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Research article

The eigenvalues of β-Laplacian of slant submanifolds in complex space forms

Lamia Saeed Alqahtani[1,*] **and Akram Ali**[2]

[1] Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia

[2] Department of Mathematics, College of Sciences, King Khalid University, Abha, Saudi Arabia

* **Correspondence:** Email: Lalqahtani@kau.edu.sa.

Abstract: In this paper, we provided various estimates of the first nonzero eigenvalue of the β-Laplacian operator on closed orientated p-dimensional slant submanifolds of a $2m$-dimensional complex space form $\widetilde{\mathbb{V}}^{2m}(4\epsilon)$ with constant holomorphic sectional curvature 4ϵ. As applications of our results, we generalized the Reilly-inequality for the Laplacian to the β-Laplacian on slant submanifolds of a complex Euclidean space and a complex projective space.

Keywords: eigenvalues; elliptical Laplacian operators; complex space forms; slant submanifolds; β-Laplcians operators
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1. Introduction

A crucial part of Riemannian geometry is determining the bound of the eigenvalue of the Laplacian on a particular manifold. The study of eigenvalues, that show up as solutions to the Dirichlet or Neumann boundary value problems for curvature functions, is a key goal of this purpose. Due to the diversity of boundary conditions on a manifold, and from the perspective of the Dirichlet boundary condition, one can consider determining the upper bound of the eigenvalue as a method of locating the proper bound of the Laplacian on the particular manifold. Finding the eigenvalues of the β-Laplace and Laplace operators has attracted more attention in recent years. Now, if the first eigenvalue of the Dirichlet boundary condition is denoted by $\upsilon_1(\Sigma) > 0$ on a complete noncompact Riemannian manifold $\boldsymbol{V}^p$ with the compact domain Σ in $\boldsymbol{V}^p$, then we have

$$\Delta\sigma + \upsilon\sigma = 0 \text{ on } \Sigma \text{ and } \sigma = 0 \text{ on } \partial\Sigma, \tag{1.1}$$

where Δ is the Laplacian on $\boldsymbol{V}^p$ and σ is a nonzero function defined on $\boldsymbol{V}^p$, then, $\upsilon_1(\boldsymbol{V}^p)$ expressed as $inf_{\Sigma}\upsilon_1(\Sigma)$. The Dirichlet eigenvalues are the eigenvalues of the Laplace operator on a domain with Dirichlet boundary conditions. They have many important consequences in various areas of mathematics, including differential geometry, number theory, and mathematical physics. The Dirichlet eigenvalues determine the geometry of a domain. For example, the first Dirichlet eigenvalue of a domain is related to the diameter of the domain. The higher eigenvalues are related to the curvature of the domain and the way it is embedded in Euclidean space. In this sequel, the Dirichlet eigenvalues appear in the solution of the heat equation on a domain. The eigenvalues and the corresponding eigenfunctions determine the solution's decay rate. Also, the Dirichlet eigenvalues are an important tool in spectral theory, which deals with studying the spectrum of operators. The spectrum of the Laplace operator with Dirichlet boundary conditions contains the Dirichlet eigenvalues, and the behavior of the eigenvalues can reveal information about the underlying geometry of the domain. Therefore, it has been studied on a large scale [1–3].

The Reilly formula only applies to the manifold's inherent geometry and most definitely to the particular PDE being examined in the next equation. One can easily comprehend this by the following example: Let $(\boldsymbol{V}^p, g)$ be a compact p-dimensional Riemannian manifold and $\upsilon_1 \neq 0$ is the first eigenvalue of the Neumann boundary condition on $\boldsymbol{V}^p$, and we have

$$\Delta\sigma + \upsilon_1\sigma = 0 \ \text{ on } \boldsymbol{V}^p \quad \text{and} \quad \frac{\partial\sigma}{\partial N} = 0 \quad \text{on } \partial\boldsymbol{V}^p, \tag{1.2}$$

where N is the outward normal on $\partial\boldsymbol{V}^p$. The Neumann eigenvalue problem is a classical mathematical physics problem with a wide range of applications in various fields, including acoustics, electromagnetics, quantum mechanics and fluid dynamics. Also, the Neumann eigenvalues of the Laplacian operator correspond to the energy levels of a quantum mechanical system. This is used, for example, in the study of the Schrödinger equation and the calculation of the electronic structure of molecules. Furthermore, the Neumann eigenvalues of the Laplacian operator correspond to the frequencies of small oscillations of a fluid in a closed container. This is used in the study of fluid dynamics, where the resonant frequencies determine the stability of the fluid flow.

In [4], a result of Reilly proved the following famous upper bound inequality of the Laplacian associated with the first nonzero eigenvalue υ_1^{∇}:

$$\upsilon_1^{\nabla} \leq \frac{p}{Vol(\boldsymbol{V}^p)} \int_{\boldsymbol{V}^p} |\mathcal{H}|^2 dV, \tag{1.3}$$

for a Riemannian submanifold $\boldsymbol{V}^p$ isometrically embedded in the Euclidean space $\mathbb{R}^{2m}$ included the mean curvature $\mathcal{H}$ with dimension denoted by p of $\boldsymbol{V}^p$. In this case, the submanifold $\boldsymbol{V}^p$ is connected, closed, and oriented and the boundary satisfies $\partial\boldsymbol{V}^p = 0$.

The inequality (1.3) leads to the great inspiration for several authors in this field, as they have created such problems for various ambient settings. For example, on Minkowski spaces [3], on closed Riemannian manifolds [5] of the β-Laplacian under integral curvature conditions and on the hyperbolic spaces [6] with some integral conditions imposed on the mean curvature. Also, it was studied on product manifolds [7] of the Hodge Laplacian, on projective spaces [8] for the β-Laplacian that generalized (1.3), on Kaehler manifolds in [9] and for the Wentzel-Laplace operator in Euclidean space [10]. Motivated by literature, the upper bound of the first eigenvalue $\upsilon_1^{\nabla} > 0$ of the Laplacian is established in [4, 11] for connected space form $\boldsymbol{V}^p(\epsilon)$ with constant curvature ϵ and is given as follows:

$$\upsilon_1^{\nabla} \leq \frac{p}{Vol(\boldsymbol{V})} \int_{\boldsymbol{V}^p} \Big(|\mathbb{H}|^2 + \epsilon\Big) dV, \tag{1.4}$$

where $\boldsymbol{V}^p$ is a closed orientable submanifold of dimension p in $\boldsymbol{V}^p(\epsilon)$. It is easy to study that the inequality (1.4) generalized for the Euclidean space $\mathbb{R}^{2m}$ with $\epsilon = 0$, the unit sphere $\mathbb{S}^{2m}(1)$ with $\epsilon = 1$, and the hyperbolic space $\mathbb{H}^{2m}(-1)$ $\epsilon = -1$, respectively. The equality case holds in (1.4) if, and only if, $\boldsymbol{V}^p$ is minimal in a geodesic sphere of radius r_ϵ of $\widetilde{\mathbb{V}}^m(\epsilon)$ with $r_0 = (p/\upsilon_1^{\Delta})^{1/2}, r_1 = \arcsin r_0$, and $r_{-1} = \operatorname{arcsinh} r_0$. Next, inequality (1.4) was extended for the β-Laplacian in [12, 13] as given the results in [4] that assumed expanded applications. Similar results can be found in [1–3, 14–22] through the work of [4]. Now, we are defining the β-Laplacian operator for $\beta > 1$ which satisfies the following differential equation:

$$\Delta_\beta \sigma = div(|\nabla\sigma|^{\beta-2}\nabla\sigma). \tag{1.5}$$

If we substitute $\beta = 2$ in (1.5), then it becomes the usual Laplacian. Similarly, the eigenvalue Λ of Δ_β is as follows:

$$\Delta_\beta \sigma = -\upsilon|\sigma|^{\beta-2}\sigma, \tag{1.6}$$

for the Dirichlet boundary condition (1.1) (or Neumann boundary condition (1.2)).

The first nonzero eigenvalue $\upsilon_{1,\beta}$ of Δ_β on a Riemannian manifold $\boldsymbol{V}^p$ with no boundary demonstrates the variational characteristic of the Rayleigh type [23]:

$$\upsilon_{1,\beta} = \inf\left\{ \frac{\int_{\boldsymbol{V}} |\nabla\sigma|^\beta}{\int_{\boldsymbol{V}} ||\sigma||^\beta} |\sigma \in W^{1,\beta}(\boldsymbol{V}^p)\ \{0\}, \int_{\boldsymbol{V}} |\sigma|^{\beta-2}\sigma = 0 \right\}. \tag{1.7}$$

The elliptical β-Laplacian is a nonlinear generalization of the standard Laplace operator that arises in various areas of mathematics and physics. It is a partial differential operator that appears in the study of nonlinear elliptic equations. It is used to model various physical phenomena, such as the behavior of fluids, electromagnetism, and elasticity. Moreover, the standard Laplace operator is linear, and the elliptical β-Laplace operator is nonlinear. It exhibits a wide range of interesting and complex behavior. It is given the name β-Laplacian operator because it involves the β^{th} power of the gradient of a function. Therefore, the study of the elliptical β-Laplacian operator is an active area of research in both pure and applied mathematics. It has important applications in fields such as engineering, physics, and biology. Hence, influenced by the studies in [10, 12, 13], we provide a sharp estimate to the first nonzero eigenvalue of the β-Laplacian on a slant submanifold $\boldsymbol{V}^p$ of a complex space form $\widetilde{\mathbb{V}}^{2m}(4\epsilon)$. Now, we announce our first result:

Theorem 1.1. *Let* $\mathbf{V}^p$ *be a* $(p \geq 2)$*-dimensional closed orientated slant submanifold of an m-dimensional complex space form* $\widetilde{\mathbb{V}}^{2m}(4\epsilon)$*. The first nonzero eigenvalue* $\upsilon_{1,\beta}$ *of the* β*-Laplacian satisfies*

$$\upsilon_{1,\beta} \leq \frac{(2m+1)^{(1-\frac{\beta}{2})} p^{\frac{\beta}{2}}}{(Vol(\mathbf{V}))^{\beta/2}} \left\{ \int_{\mathbf{V}} \Big(\epsilon + \frac{3\epsilon\cos^2\theta}{(p-1)} + |\mathcal{H}|^2\Big) dV \right\}^{\beta/2}, \quad for\ \ 1 < \beta \leq 2, \tag{1.8}$$

$$and \quad \upsilon_{1,\beta} \leq \frac{(2m+1)^{(\frac{\beta}{2}-1)} p^{\frac{\beta}{2}}}{Vol(\mathbf{V})} \int_{\mathbf{V}} \Big(\epsilon + \frac{3\epsilon\cos^2\theta}{(p-1)} + |\mathcal{H}|^2\Big)^{\beta/2} dV, \quad for\ \ 2 < \beta \leq \frac{p}{2} + 1, \tag{1.9}$$

where $\mathcal{H}$ is the mean curvature vector of $\mathbf{V}^p$ in $\mathbb{V}^{2m}(4\epsilon)$ and $Vol(\mathbf{V})$ is the volume of $\mathbf{V}^p$. Moreover, the equality holds if, and only if, $\beta = 2$ and $\mathbf{V}^p$ is minimally immersed in a geodesic sphere of radius r_ϵ of $\widetilde{\mathbb{V}}^{2m}(4\epsilon)$, with $r_0 = (p/\upsilon_1^\Delta)^{1/2}$, $r_1 = \arcsin r_0$, and $r_{-1} = arcsinh\, r_0$.

An immediate application of Theorem 1.1 for $\epsilon = 0$ is the complex Euclidean space; that is, for the scalar flat case, then:

Corollary 1.1. *Let $\mathbf{V}^p$ be a $(p \geq 2)$-dimensional closed orientated slant submanifold of an m-dimensional complex Euclidean space $\mathbb{R}^{2m}$, then the first nonzero eigenvalue $\upsilon_{1,\beta}$ of the β-Laplacian satisfies*

$$\upsilon_{1,\beta} \leq \frac{(2m+1)^{(1-\frac{\beta}{2})} p^{\frac{\beta}{2}}}{(Vol(\mathbf{V}))^{\beta/2}} \left(\int_{\mathbf{V}} |\mathcal{H}|^2 dV \right)^{\frac{\beta}{2}} \quad for \;\; 1 < \beta \leq 2, \tag{1.10}$$

$$and \;\; \upsilon_{1,\beta} \leq \frac{(2m+1)^{(\frac{\beta}{2}-1)} p^{\frac{\beta}{2}}}{Vol(\mathbf{V})} \int_{\mathbf{V}} |\mathcal{H}|^\beta dV \quad for \;\; 2 < \beta \leq \frac{p}{2} + 1. \tag{1.11}$$

Remark 1.1. Inequalities (1.10) and (1.11) characterized by $\beta = 2$ generalize the Reilly-type inequality (1.3). In other words, the estimates of Reilly-type for the first eigenvalue of the Laplacian in [4] are defined to be cases of the results in Theorem 1.1 for $\epsilon = 0$ and $\beta = 2$.

The next result we will state as a particular version of Theorem 1.1. The following result is obtained precisely by substituting $\epsilon = 1$ in (1.8) and (1.9).

Corollary 1.2. *Let $\mathbf{V}^p$ be a $(p \geq 2)$-dimensional closed orientated slant submanifold in an m-dimensional complex projective space $\mathbb{C}P^{2m}(4)$, then, the first nonzero eigenvalue $\upsilon_{1,\beta}$ of the β-Laplacian satisfies*

$$\upsilon_{1,\beta} \leq \frac{(2m+1)^{(1-\frac{\beta}{2})} p^{\frac{\beta}{2}}}{(Vol(\mathbf{V}))^{\beta/2}} \left(\int_{\mathbf{V}} \left(1 + \frac{3\cos^2\theta}{(p-1)} + |\mathcal{H}|^2\right) dV \right)^{\frac{\beta}{2}}, \quad for \;\; 1 < \beta \leq 2, \tag{1.12}$$

$$and \;\; \upsilon_{1,\beta} \leq \frac{(2m+1)^{(\frac{\beta}{2}-1)} p^{\frac{\beta}{2}}}{Vol(\mathbf{V})} \int_{\mathbf{V}} \left(1 + \frac{3\cos^2\theta}{(p-1)} + |\mathcal{H}|^2\right)^{\frac{\beta}{2}} dV, \quad for \;\; 2 < \beta \leq \frac{p}{2} + 1. \tag{1.13}$$

This paper is organized as follows. In Section 2, we recall the structure equations of a slant submanifold $\boldsymbol{V}^p$ in $\mathbb{V}^{2m}(4\epsilon)$. Also, we show the consequences of change on some geometric quantities due to changing the metric on $\mathbb{V}^{2m}(4\epsilon)$ under the conformal transformation. In Section 3, we prove Theorem 1.1. Furthermore, as the method in [8] does not work for $\epsilon = -1$, we find suitable test functions to estimate the upper bound of $\upsilon_{1,\beta}$ by conformal transformation to a unit sphere.

2. Preliminaries

Let $\widetilde{\mathbb{V}}^{2m}(4\epsilon)$ be a complex space form of constant holomorphic sectional curvature 4ϵ endowed bt the Kaehler manifold, then, the curvature tensor $\widetilde{R}$ of $\widetilde{\mathbb{V}}^{2m}(4\epsilon)$ can be expressed as

$$\begin{aligned}\widetilde{R}(\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \mathcal{U}_4) =& \epsilon \Big\{ g(\mathcal{U}_1, \mathcal{U}_3) g(\mathcal{U}_2, \mathcal{U}_4) - g(\mathcal{U}_2, \mathcal{U}_3) g(\mathcal{U}_1, \mathcal{U}_4) + g(\mathcal{U}_1, \mathcal{J}\mathcal{U}_3) g(\mathcal{J}\mathcal{U}_2, \mathcal{U}_4) \\ & - g(\mathcal{U}_2, \mathcal{J}\mathcal{U}_3) g(\mathcal{J}\mathcal{U}_1, \mathcal{U}_4) + 2g(\mathcal{U}_1, \mathcal{J}\mathcal{U}_2) g(\mathcal{J}\mathcal{U}_3, \mathcal{U}_4) \Big\},\end{aligned} \tag{2.1}$$

for any $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \mathcal{U}_4 \in \Gamma(T\widetilde{\mathbb{V}}^{2m})$. A p-dimensional Riemannian submanifold $\boldsymbol{V}^p$ of $\widetilde{\mathbb{V}}^{2m}(4\epsilon)$ is called totally real if the standard complex structure $\mathcal{J}$ of $\widetilde{\mathbb{V}}^{2m}(4\epsilon)$ maps any tangent space of $\boldsymbol{V}^p$ into the corresponding normal space [24].

Definition 2.1. Let $\boldsymbol{V}^p$ be a Riemannian submanifold of a K*ä*hler manifold $\widetilde{\mathbb{V}}^{2m}$, then $\boldsymbol{V}^p$ is a real submanifold if $\mathcal{J}(T\boldsymbol{V}) \subset T\boldsymbol{V}$ and $\boldsymbol{V}^p$ is a complex submanifold if $\mathcal{J}(T\boldsymbol{V}) \subset T\boldsymbol{V}^{\perp}$.

Slant submanifolds are a class of submanifolds in Riemannian geometry that satisfy a certain condition related to the angle between their tangent spaces and a fixed complex structure. A slant submanifold $\boldsymbol{V}^p$ of a complex space form $\widetilde{\mathbb{V}}^{2m}$ is a submanifold $\boldsymbol{V}^p$ of $\widetilde{\mathbb{V}}^{2m}$, such that the angle between the tangent space $T\boldsymbol{V}^p$ and the complex structure $\mathcal{J}$ of $\boldsymbol{V}^p$ satisfies the equation $\cos^2\theta + \sin^2\theta = \kappa$, where κ is a constant between zero and one. Here, θ is the angle between the tangent space $T\boldsymbol{V}^p$ and the complex structure $\mathcal{J}$. There are several ways to classify slant submanifolds according to their geometry. One common classification is based on the shape of the mean curvature vector. In particular, slant submanifolds can be classified as follows. The classification of slant submanifolds is an active area of research in differential geometry, with many open questions and directions for further study. Let $\mathbb{U} \in \Gamma(T\mathbb{V})$, and we have

$$\mathcal{J}\mathbb{U} = \mathcal{P}\mathbb{U} + \mathcal{F}\mathbb{U}, \tag{2.2}$$

where $\mathcal{P}\mathbb{U}$ and $\mathcal{F}\mathbb{U}$ are tangential and normal components of $\mathcal{J}\mathbb{U}$. It is known that $\boldsymbol{V}^p$ is a slant submanifold of $\widetilde{\mathbb{V}}$ if, and only if,

$$\mathcal{P}^2 = \kappa I \tag{2.3}$$

for some $\lambda \in [-1, 0]$ (see [25]), where I denotes the identity transformation of $T\boldsymbol{V}$. Moreover, if $\boldsymbol{V}^p$ is a slant submanifold and θ is the slant angle of $\boldsymbol{V}^p$, then $\kappa = -\cos^2\theta$. Thus, we obtain the following characterization theorem.

Lemma 2.1. *Let* $\mathbf{V}^p$ *be a slant submanifold of a Kaehler manifold* $\widetilde{V}^{2m}$*,*

$$g(\mathcal{P}\mathcal{U}_1, \mathcal{P}\mathcal{U}_2) = \cos^2\theta g(\mathcal{U}_1, \mathcal{U}_2), \tag{2.4}$$

$$g(\mathcal{F}\mathcal{U}_1, \mathcal{F}\mathcal{U}_2) = \sin^2\theta g(\mathcal{U}_1, \mathcal{U}_2), \tag{2.5}$$

for $\mathcal{U}_1, \mathcal{U}_2 \in \Gamma(\mathcal{D}^\theta)$.

Using the moving frame method, we recall some well-known facts about submanifold geometry and conformal geometry. On indices other than special declarations, we use convection as follows:

$$1 \le i, j, k, \cdots \le p, \quad p+1 \le \alpha, q, \gamma, \cdots \le 2m, \quad 1 \le a, b, c, \cdots \le 2m.$$

2.1. Structure equations for slant submanifolds

Following the same method as appeared in [26], by submitting $\mathcal{U}_1 = \mathcal{U}_3 = v_i$ and $\mathcal{U}_2 = \mathcal{U}_4 = v_j$ in (2.1), and taking the trace of Riemannian metric with v_i, we have

$$\begin{aligned}\widetilde{R}(v_i, v_j, v_i, v_j) = \epsilon\Big\{ & g(v_i, v_i)g(v_j, v_j) - g(v_i, v_j)g(v_i, v_j) + g(v_i, \mathcal{J}v_j)g(\mathcal{J}v_j, v_i) \\ & - g(v_i, \mathcal{J}v_i)g(v_j, \mathcal{J}v_j) + 2g^2(\mathcal{J}v_j, v_i)\Big\}.\end{aligned} \tag{2.6}$$

Taking the summation in (2.6) over the basis vector fields of $T\boldsymbol{V}^p$ such that $1 \leq i \neq j \leq p$, one shows that

$$2\widetilde{\tau}(T\boldsymbol{V}^p) = \epsilon\Big\{p(p-1) + 3\sum_{1\leq i\neq j\leq p} g^2(\mathcal{J}v_i, v_j)\Big\}. \tag{2.7}$$

Thus, it is easily seen that for a slant submanifold $\boldsymbol{V}^p$

$$\sum_{i,j=1}^{p} g^2(\mathcal{P}v_i, v_j) = p\cos^2\theta. \tag{2.8}$$

From (2.7) and (2.8), it follows that

$$2\widetilde{\tau}(T\boldsymbol{V}^p) = \epsilon\Big\{p(p-1) + 3p\cos^2\theta\Big\}. \tag{2.9}$$

Take a trace of the above equation. Implementing Eqs (2.2) to (2.5) and by the Gauss equation for a slant submanifold in a complex space form $\widetilde{\mathbb{V}}^{2m}(4\epsilon)$ that is defined in detail [26], we get

$$\mathcal{R} = \epsilon\Big\{p(p-1) + 3p\cos^2\theta\Big\} + p^2|\mathcal{H}|^2 - S, \tag{2.10}$$

where $\mathcal{R}$ is the scalar curvature of $\boldsymbol{V}^p$, $S = \sum_{\alpha,i,j}(\sigma_{ij}^{\alpha})^2$ is the squared norm of the second fundamental form, and $\mathcal{H} = \sum_{\alpha}\mathcal{H}^{\alpha}v_{\alpha} = \frac{1}{p}\sum_{\alpha}(\sum_{i}\sigma_{ii}^{\alpha})v_{\alpha}$ is the mean curvature vector of $\boldsymbol{V}^p$.

2.2. Conformal relations

Although these relations are well-known (cf. [26–28]), we will use directly all related equations from [26] for the curvature and the second fundamental form change under conformal transformations. We have

$$\begin{aligned} e^{2\lambda}\widetilde{\mathcal{R}}_{ijkl} =& \mathcal{R}_{ijkl} - \Big(\lambda_{ik}\delta_{jl} + \lambda_{jl}\delta_{ik} - \lambda_{il}\delta_{jk} - \lambda_{jk}\delta_{il}\Big) \\ &+ \Big(\lambda_i\lambda_k\delta_{jl} + \lambda_j\lambda_l\delta_{ik} - \lambda_j\lambda_k\delta_{il} - \lambda_i\lambda_l\delta_{jk}\Big) - |\nabla_{\alpha}|^2\Big(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\Big), \end{aligned} \tag{2.11}$$

where λ_{α} is the covariant derivative of λ with respect to v_{α} and $|\nabla_{\alpha}|^2$ stands for the norm of Levi-Civita with respect to indices α. By pulling back to $\boldsymbol{V}^p$ by x, we have

$$\tilde{\sigma}_{ij}^{\alpha} = e^{-\lambda}(\sigma_{ij}^{\alpha} - \lambda_{\alpha}\delta_{jl}), \text{ and } \widetilde{\mathcal{H}}^{\alpha} = e^{-\alpha}(\widetilde{\mathcal{H}}^{\alpha} - \lambda_{\alpha}). \tag{2.12}$$

From this, it is easy to obtain the useful relation

$$e^{2\lambda}(\widetilde{S} - p|\widetilde{\mathcal{H}}|^2) = S - p|\mathcal{H}|^2. \tag{2.13}$$

3. Proof of Theorem 1.1

Here, we demonstrate the proof of Theorem 1.1, which is stated in part one of this paper. We begin by outlining some fundamental formulas, and we offer a key lemma that is relevant to the study and is motivated by [1, 12, 13, 26].

Lemma 3.1. *Let $x : \mathbf{V}^p \to \widetilde{\mathbb{V}}^{2m}(4\epsilon)$ be the immersion from a p-dimensional closed-oriented slant submanifold to a complex space form $\widetilde{\mathbb{V}}^{2m}(4\epsilon)$, then, for $\beta > 1$, there exists a regular conformal map $\Gamma : \widetilde{\mathbb{V}}^{2m}(4\epsilon) \to \mathbb{C}P^{2m}(4)$ such that the immersion $\varphi = \Gamma \circ x = (\varphi^1, \cdots \varphi^{2m+1})$ satisfies*

$$\int_{\mathbf{V}^p} |\varphi^a|^{\beta-2} \varphi^a dV_M = 0, \quad a = 1, \cdots 2m+1, \tag{3.1}$$

where the manifold $\mathbb{C}P^{2m}(4)$ carries a natural metric by the Hopf fibration $\pi : \mathbb{S}^{(2m-1)} \subset \mathbb{R}^{(2m-1)+1} \longrightarrow \mathbb{C}P^{2m}(4)$.

Proof. The main idea of Lemma 3.1 originates from $\beta = 2$ in (cf. [12, 13, 26, 29, 30]). The detailed proof from above is given in [13]. □

The test function in Lemma 3.1 [1] provides an upper bound on $\upsilon_{1,\beta}$ based on the conformal function.

Lemma 3.2. *Let $\mathbf{V}^p$ be a ($p \geq 2$)-dimensional closed orientated slant submanifold of a $\mathbf{V}^p$-dimensional complex space form $\widetilde{\mathbb{V}}^{2m}(4\epsilon)$. Let Υ_ε denote he standard metric on $\widetilde{\mathbb{V}}^{2m}(4\epsilon)$ and $\Gamma^*\Upsilon_1 = e^{2\lambda}\Upsilon_\varepsilon$, where Γ is the conformal map in Lemma 3.1. We have for all $\beta > 1$,*

$$\upsilon_{1,\beta} Vol(\mathbf{V}^p) \leq (2m+1)^{|1-\frac{\beta}{2}|} p^{\frac{\beta}{2}} \int_{\mathbf{V}} (e^{2\lambda})^{\frac{\beta}{2}} dV. \tag{3.2}$$

Proof. With Lemma 3.1 in mind and φ^a as the test function, then

$$\upsilon_{1,\beta} \int_{\mathbf{V}^p} |\varphi^a|^\beta \leq |\nabla \varphi^a|^\beta dV, \quad 1 \leq a \leq 2m+1. \tag{3.3}$$

Note that $\sum\limits_{a=1}^{2m+1} |\varphi^a|^2 = 1$, then $|\varphi^a| \leq 1$. Thus, we arrive at

$$\sum_{a=1}^{2m+1} |\nabla \varphi^a|^2 = \sum_{i=1}^{p} |\nabla_{v_i} \varphi|^2 = p e^{2\lambda}. \tag{3.4}$$

Considering $1 < \beta \leq 2$, then we derive

$$|\varphi^a|^2 \leq |\varphi^a|^\beta. \tag{3.5}$$

Using (3.3)–(3.5) and the Hölder inequality, we find

$$\upsilon_{1,\beta} Vol(M) = \upsilon_{1,\beta} \sum_{a=1}^{2m+1} \int_V |\varphi^a|^2 dV \leq \upsilon_{1,\beta} \sum_{a=1}^{2m+1} \int_V |\varphi^a|^\beta dV$$

$$\leq \upsilon_{1,\beta} \int_{\boldsymbol{V}} \sum_{a=1}^{2m+1} |\nabla\varphi^a|^\beta dV \leq (2m+1)^{1-\beta/2} \int_{\boldsymbol{V}} \Big(\sum_{a=1}^{p} |\nabla\varphi^a|^2 \Big)^{\frac{\beta}{2}} dV$$
$$= (2m+1)^{1-\frac{\beta}{2}} \int_{\boldsymbol{V}} \left(pe^{2\lambda}\right)^{\frac{\beta}{2}} dV.$$

It implies that (3.2) must be true. However, if we choose $\beta \geq 2$, it implies that the Holder equality can be proved.

$$1 = \sum_{a=1}^{2m+1} |\varphi^a|^2 \leq (2m+1)^{1-\frac{2}{\beta}} \left(\sum_{a=1}^{2m+1} |\varphi^a|^\beta \right)^{\frac{2}{\beta}}, \tag{3.6}$$

from which we obtain

$$\upsilon_{1,\beta} Vol(\boldsymbol{V}^p) \leq (2m+1)^{\frac{\beta}{2}-1} \Big(\sum_{a=1}^{2m+1} \upsilon_{1,\beta} \int_{\boldsymbol{V}^p} |\varphi^a|^\beta dV \Big). \tag{3.7}$$

Also, by Minkowski's inequality, we have

$$\sum_{a=1}^{2m+1} |\nabla\varphi^a|^\beta \leq \left(\sum_{a=1}^{2m+1} |\nabla\varphi^a|^2 \right)^{\frac{\beta}{2}} = \left(pe^{2\lambda}\right)^{\frac{\beta}{2}}. \tag{3.8}$$

Hence (3.2) follows from (3.3), (3.7) and (3.8). This completes the proof of the lemma. □

Now, we can demonstrate the proof of Theorem 1.1.

3.1. Proof of Theorem 1.1

We begin with the case $1 < \beta \leq 2$. By using Lemma 3.2 and the Hölder inequality, we have

$$\upsilon_{1,\beta} Vol(\boldsymbol{V}^p) \leq (2m+1)^{1-\frac{\beta}{2}} p^{\frac{\beta}{2}} \int_{\boldsymbol{V}} \left(e^{2\lambda}\right)^{\frac{\beta}{2}} dV$$
$$\leq (2m+1)^{1-\frac{\beta}{2}|} p^{\frac{\beta}{2}} (Vol(\boldsymbol{V}))^{1-\frac{\beta}{2}} \left(\int_{\boldsymbol{V}} e^{2\lambda} dV \right)^{\frac{\beta}{2}}.$$

Note that we can compute $e^{2\lambda}$ using the conformal relations and the Gauss equation as follows:

We assume that $\widetilde{\mathbb{V}}^{2m} = \widetilde{\mathbb{V}}^{2m}(4\varepsilon)$, $\tilde{g} = \Upsilon_\epsilon$, $\tilde{\tilde{g}} = \Gamma^*\Upsilon_1$ in previous. From (2.10), the Gauss equations for the immersion x and the slant immersion $\varphi = \Gamma \circ x$, respectively, are:

$$\mathcal{R} = \epsilon \Big\{ p(p-1) + 3p\cos^2\theta \Big\} + p(p-1)|\mathcal{H}|^2 + (p|\mathcal{H}|^2 - S), \tag{3.9}$$

$$\tilde{\mathcal{R}} = \Big\{ p(p-1) + 3p\cos^2\theta \Big\} + p(p-1)|\tilde{\mathcal{H}}|^2 + (p|\tilde{\mathcal{H}}|^2 - \tilde{S}). \tag{3.10}$$

Tracing (2.11), it can be found that

$$e^{2\lambda}\tilde{\mathcal{R}} = \mathcal{R} - (p-2)(p-1)|\nabla_\lambda|^2 - 2(p-1)\Delta_\lambda, \tag{3.11}$$

which substituted jointly with (3.9) and (3.10) into (3.11) gives

$$\begin{aligned}&e^{2\lambda}\Big\{p(p-1)+3p\cos^2\theta+p(p-1)|\tilde{\mathcal{H}}|^2+(p|\tilde{\mathcal{H}}|^2-\tilde{S})\Big\}\\=&\epsilon\Big\{p(p-1)+3p\cos^2\theta\Big\}+p(p-1)|\mathcal{H}|^2+(p|\mathcal{H}|^2-S)-(p-2)(p-1)|\nabla_\lambda|^2-2(p-1)\Delta_\lambda.\end{aligned}$$

From this, it follows that

$$\begin{aligned}&p(p-1)\Big\{\Big(e^{2\lambda}-\epsilon\Big\{1+\frac{3\cos^2\theta}{p-1}\Big\}\Big)+(e^{2\lambda}|\tilde{\mathcal{H}}|^2-|\mathcal{H}|^2)\Big\}\\=&p(|\mathcal{H}|^2-e^{2\lambda}|\widetilde{\mathcal{H}}|^2)+e^{2\lambda}\tilde{S}-S-(p-2)(p-1)|\nabla_\lambda|^2-2(p-1)\Delta_\lambda.\end{aligned}\tag{3.12}$$

Now, from (2.12) and (2.13), we derive

$$\begin{aligned}&p(p-1)\Big(e^{2\lambda}-\epsilon\Big\{1+\frac{3\cos^2\theta}{p-1}\Big\}\Big)+p(p-1)\sum_\alpha(\mathcal{H}^\alpha-\lambda_\alpha)^2-p(p-1)|\mathcal{H}|^2\\=&-(p-2)(p-1)|\nabla_\lambda|^2-2(p-1)\Delta_\lambda.\end{aligned}$$

Multiplying with $\frac{1}{p(p-1)}$ in the proceeding equation, we imply that

$$e^{2\lambda}=\Big(\epsilon\Big\{1+\frac{3\cos^2\theta}{p-1}\Big\}+|\mathcal{H}|^2\Big)-\frac{2\Delta_\lambda}{\epsilon\Big\{1+\frac{3\cos^2\theta}{p-1}\Big\}}-\frac{p-2}{p}|\Delta_\lambda|^2-|(\bar{\nabla}_\lambda)^\perp-\mathcal{H}|^2.\tag{3.13}$$

By integration, it is not difficult to check that

$$\begin{aligned}\upsilon_{1,\beta}\,Vol(\boldsymbol{V}^p)&\le(2m+1)^{1-\frac{\beta}{2}|}p^{\frac{\beta}{2}}(Vol(\boldsymbol{V}^p))^{1-\frac{\beta}{2}}\left(\int_{\boldsymbol{V}}e^{2\lambda}dV\right)^{\frac{\beta}{2}}\\&\le\frac{(2m+1)^{1-\frac{\beta}{2}|}p^{\frac{\beta}{2}}}{(Vol(\boldsymbol{V}))^{\frac{\beta}{2}-1}}\left\{\int_{\boldsymbol{V}}\Big(\epsilon\Big\{1+\frac{3\cos^2\theta}{p-1}\Big\}+|\mathcal{H}|^2\Big)dV\right\}^{\frac{\beta}{2}}.\end{aligned}$$

This is equivalent to (1.8), as we wanted to prove.

For the cas $\beta>2$, we cannot use $\int_{\boldsymbol{V}}(e^{2\lambda})$ to govern $\int_{\boldsymbol{V}}(e^{2\lambda})^{\frac{\beta}{2}}$ by applying the Hölder inequality directly. Instead by multiplying $e^{(\beta-2)\lambda}$ on both sides of (3.13), and then integrating on $\boldsymbol{V}^p$ (cf. [31]), we obtain

$$\begin{aligned}\int_{\boldsymbol{V}}e^{\beta\lambda}dV&\le\int_{\boldsymbol{V}}\Big(\epsilon\Big\{1+\frac{3\cos^2\theta}{p-1}\Big\}+|\mathcal{H}|^2\Big)e^{(\beta-2)\lambda}dV-\int_{\boldsymbol{V}}\Big(\frac{p-2-2\beta+4}{p}\Big)e^{(\beta-2)}|\Delta_\lambda|^2dV\\&\le\int_{\boldsymbol{V}}\Big(\epsilon\Big\{1+\frac{3\cos^2\theta}{p-1}\Big\}+|\mathcal{H}|^2\Big)e^{(\beta-2)\lambda}dV.\end{aligned}\tag{3.14}$$

Next, following from the assumption that $\beta\ge2\beta-2$, we apply Young's inequality, then

$$\int_{\boldsymbol{V}}\Big(\epsilon\Big\{1+\frac{3\cos^2\theta}{p-1}\Big\}+|\mathcal{H}|^2\Big)e^{(\beta-2)\lambda}dV\le\frac{2}{\beta}\int_{\boldsymbol{V}}\Big(\Big|\epsilon\Big\{1+\frac{3\cos^2\theta}{p-1}\Big\}+|\mathcal{H}|^2\Big|\Big)^{\frac{\beta}{2}}dV+\frac{(\beta-2)}{\beta}\int_{\boldsymbol{V}}e^{\frac{\beta}{\lambda}}dV.\tag{3.15}$$

From this, we deduce that

$$\int_V e^{\beta\lambda} dV \leq \int_V \left(\left|\epsilon\left\{1 + \frac{3\cos^2\theta}{p-1}\right\} + |\mathcal{H}|^2\right|\right)^{\frac{\beta}{2}} dV, \tag{3.16}$$

and from (3.14) and (3.15) and by using (3.16) in (3.2), we obtain (1.9).

Now, suppose the equality holds in (1.8), then by considering the cases in (3.3) and (3.5), we get

$$\begin{aligned} |\varphi^a|^2 =& |\varphi^a|^\beta, \\ \Delta_\beta \varphi^a =& - \upsilon_{1,\beta} |\varphi^a|^{\beta-2} \varphi^a, \end{aligned}$$

for each $a = 1, \cdots, 2m+1$. If $1 < \beta < 2$, then $|\varphi^a| = 0$ or 1. However, $\sum_{a=1}^{2m+1} |\varphi^a|^2 = 1$, so there is exactly one a such that $|\varphi^a| = 1$, then $\upsilon_{1,\beta} = 0$, which is a contradiction. Hence, $\beta = 2$, and it reduces to the Laplacian case, then, we are in a position to use the theorem in [4, 8].

Suppose that the equality holds in (1.9) and $\beta > 2$, then (3.7) and (3.8) must become equalities, which means that

$$|\varphi^1|^\beta = \cdots = |\varphi^{2m+1}|^\beta,$$

and so, there exists some a such that $|\nabla\varphi^a| = 0$. This means that φ^a is constant and $\upsilon_{1,\beta} = 0$, which leads to a contradiction that $\upsilon_{1,\beta}$ is a nonzero eigenvalue. As a result, the theorem has been proved.

The Reilly inequality (1.3) is generalized now to all β-Laplacian expressions.

Remark 3.1. In the case of $\beta = 2$, the corollary is recovered.

Corollary 3.1. *Let* $\mathbf{V}^p$ *be a* ($p \geq 2$)*-dimensional closed orientated slant submanifold of an m-dimensional complex space form* $\widetilde{\mathbb{V}}^{2m}(4\epsilon)$*, then, the first nonzero eigenvalue* υ_1^Δ *of the Laplacian satisfies*

$$\upsilon_1^\Delta \leq \frac{p}{Vol(\mathbf{V})} \int_{\mathbf{V}} \left(\epsilon + \frac{3\epsilon\cos^2\theta}{(p-1)} + |\mathcal{H}|^2\right) dV. \tag{3.17}$$

Moreover, the equality holds in (3.17) *if, and only if,* $\mathbf{V}^p$ *is minimally immersed in a geodesic sphere of radius* r_ϵ *of* $\widetilde{\mathbb{V}}^{2m}(4\epsilon)$ *with* $r_0 = (p/\upsilon_1^\Delta)^{1/2}$, $r_1 = \arcsin r_0$ *and* $r_{-1} = arcsinh\, r_0$.

Remark 3.2. By assuming that $1 < \beta \leq 2$, we have $\frac{\beta}{2(\beta-1)} \geq 1$, then, the by Hölder inequality, we have

$$\begin{aligned} &\int_V \left(\epsilon + \frac{3\epsilon\cos^2\theta}{(p-1)} + |\mathcal{H}|^2\right) dV \\ \leq &(Vol(\boldsymbol{V}))^{1-\frac{2(\beta-1)}{\beta}} \left(\int_M \left(\epsilon + \frac{3\epsilon\cos^2\theta}{(p-1)} + |\mathcal{H}|^2\right)^{\frac{\beta}{2(\beta-1)}} dV\right)^{\frac{2(\beta-1)}{\beta}}. \end{aligned} \tag{3.18}$$

The upper bound in (1.8) is better than the upper bound given in Theorem 1.5 in [8] for $\epsilon = 1$.

Inspired by Remark 3.2, we provide the following result.

Corollary 3.2. *Let $\mathbf{V}^p$ be a $(p \geq 2)$-dimensional closed orientated slant submanifold of an m-dimensional complex space form $\widetilde{\mathbb{V}}^{2m}(4\epsilon)$. The first nonzero eigenvalue $\upsilon_{1,\beta}$ of the β-Laplacian satisfies*

$$\upsilon_{1,\beta} \leq \frac{(2m+1)^{(1-\frac{\beta}{2})} p^{\frac{\beta}{2}}}{(Vol(\mathbf{V}))^{(\beta-1)}} \left(\int_{\mathbf{V}} \left(\epsilon \Big\{ 1 + \frac{3\cos^2\theta}{\beta - 1} \Big\} + |\mathcal{H}|^2 \right)^{\frac{\beta}{2(\beta-1)}} dV \right)^{(\beta-1)} \tag{3.19}$$

for $1 < \beta \leq 2$.

Proof. From (1.8) and (3.18), we get the required result. □

4. Conclusions

The eigenvalues of elliptic Laplace operators, also known as the Laplace-Beltrami operator, have many applications in mathematics and physics. Provided are a few examples in the main three areas. Geometrically, the eigenvalues of the Laplace-Beltrami operator on a Riemannian manifold are closely related to the geometry of the manifold. In particular, the first eigenvalue is related to the size and curvature of the manifold. In the spectral theory, the eigenvalues of the Laplace-Beltrami operator can be used to study the spectrum of other differential operators on the same manifold. For example, the eigenvalues of the Laplace-Beltrami operator on a surface can be used to study the spectrum of the Dirac operator on the same surface. Lastly, in physics, the Laplace-Beltrami operator appears in many physical problems, such as the study of heat flow, electrostatics, and quantum mechanics. In particular, the eigenvalues of the Laplace-Beltrami operator on a bounded domain are important in the study of the eigenfunctions of the Schrödinger equation [23,32]. The Dirichlet eigenvalues have connections to the distribution of prime numbers. The Riemann hypothesis, one of the most famous unsolved problems in mathematics, is closely related to the eigenvalues of a certain operator, called the Riemann zeta function operator, which is related to the Dirichlet eigenvalues. It also can be used to solve inverse problems, such as determining the shape of a domain from its Dirichlet-to-Neumann map. This has applications, for example, in medical imaging, where the shape of an organ can be determined from measurements of the electromagnetic fields it produces. Next, the Neumann eigenvalues encode important geometric and analytic information about the underlying manifold and domain. For example, they are related to the isoperimetric inequality, the spectrum of the Laplace-Beltrami operator on the entire manifold, and the asymptotic behavior of heat kernels. It is a well-studied topic in spectral geometry, and there are many results concerning the existence, uniqueness, and asymptotic behavior of Neumann eigenfunctions and eigenvalues. In particular, the Courant-Friedrichs-Lewy (CFL) inequality implies that the Neumann eigenvalues grow at least linearly concerning the index, and Weyl's law gives an asymptotic estimate for the counting function of the Neumann eigenvalue. Overall, the Dirichlet, Neumann, and Laplace-Beltrami operator eigenvalue problems have many important applications in mathematical physics and provide a powerful tool for understanding the behavior of physical systems [33–36].

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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