix $x \in Q$, in view of the condition (1.6) with $m = 0$, we have

$$T(f\chi_{(2Q)^c})(x) \le C_1\int_{(2Q)^c}\frac{|f(z)|}{|x-z|^{n-\alpha}}dz. \tag{3.4}$$

Note that

$$|x-z| \ge |x-z|_\infty \ge |z-x_0|_\infty - |x-x_0|_\infty \ge \frac{1}{2}|z-x_0|_\infty$$

for $z \in (2Q)^c$. By (3.4), we have

$$\begin{aligned} T(f\chi_{(2Q)^c})(x) &\le 2^{n-\alpha}C_1\sum_{l=0}^{\infty}\int_{2^l r\le|z-x_0|_\infty<2^{l+1}r}\frac{|f(z)|}{|z-x_0|_\infty^{n-\alpha}}dz \\ &\le 2^{n-\alpha}C_1\sum_{l=0}^{\infty}(2^l r)^{\alpha-n}\int_{2^{l+1}Q}|f(z)|dz. \end{aligned} \tag{3.5}$$

Fix $l \in \mathbb{N}$. Using Hölder's inequality, one has

$$\begin{aligned} \int_{2^{l+1}Q}|f(z)|dz &\le \Big(\int_{2^{l+1}Q}|f(z)|^p w^p(z)dz\Big)^{1/p}\Big(\int_{2^{l+1}Q}w^{-p'}(z)dz\Big)^{1/p'} \\ &\le w^q(2^{l+1}Q)^{\beta/p}\|f\|_{\widetilde{M}^{p,\beta}(w^p,w^q)}\Big(\int_{2^{l+1}Q}w^{-p'}(z)dz\Big)^{1/p'}. \end{aligned} \tag{3.6}$$

Since $w \in A_{p,q}(\mathbb{R}^n)$, then

$$\Big(\int_{2^{l+1}Q}w^{-p'}(z)dz\Big)^{1/p'} \le [w]_{A_{p,q}}^{1/q}|2^{l+1}Q|^{1-\frac{\alpha}{n}}w^q(2^{l+1}Q)^{-1/q}. \tag{3.7}$$

Combining (3.7) with (3.6) yields

$$\int_{2^{l+1}Q}|f(z)|dz \le [w]_{A_{p,q}}^{1/q}|2^{l+1}Q|^{1-\frac{\alpha}{n}}w^q(2^{l+1}Q)^{\frac{q\beta/p-1}{q}}\|f\|_{\widetilde{M}^{p,\beta}(w^p,w^q)}. \tag{3.8}$$

In light of (3.5) and (3.8) we have

$$\begin{aligned} T(f\chi_{(2Q)^c})(x) &\le 2^{n-\alpha}C_1\sum_{l=0}^{\infty}(2^l r)^{\alpha-n}\int_{2^{l+1}Q}|f(z)|dz \\ &\le C(C_1,n,\alpha,[w]_{A_{p,q}})\|f\|_{\widetilde{M}^{p,\beta_1}(w^p,w^q)} \\ &\quad\times\sum_{l=0}^{\infty}(2^l r)^{\alpha-n}|2^{l+1}Q|^{1-\frac{\alpha}{n}}w^q(2^{l+1}Q)^{\frac{q\beta/p-1}{q}} \\ &\le C(C_1,n,\alpha,[w]_{A_{p,q}})\|f\|_{\widetilde{M}^{p,\beta}(w^p,w^q)}\sum_{l=0}^{\infty}w^q(2^{l+1}Q)^{\frac{q\beta/p-1}{q}}. \end{aligned} \tag{3.9}$$

Note that $q\beta/p < 1$. Invoking Lemma 2.1(iv) and (3.9), we have

$$\begin{aligned} I_2 &\le C(C_1,n,\alpha,p,q,[w]_{A_{p,q}})\|f\|_{\widetilde{M}^{p,\beta}(w^p,w^q)}\sum_{l=0}^{\infty}\Big(\frac{w^q(2^{l+1}Q)}{w^q(Q)}\Big)^{\frac{q\beta/p-1}{q}} \\ &\le C(C_1,n,\alpha,p,q,[w]_{A_{p,q}})\|f\|_{\widetilde{M}^{p,\beta}(w^p,w^q)}\sum_{l=0}^{\infty}\gamma_{w^q}^{-\frac{(1-q\beta/p)(l+1)}{q}} \\ &\le C(C_1,n,\alpha,p,q,[w]_{A_{p,q}})\|f\|_{\widetilde{M}^{p,\beta}(w^p,w^q)}. \end{aligned}$$

Combining this with (3.2) and (3.3) implies that

$$\Big(\frac{1}{w^q(Q)^{q\beta/p}}\int_Q|T(f)(x)|^q w^q(x)dx\Big)^{1/q} \le C(C_1,C_2,n,\alpha,p,q,[w]_{A_{p,q}})\|f\|_{\widetilde{M}^{p,\beta}(w^p,w^q)}.$$

This proves (3.1) and completes the proof of the case for $m = 0$.

Step 2: Proof of (1.8) **for** $m \in \mathbb{N}\setminus\{0\}$. Let $f \in \widetilde{M}^{p,\beta}(w^p,w^q)$ and $\beta \in (0,p/q)$. Fix a cube $Q = Q(x_0,r)$. By Remark 2.1, to prove (1.8) for $m \in \mathbb{N}\setminus\{0\}$, it suffices to show that

$$\Big(\frac{1}{w^q(Q)^{q\beta/p}}\int_Q|T_m(f)(x)|^q w^q(x)dx\Big)^{1/q} \le C\prod_{j=1}^{m}\|b_j\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_{\widetilde{M}^{p,\beta}(w^p,w^q)}, \tag{3.10}$$

where $C > 0$ is independent of x_0, r and $\vec{b}$.

Decompose f as follows

$$f = f\chi_{2Q} + f\chi_{(2Q)^c}.$$

We can write

$$\begin{aligned} &\Big(\frac{1}{w^q(Q)^{q\beta/p}}\int_Q|T_m f(x)|^q w^q(x)dx\Big)^{1/q} \\ &\le \Big(\frac{1}{w^q(Q)^{q\beta/p}}\int_Q|T_m(f\chi_{2Q})(x)|^q w^q(x)dx\Big)^{1/q} \\ &\quad+\Big(\frac{1}{w^q(Q)^{q\beta/p}}\int_Q|T_m(f\chi_{(2Q)^c})(x)|^q w^q(x)dx\Big)^{1/q} \\ &=: J_1 + J_2. \end{aligned} \tag{3.11}$$

By Theorem A, (1.7) and the fact that

$$\frac{w^q(2Q)}{w^q(Q)} \le [w^q]_{A_q}2^{nq},$$

we have

$$\begin{aligned}
J_1 &\le C_2 \prod_{j=1}^m \|b_j\|_{\mathrm{BMO}(\mathbb{R}^n)} \frac{1}{w^q(Q)^{\beta/p}} \Big(\int_{2Q} |f(x)|^p w^p(x) dx \Big)^{1/p} \\
&= C_2 \prod_{j=1}^m \|b_j\|_{\mathrm{BMO}(\mathbb{R}^n)} \Big(\frac{1}{w^q(Q)^{\beta}} \int_{2Q} |f(x)|^p w^p(x) dx \Big)^{1/p} \\
&\le C_2 \prod_{j=1}^m \|b_j\|_{\mathrm{BMO}(\mathbb{R}^n)} \\
&\quad \times \Big(\Big(\frac{w^q(2Q)}{w^q(Q)} \Big)^{\beta} \frac{1}{w^q(2Q)^{\beta}} \int_{2Q} |f(x)|^p w^p(x) dx \Big)^{1/p} \\
&\le C(C_2, n, p, q, \beta, [w^q]_{A_q}) \prod_{j=1}^m \|b_j\|_{\mathrm{BMO}(\mathbb{R}^n)} \|f\|_{\widetilde{M}^{p,\beta}(w^p, w^q)}.
\end{aligned} \tag{3.12}$$

Next we estimate J_2. Fix $x \in Q$. By (1.6) and the fact that

$$|x - z| \ge \frac{1}{2} |z - x_0|_\infty$$

for $z \in (2Q)^c$, one has

$$\begin{aligned}
|T_m(f\chi_{(2Q)^c})| &\le C_1 \int_{(2Q)^c} \prod_{j=1}^m |b_j(x) - b_j(z)| \frac{|f(z)|}{|x - z|^{n-\alpha}} dz \\
&\le C_1 2^{n-\alpha} \sum_{l=0}^{\infty} \int_{2^l r \le |z - x_0|_\infty \le 2^{l+1} r} \prod_{j=1}^m |b_j(x) - b_j(z)| \frac{|f(z)|}{|z - x_0|_\infty^{n-\alpha}} dz \\
&\le C_1 2^{n-\alpha} \sum_{l=0}^{\infty} (2^l r)^{\alpha - n} \int_{2^{l+1}Q} |f(z)| \prod_{j=1}^m |b_j(x) - b_j(z)| dz.
\end{aligned}$$

For convenience, we set $E = \{1, \cdots, m\}$. For any $j \in \{1, 2, \cdots, m\}$ and $l \in \mathbb{N}$, we let

$$b_{j,2^{l+1}Q} = \frac{1}{|2^{l+1}Q|} \int_{2^{l+1}Q} b_j(z) dz.$$

Note that

$$\begin{aligned}
\prod_{j=1}^m |b_j(x) - b_j(z)| &\le \prod_{j=1}^m (|b_j(x) - b_{j,2^{l+1}Q}| + |b_j(z) - b_{j,2^{l+1}Q}|) \\
&\le \sum_{\tau \subset E} \Big(\prod_{\mu \in \tau} |b_\mu(x) - b_{\mu,2^{l+1}Q}| \Big) \Big(\prod_{\nu \in E \setminus \tau} |b_\nu(z) - b_{\nu,2^{l+1}Q}| \Big).
\end{aligned}$$

Then we have

$$\begin{aligned}
T_m(f\chi_{(2Q)^c})(x) &\le 2^{n-\alpha} C_1 \sum_{\tau \subset E} \Big(\prod_{\mu \in \tau} |b_\mu(x) - b_{\mu,2^{l+1}Q}| \Big) \\
&\quad \times \sum_{l=0}^{\infty} (2^l r)^{\alpha - n} \int_{2^{l+1}Q} |f(z)| \Big(\prod_{\nu \in E \setminus \tau} |b_\nu(z) - b_{\nu,2^{l+1}Q}| \Big) dz.
\end{aligned}$$

Fix $\tau \subset E$. Let

$$t = \frac{(1+\epsilon)p'}{(1+\epsilon)p' - \epsilon}.$$

Clearly, $t \in (1, p)$. By Hölder's inequality, we have

$$\begin{aligned}
&\sum_{l=0}^{\infty}(2^l r)^{\alpha-n}\int_{2^{l+1}Q}|f(z)|\Big(\prod_{\nu\in E\setminus\tau}|b_\nu(z)-b_{\nu,2^{l+1}Q}|\Big)dz\\
&\le \sum_{l=0}^{\infty}(2^l r)^{\alpha-n}\Big(\int_{2^{l+1}Q}|f(z)|^t dz\Big)^{1/t}\\
&\quad\times\Big(\int_{2^{l+1}Q}\Big(\prod_{\nu\in E\setminus\tau}|b_\nu(z)-b_{\nu,2^{l+1}Q}|\Big)^{t'}dz\Big)^{1/t'}.
\end{aligned}\tag{3.13}$$

On the other hand, we can choose $\{s_i\}_{i\in E\setminus\tau} \subset (1, \infty)$ such that

$$\sum_{i\in E\setminus\tau} 1/s_i = 1.$$

By Hölder's inequality and the property of $\mathrm{BMO}(\mathbb{R}^n)$, one has

$$\begin{aligned}
&\Big(\int_{2^{l+1}Q}\Big(\prod_{\nu\in E\setminus\tau}|b_\nu(z)-b_{\nu,2^{l+1}Q}|\Big)^{t'}dz\Big)^{1/t'}\\
&\le \prod_{\nu\in E\setminus\tau}\Big(\int_{2^{l+1}Q}|b_\nu(z)-b_{\nu,2^{l+1}Q}|^{s_\nu t'}dz\Big)^{1/(s_\nu t')}\\
&\le \prod_{\nu\in E\setminus\tau}\|b_\nu\|_{\mathrm{BMO}(\mathbb{R}^n)}|2^{l+1}Q|^{1/(s_\nu t')}\\
&\le |2^{l+1}Q|^{1/t'}\prod_{\nu\in E\setminus\tau}\|b_\nu\|_{\mathrm{BMO}(\mathbb{R}^n)}.
\end{aligned}\tag{3.14}$$

Let $s = p/t$. Then

$$1/(s't) = 1/t - 1/p = 1/(p'(1+\epsilon)).$$

By Hölder's inequality, one has

$$\begin{aligned}
\Big(\int_{2^{l+1}Q}|f(z)|^t dz\Big)^{1/t} &\le \Big(\int_{2^{l+1}Q}|f(z)|^p w^p(z)dz\Big)^{1/p}\Big(\int_{2^{l+1}Q}w^{-s't}(z)dz\Big)^{1/(s't)}\\
&\le w(2^{l+1}Q)^{\beta/p}\|f\|_{\widetilde{M}^{p,\beta}(w^p,w^q)}\Big(\int_{2^{l+1}Q}w^{-p'(1+\epsilon)}(z)dz\Big)^{1/(p'(1+\epsilon))}.
\end{aligned}\tag{3.15}$$

Since $w \in A_{p,q}(\mathbb{R}^n)$, by Lemma 2.2, we have that $w^{-p'} \in A_{1+\frac{p'}{q}}(\mathbb{R}^n)$. By Lemma 2.1 (i), there exists a constant $\epsilon \in (0, 1)$ such that

$$w^{-p'(1+\epsilon)} \in A_{1+\frac{p'}{q}}(\mathbb{R}^n) \subset A_{1+\frac{p'(1+\epsilon)}{q}}(\mathbb{R}^n).$$

Then we have

$$\Big(\int_{2^{l+1}Q}w^{-p'(1+\epsilon)}(z)dz\Big)^{1/(p'(1+\epsilon))} \le [w^{-p'(1+\epsilon)}]_{A_{1+\frac{p'(1+\epsilon)}{q}}}^{\frac{1}{p'(1+\epsilon)}}|2^{l+1}Q|^{\frac{1}{1+\epsilon}+\frac{\epsilon}{p(1+\epsilon)}-\frac{\alpha}{n}}w^q(2^{l+1}Q)^{-\frac{1}{q}}\|f\|_{\widetilde{M}^{p,\beta}(w^p,w^q)}. \tag{3.16}$$

Note that

$$\frac{1}{1+\epsilon}+\frac{\epsilon}{p(1+\epsilon)}=\frac{1}{t}.$$

It follows from (3.13)–(3.16) that

$$\begin{aligned}
&\sum_{l=0}^{\infty}(2^l r)^{\alpha-n}\int_{2^{l+1}Q}|f(z)|\Big(\prod_{\nu\in E\backslash\tau}|b_\nu(z)-b_{\nu,2^{l+1}Q}|\Big)dz\\
&\le\sum_{l=0}^{\infty}(2^l r)^{\alpha-n}[w^{-p'(1+\epsilon)}]_{A_{1+\frac{p'(1+\epsilon)}{q}}(\mathbb{R}^n)}^{\frac{1}{p'(1+\epsilon)}}|2^{l+1}Q|^{\frac{1}{t}-\frac{\alpha}{n}}\\
&\quad\times w^q(2^{l+1}Q)^{\beta/p-1/q}\|f\|_{\widetilde{M}^{p,\beta}(w^p,w^q)}|2^{l+1}Q|^{\frac{1}{t'}}\prod_{\nu\in E\backslash\tau}\|b_\nu\|_{\mathrm{BMO}(\mathbb{R}^n)}\\
&\le[w^{-p'(1+\epsilon)}]_{A_{1+\frac{p'(1+\epsilon)}{q}}}^{\frac{1}{p'(1+\epsilon)}}\prod_{\nu\in E\backslash\tau}\|b_\nu\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_{\widetilde{M}^{p,\beta}(w^p,w^q)}\\
&\quad\times\sum_{l=0}^{\infty}w(2^{l+1}Q)^{\beta/p-1/q}.
\end{aligned}\tag{3.17}$$

By (3.13), (3.17), Lemma 2.1(iv) and the fact that $q\beta/p<1$, one has

$$\begin{aligned}
J_2&\le C\sum_{\tau\in E}\prod_{\nu\in E\backslash\tau}\|b_\nu\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_{\widetilde{M}^{p,\beta}(w^p,w^q)}\\
&\quad\times\Big(\frac{1}{w^q(Q)^{q\beta/p}}\int_Q\Big(\sum_{l=0}^{\infty}\Big(\prod_{\mu\in\tau}|b_\mu(x)-b_{\mu,2^{l+1}Q}|\Big)\times w^q(2^{l+1}Q)^{\frac{q\beta/p-1}{q}}\Big)^q w^q(x)dx\Big)^{1/q}\\
&\le C\sum_{\tau\in E}\prod_{\nu\in E\backslash\tau}\|b_\nu\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_{\widetilde{M}^{p,\beta}(w^p,w^q)}\\
&\quad\times\Big(\frac{1}{w^q(Q)}\int_Q\Big(\sum_{l=0}^{\infty}\Big(\prod_{\mu\in\tau}|b_\mu(x)-b_{\mu,2^{l+1}Q}|\Big)\times\Big(\frac{w^q(2^{l+1}Q)}{w^q(Q)}\Big)^{\frac{q\beta/p-1}{q}}\Big)^q w^q(x)dx\Big)^{1/q}\\
&\le C\sum_{\tau\in E}\prod_{\nu\in E\backslash\tau}\|b_\nu\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_{\widetilde{M}^{p,\beta}(w^p,w^q)}w^q(Q)^{-1/q}\\
&\quad\times\Big(\int_Q\Big(\sum_{l=0}^{\infty}\gamma_{w^q}^{-\frac{(1-q\beta/p)(l+1)}{q}}\Big(\prod_{\mu\in\tau}|b_\mu(x)-b_{\mu,2^{l+1}Q}|\Big)\Big)^q w^q(x)dx\Big)^{1/q}.
\end{aligned}\tag{3.18}$$

We can choose $\{t_i\}_{i\in\tau}\subset(1,\infty)$ such that $\sum_{i\in\tau}1/t_i=1$. By Minkowski's inequality and Hölder's inequality, one has

$$\begin{aligned}
&\Big(\int_Q\Big(\sum_{l=0}^{\infty}\gamma_{w^q}^{-\frac{(1-q\beta/p)(l+1)}{q}}\Big(\prod_{\mu\in\tau}|b_\mu(x)-b_{\mu,2^{l+1}Q}|\Big)\Big)^q w^q(x)dx\Big)^{1/q}\\
&\le\sum_{l=0}^{\infty}\gamma_{w^q}^{-\frac{(1-q\beta/p)(l+1)}{q}}\Big(\int_Q\Big(\prod_{\mu\in\tau}|b_\mu(x)-b_{\mu,2^{l+1}Q}|\Big)^q w^q(x)dx\Big)^{1/q}\\
&\le\sum_{l=0}^{\infty}\gamma_{w^q}^{-\frac{(1-q\beta/p)(l+1)}{q}}\prod_{\mu\in\tau}\Big(\int_Q\Big(|b_\mu(x)-b_{\mu,2^{l+1}Q}|\Big)^{qt_\mu}w^q(x)dx\Big)^{1/(qt_\mu)}.
\end{aligned}\tag{3.19}$$

Note that $w^q \in A_q(\mathbb{R}^n)$. By Lemma 2.1(v), Minkowski's inequality and the fact that

$$|b_{\mu,Q} - b_{\mu,2^{l+1}Q}| \leq C(l+1)\|b_\mu\|_{\mathrm{BMO}(\mathbb{R}^n)},$$

we obtain

$$\begin{aligned}\Big(\int_Q \Big(|b_\mu(x) - b_{\mu,2^{l+1}Q}|\Big)^{qt_\mu} w^q(x)dx\Big)^{1/(qt_\mu)} &\leq |b_{\mu,Q} - b_{\mu,2^{l+1}Q}| w^q(Q)^{1/(qt_\mu)} \\ &\quad + \Big(\int_Q \Big(b_\mu(x) - b_{\mu,Q}\Big)^{qt_\mu} w^q(x)dx\Big)^{1/(qt_\mu)} \\ &\leq C(l+1)\|b_\mu\|_{\mathrm{BMO}(\mathbb{R}^n)} w^q(Q)^{1/(qt_\mu)}.\end{aligned} \tag{3.20}$$

It follows from (3.18)–(3.20) that

$$\begin{aligned} J_2 &\leq \prod_{j=1}^m \|b_j\|_{\mathrm{BMO}(\mathbb{R}^n)} \|f\|_{\widetilde{M}^{p,\beta}(w^p,w^q)} \|f\|_{\widetilde{M}^{p,\beta}(w^p,w^q)} \sum_{l=0}^\infty \frac{l+1}{\gamma_{w^q}^{\frac{(1-q\beta/p)(l+1)}{q}}} \\ &\leq C \prod_{j=1}^m \|b_j\|_{\mathrm{BMO}(\mathbb{R}^n)} \|f\|_{\widetilde{M}^{p,\beta}(w^p,w^q)} \end{aligned} \tag{3.21}$$

since $\gamma_{w^q} > 1$ and $q\beta/p < 1$. Then (3.10) follows from (3.11), (3.12) and (3.21). This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. The boundedness part of Theorem 1.1 follows easily from Corollary 1.1. We prove the compactness part of Theorem 1.1 by considering five steps:

Step 1. Reduction via approximation argument. For a fixed $b \in \mathrm{CMO}(\mathbb{R}^n)$ and $\epsilon \in (0,1)$, there exists $b_\epsilon \in C_c^\infty(\mathbb{R}^n)$ such that $\|b_\epsilon - b\|_{\mathrm{BMO}(\mathbb{R}^n)} < \epsilon$. It is clear that

$$b_\epsilon^m - b^m = (b_\epsilon - b)(b_\epsilon^{m-1} + b_\epsilon^{m-2} b + \cdots + b^{m-1}).$$

For convenience, we set

$$\vec{b_1} = (b_\epsilon - b, \overbrace{b_\epsilon, \cdots, b_\epsilon}^{m-1}), \vec{b_2} = (b_\epsilon - b, \overbrace{b_\epsilon, \cdots, b_\epsilon}^{m-2}, b), \cdots, \vec{b_m} = (b_\epsilon - b, \overbrace{b, \cdots, b}^{m-1}).$$

We can write

$$|(T_{K_\alpha})_{b_\epsilon}^m(f)(x) - (T_{K_\alpha})_b^m(f)(x)| \leq \sum_{j=1}^m (T_{K_\alpha})_{\vec{b}_j}^m(f)(x),$$

which combined with Corollary 1.2 and Minkowski's inequality implies that

$$\|(T_{K_\alpha})_{b_\epsilon}^m(f) - (T_{K_\alpha})_b^m(f)\|_{M^{q,q\beta/p}(w^q)} \leq \sum_{j=1}^m \|(T_{K_\alpha})_{\vec{b}_j}^m(f)\|_{M^{q,q\beta/p}(w^q)} \leq C\epsilon \|f\|_{M^{p,\beta}(w^p,w^q)}.$$

This together with [33, Theorem (iii)] imply that to obtain the compactness for $(T_{K_\alpha})_b^m$ with $b \in \mathrm{CMO}(\mathbb{R}^n)$, it suffices to prove the compactness for $(T_{K_\alpha})_b^m$ with $b \in C_c^\infty(\mathbb{R}^n)$.

In what follows, we let $b \in (C)_c^\infty(\mathbb{R}^n)$. We want to show that $(T_{K_\alpha})_b^m$ is compact from $M^{p,\beta}(w^p, w^q) \to M^{q,q\beta/p}(w^q)$.

Step 2. Reduction via smooth truncated techniques. We shall adopt the truncated techniques followed from [15] to prove the compactness part. Let $\varphi \in C^\infty([0,\infty))$ satisfy that $0 \le \varphi \le 1$, $\varphi(t) \equiv 1$ if $t \in [0,1]$ and $\varphi(t) \equiv 0$ if $t \in [2,\infty)$. For any $\eta > 0$, we define the function $K_{\alpha,\eta}$ by

$$K_{\alpha,\eta}(x,y) = K_\alpha(x,y)(1 - \varphi(2\eta^{-1}|x-y|)).$$

By (1.2), we have

$$\begin{aligned} |(T_{K_{\alpha,\eta}})_b^m(f) - (T_{K_\alpha})_b^m(f)| &\le \int_{\mathbb{R}^n} |(b(x)-b(z))^m f(z)||(K_{\alpha,\eta}(x,z) - K_\alpha(x,z))|dz \\ &= \int_{\mathbb{R}^n} |(b(x)-b(z))^m f(z)||K(x,z)|\varphi(2\eta^{-1}|x-z|)dz \\ &\le C_{K_\alpha}(\|b\|_{L^\infty(\mathbb{R}^n)} + |b(x)|)^{m-1}\|\nabla b\|_{L^\infty(\mathbb{R}^n)} \int_{|x-z|\le\eta} \frac{|f(z)|}{|x-z|^{n-\alpha-1}}dz \\ &\le C_{K_\alpha}(\|b\|_{L^\infty(\mathbb{R}^n)} + |b(x)|)^{m-1}\|\nabla b\|_{L^\infty(\mathbb{R}^n)} 2^{n-\alpha}\omega_n \eta M_\alpha(f)(x) \end{aligned} \tag{3.22}$$

for every $x \in \mathbb{R}^n$, where $\omega_n = |B(0,1)|$. Here M_α with $0 < \alpha < n$ is the usual fractional maximal operator defined by

$$M_\alpha(f)(x) = \sup_{r>0} \frac{1}{|B(0,r)|^{1-\alpha/n}} \int_{|y|\le r} |f(x-y)|dy.$$

Combining (3.22) with the $M^{p,\beta}(w^p,w^q) \to M^{q,q\beta/p}(w^q)$ boundedness for M_α implies that

$$\|(T_{K_{\alpha,\eta}})_b^m(f) - (T_{K_\alpha})_b^m(f)\|_{M^{q,q\beta/p}(w^q)} \le C\eta\|f\|_{M^{p,\beta}(w^p,w^q)}, \ \forall f \in M^{p,\beta}(w^p,w^q). \tag{3.23}$$

By (3.23) and [33, Theorem (iii)], the compactness for $(T_{K_\alpha})_b^m$ reduces to the compactness for $(T_{K_{\alpha,\eta}})_b^m$ when $\eta > 0$ is small enough. We set

$$\mathcal{F} := \{(T_{K_{\alpha,\eta}})_b^m(f) : \|f\|_{M^{p,\beta}(w^p,w^q)} \le 1\}.$$

To prove the compactness of $(T_{K_{\alpha,\eta}})_b^m$, it is enough to show that $\mathcal{F}$ is pre-compact when $\eta > 0$ is small enough. By Proposition 2.1, it is enough to verify that $\mathcal{F}$ satisfies conditions (i)–(iii) of Proposition 2.1.

Step 3. A verification for condition (i) of Proposition 2.1. Let $\eta \in (0,1)$. By (3.23) and the boundedness part of Theorem 1.1, one gets

$$\begin{aligned} \|(T_{K_{\alpha,\eta}})_b^m(f)\|_{M^{q,q\beta/p}(w^q)} &\le \|(T_{K_{\alpha,\eta}})_b^m(f) - (T_{K_\alpha})_b^m(f)\|_{M^{q,q\beta/p}(w^q)} + \|(T_{K_\alpha})_b^m(f)\|_{M^{q,q\beta/p}(w^q)} \\ &\le C\|f\|_{M^{p,\beta}(w^p,w^q)} \le C, \end{aligned}$$

when

$$\|f\|_{M^{p,\beta}(w^p,w^q)} \le 1.$$

This yields that $\mathcal{F}$ satisfies condition (i) of Proposition 2.1.

Step 4. A verification for condition (ii) of Proposition 2.1. Assume that $b \in C_0^\infty(\mathbb{R}^n)$ and is supported in a cube $Q = Q(0,r)$. Fix $f \in M^{p,\beta}(w^p,w^q)$ with

$$\|f\|_{M^{p,\beta}(w^p,w^q)} \le 1$$

and

$$E_N := \{x \in \mathbb{R}^n : |x| > N\}$$

with $N \geq \max\{nr, 1\}$. Note that

$$|z| \leq n|z|_\infty \leq \frac{1}{2}nr \leq \frac{1}{2}N \leq \frac{1}{2}|x|$$

when $x \in E_N$ and $z \in Q$. Then we have

$$|x - z| \geq |x| - |z| \geq \frac{1}{2}|x|$$

when $x \in E_N$ and $z \in Q$. By (1.2), we have

$$|K_{\alpha,\eta}(x, y)| \leq |K_\alpha(x, y)| \leq \frac{C_{K_\alpha}}{|x - y|^{n-\alpha}}, \quad \text{for } x \neq y. \tag{3.24}$$

Note that $b(x) = 0$ when $x \in E_N$ since $N \geq nr$. By (3.24), we have

$$\begin{aligned}(T_{K_{\alpha,\eta}})^m_b(f)(x) &\leq C_{K_\alpha} \int_{\mathbb{R}^n} \frac{|(b(x) - b(z))^m f(z)|}{|x - z|^{n-\alpha}} dz \\ &\leq 2^{n-\alpha} C_{K_\alpha} \|b\|^m_{L^\infty(\mathbb{R}^n)} |x|^{\alpha-n} \int_Q |f(z)| dz\end{aligned} \tag{3.25}$$

for every $x \in E_N$. By arguments similar to those used to derive (3.8), we have

$$\int_Q |f(z)| dz \leq [w]^{1/q}_{A_{p,q}} w^q(Q)^{\beta/p - 1/q} |Q|^{1-\alpha/n} \|f\|_{\widetilde{M}^{p,\beta}(w^p, w^q)}. \tag{3.26}$$

For a fixed cube $\tilde{Q} = \tilde{Q}(x_0, t)$, we get from (3.25) and (3.26) that

$$\begin{aligned}&\frac{1}{w^q(\tilde{Q})^{q\beta/p}} \int_{\tilde{Q}} |(T_{K_{\alpha,\eta}})^m_b(f)(x)\chi_{E_N}(x)|^q w^q(x) dx \\ &\leq C_1 w^q(Q)^{q\beta/p-1} |Q|^{q\frac{n-\alpha}{n}} \frac{1}{w^q(\tilde{Q})^{q\beta/p}} \int_{\tilde{Q}\cap E_N} |x|^{-(n-\alpha)q} w^q(x) dx \\ &\leq C_1 w^q(Q)^{q\beta/p-1} |Q|^{q\frac{n-\alpha}{n}} \\ &\quad \times \frac{1}{w^q(\tilde{Q})^{q\beta/p}} \sum_{j=0}^\infty \int_{\tilde{Q}\cap(B(0,2^{j+1}N)\setminus B(0,2^jN))} |x|^{-(n-\alpha)} w^q(x) dx \\ &\leq C_1 w^q(Q)^{q\beta/p-1} |Q|^{q\frac{n-\alpha}{n}} \frac{1}{w^q(\tilde{Q})^{q\beta/p}} \\ &\quad \times \sum_{j=0}^\infty (2^j N)^{-(n-\alpha)q} w^q(\tilde{Q} \cap (B(0, 2^{j+1}N) \setminus B(0, 2^j N))) \\ &\leq C_1 w^q(Q)^{q\beta/p-1} |Q|^{q\frac{n-\alpha}{n}} \\ &\quad \times \sum_{j=0}^\infty (2^j N)^{-(n-\alpha)q} w^q(\tilde{Q} \cap (B(0, 2^{j+1}N) \setminus B(0, 2^j N)))^{1-q\beta/p},\end{aligned} \tag{3.27}$$

where

$$C_1 = (2^{n-\alpha}C_{K_\alpha}\|b\|^m_{L^\infty(\mathbb{R}^n)}\|f\|_{\widetilde{M}^{p,\beta}(w^p,w^q)})^q[w]_{A_{p,q}}.$$

Invoking Lemma 2.2, we see that $w^q \in A_{q\frac{n-\alpha}{n}}(\mathbb{R}^n)$. Applying Lemma 2.1 (ii), there exists $\epsilon > 0$ such that $w^q \in A_{q\frac{n-\alpha}{n}-\epsilon}(\mathbb{R}^n)$. Then by Lemma 2.1 (iii) we have

$$\begin{aligned} w^q(\tilde{Q}\cap(B(0,2^{j+1}N)\setminus B(0,2^jN))) &\leq w^q(B(0,2^{j+1}N)) \leq w^q(Q(0,2^{j+2}N)) \\ &\leq [w^q]_{A_{q\frac{n-\alpha}{n}-\epsilon}}(2^{j+2}N)^{q(n-\alpha)-n\epsilon}w^q(Q(0,1)). \end{aligned}$$

This together with (3.27) yields that

$$\begin{aligned} &\frac{1}{w^q(\tilde{Q})^{q\beta/p}}\int_{\tilde{Q}}|(T_{K_{\alpha,\eta}})^m_b(f)(x)\chi_{E_N}(x)|^q w^q(x)dx \\ &\leq C_1[w^q]^{1-q\beta/p}_{A_{q\frac{n-\alpha}{n}-\epsilon}(\mathbb{R}^n)}w^q(Q)^{q\beta/p-1}|Q|^{q\frac{n-\alpha}{n}}w^q(Q(0,1))^{1-q\beta/p} \\ &\quad\times\sum_{j=0}^{\infty}(2^jN)^{-(n-\alpha)q}(2^{j+2}N)^{(q(n-\alpha)-n\epsilon)(1-q\beta/p)} \\ &\leq C_1[w^q]^{1-q\beta/p}_{A_{q\frac{n-\alpha}{n}-\epsilon}(\mathbb{R}^n)}w^q(Q)^{q\beta/p-1}|Q|^{q\frac{n-\alpha}{n}}w(Q(0,1))^{1-q\beta/p} \\ &\quad\times\sum_{j=0}^{\infty}(2^jN)^{-q^2\beta(n-\alpha)/p-n\epsilon(1-q\beta/p)} \\ &\leq C_1[w^q]^{1-q\beta/p}_{A_{q\frac{n-\alpha}{n}-\epsilon}(\mathbb{R}^n)}w^q(Q)^{q\beta/p-1}|Q|^{q\frac{n-\alpha}{n}}w(Q(0,1))^{1-q\beta/p} \\ &\quad\times N^{-q^2\beta(n-\alpha)/p-n\epsilon(1-q\beta/p)}, \end{aligned}$$

which gives

$$\begin{aligned} \|(T_{K_{\alpha,\eta}})^m_b(f)\chi_{E_N}\|_{M^{q,q\beta/p}(w^q)} &\leq 2^{n-\alpha}C_{K_\alpha}\|b\|^m_{L^\infty(\mathbb{R}^n)}\|f\|_{\widetilde{M}^{p,\beta}(w^p,w^q)}[w]^{1/q}_{A_{p,q}}[w^q]^{\frac{1-q\beta/p}{q}}_{A_{q\frac{n-\alpha}{n}-\epsilon}}w^q(Q)^{\frac{q\beta/p-1}{q}}|Q|^{\frac{n-\alpha}{n}} \\ &\quad\times w^q(Q(0,1))^{\frac{1-q\beta/p}{q}}N^{-q\beta(n-\alpha)/p-n\epsilon(1/q-\beta/p)}. \end{aligned}$$

This, together with (3.27), implies that $\mathcal{F}$ satisfies the condition (ii) of Proposition 2.1.

Step 5. A verification for condition (iii) of Proposition 2.1. It suffices to show that

$$\lim_{|h|\to 0}\|(T_{K_{\alpha,\eta}})^m_b(f)(\cdot+h)-(T_{K_{\alpha,\eta}})^m_b(f)(\cdot)\|_{M^{q,q\beta/p}(w^q)} = 0 \tag{3.28}$$

for a fixed $\eta \in (0,1)$.

At first we shall prove that

$$|K_{\alpha,\eta}(x,y)-K_{\alpha,\eta}(z,y)| \leq C\tilde{\theta}\Big(\frac{|x-z|}{|x-y|}\Big)\frac{1}{|x-y|^{n-\alpha}} \tag{3.29}$$

for all $|x-y| > 2|x-z|$, where $\tilde{\theta} := \theta(t)+t$ and the constant C is independent of η.

When $|x-y| > 2|x-z|$, we consider the following different cases:

Case 1: ($|x-y| \geq \eta$ and $|z-y| \geq \eta$). In this case we have that $K_{\alpha,\eta}(x,y) = K_\alpha(x,y)$ and $K_{\alpha,\eta}(z,y) = K_\alpha(z,y)$. This together with (1.3) yields (3.29).

Case 2: ($|x-y| < \eta$ and $|z-y| < \eta$). Without loss of generality we may assume that $|x-y| \geq |z-y|$. It is clear that $|y-z| > \frac{1}{2}|x-y|$. We have

$$|K_{\alpha,\eta}(x,y) - K_{\alpha,\eta}(z,y)| \leq |K_\alpha(x,y) - K_\alpha(z,y)| + |K_\alpha(x,y) - K_\alpha(z,y)|\varphi(2\eta^{-1}|x-y|) \\ + |K_\alpha(z,y)||\varphi(2\eta^{-1}|x-y|) - \varphi(2\eta^{-1}|z-y|)|.$$

Similarly,

$$|K_{\alpha,\eta}(y,x) - K_{\alpha,\eta}(y,z)| \leq |K_\alpha(y,x) - K_\alpha(y,z)| + |K_\alpha(y,x) - K_\alpha(y,z)|\varphi(2\eta^{-1}|x-y|) \\ + |K_\alpha(y,z)||\varphi(2\eta^{-1}|x-y|) - \varphi(2\eta^{-1}|z-y|)|$$

The above facts, together with (1.2) and (1.3), imply that

$$\begin{aligned} &|K_{\alpha,\eta}(x,y) - K_{\alpha,\eta}(z,y)| + |K_{\alpha,\eta}(y,x) - K_{\alpha,\eta}(y,z)| \\ &\leq 2(|K_\alpha(x,y) - K_\alpha(z,y)| + |K_\alpha(y,x) - K_\alpha(y,z)|) \\ &\quad + (|K_\alpha(z,y)| + |K_\alpha(y,z)|)|\varphi(2\eta^{-1}|x-y|) - \varphi(2\eta^{-1}|z-y|)| \\ &\leq 2\theta\Big(\frac{|x-z|}{|x-y|}\Big)\frac{1}{|x-y|^{n-\alpha}} + \frac{2C_{K_\alpha}}{|y-z|^{n-\alpha}}|\varphi(2\eta^{-1}|x-y|) - \varphi(2\eta^{-1}|z-y|)|. \end{aligned}$$

Note that $|\varphi'(t)| \leq C\chi_{1\leq t\leq 2}(t)$ for all $t > 0$. Then we have

$$|\varphi(2\eta^{-1}|x-y|) - \varphi(2\eta^{-1}|z-y|)| \leq \frac{2}{\eta}|\varphi'(t)||x-z| \leq C\frac{2}{\eta}\chi_{1\leq t\leq 2}(t) \leq C\frac{4|x-z|}{\eta t} \leq C\frac{|x-z|}{|x-y|}, \tag{3.30}$$

where $t \in (\frac{2}{\eta}|z-y|, \frac{2}{\eta}|x-y|)$. Therefore, we get

$$|K_{\alpha,\eta}(x,y) - K_{\alpha,\eta}(z,y)| + |K_{\alpha,\eta}(y,x) - K_{\alpha,\eta}(y,x)| \leq C\widetilde{\theta}\Big(\frac{|x-z|}{|x-y|}\Big)\frac{1}{|x-y|^{n-\alpha}},$$

which gives (3.29) in this case.

Case 3: ($|x-y| \geq \eta$ and $|z-y| < \eta$). In this case we have that $K_{\alpha,\eta}(x,y) = K_\alpha(x,y)$ and $|z-y| > \frac{1}{2}|x-y|$ since $|x-y| > 2|x-z|$. This together with (1.2), (1.3) and (3.30) implies that

$$\begin{aligned} &|K_{\alpha,\eta}(x,y) - K_{\alpha,\eta}(z,y)| + |K_{\alpha,\eta}(y,x) - K_{\alpha,\eta}(y,z)| \\ &= |K_\alpha(x,y) - K_{\alpha,\eta}(z,y)| + |K_\alpha(y,x) - K_{\alpha,\eta}(y,z)| \\ &\leq |K_\alpha(x,y) - K_\alpha(z,y)| + |K_\alpha(y,x) - K_\alpha(y,z)| \\ &\quad + (|K_\alpha(z,y)| + |K_\alpha(y,z)|)\varphi(2\eta^{-1}|z-y|) \\ &\leq |K_\alpha(x,y) - K_\alpha(z,y)| + |K_\alpha(y,x) - K_\alpha(y,z)| + (|K_\alpha(z,y)| \\ &\quad + |K_\alpha(y,z)|)|\varphi(2\eta^{-1}|z-y|) - \varphi(2\eta^{-1}|x-y|)| \\ &\leq \theta\Big(\frac{|x-z|}{|x-y|}\Big)\frac{1}{|x-y|^{n-\alpha}} + \frac{2C_{K_\alpha}}{|y-z|^{n-\alpha}}\frac{|x-z|}{|x-y|}, \end{aligned}$$

which proves (3.29) in this case.

Case 4: ($|x-y| < \eta$ and $|z-y| \geq \eta$). The case is similar to Case 3.

In what follows, we set $|h| < \frac{\eta}{8}$ and $\eta \in (0, 1)$. By the definition of $(T_{K_{\alpha,\eta}})^m_b$,

$$\begin{aligned}
&|(T_{K_{\alpha,\eta}})^m_b(f)(x+h) - (T_{K_{\alpha,\eta}})^m_b(f)(x)| \\
&\le \int_{\mathbb{R}^n} |(b(x+h) - b(y))^m (K_{\alpha,\eta}(x+h, y) - K_{\alpha,\eta}(x, y)) f(y)| dy \\
&\quad + \int_{\mathbb{R}^n} |((b(x+h) - b(y))^m - (b(x) - b(y))^m) K_{\alpha,\eta}(x, y) f(y)| dy \\
&=: L_1 + L_2.
\end{aligned} \tag{3.31}$$

For L_1. Because $|h| < \frac{\eta}{8}$, we have

$$K_{\alpha,\eta}(x+h, y) = K_{\alpha,\eta}(x, y) = 0$$

when $|x - y| \le \frac{\eta}{4}$. Moreover, $|x - y| > 2|h|$ when $|x - y| > \frac{\eta}{4}$. By (3.29), we have that for almost every $x \in \mathbb{R}^n$,

$$\begin{aligned}
L_1 &\le \int_{|x-y|>\frac{\eta}{4}} |b(x+h) - b(y)|^m |K_{\alpha,\eta}(x+h, y) - K_{\alpha,\eta}(x, y)||f(y)| dy \\
&\le C \int_{|x-y|>\frac{\eta}{4}} \frac{1}{|x-y|^{n-\alpha}} \widetilde{\theta}\Big(\frac{|h|}{|x-y|}\Big)|f(y)| dy \\
&\le C \sum_{j=0}^{\infty} \widetilde{\theta}\Big(\frac{2^{2-j}|h|}{\eta}\Big) \int_{2^{j-2}\eta \le |x-y| \le 2^{j-1}\eta} \frac{1}{|x-y|^{n-\alpha}} |f(y)| dy \\
&\le C \sum_{j=0}^{\infty} \widetilde{\theta}\Big(\frac{2^{2-j}|h|}{\eta}\Big) M_\alpha f(x).
\end{aligned}$$

Note that

$$\begin{aligned}
\sum_{j=0}^{\infty} \widetilde{\theta}\Big(\frac{2^{2-j}|h|}{\eta}\Big) \le \sum_{j=0}^{\infty} \int_{2^{-j}}^{2^{-j+1}} \frac{\widetilde{\theta}(4t|h|/\eta)}{t} dt \le C \int_0^2 \frac{\widetilde{\theta}(4t|h|/\eta)}{t} dt \\
\le C \int_0^{8|h|/\eta} \frac{\widetilde{\theta}(t)}{t} dt < \infty.
\end{aligned}$$

This, together with the boundedness for M_α: $M^{p,\beta}(w^p, w^q) \to M^{q,q\beta/p}(w^q)$, implies that

$$\begin{aligned}
&\|L_1\|_{M^{q,\beta_2}(w^q)} \\
&\le C\Big(\int_0^{8|h|/\eta} \frac{\widetilde{\theta}(t)}{t} dt\Big) \|M_\alpha(f)\|_{M^{q,q\beta/p}(w^q)} \le C\Big(\int_0^{8|h|/\eta} \frac{\widetilde{\theta}(t)}{t} dt\Big) \|f\|_{M^{p,\beta}(w^p, w^q)} \\
&\le C \int_0^{8|h|/\eta} \frac{\widetilde{\theta}(t)}{t} dt,
\end{aligned}$$

which yields that $\|L_1\|_{M^{q,q\beta/p}(w^q)} \to 0$ as $|h| \to 0$.

Divide the second term L_2 as follows

$$\begin{aligned}L_2 &= \int_{\mathbb{R}^n} |(b(x+h)-b(y))^m - (b(x)-b(y))^m||K_{\alpha,\eta}(x,y)f(y)|dy\\ &= \int_{|x-y|>\eta} |(b(x+h)-b(y))^m - (b(x)-b(y))^m||K_{\alpha,\eta}(x,y)f(y)|dy\\ &\quad + \int_{\eta/2\le|x-y|\le\eta} |(b(x+h)-b(y))^m - (b(x)-b(y))^m||K_{\alpha,\eta}(x,y)f(y)|dy\\ &=: L_{2,1} + L_{2,2}.\end{aligned}$$

We write

$$\begin{aligned}&(b(x+h)-b(y))^m - (b(x)-b(y))^m\\ &= (b(x+h)-b(x)+b(x)-b(y))^m - (b(x)-b(y))^m\\ &= \sum_{i=1}^{m} C_m^i (b(x+h)-b(x))^i (b(x)-b(y))^{m-i}\\ &= \sum_{i=1}^{m} C_m^i (b(x+h)-b(x))^i \sum_{j=0}^{m-i} C_{m-i}^j b(x)^j (-b(y))^{m-i-j},\end{aligned}$$

where

$$C_N^r = \frac{N!}{r!(N-r)!}$$

for any r, $N \in \mathbb{N}$ with $r \le N$. Hence, we obtain

$$\begin{aligned}L_{2,1} &\le \sum_{i=1}^{\infty} C_m^i |b(x+h)-b(x)|^i \sum_{j=0}^{m-i} C_{m-i}^j |b(x)|^j\\ &\quad \times \Big| \int_{|x-y|>\eta} K_\alpha(x,y) b(y)^{m-i-j} f(y)dy \Big|\\ &\le \sum_{i=1}^{m} C_m^i |b(x+h)-b(x)|^i \sum_{j=0}^{m-i} C_{m-i}^j |b(x)|^j |T_{K_\alpha}(b^{m-i-j}f)(x)|\\ &\le C|h||T_{K_\alpha}(f)(x)|.\end{aligned}$$

From this and the $M^{p,\beta}(w^p, w^q) \to M^{q,q\beta/p}(w^q)$ boundedness of T_{K_α}, we obtain

$$\|L_{2,1}\|_{M^{q,q\beta/p}(w^q)} \le C|h|\|T_{K_\alpha} f\|_{M^{q,q\beta/p}(w^q)} \le C|h|\|f\|_{M^{p,\beta}(w^p,w^q)} \le C|h|.$$

On the other hand, one has

$$\begin{aligned}\Big| \int_{\eta/2\le|x-y|\le\eta} K_{\alpha,\eta}(x,y) b(y)^{m-i-j} f(y)dy \Big| &\le C \int_{\eta/2\le|x-y|\le\eta} |K_{\alpha,\eta}(x,y)||f(y)|dy\\ &\le C\frac{1}{\eta^{n-\alpha}} \int_{\eta/2\le|x-y|\le\eta} |f(y)|dy \le CM_\alpha(f)(x).\end{aligned}$$

It yields that

$$
\begin{aligned}
L_{2,2} \leq & \sum_{i=1}^{i} C_m^i |b(x+h) - b(x)|^i \sum_{j=0}^{m-i} C_{m-i}^j |b(x)|^j \\
& \times \Big| \int_{\eta/2 \leq |x-y| \leq \eta} K_{\alpha,\eta}(x,y) b(y)^{m-i-j} f(y) dy \Big| \\
\leq & C|h| M_\alpha(f)(x).
\end{aligned}
$$

It follows that

$$\|L_{2,2}\|_{M^{q,q\beta/p}(w^q)} \leq C|h| \|M_\alpha(f)\|_{M^{q,q\beta/p}(w^q)} \leq C|h| \|f\|_{M^{p,\beta}(w^p,w^q)} \leq C|h|.$$

It follows from above estimates of $L_1, L_{2,1}$ and $L_{2,2}$ that

$$\|(T_{K_{\alpha,\eta}})_b^m(f)(\cdot + h) - (T_{K_{\alpha,\eta}})_b^m(f)(\cdot)\|_{M^{q,q\beta/p}(w^q)} \to 0$$

as $|h| \to 0$ uniformly for all f with $\|f\|_{M^{p,\beta}(w^p,w^q)} \leq 1$. This verifies the condition (iii) of Proposition 2.1. Theorem 1.2 is now proved. □

4. Conclusions

This paper is devoted to establishing some new results focusing on the boundedness and compactness for the iterated commutators of the θ-type Calderón-Zygmund singular integral and its fractional variant on the weighed Morrey spaces. The main results improve and extend some known ones.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All of the authors of this article declare no conflicts of interest.

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