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Research article

The core of the unit sphere of a Banach space

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Abstract: A geometric invariant or preserver is essentially a geometric property of the unit sphere of a real Banach space that remains invariant under the action of a surjective isometry onto the unit sphere of another real Banach space. A new geometric invariant of the unit ball of a real Banach space was introduced and analyzed in this manuscript: The core of the unit sphere. This geometric invariant consists of all points in the unit sphere of a real Banach space, which are contained in a unique maximal face. It is, in a geometrical sense, the opposite of fractal-like sets such as starlike sets. Classical geometric properties, such as smoothness and strict convexity, were employed to characterize the core of the unit sphere. Also, the core was related to a recently introduced new index: the index of strong rotundity. A characterization of the core in terms of the index of strong rotundity was provided. Finally, applications to longstanding open problems, such as Tingley's problem, were provided by presenting a new notion: Mazur-Ulam classes of Banach spaces.

Keywords: Tingley's Problem; Mazur-Ulam property; surjective isometry; geometric invariant; extreme point; exposed point; face; facet; strict convexity, smoothness
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1. Introduction

Geometric invariants of the unit ball of a real Banach space have recently played an important role in longstanding open problems such as the Banach-Mazur conjecture for rotations (is every transitive and separable Banach space a Hilbert space?) [1, 6, 22, 28, 36] and Tingley's problem (is it always possible to extend a surjective isometry defined between the unit spheres of two real Banach spaces to a surjective linear isometry between the whole spaces?) [12, 20, 23, 32, 39–47]. Special cases of geometric invariants are the so-called indices or moduli, such as the classical modulus of convexity [13] and modulus of smoothness [33], as well as the recently new introduced index of rotundity [26] and index of strong rotundity [29]. More geometric invariants, such as maximal faces, facets, and the frame of the unit ball, can be found in the literature of Tingley's problem [10, 43]. Nevertheless, the Banach-Mazur conjecture for rotations usually produces geometric preservers under surjective linear isometries because transitivity involves the action on the unit sphere of the group of surjective linear isometries of a Banach space [22, 28].

Since the appearance of remarkable results such as the Mazur-Ulam theorem [37] and Mankiewicz theorem [35], the employment of linear and nonlinear isometries and their corresponding geometric invariants have been an extremely prolific topic. For instance, fractal-like sets such as starlike sets have gained severe importance in approaching both the Banach-Mazur conjecture for rotations and Tingley's problem. The behavior of fractals contained in the unit sphere of infinite-dimensional Banach spaces is clear under the action of surjective linear isometries, but it is not so clear under the action of surjective isometries between unit spheres. This manuscript pushes forward the edge of this research field by finding a new geometric invariant that serves to characterize and better understand the geometry of the unit ball of a real Banach space.

2. Methodology

Only nonzero real vector spaces will be considered throughout this manuscript by default (many of the results of this work can be easily readapted to complex spaces). For a normed space X, $\mathsf{B}_X, \mathsf{U}_X, \mathsf{S}_X$ stand for the (closed) unit ball, the open unit ball, and the unit sphere, respectively. For $x \in X$ and $r > 0$, $\mathsf{B}_X(x, r), \mathsf{U}_X(x, r), \mathsf{S}_X(x, r)$ denote the (closed) ball of center x and radius r, the open ball of center x and radius r, and the sphere of center x and radius r. Now, let X denote a topological space and $A \subseteq X$, then $\mathrm{int}(A), \mathrm{cl}(A), \mathrm{bd}(A)$ stand for the interior of A, the closure of A, and the boundary of A, respectively. If $B \subseteq A$, then $\mathrm{int}_A(B), \mathrm{cl}_A(B), \mathrm{bd}_A(B)$ stand for the relative interior of B with respect to A, the relative closure of B with respect to A, and the relative boundary of B with respect to A, respectively.

The upcoming definitions are very well known among the Banach space geometers and belong to the folklore of the classic literature of Banach space theory. For further reading on these topics, we refer the reader to the classical texts [17, 18, 38].

Let X be a vector space. Let $E \subseteq F \subseteq X$. We say that E satisfies the extremal condition with respect to F provided that the following property is satisfied: $\forall x, y \in F\ \forall t \in (0, 1)\ \ tx+(1-t)y \in E \Rightarrow x, y \in E$. Under this situation, we say that E is extremal in F. When an extremal subset $E = \{e\}$ is a singleton, then e is called an extremal point of F. The set of extremal points of F is denoted by $\mathrm{ext}(F)$. If both E and F are convex, then E is called a face of F if it is extremal in F. Extremal points of convex sets are called extreme points and denoted also by $\mathrm{ext}(F)$.

If X is a Banach space, then the set of maximal (proper) faces of the unit ball B_X will be denoted by C_X. If F is any convex subset of the unit sphere S_X, then $C_F := \{C \in C_X : F \subseteq C\}$. A point $x \in \mathsf{S}_X$ is said to be an exposed point of B_X if there exists x^* in the unit sphere S_{X^*} of the dual space X^* in such a way that $(x^*)^{-1}(\{1\}) \cap \mathsf{B}_X = \{x\}$ (the functional x^* is called a supporting functional that exposes x on B_X). On the other hand, $x \in \mathsf{S}_X$ is said to be a strongly exposed point of B_X if there exists $x^* \in \mathsf{S}_{X^*}$ verifying the following property: If $(x_n)_{n\in\mathbb{N}} \subseteq \mathsf{B}_X$ is such that $(x^*(x_n))_{n\in\mathbb{N}}$ converges to 1, then $(x_n)_{n\in\mathbb{N}}$ converges to x (the functional x^* is said to strongly expose x on B_X). Special attention will be paid to the sets $\Pi_X := \{(x, x^*) \in \mathsf{S}_X \times \mathsf{S}_{X^*} : x^*(x) = 1\}$, $\Pi_X^e := \{(x, x^*) \in \mathsf{S}_X \times \mathsf{S}_{X^*} : x^* \text{ exposes } x \text{ on } \mathsf{B}_X\}$, and $\Pi_X^{se} := \{(x, x^*) \in \mathsf{S}_X \times \mathsf{S}_{X^*} : x^* \text{ strongly exposes } x \text{ on } \mathsf{B}_X\}$. Notice that $\Pi_X^{se} \subseteq \Pi_X^e \subseteq \Pi_X$. The set of rotund points of B_X is defined as $\operatorname{rot}(\mathsf{B}_X) = \{x \in \mathsf{S}_X : \{x\} \text{ is a maximal face of } \mathsf{B}_X\}$. In view of the Hahn-Banach separation theorem, the set of rotund points can be described as $\operatorname{rot}(\mathsf{B}_X) = \{x \in \mathsf{S}_X :$ if $x^* \in \mathsf{S}_{X^*}$ is so that $(x, x^*) \in \Pi_X$, then $(x, x^*) \in \Pi_X^e\}$. We refer the reader to [4, 5] for a wider perspective on the above concepts and some other geometrical properties related with renormings. The duality mapping [7, 8] of a Banach space X is the set-valued map $J : X \to \mathcal{P}(X^*)$ defined as $J(x) := \{x^* \in X^* : \|x^*\| = \|x\| \text{ and } x^*(x) = \|x^*\|\|x\|\}$ for every $x \in X$. If $x \in \mathsf{S}_X$, then $J(x)$ is often denoted by $v(x)$ and called the spherical image of x. In this sense, $v := J|_{\mathsf{S}_X}$ is the spherical image map. A point x in the unit sphere S_X of X is said to be a smooth point [14] of the unit ball B_X of X provided that $v(x)$ is a singleton. The subset of smooth points of B_X is typically denoted by $\operatorname{smo}(\mathsf{B}_X)$. Rotund points and smooth points are somehow dual notions.

Let X be a vector space. Let M be a convex subset of X with at least two points. We define the set of inner points of M by

$$\operatorname{inn}(M) := \{x \in X : \forall m \in M \setminus \{x\}\ \exists n \in M \setminus \{m, x\} \text{ such that } x \in (m, n)\}$$

as in [24, 25, 30, 31]. The set of inner points of a convex set is the infinite dimensional version of what Tingley calls the "relative interior" of convex subsets of $\mathbb{R}^n$ in [46]. In fact, in [31, Theorem 5.1], it is proved that every nonsingleton convex subset of any finite dimensional vector space has inner points. However, in [31, Corollary 5.3], it was shown that every infinite dimensional vector space possesses a nonsingleton convex subset free of inner points. In fact, the positive face of B_{ℓ_1}, $C := \{(x_n)_{n\in\mathbb{N}} \in \mathsf{S}_{\ell_1} : x_n \geq 0\}$, is a closed convex subset satisfying that $\operatorname{inn}(C) = \varnothing$ [31, Theorem 5.4]. The idea behind this pathological result is consistent with other properties of ℓ_1 as dual of the nonbarrelled space c_{00}. For instance, there can be found unbounded sequences in ℓ_1 which are w^*-convergent to 0 as dual of c_{00}. Indeed, let $X := (c_{00}, \|\cdot\|_\infty)$, so X^* is linearly isometric to $(\ell_1, \|\cdot\|_1)$. For each $n \in \mathbb{N}$, let

$$x_n^* := \frac{1}{n}\sum_{k=1}^{n}\frac{1}{2^k}e_k + \frac{n}{\sum_{k=n+1}^{\infty}\frac{1}{2^k}}\sum_{k=n+1}^{\infty}\frac{1}{2^k}e_k.$$

For each $k \in \mathbb{N}$, $(x_n^*(e_k))_{n\in\mathbb{N}}$ converges to 0 because if $n > k$, then $x_n^*(e_k) = \dfrac{1/n}{2^k}$. As a consequence, $(x_n^*(x))_{n\in\mathbb{N}}$ converges to 0 for all $x \in c_{00}$ due to the fact that $c_{00} = \operatorname{span}\{e_n : n \in \mathbb{N}\}$. In other words, $(x_n^*)_{n\in\mathbb{N}} \xrightarrow{w^*} 0$. However, notice that

$$\begin{aligned} \left\|x_n^*\right\|_1 &= \frac{1}{n}\sum_{k=1}^{n}\frac{1}{2^k} + \frac{n}{\sum_{k=n+1}^{\infty}\frac{1}{2^k}}\sum_{k=n+1}^{\infty}\frac{1}{2^k} \\ &= \frac{1}{n}\sum_{k=1}^{n}\frac{1}{2^k} + n \geq n. \end{aligned}$$

3. Results

As we will see later on, the new geometric invariant introduced in this work (the core of the unit sphere) is intimately linked to the convexity and extremal structures of the unit sphere. In this sense, certain properties that might seem intuitively true might not hold even in finite dimensions. For instance, a subset C of the unit sphere of a Banach space X is said to be flat if its convex hull is entirely contained in the unit sphere, that is, $\mathrm{co}(C) \subseteq \mathsf{S}_X$. On the other hand, C is called almost flat provided that $[c, d] \subseteq \mathsf{S}_X$ for all $c, d \in C$. It is not intuitively trivial to think of an almost flat set that is not flat. In [10, Example 3], a novel 3-dimensional unit ball was presented containing an example of an almost flat set (of four vertices), which is not flat. This example can be simplified to three vertices within three adjacent facets. More specifically, it is enough to consider the set $E := \{(1,1,1), (-1,1,-1), (-1,-1,1)\}$ in $\mathsf{S}_{\ell_\infty^3}$, which is clearly almost flat but not flat in the unit sphere of $\ell_\infty^3 := \left(\mathbb{R}^3, \|\cdot\|_\infty\right)$. Observe that E is not connected; however, if we now take $D := [(1,1,1), (-1,1,1)] \cup [(1,1,1), (1,-1,1)] \cup [(1,1,1), (1,1,-1)]$, then D is a path-connected, almost-flat set that is not flat (see also [11, Theorem 2.1]).

A very famous result of Tingley [46, Lemmas 12 and 13] asserts that surjective isometries between finite-dimensional Banach spaces preserve antipodal points. This result has been recently transported, in any dimension, to rotund points [10, Theorem 14] and to maximal faces with inner points [10, Theorem 15]. Our first result in this manuscript goes one step further in this direction by relying on the P-property (a Banach space has the P-property whenever every proper face of the unit ball is the intersection of all maximal faces containing it). The P-property was originally introduced in [10, Definition 7], but it was motivated by [44, Definition 3.2].

Theorem 3.1. *Let X and Y be Banach spaces such that X has the P-property. Let $T : \mathsf{S}_X \to \mathsf{S}_Y$ be a surjective isometry and $F \subseteq \mathsf{S}_X$ a proper face satisfying* $\mathrm{inn}(F) \neq \varnothing$*, then $T(-F) = -T(F)$.*

Proof. Since X satisfies the P-property, $F = \bigcap_{C \in \mathcal{C}_F} C$, hence, by bearing in mind [10, Theorem 15] together with the fact that T is a homeomorphism, we have that

$$\begin{aligned} T(-F) &= T\left(-\bigcap_{C \in \mathcal{C}_F} C\right) = T\left(\bigcap_{C \in \mathcal{C}_F} -C\right) = \bigcap_{C \in \mathcal{C}_F} T(-C) = \bigcap_{C \in \mathcal{C}_F} -T(C) \\ &= -\bigcap_{C \in \mathcal{C}_F} T(C) = -T\left(\bigcap_{C \in \mathcal{C}_F} C\right) = -T(F). \end{aligned}$$

□

As mentioned above, the search for geometric invariants is a hot topic now in the theory of Banach space geometry. Here, we present a new geometric invariant.

Definition 3.2 (Core)**.** *Let X be a Banach space. The core of the unit sphere of X is defined as* $\mathrm{core}(\mathsf{S}_X) := \{x \in \mathsf{S}_X : \exists! C \in \mathcal{C}_X \;\; x \in C\}$.

Notice that $\mathrm{rot}(\mathsf{B}_X) \cup \mathrm{smo}(\mathsf{B}_X) \subseteq \mathrm{core}(\mathsf{S}_X)$. Our next result characterizes the core. Recall that U_X stands for the open unit ball of X and $\mathsf{U}_X(x, r)$ stands for the open ball of center x and radius r.

Theorem 3.3. *Let X be a Banach space, then* $\mathrm{core}(\mathsf{S}_X) = \{x \in \mathsf{S}_X : \mathsf{B}_X \setminus \mathsf{U}_X(x, 2)$ *is convex*$\}$.

Proof.

$\subseteq$ If $x \in \mathrm{core}(\mathsf{B}_X)$ and C is the only maximal proper face of B_X containing x, then $\mathsf{B}_X \setminus \mathsf{U}_X(x, 2) = -C$. Indeed, if $y \in C$, then $[y, x] \subseteq \mathsf{S}_X$; hence, $\|y + x\| = 2$, so $-y \in \mathsf{B}_X \setminus \mathsf{U}_X(x, 2)$. This shows that $-C \subseteq \mathsf{B}_X \setminus \mathsf{U}_X(x, 2)$. Next, if $y \in \mathsf{B}_X \setminus \mathsf{U}_X(x, 2)$, then $\|y - x\| = 2$, so $\|-y + x\| = 2$; hence, $[-y, x] \subseteq \mathsf{S}_X$, so $-y \in C$, that is, $y \in -C$. This proves that $-C \supseteq \mathsf{B}_X \setminus \mathsf{U}_X(x, 2)$.

- $\supseteq$ Conversely, assume that $D := \mathsf{B}_X \setminus \mathsf{U}_X(x,2)$ is convex. Let $C := -D$. We will show that C is the only maximal proper face of B_X containing x. Indeed, notice that $D \subsetneq \mathsf{S}_X$. Fix an arbitrary $y \in \mathsf{S}_X$ so that $[y,x] \subseteq \mathsf{S}_X$, then $\|y+x\| = 2$, so $-y \in \mathsf{B}_X \setminus \mathsf{U}_X(x,2) = D = -C$, that is, $y \in C$. This concludes the proof.

□

In [29], for every $(x,x^*) \in \Pi_X$, the following indices are introduced:

$$\begin{array}{rcl} \upsilon_X(\cdot,(x,x^*)): \ [0,2] & \to & [0,2] \\ \varepsilon & \mapsto & \upsilon_X(\varepsilon,(x,x^*)) := \inf\{1 - x^*(y) : \|y\| \leq 1, \ \|x-y\| \geq \varepsilon\} \end{array}$$

and

$$\begin{array}{rcl} \eta_X(\cdot,(x,x^*)): \ [0,2] & \to & [0,2] \\ \varepsilon & \mapsto & \eta_X(\varepsilon,(x,x^*)) := d\left((x^*)^{-1}(\{1\}), \mathsf{B}_X \setminus \mathsf{U}_X(x,\varepsilon)\right) \end{array}$$

The latter one, $\eta_X(\cdot,(x,x^*))$, is denominated as index of strong rotundity [29]. It is noticed that $0 \leq \upsilon_X(\varepsilon,(x,x^*)) \leq \eta_X(\varepsilon,(x,x^*)) \leq 2$ for all $\varepsilon \in [0,2]$, and the index of strong rotundity characterizes whether a Banach space is strongly rotund since $\Pi_X^{se} = \{(x,x^*) \in \Pi_X : \forall \varepsilon \in (0,2] \ \ \eta_X(\varepsilon,(x,x^*)) > 0\}$. On the other hand, the index of rotundity [26] is defined as $\zeta_X := \sup\{\operatorname{diam}(C) : C \subseteq \mathsf{S}_X \text{ is convex}\}$. The next results relate the previous indices.

Theorem 3.4. *Let X be a Banach space. For every $\varepsilon \in [0,\zeta_X)$, there exists $(x,x^*) \in \Pi_X$ such that $\upsilon_X(\varepsilon,(x,x^*)) = \eta_X(\varepsilon(x,x^*)) = 0$.*

Proof. In first place, by [29, Theorem 2.4], $0 \leq \upsilon_X(\varepsilon,(x,x^*)) \leq \eta_X(\varepsilon,(x,x^*)) \leq 2$ for all $\varepsilon \in [0,2]$ and all $(x,x^*) \in \Pi_X$; thus, it only suffices to show that, for every $\varepsilon \in [0,\zeta_X)$, there exists $(x,x^*) \in \Pi_X$ such that $\eta_X(\varepsilon(x,x^*)) = 0$. Fix an arbitrary $\varepsilon \in [0,\zeta_X)$. There exists $C \in \mathcal{C}_X$ such that $\varepsilon < \operatorname{diam}(C) \leq \zeta_X$. There exists $x^* \in \mathsf{S}_{X^*}$ such that $C = (x^*)^{-1}(\{1\}) \cap \mathsf{B}_X$. We can find $x,y \in C$ satisfying that $\|x-y\| \geq \varepsilon$. Note that $y \in (x^*)^{-1}(\{1\}) \cap (\mathsf{B}_X \setminus \mathsf{U}_X(x,\varepsilon))$, meaning that $\eta_X(\varepsilon,(x,x^*)) = d\left((x^*)^{-1}(\{1\}), \mathsf{B}_X \setminus \mathsf{U}_X(x,\varepsilon)\right) = 0$. □

Previous indices may be used to characterize the core of the unit sphere.

Theorem 3.5. *Let X be a Banach space, then*

$$\operatorname{core}(\mathsf{S}_X) = \{x \in \mathsf{S}_X : \exists x^* \in \nu(x) \ \upsilon_X(2,(x,x^*)) = \eta_X(2,(x,x^*)) = 2\}.$$

Proof.

- $\subseteq$ Fix an arbitrary $x \in \operatorname{core}(\mathsf{S}_X)$. Let C be the only maximal proper face of B_X containing x. Take $x^* \in \nu(x)$ such that $C = (x^*)^{-1}(\{1\}) \cap \mathsf{B}_X$. We already know from [29, Theorem 2.4] that $0 \leq \upsilon_X(\varepsilon,(x,x^*)) \leq \eta_X(\varepsilon,(x,x^*)) \leq 2$ for all $\varepsilon \in [0,2]$. Thus, it only suffices to prove that $\upsilon_X(2,(x,x^*)) = 2$. In accordance with Theorem 3.3, we have that $\mathsf{B}_X \setminus \mathsf{U}_X(x,2) = -C$; therefore, $\upsilon_X(2,(x,x^*)) = \inf\{1 - x^*(y) : y \in \mathsf{B}_X \setminus \mathsf{U}_X(x,2)\} = \inf\{1 - x^*(y) : y \in -C\} = 2$.
- $\supseteq$ Conversely, take any $x \in \mathsf{S}_X$ for which there exists $x^* \in \nu(x)$ with $\upsilon_X(2,(x,x^*)) = \eta_X(2,(x,x^*)) = 2$. Since $-1 \leq x^*(y) \leq 1$ for all $y \in \mathsf{B}_X \setminus \mathsf{U}_X(x,2)$, we have that $2 = \upsilon_X(2,(x,x^*)) \leq 1 - x^*(y) \leq 2$ for each $y \in \mathsf{B}_X \setminus \mathsf{U}_X(x,2)$. Therefore, $x^*(y) = -1$ for each $y \in \mathsf{B}_X \setminus \mathsf{U}_X(x,2)$, meaning that $\mathsf{B}_X \setminus \mathsf{U}_X(x,2) \subseteq (-x^*)^{-1}(\{1\}) \cap \mathsf{B}_X$. Let us show next that $(x^*)^{-1}(\{1\}) \cap \mathsf{B}_X$ is the only maximal

proper face of B_X containing x. Indeed, fix any arbitrary $z \in \mathsf{S}_X$ such that $[z, x] \subseteq \mathsf{S}_X$, then $\|z + x\| = 2$, so $\| - z - x\| = 2$. Hence, $-z \in \mathsf{B}_X \setminus \mathsf{U}_X(x, 2) \subseteq (-x^*)^{-1}(\{1\}) \cap \mathsf{B}_X$, that is, $x^*(z) = 1$, meaning that $z \in (x^*)^{-1}(\{1\}) \cap \mathsf{B}_X$.

□

In the literature of Tingley's problem and more generally in the literature of Banach space geometry, the notion of the starlike set is very much employed. Let X be a normed space. The starlike set of a point $x \in \mathsf{S}_X$ is defined as $\mathrm{st}(x, \mathsf{B}_X) := \{y \in \mathsf{B}_X : \|x + y\| = 2\}$. Notice that $\mathrm{st}(x, \mathsf{B}_X) \subseteq \mathsf{S}_X$. Also, $\mathrm{st}(x, \mathsf{B}_X) = \{y \in \mathsf{S}_X : [y, x] \subseteq \mathsf{S}_X\} = \bigcup\{C \subseteq \mathsf{S}_X : C \text{ is a maximal face of } \mathsf{B}_X \text{ containing } x\} = \mathsf{B}_X \setminus \mathsf{U}_X(-x, 2)$. According to [10, Theorem 9], $\mathrm{st}(x, \mathsf{B}_X)$ satisfies the extremal condition with respect to B_X for each $x \in \mathsf{S}_X$. The following lemma improves [10, Theorem 9].

Lemma 3.6. *Let X be a Banach space. Let $x \in \mathsf{S}_X$. If $\mathrm{st}(x, \mathsf{B}_X)$ is flat, then $\mathrm{st}(x, \mathsf{B}_X)$ is convex; hence, it is the only maximal face of B_X containing x.*

Proof. If $\mathrm{st}(x, \mathsf{B}_X)$ is flat, then $\mathrm{co}(\mathrm{st}(x, \mathsf{B}_X)) \subseteq \mathsf{S}_X$; therefore, there exists $x^* \in \mathsf{S}_{X^*}$ satisfying that $\mathrm{st}(x, \mathsf{B}_X) \subseteq (x^*)^{-1}(\{1\}) \cap \mathsf{B}_X$. Take any arbitrary $z \in (x^*)^{-1}(\{1\}) \cap \mathsf{B}_X$. The convexity of $(x^*)^{-1}(\{1\}) \cap \mathsf{B}_X$ allows that $\|z + x\| = 2$, so $z \in \mathrm{st}(x, \mathsf{B}_X)$. As a consequence, $\mathrm{st}(x, \mathsf{B}_X) = (x^*)^{-1}(\{1\}) \cap \mathsf{B}_X$, so $\mathrm{st}(x, \mathsf{B}_X)$ is the only maximal face of B_X containing x. □

A direct consequence of Lemma 3.6 is the following dichotomy theorem.

Theorem 3.7. *Let X be a Banach space. For every $x \in \mathsf{S}_X$, only one of the following two (disjoint) possibilities can happen:*

1) $\mathrm{st}(x, \mathsf{B}_X)$ *is not convex.*
2) $\mathrm{st}(x, \mathsf{B}_X)$ *is a maximal face of* B_X.

Previous dichotomy theorem has the following consequence on Tingley's problem.

Theorem 3.8. *Let X and Y be Banach spaces. If $T : \mathsf{S}_X \to \mathsf{S}_Y$ is a surjective isometry, then T maps non-convex starlike sets of B_X to non-convex starlike sets of B_Y, and maximal-face starlike sets of B_X to maximal-face starlike sets of B_Y.*

Proof. Fix an arbitrary $x \in \mathsf{S}_X$. By [10, Theorem 3], $T(\mathrm{st}(x, \mathsf{B}_X)) = \mathrm{st}(T(x), B_Y)$. Suppose first that $\mathrm{st}(x, \mathsf{B}_X)$ is not convex. If so is $\mathrm{st}(T(x), B_Y)$, then it is a maximal face of B_Y by the dichotomy theorem, reaching the contradiction that $\mathrm{st}(x, \mathsf{B}_X)$ is a maximal face of B_X by relying on T^{-1} and on [10, Theorem 1]. As a consequence, $\mathrm{st}(T(x), \mathsf{B}_Y)$ is not convex. Finally, if $\mathrm{st}(x, \mathsf{B}_X)$ is a maximal face of B_X, then so is $\mathrm{st}(T(x), B_Y)$ by bearing in mind [10, Theorem 1]. □

In [11, Definition 5], a new geometrical type of maximal face was introduced in the literature: strongly maximal faces. Given a Banach space X, we say that a convex subset $F \subseteq \mathsf{S}_X$ is a strongly maximal face of B_X provided that $\bigcup_{f \in F} \mathrm{st}(f, \mathsf{B}_X) = F$. Trivial examples of strongly maximal faces are rotund points. In [11, Lemma 5.6], it was shown that every strongly maximal face is a maximal face.

Theorem 3.9. *Let X be a Banach space. If $F \subseteq \mathsf{S}_X$ is a strongly maximal face of B_X, then $F = \mathrm{st}(f, \mathsf{B}_X)$ for all $f \in F$.*

Proof. Fix an arbitrary $f \in F$. By definition, $\mathrm{st}(f, \mathsf{B}_X) \subseteq F$ and F is convex; therefore, $\mathrm{st}(f, \mathsf{B}_X)$ is flat. By applying Lemma 3.6, we conclude that $\mathrm{st}(f, \mathsf{B}_X)$ is the only maximal face containing f. Finally, [11, Lemma 5.6] assures that F is a maximal face, thus $F = \mathrm{st}(f, \mathsf{B}_X)$. $\square$

The converse to Theorem 3.9 does not hold in the sense described in the following example.

Example 3.10. *Let $X := \ell_\infty^2$. Fix $x := (1, 0)$, then $\mathrm{st}(x, \mathsf{B}_X)$ is flat. Hence, it is the only maximal face of B_X containing x. However, $\mathrm{st}(x, \mathsf{B}_X)$ is not a strongly maximal face of B_X because $\mathrm{st}(x, \mathsf{B}_X)$ does not contain $\mathrm{st}(y, \mathsf{B}_X)$, where $y := (1, 1) \in \mathrm{st}(x, \mathsf{B}_X)$.*

The following corollary may be understood as a reformulation of Theorem 3.3.

Corollary 3.11. *Let X be a Banach space, then $\mathrm{core}(\mathsf{S}_X) = \{x \in \mathsf{S}_X : \mathrm{st}(x, \mathsf{B}_X)$ is flat$\}$.*

Proof.

- $\subseteq$ If $x \in \mathrm{core}(\mathsf{S}_X)$, then Theorem 3.3 assures that $\mathsf{B}_X \setminus \mathsf{U}_X(x, 2)$ is convex, meaning that $\mathrm{st}(x, \mathsf{B}_X)$ is convex as well.
- $\supseteq$ Conversely, if $\mathrm{st}(x, \mathsf{B}_X)$ is flat, then $\mathrm{st}(x, \mathsf{B}_X)$ is convex by Lemma 3.6. Therefore, it is the only maximal face of B_X containing x.

$\square$

The following corollary highlights the core as a geometric invariant.

Corollary 3.12. *Let X and Y be Banach spaces. If $T : \mathsf{S}_X \to \mathsf{S}_Y$ is a surjective isometry, then $T(\mathrm{core}(\mathsf{S}_X)) = \mathrm{core}(\mathsf{S}_Y)$.*

Proof. Since $T^{-1} : \mathsf{S}_Y \to \mathsf{S}_X$ is a surjective isometry as well, it only suffices to show that $T(\mathrm{core}(\mathsf{S}_X)) \subseteq \mathrm{core}(\mathsf{S}_Y)$. Indeed, pick any $x \in \mathrm{core}(\mathsf{S}_X)$. Notice that $-x \in \mathrm{core}(\mathsf{S}_X)$. By relying on Corollary 3.11, $\mathrm{st}(-x, \mathsf{B}_X)$ is flat. Next, flatness is a geometric invariant [10, Theorem 12(4)], that is, $T(\mathrm{st}(-x, \mathsf{B}_X))$ is flat in S_Y. Next, by bearing in mind [10, Remark 4], $T(\mathrm{st}(-x, \mathsf{B}_X)) = \mathrm{st}(-T(x), B_Y)$. As a consequence, by applying Corollary 3.11 once more, $-T(x) \in \mathrm{core}(\mathsf{S}_Y)$, meaning that $T(x) \in \mathrm{core}(\mathsf{S}_Y)$. $\square$

The frame of the unit ball is another important geometric invariant involved in Tingley's problem [42, 43]. If X is a Banach space, then the frame of B_X is characterized [10, Theorem 7] as $\mathrm{frm}(\mathsf{B}_X) = \bigcup\{\mathrm{bd}_{\mathsf{S}_X}((x^*)^{-1}(\{1\}) \cap \mathsf{B}_X) : x^* \in \bigcup_{x \in \mathsf{S}_X} v(x)\}$. In particular, $\mathrm{frm}(\mathsf{B}_X) = \mathsf{S}_X$ if, and only if, for every proper face $C \subseteq \mathsf{S}_X$, then $\mathrm{int}_{\mathsf{S}_X}(C) = \varnothing$.

Lemma 3.13. *Let X be a Banach space, then*

1) $\bigcup_{C \in \mathcal{C}_X} \mathrm{inn}(C) \subseteq \mathrm{core}(\mathsf{S}_X)$.

2) If $C \in \mathcal{C}_X$ is separable and $(c_n)_{n \in \mathbb{N}}$ is dense in C, then $c := \displaystyle\sum_{n=1}^{\infty} \frac{c_n}{2^n} \in \mathrm{core}(\mathsf{S}_X)$.

3) $\mathrm{core}(\mathsf{S}_X) \supseteq \mathsf{S}_X \setminus \mathrm{frm}(\mathsf{B}_X)$.

Proof.

1) Fix an arbitrary $C \in \mathcal{C}_X$ and an arbitrary $c \in \mathrm{inn}(C)$. Let $D \in \mathcal{C}_X$ such that $c \in D$. Since $\mathrm{inn}(C) \cap D \neq \varnothing$, in virtue of [25, Lemma 2.1], we have that $C \subseteq D$. By maximality, $C = D$. This shows that $c \in \mathrm{core}(\mathsf{S}_X)$.

2) Let $D \in C_X$ such that $c \in D$. There exists a functional $x^* \in \mathsf{S}_{X^*}$ such that $D = (x^*)^{-1}(\{1\}) \cap \mathsf{B}_X$. Since $c \in D$, we have that $x^*(c) = 1$, which implies that $x^*(c_n) = 1$ for all $n \in \mathbb{N}$. The density of $(c_n)_{n\in\mathbb{N}}$ in C assures that $x^*(C) = \{1\}$; in other words, $C \subseteq (x^*)^{-1}(\{1\}) \cap \mathsf{B}_X = D$. By maximality, $C = D$. As a consequence, $c \in \mathrm{core}(\mathsf{S}_X)$.
3) If $x \in \mathsf{S}_X \setminus \mathrm{frm}(\mathsf{B}_X)$, then there exists a facet $C \in C_X$ satisfying that $x \in \mathrm{int}_{\mathsf{S}_X}(C)$. In accordance with [10, Lemma 5(5)] and Proposition 3.13(1), $\mathrm{int}_{\mathsf{S}_X}(C) = \mathrm{inn}(C) \subseteq \mathrm{core}(\mathsf{S}_X)$.

□

Notice that there are examples of Banach spaces for which $\mathrm{core}(\mathsf{S}_X) \neq \mathsf{S}_X \setminus \mathrm{frm}(\mathsf{B}_X)$. Indeed, if X is strictly convex and $\dim(X) \geq 2$, then $\mathsf{S}_X = \mathrm{core}(\mathsf{S}_X) = \mathrm{frm}(\mathsf{B}_X)$.

A variation of Tingley's problem was introduced in [12] and it is known as the Mazur-Ulam property. A Banach space X satisfies the Mazur-Ulam property if for an arbitrary Banach space Y, any surjective isometry between the unit spheres of X and Y is the restriction of a surjective linear isometry between the whole spaces. There are plenty of examples of Banach spaces satisfying the Mazur-Ulam property [2, 3, 9, 15, 16, 19–21, 32, 34]. The Mazur-Ulam property motivates the upcoming definition.

We will denote by $\mathscr{B}$ to the class of all real Banach spaces. A subclass $\mathscr{C} \subseteq \mathscr{B}$ is said to be isometric (isomorphic) if $\mathscr{C}$ is invariant under surjective linear isometries (isomorphisms), that is, if $X \in \mathscr{C}$, $Y \in \mathscr{B}$, and $T : X \to Y$ is a surjective linear isometry (isomorphism), then $Y \in \mathscr{C}$.

Definition 3.14 (Mazur-Ulam class)**.** *A subclass $\mathscr{C} \subseteq \mathscr{B}$ is said to be a Mazur-Ulam class if $\mathscr{C}$ is invariant under surjective isometries between unit spheres; that is, if $X \in \mathscr{C}$, $Y \in \mathscr{B}$, and $T : \mathsf{S}_X \to \mathsf{S}_Y$ is a surjective isometry, then $Y \in \mathscr{C}$.*

Notice that every Mazur-Ulam class is an isometric class. By bearing in mind [10, Corollary 6], the class of all strictly convex Banach spaces is a Mazur-Ulam. We will identify more Mazur-Ulam classes of Banach spaces.

Theorem 3.15. *The class of Banach spaces whose unit sphere contains a dense amount of rotund points is a Mazur-Ulam class.*

Proof. Let $\mathscr{C}$ denote the class of Banach spaces whose unit sphere contains a dense amount of rotund points. Let $X \in \mathscr{C}$, $Y \in \mathscr{B}$, and $T : \mathsf{S}_X \to \mathsf{S}_Y$ be a surjective isometry. We will show that $Y \in \mathscr{C}$. Indeed, in virtue of [10, Theorem 14], $T(\mathrm{rot}(\mathsf{B}_X)) = \mathrm{rot}(\mathsf{B}_Y)$; thus, since T is a homeomorphism, $\mathrm{rot}(\mathsf{B}_Y)$ is dense in S_Y. □

In [27], a three-dimensional Banach space is constructed in such a way that its unit sphere consists of extreme points, except for two nontrivial maximal segments (opposite to each other).

Theorem 3.16. *The class of Banach spaces whose unit sphere consists of extreme points, except for two nontrivial maximal segments (opposite to each other), is a Mazur-Ulam class.*

Proof. Let $\mathscr{C}$ denote the class of Banach spaces whose unit sphere consists of extreme points, except for two nontrivial maximal segments (opposite to each other). Let $X \in \mathscr{C}$, $Y \in \mathscr{B}$, and $T : \mathsf{S}_X \to \mathsf{S}_Y$ be a surjective isometry. We will show that $Y \in \mathscr{C}$. In the first place, note that every extreme point of B_X is indeed a rotund point of B_X, except for the four extremes of the two opposite segments. So, essentially, if S and $-S$ denote the opposite nontrivial maximal segments, then $\mathsf{S}_X = \mathrm{rot}(\mathsf{B}_X) \cup S \cup -S$. Also, notice that both S and $-S$ must be maximal faces of B_X. Therefore, by applying [10, Theorem 14], we have that $T(\mathrm{rot}(\mathsf{B}_X)) = \mathrm{rot}(\mathsf{B}_Y)$. In view of [10, Corollary 8], $T(S)$ is both a segment of S_Y and a maximal face of B_Y, and the same goes for $T(-S)$. Next, $\mathrm{inn}(S) \neq \varnothing$ because S is a nontrivial segment; thus, according to [10, Theorem 15(1)], $T(-S) = -S$. Finally, we conclude that $\mathsf{S}_Y = \mathrm{rot}(\mathsf{B}_Y) \cup T(S) \cup -T(S)$, meaning that $Y \in \mathscr{C}$. □

4. Discussion

By looking at the proof of Theorem 3.16, it is noticeable that the class of Banach spaces with a dimension greater than or equal to 3 whose unit sphere consists of extreme points, except for two nontrivial maximal segments (opposite to each other), is contained in the class of Banach spaces whose unit sphere contains a dense amount of rotund points. These two classes, even though they have been proved to be Mazur-Ulam classes in Theorems 3.15 and 3.16, might seem to be small classes; in other words, one could think that there are not many examples of Banach spaces that belong to the previous classes. On the contrary, we will discuss how to possibly construct many examples of Banach spaces whose unit sphere consists of extreme points, except for two nontrivial maximal segments (opposite to each other). We will begin by relying on the following two technical lemmas, which are well known in the literature of Banach space geometry, but whose proof we include for the sake of completeness.

Lemma 4.1. *Let $\alpha_1, \alpha_2, \beta_1, \beta_2$ be positive numbers such that $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in \mathsf{S}^1 = \mathsf{S}_{\ell_2^2}$ and $(\alpha_1\alpha_2, \beta_1\beta_2) \in \mathsf{S}_{\ell_1^2}$, then $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$.*

Proof. Since $1 = \alpha_1\alpha_2 + \beta_1\beta_2$, we have that $(\alpha_1+\alpha_2)^2 + (\beta_1+\beta_2)^2 = 4$; in other words, $\left(\frac{\alpha_1+\alpha_2}{2}, \frac{\beta_1+\beta_2}{2}\right) \in \mathsf{S}^1$. Since S^1 is strictly convex, we conclude the result. □

Lemma 4.2. *Let X and Y be normed spaces. If $(x_1, y_1), (x_2, y_2), \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) \in \mathsf{S}_{X\oplus_2 Y}$, then $\|x_1 + x_2\| = \|x_1\| + \|x_2\|$, $\|x_1\| = \|x_2\|$, $\|y_1 + y_2\| = \|y_1\| + \|y_2\|$, $\|y_1\| = \|y_2\|$. In particular, $\left[\frac{x_1}{\|x_1\|}, \frac{x_2}{\|x_2\|}\right] \subseteq \mathsf{S}_X$ and $\left[\frac{y_1}{\|y_1\|}, \frac{y_2}{\|y_2\|}\right] \subseteq \mathsf{S}_Y$.*

Proof. By Hölder's inequality,

$$\begin{aligned} 4 &= \|x_1 + x_2\|^2 + \|y_1 + y_2\|^2 \\ &\leq \|x_1\|^2 + \|x_2\|^2 + 2\|x_1\|\|x_2\| + \|y_1\|^2 + \|y_2\|^2 + 2\|y_1\|\|y_2\| \\ &= 2 + 2\left(\|x_1\|\|x_2\| + \|y_1\|\|y_2\|\right) \\ &\leq 2 + 2\sqrt{\|x_1\|^2 + \|y_1\|^2}\sqrt{\|x_2\|^2 + \|y_2\|^2} \\ &= 4 \end{aligned}$$

which forces that $\|x_1 + x_2\| = \|x_1\| + \|x_2\|$, $\|y_1 + y_2\| = \|y_1\| + \|y_2\|$ and $\|x_1\|\|x_2\| + \|y_1\|\|y_2\| = 1$. In view of Lemma 4.1, we have that $\|x_1\| = \|x_2\|$ and $\|y_1\| = \|y_2\|$. Finally,

$$\left\| \frac{\|x_1\|}{\|x_1\| + \|x_2\|} \frac{x_1}{\|x_1\|} + \frac{\|x_2\|}{\|x_1\| + \|x_2\|} \frac{x_2}{\|x_2\|} \right\| = 1,$$

so $\left[\frac{x_1}{\|x_1\|}, \frac{x_2}{\|x_2\|}\right] \subseteq \mathsf{S}_X$. In a similar way, it can be shown that $\left[\frac{y_1}{\|y_1\|}, \frac{y_2}{\|y_2\|}\right] \subseteq \mathsf{S}_Y$. □

A direct consequence of Lemma 4.2 is that, under the settings of that lemma, if $x \in \mathrm{ext}(\mathsf{B}_X)$ and $y \in \mathrm{ext}(\mathsf{B}_Y)$, then $\frac{(x,y)}{\sqrt{2}} \in \mathrm{ext}\left(\mathsf{B}_{X \oplus_2 Y}\right)$. Let $\mathscr{C}$ denote the class of Banach spaces whose unit sphere consists of extreme points, except for two nontrivial maximal segments (opposite to each other). If $X \in \mathscr{C}$ and Y is a strictly convex Banach space, then we will show that $X \oplus_2 Y \notin \mathscr{C}$. Notice that if S and $-S$ denote the opposite nontrivial maximal segments of S_X, then we already know from Theorem 3.16 that $\mathsf{S}_X = \mathrm{rot}(\mathsf{B}_X) \cup S \cup -S$. Observe that, in view of Lemma 4.2, $S \times \{0\}$ and $-S \times \{0\}$ are opposite nontrivial maximal segments of $\mathsf{S}_{X \oplus_2 Y}$. Nevertheless, for every $y \in \mathsf{S}_Y$, by relying on Lemma 4.2 again, $\frac{S}{\sqrt{2}} \times \left\{\frac{y}{\sqrt{2}}\right\}$ and $\frac{-S}{\sqrt{2}} \times \left\{\frac{y}{\sqrt{2}}\right\}$ are also opposite nontrivial maximal segments of $\mathsf{S}_{X \oplus_2 Y}$. As a consequence, $X \oplus_2 Y \notin \mathscr{C}$.

5. Conclusions

The core of the unit sphere is a geometric invariant, which is a key factor in understanding the geometry of the unit ball of a real Banach space. It is invariant under surjective isometries of unit spheres and it has strong connections to strict convexity and smoothness in real Banach spaces. It can also be characterized through the index of strong rotundity.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that there is no conflict of interest.

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