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Research article

A decision-making framework based on the Fermatean hesitant fuzzy distance measure and TOPSIS

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Abstract: A particularly useful assessment tool for evaluating uncertainty and dealing with fuzziness is the Fermatean fuzzy set (FFS), which expands the membership and non-membership degree requirements. Distance measurement has been extensively employed in several fields as an essential approach that may successfully disclose the differences between fuzzy sets. In this article, we discuss various novel distance measures in Fermatean hesitant fuzzy environments as research on distance measures for FFS is in its early stages. These new distance measures include weighted distance measures and ordered weighted distance measures. This justification serves as the foundation for the construction of the generalized Fermatean hesitation fuzzy hybrid weighted distance (D_{GFHFHWD}) scale, as well as the discussion of its weight determination mechanism, associated attributes and special forms. Subsequently, we present a new decision-making approach based on D_{GFHFHWD} and TOPSIS, where the weights are processed by exponential entropy and normal distribution weighting, for the multi-attribute decision-making (MADM) issue with unknown attribute weights. Finally, a numerical example of choosing a logistics transfer station and a comparative study with other approaches based on current operators and FFS distance measurements are used to demonstrate the viability and logic of the suggested method. The findings illustrate the ability of the suggested MADM technique to completely present the decision data, enhance the accuracy of decision outcomes and prevent information loss.

Keywords: Fermatean hesitant fuzzy set; hybrid weighted distance measure; exponential entropy; normal distribution weighting
Mathematics Subject Classification: 90B50

List of Abbreviations:

Sr. No	Complete Word	Abbreviation
1	Fuzzy sets	FSs
2	Multi-attribute decision making	MADM
3	Analytical hierarchy process	AHP
4	Simple additive weighting	SAW
5	Technique for order of preference by similarity to ideal solution	TOPSIS
6	Multivariate adaptive regression splines	MARS
7	Linear regression	LR
8	Intuitionistic fuzzy set	IFS
9	Pythagorean fuzzy set	PFS
10	Fermatean fuzzy set	FFS
11	Intuitionistic hesitant fuzzy set	IHFS
12	Interval-valued hesitant fuzzy set	IVHFS
13	Pythagorean hesitant fuzzy set	PHFS
14	Fermatean hesitant fuzzy sets	FHFSs
15	Multi-criteria decision-making	MCDM
16	Interval-valued Fermatean hesitant fuzzy sets	IVFHFSs
17	Combination compromise solution	Cocoso
18	Fermatean fuzzy linguistic term set	FHLTS
19	Ordered weighted distance measure	OWD
20	Hybrid weighted distance	HWD
21	Tomada de decisao interativa multicriterio	TODIM
22	Generalized Fermatean hesitation fuzzy hybrid weighted distance	GFHFHWD
23	Fermatean hesitant fuzzy weighted distance measure	FHFWD
24	Fermatean hesitant fuzzy ordered weighted distance measure	FHFOWD
25	Fermatean hesitant fuzzy hybrid weighted distance measure	FHFHWD
26	Intuitionistic fuzzy weighted average operator	IFWA
27	Intuitionistic fuzzy ordered weighted average operator	IFOWA
28	Criteria importance through intercriteria correlation	CRITIC
29	Best-worst method	BWM
30	Coefficient of variation method	COV
31	Fermatean fuzzy integrated weighted average distance measure	FFIWAD
32	Fermatean fuzzy Hamacher hybrid weight averaging operator	FFHHWA
33	Fermatean fuzzy weighted average operator	FFWA
34	Fermatean fuzzy weighted geometric operators	FFWG

1. Introduction

1.1. Uncertainty

In mathematics, fuzzy sets (FSs) are an extension of the notion of a crisp (or classical) set that allows for partial membership. An element in traditional set theory either belongs to a set or it does not. Many notions, however, are not that cut-and-dry in reality, and whether or not a concept belongs to a set may just be a question of degree. However, in the actual world, many notions are not as cut and dry, and belonging to a group may be dependent on one's degree of involvement in that set. FSs provide us the ability to describe this kind of partial membership by allowing us to give each element in the set a degree of membership that may range from 0 (not a member at all) to 1 (completely a member). Although a number between 0 and 1 is often used to express the degree of membership, it is possible to use other scales as well. When there is no obvious border between members and non-members, as there is with fuzzy sets, it is beneficial to depict complicated or ambiguous notions using fuzzy sets. Some examples of such concepts are "tall" and "heavy". Artificial intelligence [1], decision analysis [2], control theory [3] and pattern recognition [4] are just a few of the numerous domains in which they are employed.

A decision-making process known as multi-attribute decision-making (MADM) involves contrasting and evaluating several possibilities in light of various specific criteria or characteristics. Business, engineering and public policy are just a few of the fields where the MADM is helpful. Using the MADM approach, decision-makers first determine the pertinent features that are essential for evaluating the options under discussion. These factors include, but are not restricted to, things like cost, quality, dependability, availability and a host of other things. How important a feature is to the decision-making process will decide the weight or significance score that is often assigned to each characteristic. Decision-makers then assess the alternatives in relation to each aspect using a variety of methodologies, such as scoring [5], ranking [6] and pairwise comparison [7]. The selections are then ranked or given a final score based on the combined findings of these investigations. With MADM, a variety of techniques and tactics are employed. Some of these are the analytical hierarchy process (AHP) [8], simple additive weighting (SAW) [9] and technique for order Preference by similarity to an ideal solution (TOPSIS) [10]. When there are several options and factors to consider, the MADM process may be challenging. However, it is a potent instrument for making decisions that might assist ensure that conclusions are founded on a thorough and rigorous evaluation of all pertinent factors.

Mathematical models, such as multivariate adaptive regression splines (MARS) [11], artificial neural networks [12] and linear regression (LR) [13], are the most widely used techniques for processing data and making predictions. It may be a challenging procedure for decision-makers to measure a collection of qualities with accurate values because of the growing uncertainty and ambiguity of the information, as well as the complexity and changeability of the decision-making environment in real life. As a consequence of this, more decision-makers have a tendency to employ fuzzy values rather than actual numbers to analyze the information before making judgments. Fuzziness was integrated with a variety of mathematical models or approaches, which led to the development of new fuzzy mathematical methods. Zadeh [14] is credited with being the first person to suggest the idea of FSs. The research of fuzzy theory has made progress, and numerous new concepts and models of expansions of FSs have been developed. Atanassov [15] created the idea of intuitionistic fuzzy set (IFS). He included the non-membership degree to represent the disagreement of evaluators, which is referred to as a generalization of FSs. The extensive implementation of IFS theory in a variety of different disciplines has proved the advantages of the theory when it comes to coping with uncertainty [16–18]. Then, Yager and Abbasov [19] presented Pythagorean fuzzy set (PFS), extended the constraint requirement of the IFS to the sum of the squares of the membership and non-membership degrees being less than 1, and improved the capacity of FSs to characterize uncertainty. PFS theory has been extensively applied to a wide variety of MADM disciplines, with several publications focused on information aggregation operators [20,21], decision-making methods [2,22], correlation coefficient [23], information measures [4,24] and other related topics.

1.2. Related works

The limitations of IFS and PFS become more noticeable when the difficulty of the issue and the reluctance of decision-makers increases. Assume that a person chooses to communicate his choice for one option over another in such a manner that the degree of satisfaction is 0.8 while the degree of discontent is 0.7. In this scenario, it is very clear that 0.8 + 0.7>1, as is the case with 0.8^2 + 0.7^2>1. The requirements of IFS and PFS make it difficult to resolve such situations. Therefore, the Fermatean fuzzy set (FFS), which is a useful tool for representing uncertainty and was first presented by Senapati and Yager [25], has emerged as a more generic fuzzy set for the purpose of addressing MADM issues. The FFS extends the requirement to include situations in which the total number of cubes representing degrees of membership and non-membership is less than or equal to one. When compared to IFS and PFS, it can lessen the limits placed on the degrees of membership and non-membership, enhance the capacity to capture uncertainties and cope with a greater degree of ambiguity. Thus, a growing number of scholars have shown interest in FFS as it is considered a more effective tool for solving problems with insufficient information. It has been effectively implemented in a number of different study domains, such as aggregation operators [26,27], distance measures [28,29], similarity measures [30], decision-making procedures [31,32] and so on. From the existing literature, FFS has made significant advances in theory and practice in recent years, although a comprehensive theoretical system need to be developed. The research filed should be enlarged and developed, and the decision-making method of FFS needs to be further improved. To enrich the theoretical research of FFS, this work discusses different distance measures under the Fermatean hesitant fuzzy environment and creates a new MADM approach based on the provided distance measure.

Despite FFS has addressed some of the issues that plagued IFS and PFS, they are still unable to accurately portray the reluctance of decision-makers to provide assessment data. Torra [33] initially developed the notion of the hesitant fuzzy set (HFS), which enables the membership degree of an attribute belonging to a HFS to include a set of potential values. HFS can solve the subjective hesitation of evaluators and is widely used in MADM problems [34]. Since its appearance, various traditional fuzzy sets have been combined with HFS to propose new fuzzy concepts and models, such as an intuitionistic hesitant fuzzy set (IHFS) [35], interval-valued hesitant fuzzy set (IVHFS) [36], a Pythagorean hesitant fuzzy set (PHFS) [37] and hesitant q-rung orthopair fuzzy set [38,39]. Recently, some scholars introduced the HFS into Fermatean fuzzy environment and achieved some results, which are listed in Table 1. From Table 1, the combination of FFS and HFS has received the interest of some scholars, but it is in the initial stage. To expand the theoretical research of Fermatean hesitant fuzzy sets (FHFSs), we provide several distance measures of FHFSs and establishes a decision-making model based on the developed distance measure to solve the MADM problem with unknown attribute weights.

Table 1. Some related works under the Fermatean hesitant fuzzy environment in recent years.

Content	Method	Attribute weights	References
Hamming distance measure of FHFS	FHF-VIKOR MADM method	Given in advance	Mishra et al. [40]
Fermatean hesitant fuzzy weighted average and geometric operator	MCDM method	The maximum deviation principle	Kirisci [41]
Some weighted operators of FHFSs	TOPSIS method	Given in advance	Khan et al. [42]
A new score function of FFS and the definition of hesitant Fermatean fuzzy sets (HFFSs)	A hesitant Fermatean fuzzy CoCoSo method	Given in advance	Lai et al. [43]
The weighted Bonferroni mean and Einstein weighted Bonferroni mean operators of HFFSs	MADM method	Given in advance	Wang et al. [44]
The prioritized Heronian mean operator of FHFSs	MADM method	The weights with priority information are calculated using special formula	Ruan et al. [45]
Some Choquet integral ordered aggregation operators of FHFSs	MADM method	The λ-fuzzy measure is used to observe the impact of different values of lambda on the decision result	Sha and Shao [46]
Various distance measures and aggregation operators of interval-valued HFFSs	A hybrid conventional complex proportional assessment (COPRAS) method	Given in advance	Mishra et al. [47]
Four types of correlation coefficients for FFSs were developed and expanded to include correlation coefficients and of interval-valued FHFSs	MCDM method	Given in advance	Demir [48]

As an important information integration tool for MADM, distance measure can reflect the difference between complex fuzzy information or different fuzzy sets. Therefore, there is a lot of discussion about how to provide actual distance measure for FSs. Distance measure has been combined with various classical decision-making techniques and often used in many fuzzy settings and fields, such as pattern analysis [49], medical diagnosis [50], linear Diophantine fuzzy sets [51], Fermatean fuzzy linguistic term set (FFLTS) [52], etc. In particular, linear Diophantine fuzzy sets are a new extension of FSs. By taking additional reference or control parameters into account, the sum of membership and non-membership degrees in linear Diophantine fuzzy sets can be greater than 1. Therefore, the research on this fuzzy set has achieved some results in recent years, and several scholars have studied its application in decision-making. Under the Fermatean fuzzy environment, Liu et al. [52] constructed a new similarity measure between FFLTS and combined the cosine similarity measure and Euclidean distance measure to obtain a corresponding distance measure. Yang et al. [53] presented a novel decision-making framework based on Fermatean fuzzy integrated weighted distance measure and TOPSIS approach. Kirisci [54] presented new metrics of distance and cosine similarity amongst FFS. Genie [55] established a number of different innovative distance measurements for FFSs by using t-conorms. Deng and Wang [56] suggested two unique distance measures for FFSs, one based on the Hellinger distance, and the other based on the triangular divergence. Both of these measures are based on the Hellinger distance. In their study on the quality of low-carbon cities, Zeng et al. [57] established a unique comprehensive framework that was based on TOPSIS and Fermatean fuzzy hybrid weighted distance measure. Therefore, there is a significant theoretical void in regards to the distance measurements for FFSs, and very few articles have investigated it; therefore, it is an area that merits being discussed and investigated further. In conclusion, some studies on FFSs in recent years are shown in Table 2.

Table 2. Some related works on information measures for FFSs in recent years.

Information measures	Method	References
Some similarity and distance measures between FFLTS, including the weighted cosine similarity measure, the weighted generalized distance measure and the weighted distance measure	MCDM with TODIM and TOPSIS methods	Liu et al.[52]
The Fermatean fuzzy ordered weighted average and Fermatean fuzzy integrated weighted average distance measures	MCDM with TOPSIS method	Yang et al.[53]
Several new cosine similarity and distance measures for FFS	MADM with TOPSIS method	Kirisci [54]
Some novel distance measures using t-conorms and knowledge measures for FFS	MCDM method	Ganie [55]
The distance measure based on the Hellinger distance and the triangular divergence of FFS	MADM method	Deng and Wang [56]
The hybrid weighed distance measure between different FFS	MADM with TOPSIS method	Zeng et al. [57]

The ordered weighted distance measure (OWD) and a method for determining the weights of OWD measures were first developed by Xu and Chen [58]. Unlike traditional distance measures, OWD takes into account the impact of position weights on the aggregation results and it has the ability to dynamically weaken or strengthen certain positions to arrive at a choice that is more logical. OWD measure has been further developed into other forms and extended to various fuzzy settings as a result of its efficiency in data processing. These forms include linguistic continuous OWD [59], induced OWD [60], ordered weighted averaging distance measure [61] and so on. However, OWD measure only considers the importance of attribute weights, so it is limited to decision-making problems with attribute location information. Then, the hybrid weighted distance (HWD) measure developed by Xu and Xia [62] accurately captures the significance of aggregated data and its location and effectively expands the application of OWD. A novel risk prioritization model was developed by Liu et al. [63] and it makes use of interval 2-tuple HWD measure. The hybrid weighted Pythagorean fuzzy distance measure was introduced by Zeng et al. [64], who then used it to study a new TOPSIS model. Ding et al. [65] created a Tomada de decisao interativa multicriterio (TODIM) dynamic emergency decision-making approach based on HWD that operates in an environment that is probabilistically uncertain and reluctant. Zeng et al. [66] develops a novel comprehensive framework based TOPSIS and Fermatean fuzzy hybrid weighted distance measure for low-carbon city quality.

1.3. Motivation

To summarize, the investigation into FFSs is only at its preliminary phases at this point. The vast majority of theoretical accomplishments are centered on aggregation operators, whereas just a few research are concerned with distance measurements. There has not been much study done on FHFS, and the theoretical underpinnings of the system have not even been defined yet. It is of major academic value to research novel distance measures for FHFSs, particularly distance measures that include attribute weight and position weight. This is because the distance measure is a crucial instrument for expressing the degree of difference that exists between a number of different FSs. In this study, we extend the HWD measure to the Fermatean hesitant fuzzy environment and develops different distance measures based on the FHFS in order to cover the necessary theoretical gaps that have been identified. Following this step, a combination attribute weighting mechanism that makes use of the exponential entropy is built. The weighting of the normal distribution is also taken into consideration when determining the position weight. After that, a new generalized Fermatean hesitant fuzzy hybrid weighted distance measure is created, and its relevant features and special forms are examined. This distance measure is denoted by the acronym D_{GFHFHWD}. The exponential entropy and the normal distribution weighting both contribute to the development of this novel distance measure. It is able to remove the impact of excessively big or excessively tiny variations on the outcomes of the decision-making process, as well as handle the issue of subjective and objective weight allocation. In addition to that, we put out an original MADM methodology that is derived from D_{GFHFHWD} and the TOPSIS method. This D_{GFHFHWD}-TOPSIS approach is able to effectively make full use of the existing information from the original data, take into consideration the effects of attribute and position weights and provide results that appropriately represent the gaps between the evaluation schemes. In conclusion, a numerical example taken from the industry of logistics and transportation is used to demonstrate the viability of the suggested technique as well as the rationale behind it. The discussion then moves on to examine how much of a role the parameter had in determining the rankings.

The following is a list of the primary contributions that this paper makes:

In this paper, we use Fermatean hesitant fuzzy numbers to describe the evaluation information and considers the subjective hesitancy of experts to make the decision more realistic.

New distance measures for FHFSs are developed, such as the weighted distance measure (D_{FHFWD}), the OWD measure (D_{FHFOWD}) and the HWD measure (D_{FHFHWD}). These new distance measures are all denoted by their respective abbreviations.

For the purposes of weight processing, we make use of the exponential entropy weighting approach for determining attribute weight and the normal distribution weighting method for determining position weight, respectively.

A generalized Fermatean hesitant fuzzy hybrid weighted distance, abbreviated as D_{GFHFHWD}, is described. This distance measure takes into account the significance of aggregating data as well as the location of the data, and its associated features and special forms are examined.

A MADM model that is constructed on the basis of D_{GFHFHWD} and TOPSIS is what is going to supply an acceptable assessment approach for picking out the logistics transfer station.

The following describes the structure of this paper's organization. In Section 2, we go over some of the most fundamental definitions of the various forms of FSs. Previous distance measurements between HFS are discussed in Section 3, as well as some new distance measures that have been defined in the context of the Fermatean hesitant fuzzy environment. The fourth section provides an illustration of a novel Fermatean hesitant fuzzy TOPSIS MADM technique based on a generalized Fermatean hesitant fuzzy hybrid weighted distance measure as well as a demonstration of the method's particular phases. Following this, a numerical example pertaining to the logistics and transportation business as well as a comparative study of parameter are carried out in order to test the rationale and efficiency of the technique that are discussed.

2. Generalized Fermatean hesitant fuzzy hybrid weighted distance measure

2.1. Basic concepts

In this section, some basic concepts related FHFS for the understanding of proposed work are discussed.

Definition 1. [40] Let X be a universe of discourse. A FHFS A in X is an object of the following form

$$A = \{\langle x, U_A(x), V_A(X)\rangle | x \in X\} \tag{1}$$

where $U_A(x) = \{\mu_A(x) | \mu_A(x) \in U_A(x)\}$ and $V_A(x) = \{\nu_A(x) | \nu_A(x) \in V_A(x)\}$ are nonempty finite subsets of $[0,1]$, denoting several possible membership and non-membership degrees of x belonging to A. The condition $\mu_A^3(x) + \nu_A^3(x) \leq 1$ holds for $\forall x \in X$ with $\mu_A(x) \in U_A(x)$ and $\nu_A(x) \in V_A(x)$.

$$\Pi_A(x) = \{\pi_A(x) = \sqrt[3]{1 - \mu_A^3(x) - \nu_A^3(x)} | \mu_A(x) \in U_A(x), \nu_A(x) \in V_A(x)\},$$

is named as the set of membership degrees of x on A.

For convenience, this paper regards $\langle \mathrm{U_A(x)}, \mathrm{V_A(x)}\rangle$ as a FHFN and takes Ω for the set of all FHFNs.

To compare different FHFNs further, the scoring function and the accuracy function were defined, and specific comparison methods were proposed accordingly.

Definition 2. [41] Let $\alpha = \langle U_\alpha, V_\alpha\rangle \in \Omega$, $|U_\alpha|$ and $|V_\alpha|$ are the cardinal numbers of the set U_α and V_α.

Then the scoring function S_α and the accuracy function P_α are defined as follows:

$$S_\alpha = \frac{1}{|U_\alpha|}\sum_{\mu\in U_\alpha}\mu^3 - \frac{1}{|V_\alpha|}\sum_{\nu\in V_\alpha}\nu^3 \tag{2}$$

$$P_\alpha = \frac{1}{|U_\alpha|}\sum_{\mu\in U_\alpha}\mu^3 + \frac{1}{|V_\alpha|}\sum_{\nu\in V_\alpha}\nu^3 \tag{3}$$

Let $\alpha = \langle U_\alpha, V_\alpha\rangle$, $\beta = \langle U_\beta, V_\beta\rangle \in \Omega$, then

(1) If $S_\alpha > S_\beta$, then $\alpha > \beta$.

(2) If $S_\alpha = S_\beta$, then the accuracy function is further compared.

(i) If $P_\alpha > P_\beta$, then $\alpha > \beta$;

(ii) If $P_\alpha = P_\beta$, then $\alpha \sim \beta$.

2.2. Fermatean hesitant fuzzy distance measures

This paper first develops several distance measures of FHFSs and supposes $\alpha = \langle U_\alpha, V_\alpha\rangle$, $\beta = \langle U_\beta, V_\beta\rangle \in \Omega$. In general, the cardinal number of different sets is not equal, that is, $|U_\alpha| \neq |U_\beta|, |V_\alpha| \neq |V_\beta|, |\Pi_\alpha| \neq |\Pi_\beta|$. The above sets should have the same cardinal number to ensure the rationality of the calculation. To solve this problem, considering the varying risk preferences of decision-makers, Xu and Xia [62] proposed some fuzzy sets with small extending cardinal numbers and equalized the cardinal number of the two sets by adding new elements. The details are as follows:

From the optimistic perspective: adding multiple large elements to the set with a small cardinal number.

From the pessimistic perspective: adding multiple small elements to the set with a small cardinal number.

The following distance measures for HFSs are defined through the above processing.

Definition 3. [34] Let M and N be two HFSs in a nonempty set $X = \{x_1, x_2, \cdots, x_n\}$.

The normalized hesitant Hamming distance between M and N is defined as follows:

$$d_{HH}(M,N) = \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{l_{x_i}}\sum_{j=1}^{l_{x_i}}\left|h_M^j(x_i) - h_N^j(x_i)\right|\right] \tag{4}$$

The normalized hesitant Euclidean distance between M and N is defined as follows:

$$d_{HE}(M,N)=\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{l_{x_i}}\sum_{j=1}^{l_{x_i}}\left|h_M^j(x_i)-h_N^j(x_i)\right|^2\right]^{\frac{1}{2}} \tag{5}$$

The normalized generalized hesitant distance between M and N is defined as follows:

$$d_{GH}(M,N)=\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{l_{x_i}}\sum_{j=1}^{l_{x_i}}\left|h_M^j(x_i)-h_N^j(x_i)\right|^\lambda\right]^{\frac{1}{\lambda}} \tag{6}$$

where $h_M^j(x_i)$ and $h_N^j(x_i)$ denote the $j\text{th}$ largest element of the set $h_M(x_i)$ and $h_N(x_i)$. l_{x_i} is termed as the cardinal number of $h_M(x_i)$, $h_N(x_i)$ and $\lambda > 0$.

The above-mentioned distance measures of HFSs all involve the difference operation of the $j\text{th}$ largest element in $h_M(x_i)$ and $h_N(x_i)$, indicating that the elements in the sets are sorted in ascending or descending order and subtracted from the elements of the same position. Therefore, to maintain the overall coordination, the elements in the rest of this paper are added by default from a pessimistic perspective and represented by simplified symbols. The elements in the sets of the membership, non-membership and hesitancy degrees are arranged in ascending order.

Example 1. Let $\alpha=\langle\{0.7,0.9\},\{0.4,0.6\}\rangle$ and $\beta=\langle\{0.3,0.5,0.7\},\{0.8\}\rangle$ be two FHFNs. To fully reflect the fuzzy information and consider the hesitancy of α and β, this paper adds the information of indeterminacy degree, then

$$\alpha=\langle\{0.7,0.9\},\{0.4,0.6\},\quad\{0.3803,0.5915,0.7612,0.8401\}\rangle,$$

$$\beta=\langle\{0.3,0.5,0.7\},\{0.8\},\{0.5254,0.7133,0.7725\}\rangle.$$

Add the smallest element to α and β from a pessimistic perspective, then

$$\alpha=\langle\{0.7,0.7,0.9\},\{0.4,0.6\},\{0.3803,0.5915,0.7612,0.8401\}\rangle,$$

$$\beta=\langle\{0.3,0.5,0.7\}\{0.8,0.8\},\qquad\{0.5254,0.5254,0.7133,0.7725\}\rangle.$$

Definition 4. Let $\alpha=\langle U_\alpha,V_\alpha\rangle$ and $\beta=\langle U_\beta,V_\beta\rangle$ be two FHFSs with $U_{\Theta_i}=\{\mu_{\Theta_i}|i=1,2,\ldots,k_1\}$, $V_{\Theta_i}=\{\nu_{\Theta_i}|i=1,2,\ldots,k_2\}$, $|U_\Theta|$ and $|V_\Theta|$ are, respectively, the number of elements in U_Θ and V_Θ.

Then, the distance between α and β is given as:

$$d(\alpha,\beta)=\frac{1}{2}\left(\frac{1}{k_1}\sum_{i=1}^{k_1}\left|\mu_{\alpha_i}^3-\mu_{\beta_i}^3\right|+\frac{1}{k_2}\sum_{i=1}^{k_2}\left|\nu_{\alpha_i}^3-\nu_{\beta_i}^3\right|+\frac{1}{k_3}\sum_{i=1}^{k_3}\left|\pi_{\alpha_i}^3-\pi_{\beta_i}^3\right|\right) \tag{7}$$

where $\prod_{\Theta_i}=\{\pi_{\Theta_i}|i=1,2,\ldots,k_3\}$ is the set of hesitancy degrees and satisfies the following condition that $\mathrm{k}_1=|\mathrm{U}_\Theta|,\ \mathrm{k}_2=|\mathrm{V}_\Theta|,\mathrm{k}_3=|\Pi_\Theta|$.

Example 2. Let $\alpha=\langle\{0.5,0.6,0.8\},\{0.7\}\rangle$ and $\beta=\langle\{0.6,0.9\},\{0.4,0.5\}\rangle$ be two FHFSs. Add the information of indeterminacy degree, we have

$$\alpha=\langle\{0.5,0.6,0.8\},\{0.7\},\quad\{0.5254,0.7612,0.8103\}\rangle$$
$$\beta=\langle\{0.6,0.9\},\{0.4,0.5\},\{0.5266,0.5915,0.8702,0.8963\}\rangle$$

Add the smallest element to α and β from a pessimistic perspective, then

$$\alpha=\langle\{0.5,0.6,0.8\},\{0.7,0.7\},\quad\{0.5254,0.5254,0.7612,0.8103\}\rangle$$
$$\beta=\langle\{0.6,0.6,0.9\},\{0.4,0.5\},\{0.5266,0.5915,0.8702,0.8963\}\rangle$$

Then, the distance between α and β can be calculated

$$d(\alpha,\beta)=\frac{1}{2}\left(\frac{1}{k_1}\sum_{i=1}^{k_1}\left|\mu_{\alpha_i}^3-\mu_{\beta_i}^3\right|+\frac{1}{k_2}\sum_{i=1}^{k_2}\left|\nu_{\alpha_i}^3-\nu_{\beta_i}^3\right|+\frac{1}{k_3}\sum_{i=1}^{k_3}\left|\pi_{\alpha_i}^3-\pi_{\beta_i}^3\right|\right)$$
$$=$$
$$\frac{1}{2}\left(\left(\frac{1}{3}\left(\left|0.5^3-0.6^3\right|+\left|0.6^3-0.6^3\right|+\left|0.8^3-0.9^3\right|\right)\right)+\left(\frac{1}{2}\left(\left|0.7^3-0.4^3\right|+\left|0.7^3-0.5^3\right|\right)\right)+\left(\frac{1}{4}\left(\left|0.5254^3-0.5266^3\right|+\left|0.5254^3-0.5915^3\right|+\left|0.7612^3-0.8702^3\right|+\left|0.8103^3-0.89634^3\right|\right)\right)\right)$$

$$=\frac{1}{2}(0.3757+0.2485+0.1173)=0.3598$$

Lemma 1. Let $0\le\mu_i,\nu_i\le1$, $\mu_i^3+\nu_i^3\le1(i=1,2)$, then

$$\left|\mu_1^3-\mu_2^3\right|+\left|\nu_1^3-\nu_2^3\right|+\left|\pi_1^3-\pi_2^3\right|\le2 \tag{8}$$

Proof. Let $L=\left|\mu_1^3-\mu_2^3\right|+\left|\nu_1^3-\nu_2^3\right|+\left|\pi_1^3-\pi_2^3\right|$.

If $\mu_1\ge\mu_2$, $\nu_1\ge\nu_2$, then $\mu_1^3\ge\mu_2^3$, $\nu_1^3\ge\nu_2^3$, $\mu_1^3+\nu_1^3\ge\mu_2^3+\nu_2^3$.

It is easy to get

$$L=\left|\mu_1^3-\mu_2^3\right|+\left|\nu_1^3-\nu_2^3\right|+\left|\pi_1^3-\pi_2^3\right|=\mu_1^3-\mu_2^3+\nu_1^3-\nu_2^3+\mu_1^3+\nu_1^3-(\mu_2^3+\nu_2^3)$$

$$= 2\left(\mu_1^3+\nu_1^3\right)-2\left(\mu_2^3+\nu_2^3\right)\le 2\left(\mu_1^3+\nu_1^3\right)\le 2.$$

If $\mu_1 \ge \mu_2, \quad \nu_1 \le \nu_2$, then $\mu_1^3 \ge \mu_2^3, \quad \nu_1^3 \le \nu_2^3$.

(1) If $\mu_1^3+\nu_1^3 \ge \mu_2^3+\nu_2^3$, then

$$L = \left|\mu_1^3-\mu_2^3\right|+\left|\nu_1^3-\nu_2^3\right|+\left|\pi_1^3-\pi_2^3\right| = \mu_1^3-\mu_2^3-\nu_1^3+\nu_2^3+\mu_1^3+\nu_1^3-(\mu_2^3+\nu_2^3)$$
$$= 2\mu_1^3-2\mu_2^3 = 2\left(\mu_1^3-\mu_2^3\right)\le 2\mu_1^3\le 2.$$

If $\mu_1^3+\nu_1^3 \le \mu_2^3+\nu_2^3$, then

$$L = \left|\mu_1^3-\mu_2^3\right|+\left|\nu_1^3-\nu_2^3\right|+\left|\pi_1^3-\pi_2^3\right| = \mu_1^3-\mu_2^3+\nu_2^3-\nu_1^3+\mu_2^3+\nu_2^3-(\mu_1^3+\nu_1^3)$$
$$= 2\nu_2^3-2\nu_1^3 = 2\left(\nu_2^3-\nu_1^3\right)\le 2\nu_2^3\le 2.$$

If $\mu_1 \le \mu_2, \quad \nu_1 \le \nu_2$, then $\mu_1^3 \le \mu_2^3, \quad \nu_1^3 \le \nu_2^3, \quad \mu_1^3+\nu_1^3 \le \mu_2^3+\nu_2^3$.

So we have

$$L = \left|\mu_1^3-\mu_2^3\right|+\left|\nu_1^3-\nu_2^3\right|+\left|\pi_1^3-\pi_2^3\right| = \mu_2^3-\mu_1^3+\nu_2^3-\nu_1^3+\mu_2^3+\nu_2^3-\left(\mu_1^3+\nu_1^3\right)$$
$$= 2\left(\mu_2^3+\nu_2^3\right)-2\left(\mu_1^3+\nu_1^3\right) = 2\left[\left(\mu_2^3+\nu_2^3\right)-\left(\mu_1^3+\nu_1^3\right)\right]\le 2\left(\mu_2^3+\nu_2^3\right)\le 2.$$

If $\mu_1 \le \mu_2, \quad \nu_1 \ge \nu_2$, then $\mu_1^3 \le \mu_2^3, \quad \nu_1^3 \ge \nu_2^3$.

(1) If $\mu_1^3+\nu_1^3 \ge \mu_2^3+\nu_2^3$, then

$$L = \left|\mu_1^3-\mu_2^3\right|+\left|\nu_1^3-\nu_2^3\right|+\left|\pi_1^3-\pi_2^3\right| = \mu_2^3-\mu_1^3+\nu_1^3-\nu_2^3+\mu_1^3+\nu_1^3-(\mu_2^3+\nu_2^3)$$
$$= 2\nu_1^3-2\nu_2^3 = 2\left(\nu_1^3-\nu_2^3\right)\le 2\nu_1^3\le 2.$$

(2) If $\mu_1^3+\nu_1^3 \le \mu_2^3+\nu_2^3$, then

$$L = \left|\mu_1^3-\mu_2^3\right|+\left|\nu_1^3-\nu_2^3\right|+\left|\pi_1^3-\pi_2^3\right|$$
$$= \mu_2^3-\mu_1^3+\nu_1^3-\nu_2^3+\mu_2^3+\nu_2^3-(\mu_1^3+\nu_1^3 = 2\mu_2^3-2\mu_1^3 = 2(\mu_2^3-\mu_1^3)\le 2\mu_2^3\le 2.$$

In conclusion, $L = \left|\mu_1^3-\mu_2^3\right|+\left|\nu_1^3-\nu_2^3\right|+\left|\pi_1^3-\pi_2^3\right|\le 2$ holds permanently in all of the above cases.

Theorem 1. For $\alpha = \langle U_\alpha, V_\alpha\rangle$, $\beta = \langle U_\beta, V_\beta\rangle$ and $\gamma = \langle U_\gamma, V_\gamma\rangle \in \Omega$.

(1) $0 \le d(\alpha,\beta)\le 1$ and $d(\alpha,\beta)=0$, if and only if $\alpha=\beta$;

(2) $d(\alpha,\beta)=d(\beta,\alpha)$;

(3) $d(\alpha,\beta)\le d(\alpha,\gamma)+d(\gamma,\beta)$.

Proof.

(1) For $\left|\pi_{\alpha_l}^3-\pi_{\beta_l}^3\right|=\left|\mu_{\alpha_i}^3+\nu_{\alpha_j}^3-(\mu_{\beta_i}^3+\nu_{\beta_j}^3)\right|$, it is easy to see that

$$d(\alpha,\beta)=\frac{1}{2}(\frac{1}{k_1}\sum_{i=1}^{k_1}\left|\mu_{\alpha_i}^3-\mu_{\beta_i}^3\right|+\frac{1}{k_2}\sum_{j=1}^{k_2}\left|\nu_{\alpha_j}^3-\nu_{\beta_j}^3\right|+\frac{1}{k_1k_2}\sum_{l=1}^{k_1k_2}\left|\pi_{\alpha_l}^3-\pi_{\beta_l}^3\right|)$$

$$=\frac{1}{2}(\frac{k_2}{k_1k_2}\sum_{i=1}^{k_1}\left|\mu_{\alpha_i}^3-\mu_{\beta_i}^3\right|+\frac{k_1}{k_1k_2}\sum_{i=1}^{k_1}\left|\mu_{\alpha_i}^3-\mu_{\beta_i}^3\right|+\frac{1}{k_1k_2}\sum_{j=1}^{k_2}\sum_{i=1}^{k_1}\left|\mu_{\alpha_i}^3+\nu_{\alpha_j}^3-(\mu_{\beta_i}^3+\nu_{\beta_j}^3)\right|$$

$$=\frac{1}{2}\left[\frac{1}{k_1k_2}\sum_{j=1}^{k_2}\sum_{i=1}^{k_1}|\mu_{\alpha_i}^3-\mu_{\beta_i}^3|+\frac{1}{k_1k_2}\sum_{i=1}^{k_1}\sum_{j=1}^{k_2}|\nu_{\alpha_j}^3-\nu_{\beta_j}^3|+\frac{1}{k_1k_2}\sum_{j=1}^{k_2}\sum_{i=1}^{k_1}\left|\mu_{\alpha_i}^3+\nu_{\alpha_j}^3-(\mu_{\beta_i}^3+\nu_{\beta_j}^3)\right|\right]$$

$$=\frac{1}{2}\left[\frac{1}{k_1k_2}\sum_{j=1}^{k_2}\sum_{i=1}^{k_1}(|\mu_{\alpha_i}^3-\mu_{\beta_i}^3|+|\nu_{\alpha_j}^3-\nu_{\beta_j}^3|+|\mu_{\alpha_i}^3+\nu_{\alpha_j}^3-(\mu_{\beta_i}^3+\nu_{\beta_j}^3)|)\right].$$

According to Lemma 1, $|\mu_{\alpha_i}^3-\mu_{\beta_i}^3|+|\nu_{\alpha_j}^3-\nu_{\beta_j}^3|+|\mu_{\alpha_i}^3+\nu_{\alpha_j}^3-(\mu_{\beta_i}^3+\nu_{\beta_j}^3)|\le 2$ holds for $i=1,2,\ldots,k_1, j=1,2,\ldots,k_2$. We have

$$d(\alpha,\beta)\le\frac{1}{2}(\frac{1}{k_1k_2}\sum_{i=1}^{k_1}\sum_{j=1}^{k_2}2)=1.$$

(2) The equation is obviously true and the proof process is omitted here.

(3) $$d(\alpha,\beta)=\frac{1}{2}(\frac{1}{k_1}\sum_{i=1}^{k_1}\left|\mu_{\alpha_i}^3-\mu_{\beta_i}^3\right|+\frac{1}{k_2}\sum_{i=1}^{k_2}\left|\nu_{\alpha_i}^3-\nu_{\beta_i}^3\right|+\frac{1}{k_3}\sum_{i=1}^{k_3}\left|\pi_{\alpha_i}^3-\pi_{\beta_i}^3\right|)$$

$$\le\frac{1}{2}(\frac{1}{k_1}\sum_{i=1}^{k_1}(\left|\mu_{\alpha_i}^3-\mu_{\gamma_i}^3\right|+\left|\mu_{\gamma_i}^3-\mu_{\gamma_i}^3\right|)+\frac{1}{k_2}\sum_{i=1}^{k_2}(\left|\nu_{\alpha_i}^3-\nu_{\gamma_i}^3\right|+\left|\nu_{\gamma_i}^3-\nu_{\beta_i}^3\right|)$$

$$+\frac{1}{k_3}\sum_{i=1}^{k_3}(\left|\pi_{\alpha_i}^3-\pi_{\gamma_i}^3\right|+\left|\pi_{\gamma_i}^3-\pi_{\beta_i}^3\right|))$$

$$=\frac{1}{2}(\frac{1}{k_1}\sum_{i=1}^{k_1}\left|\mu_{\alpha_i}^3-\mu_{\gamma_i}^3\right|+\frac{1}{k_2}\sum_{i=1}^{k_2}\left|\nu_{\alpha_i}^3-\nu_{\gamma_i}^3\right|+\frac{1}{k_3}\sum_{i=1}^{k_3}\left|\pi_{\alpha_i}^3-\pi_{\gamma_i}^3\right|)$$

$$+\frac{1}{2}(\frac{1}{k_1}\sum_{i=1}^{k_1}\left|\mu_{\gamma_i}^3-\mu_{\beta_i}^3\right|+\frac{1}{k_2}\sum_{i=1}^{k_2}\left|\nu_{\gamma_i}^3-\nu_{\beta_i}^3\right|+\frac{1}{k_3}\sum_{i=1}^{k_3}\left|\pi_{\gamma_i}^3-\pi_{\beta_i}^3\right|)$$

$$=d(\alpha,\gamma)+d(\gamma,\beta).$$

Then, several distance measures of FHFS are defined based on the aggregation operator theory.

Definition 5. Let $A=\{\alpha(x_i)|i=1,2,\ldots,n\}$ and $B=\{\beta(x_i)|i=1,2,\ldots,n\}$ be two FHFSs in $X=\{x_1,x_2,\ldots,x_n\}$, the weighted distance measure between A and B is a mapping D_{FHFWD}: $\Omega^n\times\Omega^n\mapsto R$ with $\alpha(x_i),\beta(x_i)\in\Omega$, which has the following form:

$$D_{\mathrm{FHFWD}}(A,B)=\sum_{i=1}^{n}\omega_i d\big(\alpha(x_i),\beta(x_i)\big) \tag{9}$$

where the weighting vector of elements is $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ with $\omega_i > 0$ and $\sum_{i=1}^{n} \omega_i = 1$

To relieve the impact of excessively large or excessively small deviations on the decision results, Xu and Chen [58], inspired by the OWA operator, proposed an ordered weighted distance measure that is extended to the Fermatean hesitant fuzzy environment in this paper.

Definition 6. Let $A = \{\alpha(x_i) | i = 1,2, \dots, n\}$ and $B = \{\beta(x_i) | i = 1,2, \dots, n\}$ be two FHFSs on a nonempty set $X = \{x_1, x_2, \dots, x_n\}$ with $\alpha(x_i), \beta(x_i) \in \Omega$. An ordered weighted distance measure between A and B is a mapping $D_{\text{FHFOWD}}: \Omega^n \times \Omega^n \mapsto R$ that has an associated weighting $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ with $\omega_i > 0$ and $\sum_{i=1}^{n} \omega_i = 1$, such that

$$D_{\text{FHFOWD}}(A, B) = \sum_{i=1}^{n} \omega_i d\left(\alpha\left(x_{\theta(i)}\right), \beta\left(x_{\theta(i)}\right)\right) \tag{10}$$

where $\theta(1), \theta(2), \dots, \theta(n)$ is any permutation of $1,2, \dots, n$, which satisfies the following condition: $d\left(\alpha\left(x_{\theta(1)}\right), \beta\left(x_{\theta(1)}\right)\right) \geq d\left(\alpha\left(x_{\theta(2)}\right), \beta\left(x_{\theta(2)}\right)\right) \geq \dots \geq \left(\alpha\left(x_{\theta(n)}\right), \beta\left(x_{\theta(n)}\right)\right)$. Here, D_{FHFOWD} considers the attribute positions and therefore weights the ordered positions of each distance measure.

The importance of the position of aggregated elements has been reflected in the weighting vector of D_{FHFOWD}, which is essentially the same as the weighting method of the OWA operator. Xu [67] summarized multiple weight-determining methods of OWA operators and proposed a weighting method based on the normal distribution. Normal distribution weighting can reduce the influence of abnormal information on decision results and obtain a more objective result by giving less weight to abnormal elements. Thus, it is a helpful tool for determining the weight of FHFOWA operators.

As two distance measures of FHFSs, D_{FHFWD} and D_{FHFOWD} are similar to the intuitionistic fuzzy weighted average (IFWA) operator and intuitionistic fuzzy ordered weighted average (IFOWA) operator proposed by Xu [16]. The difference is that the weight of the former reflects the importance of aggregated data, whereas the weight of the latter reflects the importance of positions. Because the two previously mentioned distance measures have apparent shortcomings, this paper further proposes a hybrid weighted distance measure under the Fermatean hesitant fuzzy environment by combining the attribute and position weights.

Definition 7. Let $A = \{\alpha(x_i) | i = 1,2, \dots, n\}$ and $\mathrm{B} = \{\beta(x_i) | i = 1,2, \dots, n\}$ be two FHFSs on a universe of discourse $X = \{x_1, x_2, \dots, x_n\}$ and $\alpha(x_i), \beta(x_i) \in \Omega$. Then,

$$D_{\text{FHFHWD}}(A,B)=\sum_{i=1}^{n}\omega_i\ddot{d}\left(\alpha\left(x_{\theta(i)}\right),\beta\left(x_{\theta(i)}\right)\right) \tag{11}$$

is called as a hybrid weighted distance measure between A and B on a mapping $D_{\text{FHFHWD}}: \Omega^n \times \Omega^n \mapsto R$, where $\ddot{d}\left(\alpha\left(x_{\theta(i)}\right),\beta\left(x_{\theta(i)}\right)\right)$ represents the i^{th} largest of the weighted distance $\dot{d}\left(\alpha(x_i),\beta(x_i)\right)$ with $\ddot{d}\left(\alpha(x_i),\beta(x_i)\right)=n\widetilde{\omega}_i d\left(\alpha(x_i),\beta(x_i)\right)$, $\widetilde{\omega}=(\widetilde{\omega}_1,\widetilde{\omega}_2,\ldots,\widetilde{\omega}_n)^T$ denotes the weighting vector of distance $d\left(\alpha(x_i),\beta(x_i)\right)$ with $\widetilde{\omega}_i>0$ and $\sum_{i=1}^{n}\widetilde{\omega}_i=1$. $\omega=(\omega_1,\omega_2,\ldots,\omega_n)^T$ is the position weighting vector of D_{FHFHWD} with $\omega_i>0$ and $\sum_{i=1}^{n}\omega_i=1$. For $\ddot{d}\left(\alpha(x_i),\beta(x_i)\right)=n\widetilde{\omega}_i d\left(\alpha(x_i),\beta(x_i)\right)$, $(\theta(1),\theta(2),\ldots,\theta(n))$ is any permutation of $(1,2,\ldots,n)$, which leads to the following conclusion:

$$\ddot{d}\left(\alpha\left(x_{\theta(1)}\right),\beta\left(x_{\theta(1)}\right)\right)\geq\ddot{d}\left(\alpha\left(x_{\theta(2)}\right),\beta\left(x_{\theta(2)}\right)\right)\quad\geq\ldots\geq\ddot{d}\left(\alpha\left(x_{\theta(n)}\right),\beta\left(x_{\theta(n)}\right)\right).$$

Theorem 2. (1) If the attribute weighting vector of elements is $(\widetilde{\omega}_1,\widetilde{\omega}_2,\ldots,\widetilde{\omega}_n)=\left(\frac{1}{n},\frac{1}{n},\frac{1}{n},\ldots,\frac{1}{n}\right)$, then

$$D_{\text{FHFHWD}}=D_{\text{FHFOWD}}. \tag{12}$$

(2) If the position weighting vector of elements is $(\omega_1,\omega_2,\ldots,\omega_n)=\left(\frac{1}{n},\frac{1}{n},\frac{1}{n},\ldots,\frac{1}{n}\right)$, then

$$D_{\text{FHFHWD}}=D_{\text{FHFWD}}. \tag{13}$$

Definition 8. Let $X=\{x_1,x_2,\ldots,x_n\}$ be a universe of discourse. $A=\{\alpha(x_i)|i=1,2,\ldots,n\}$ and $B=\{\beta(x_i)|i=1,2,\ldots,n\}$ are two FHFSs in X with $\alpha(x_i),\beta(x_i)\in\Omega$. A generalized hybrid weighted distance measure between A and B is a mapping $D_{\text{GFHFHWD}}: \Omega^n\times\Omega^n\mapsto R$ such that

$$D_{\text{GFHFHWD}}(A,B)=\left[\sum_{i=1}^{n}\omega_i\ddot{d}\left(\alpha\left(x_{\theta(i)}\right),\beta\left(x_{\theta(i)}\right)\right)^{\lambda}\right]^{\frac{1}{\lambda}} \tag{14}$$

where $(\omega_1,\omega_2,\ldots,\omega_n)$ represents the position weighting vector of D_{GFHFHWD} with $\omega_i>0$ and $\sum_{i=1}^{n}\omega_i=1$. Furthermore,

$$\ddot{d}\left(\alpha(x_i), \beta(x_i)\right) = n\widetilde{\omega}_i d\left(\alpha(x_i), \beta(x_i)\right) \tag{15}$$

where $\widetilde{\omega} = (\widetilde{\omega}_1, \widetilde{\omega}_2, \ldots, \widetilde{\omega}_n)^T$ denotes the weighting vector associated with distance $d\left(\alpha(x_i), \beta(x_i)\right)$ with $\widetilde{\omega}_i > 0$ and $\sum_{i=1}^{n} \widetilde{\omega}_i = 1$. $\theta(1), \theta(2), \ldots, \theta(n)$ is any permutation of $1,2,\ldots,n$, such that $\ddot{d}\left(\alpha(x_{\theta(1)}), \beta(x_{\theta(1)})\right) \geq \ddot{d}\left(\alpha(x_{\theta(2)}), \beta(x_{\theta(2)})\right) \geq \ldots \geq \ddot{d}\left(\alpha(x_{\theta(n)}), \beta(x_{\theta(n)})\right)$.

The distance D_{GFHFHWD} has some special forms when parameter λ selects different values.

Theorem 3. (1) When $\lambda = 1$, D_{GFHFHWD} will be reduced to D_{FHFHWD}. That is,

$$D_{\text{GFHFHWD}} = D_{\text{FHFHWD}} \tag{16}$$

where D_{GFHFHWD} is called a Fermatean hesitant fuzzy hybrid weighted Hamming distance.

When $\lambda = 2$, a Fermatean hesitant fuzzy hybrid weighted Euclidean distance is given as:

$$D_{\text{GFHFHWD}}(A, B) = \left[\sum_{i=1}^{n} \omega_i \ddot{d}\left(\alpha(x_{\theta(i)}), \beta(x_{\theta(i)})\right)^2\right]^{\frac{1}{2}} \tag{17}$$

In addition, Theorem 1 yields the following conclusions based on the properties of hesitant fuzzy distances.

Theorem 4. Let A and B be two FHFSs, then the D_{GFHFHWA} has the following properties:

(1) $0 \leq D_{\text{GFHFHWD}} \leq 1$;

(2) $D_{\text{GFHFHWD}}(A, B) = 0$, if and only if $A = B$;

(3) $D_{\text{GFHFHWD}}(A, B) = D_{\text{GFHFHWD}}(B, A)$.

3. Fermatean hesitant fuzzy TOPSIS method

3.1. Fermatean hesitant fuzzy MADM problems

When numerous qualities or criteria need to be taken into account at once, the process is known as multi-attribute decision-making (MADM). In many different disciplines, such as business, engineering, economics and public policy, this kind of decision-making is typical. MADM entails assessing and contrasting several options depending on how well they perform against a variety of features or criteria. AHP, TOPSIS, preference ranking organization method for enrichment evaluations (PROMETHEE) and more methodologies and approaches are available for MADM. The decision-makers' tastes and the difficulty of the situation determine the strategy to choose. By offering a structured method to take into account many factors and objectively assess options, MADM aids in addressing complicated decision-making circumstances.

The scheme provided by the invited experts is suitable for solving MADM problems where the sum of the squares of the maximum membership degree and non-membership degree is less than 1. However, when dealing with the actual situation, the provided scheme may be difficult to deal with problems where the sum of the squares of the maximum degree of membership and non-membership is greater than 1. Decision-makers may have certain biases against the DM process, be hesitant about certain vital factors, or lose significant DM information provided by experts. Therefore, it is important to consider the weights of different factors simultaneously and broaden the application scope of DM methods. To solve this problem, this paper employs FHFNs to represent attribute values, which can broaden the range of the membership and non-membership degrees and make full use of the existing evaluation information.

For MADM problems under the Fermatean hesitant fuzzy environment, let $A = \{A_1, A_2, \ldots, A_m\}$ be the finite set of alternatives and $e = \{e_1, e_2, \ldots, e_n\}$ be the set of attributes. Assume that the evaluation value of alternative $A_i (i = 1,2,\ldots,m)$ concerning attribute $e_j (j = 1,2,\ldots,n)$ is a FHFN $\alpha_{ij} = \langle U_{ij}, V_{ij} \rangle$. The associated weighting vector of attributes is $\widetilde{\omega} = (\widetilde{\omega}_1, \widetilde{\omega}_2, \ldots, \widetilde{\omega}_n)$ that satisfies $\widetilde{\omega}_i > 0$ and $\sum_{i=1}^{n} \widetilde{\omega}_i = 1$, which can be predetermined by experts and determined by an optimized model or the entropy weighting method. In this paper, the attribute weight is determined using the Fermatean hesitant fuzzy exponential entropy hybrid weighting method. The Fermatean hesitant fuzzy decision matrix $(\alpha_{ij})_{m \times n}$ can be represented as the following matrix form:

$$(\alpha_{ij})_{m \times n} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{pmatrix}.$$

Inspired by the literature [68], the attribute weights are obtained by defining the exponential entropy of FHFS and using the principle of minimizing information entropy.

Definition 9. Let $\alpha = \langle U, V \rangle$ be a FHFN, then the exponential entropy of α is defined as:

$$E(\alpha) = \frac{1}{2\left(\sqrt{e} - 1\right)} \left\{ \frac{1}{l_U} \sum_{j=1}^{l_U} \left[\mu_j e^{(1-\mu_j)} + (1-\mu_j) e^{\mu_j} - 1 \right] + \frac{1}{l_V} \sum_{j=1}^{l_V} \left[\nu_j e^{(1-\nu_j)} + (1-\nu_j) e^{\nu_j} - 1 \right] \right\} \tag{18}$$

where $U = \{\mu_1, \mu_2, \cdots, \mu_{l_U}\}$ and $V = \{\nu_1, \nu_2, \cdots, \nu_{l_V}\}$. l_U and l_V denote, respectively, the cardinal numbers of U and V.

Example 3. Let $\alpha = \langle[0.5,0.7,0.9],[0.4,0.5]\rangle$ be a FHFS, then the exponential entropy of α is given as:

$$E(\alpha) = \frac{1}{2\ (\sqrt{e}-1)}\left\{\frac{1}{3}\sum_{j=1}^{3}\left[\left(0.5e^{(1-0.5)} + (1-0.5)e^{0.5} - 1\right) + \left(0.7e^{(1-0.7)} + (1-0.7)e^{0.7} - 1\right)\right] + \left(0.9e^{(1-0.9)} + (1-0.9)e^{0.9} - 1\right)\right\} + \frac{1}{2}\sum_{j=1}^{2}\left[\left(0.4e^{(1-0.4)} + (1-0.4)e^{0.4} - 1\right) + 0.5e^{(1-0.5)} + (1-0.5)e^{0.5} - 1\right] = \frac{1}{2\ (\sqrt{e}-1)}(0.4795 + 0.6363) = 0.8600$$

3.2. The TOPSIS method and process based on generalized Fermatean hesitant fuzzy hybrid weighted distance measure

Aiming at the attribute weights in the process of information aggregation, scholars have proposed multiple methods to determine the attribute weights, including criteria importance through intercriteria correlation (CRITIC) [22], AHP [8], entropy weighting [68], best-worst method (BWM) [69], coefficient of variation method (COV) [70], social network analysis [71] and so on. AHP and BWM are common subjective weighting methods. The original data is obtained by experts according to their subjective experience and preference, lacking an objective basis and easily leading to decision bias. CRITIC comprehensively measures the weight based on the comparison strength of evaluation indicators and the conflict between different indicators, which is only applicable to the situation where the data is stable and there is a certain correlation between evaluation indicators. COV determines the weight according to the variation degree between the current value of the evaluation indicator and the target value and has certain requirements for the selection of indicators, that is, the importance of each indicator is equal.

Compared with AHP and BWM, entropy weighting method is not affected by subjective factors and reduces the influence of subjectivity on decision results. Compared with CRITIC and COV, entropy weighting is based on the idea of information entropy, and can objectively reflect the difference and importance of different indicators. It is more suitable for the situation where the correlation between indicators is weak. In this paper, the Fermatean hesitant fuzzy index is constructed according to the actual data information, and then the information entropy minimization principle and entropy-revised G1 combination weighting are used to calculate the attribute weight. According to the minimization principle of information entropy, the information entropy is

$$\bar{E}_j = \frac{1}{m}\sum_{i=1}^{m} E\left(\alpha_{ij}\right).$$

Calculate the differential coefficient G_j of the jth indicator. For the jth indicator, the smaller the difference of X_{ij}, the larger E_j. When all of the jth indicators are equal, $E_j = E_{\max} = 1$, then X_j is of no effect. Further, the more obvious the difference in the values of jth indicator, the smaller E_j, and the greater the role of this indicator. Therefore, the differential coefficient is defined as:

$$G_j = 1 - E_j. \tag{19}$$

The larger G_j, the more important this indicator.

Then, the importance of indicators in the set of alternatives is ranked by experts. After that, the differential coefficient of the kth indicator is calculated according to the ranking results, and the ratio of the importance R_k of adjacent indicators X_k and X_{k-1} is also determined.

$$R_K = \begin{cases} G_{K-1}/G_K \text{, when } G_{K-1} \geq G_K \\ 1\text{, when } G_{K-1} < G_K \end{cases} \tag{20}$$

According to the given results, the weight of the mth indicator under this criterion, which is denoted as T_m, is calculated as well.

$$T_m = \left(1 + \sum_{k=2}^{m} \prod_{i=k}^{m} R_i\right)^{-1}. \tag{21}$$

Using the weight T_m mentioned above, the weights from indicator $(m-1)$, indicator $(m-2)$,..., up to the second indicator are calculated:

$$T_{m-1} = R_m T_m \tag{22}$$

where T_{m-1} represents the entropy-revised G1 combination weighting of the indicator $(m-1)$ under this criterion.

Considering the influence of different types of attributes on decision results, the decision matrix should be normalized further. The normalized decision matrix $(\tilde{\alpha}_{ij})_{m\times n}$with $\tilde{\alpha}_{ij} = \langle \tilde{U}_{ij}, \tilde{V}_{ij} \rangle$ is determined as follows:

For cost-type attributes: $\tilde{U}_{ij} = U_{ij}$, $\tilde{V}_{ij} = V_{ij}$;

For benefit-type attributes: $\tilde{U}_{ij} = V_{ij}$, $\tilde{V}_{ij} = U_{ij}$.

For convenience, this paper adds the elements of the set with a smaller cardinal number from a pessimistic perspective while keeping the cardinal number of the membership and non-membership degrees of each alternative with respect to each attribute the same. Let Q_j and L_j $(j = 1,2,\ldots,n)$ be the cardinal numbers of the membership degree and non-membership degree of each alternative concerning attribute e_j, respectively. Then

$$\widetilde{U}_{ij} = \{\mu_{i1}, \mu_{i2}, \ldots, \mu_{iQ_j}\},\ \widetilde{V}_{ij} = \{\nu_{i1}, \nu_{i2}, \ldots, \nu_{iL_j}\}\ i = 1,2,\ldots, m,\ \ j = 1,2,\ldots, n.$$

The positive and negative ideal solutions are determined after processing the information on each attribute. Let $\mathrm{A}^+ = \{\mathrm{a}_1^+, \mathrm{a}_2^+, \ldots, \mathrm{a}_\mathrm{n}^+\}$ and $\mathrm{A}^- = \{\mathrm{a}_1^-, \mathrm{a}_2^-, \ldots, \mathrm{a}_\mathrm{n}^-\}$ be the positive and negative ideal solutions, respectively. Then

$\alpha_j^+ = \{U_j^+, V_j^+\}, j = 1,2,\ldots, n$ with $U_j^+ = \{\max\limits_{1\le i\le m}\{\mu_{i1}\}, \max\limits_{1\le i\le m}\{\mu_{i2}\}, \ldots, \max\limits_{1\le i\le m}\{\mu_{iQ_j}\}\}$ and $V_j^+ = \{\min\limits_{1\le i\le m}\{\nu_{i1}\}, \min\limits_{1\le i\le m}\{\nu_{i2}\}, \ldots, \min\limits_{1\le i\le m}\{\nu_{iL_j}\}\}$.

$\alpha_j^- = \{U_j^-, V_j^-\}, j = 1,2,\ldots, n$ with $U_j^- = \{\min\limits_{1\le i\le m}\{\mu_{i1}\}, \min\limits_{1\le i\le m}\{\mu_{i2}\}, \ldots, \min\limits_{1\le i\le m}\{\mu_{iQ_j}\}\}$ and $V_j^- = \{\max\limits_{1\le i\le m}\{\nu_{i1}\}, \max\limits_{1\le i\le m}\{\nu_{i2}\}, \ldots, \max\limits_{1\le i\le m}\{\nu_{iL_j}\}\}$.

This paper proposes a novel method based on $\mathrm{D}_{\mathrm{GFHFHWA}}$ and the TOPSIS method for solving Fermatean hesitant fuzzy MADM problems. The specific steps are listed below:

Step 1. Establish the Fermatean hesitant fuzzy decision matrix. Based on the attribute information, the exponential entropy of attributes E is obtained, then the final attribute weight $\widetilde{\omega} = (\widetilde{\omega}_1, \widetilde{\omega}_2, \ldots, \widetilde{\omega}_n)$ is calculated using the entropy-revised $G1$ combination weighting.

Step 2. Normalize the Fermatean hesitant fuzzy decision matrix and complement the cardinal numbers of the sets. From a pessimistic perspective, the smallest elements are added to the set with smaller cardinal numbers. Then, the cardinal numbers of the membership and non-membership degree sets regarding each attribute can be the same.

Step 3. Determine the positive and negative ideal solutions A^+ and A^-.

Step 4. Combining with normal distribution weighting, the distances between alternative A_i with respect to criterion e_j (denoted by α_{ij}) and the positive ideal solution A^+, α_{ij} and the negative ideal solution A^-, respectively represented as $d(\alpha_{ij}, \alpha_j^+)$ and $d(\alpha_{ij}, \alpha_j^-)(i = 1,2,\ldots, m, j = 1,2,\ldots, n)$, are calculated.

Step 5. Based on $d(\alpha_{ij}, \alpha_j^+)$ and $d(\alpha_{ij}, \alpha_j^-)$ in Step 4, the distances $D_{\mathrm{GFHFHWD}}(A_i, A^+)$ and $D_{\mathrm{GFHFHWD}}(A_i, A^-)\ (i = 1,2,\ldots, m)$ are determined according to Definition 8.

$$D_{\mathrm{GFHFHWD}}(A,B)=\left[\sum_{i=1}^{n}\omega_i\ddot{d}\Big(\alpha\big(x_{\theta(i)}\big),\beta\big(x_{\theta(i)}\big)\Big)^{\lambda}\right]^{\frac{1}{\lambda}}. \tag{14}$$

Step 6. Compute the closeness $\delta_i(i=1,2,\ldots,m)$ of the alternatives, which is defined as:

$$\delta_i=\frac{D_{\mathrm{GFHFHWD}}(A_i,A^-)}{D_{\mathrm{GFHFHWD}}(A_i,A^+)+D_{\mathrm{GFHFHWD}}(A_i,A^-)}. \tag{23}$$

The priority order of the alternatives $A_i(i=1,2,\ldots,m)$ is determined by the rankings of δ_i. The bigger the closeness δ_i, the better the alternative A_i.

4. An illustrative example of Fermatean hesitant fuzzy MADM problems

4.1. Numerical example

COVID-19 has had a significant impact on global economic development since 2020. From the overall isolation of the epidemic to today's precise isolation, it is inextricably linked to the logistics and transportation industries. As an important bloodline of the national economy, the logistics and transportation industry has made great contributions to the development of various regions and industries affected by the epidemic. With the normalization of the epidemic, the risks faced by the logistics and transportation industry have become increasingly diverse, and the control of national governments may adjust at any time. The logistics industry should actively seek new ways to adapt to the development. Suppose the risk control of a certain path is carried out in logistics planning. In that case, it is very likely to cause irreparable losses to terminal warehousing and distribution, as well as a significant impact on the enterprise and transportation industries. The advantages of traditional efficient transportation and low inventory management are diminishing in the era of the epidemic. It is worthwhile to investigate how to reduce the risks encountered by enterprises at all stages of the supply chain through management DM methods. In this section, a numerical example of evaluating a collection of transfer stations is illustrated. The project aims to select an ideal transfer station among those available and expand it into a local transportation hub center that can provide logistics companies with a choice of transfer stations.

Suppose W is a famous logistics company, needs to expand its local market share due to the rapid business development. Let $P_i=\{P_1,P_2,P_3,P_4,P_5\}$ be a set of transit stations for selection. Now some experts are invited to evaluate each transmit station from the perspective of logistics operation management according to the four attributes $\{e_1,e_2,e_3,e_4\}$ listed below: Transportation and Distribution Management (e_1), Warehousing and Materials Management (e_2), Loading, Unloading and Handling Management (e_3), Circulation and Processing

Management (e_4). The evaluation information of four transfer stations with respect to each attribute is represented by FHFNs. For example, the experts express their satisfaction degrees and dissatisfaction degrees of P_1 with respect to the attribute Transportation and Distribution Management (e_1) by some numbers between [0,1]. The higher the value of the satisfaction degree, the more satisfied, and the higher the value of the dissatisfaction degree, the less satisfied. Suppose the importance ranking of the attributes given by the experts is as follows: $e_4 \succ e_1 \succ e_2 \succ e_3$. Since different experts may have different opinions, the final result after negotiation is that the satisfaction degree is 0.2 and 0.6 and the dissatisfaction degree is 0.3 and 0.5. Therefore, the evaluation value of the alternative P_1 with respect to attribute e_1 can be denoted as a FHFN $\alpha_{11} = \langle\{0.2,0.6\},\{0.3,0.5\}\rangle$. Similarly, the assessment information of the rest alternatives is determined by $\alpha_{ij}(i = 1,2,\ldots,5, j = 1,2,\ldots,5)$, as shown in Table 3.

Table 3. Fermatean hesitant fuzzy decision matrix.

	e_1	e_2	e_3	e_4
P_1	$\langle\{0.4,0.6\},\{0.3,0.5\}\rangle$	$\langle\{0.6,0.8\},\{0.4,0.5\}\rangle$	$\langle\{0.7,0.8\},\{0.4,0.5\}\rangle$	$\langle\{0.5,0.7,0.8\},\{0.3\}\rangle$
P_2	$\langle\{0.4,0.5\},\{0.2,0.7\}\rangle$	$\langle\{0.3,0.4,0.5\},\{0.7\}\rangle$	$\langle\{0.3,0.4\},\{0.7,0.9\}\rangle$	$\langle\{0.6,0.9\},\{0.4,0.6\}\rangle$
P_3	$\langle\{0.3,0.5,0.7\},\{0.8\}\rangle$	$\langle\{0.7,0.8\},\{0.4,0.5\}\rangle$	$\langle\{0.7,0.8,0.9\},\{0.5\}\rangle$	$\langle\{0.5,0.8\},\{0.6,0.7\}\rangle$
P_4	$\langle\{0.5,0.7,0.8\},\{0.4\}\rangle$	$\langle\{0.3,0.7\},\{0.6,0.8\}\rangle$	$\langle\{0.6,0.7\},\{0.6,0.8\}\rangle$	$\langle\{0.6,0.7\},\{0.6,0.8\}\rangle$
P_5	$\langle\{0.7,0.9\},\{0.4,0.6\}\rangle$	$\langle\{0.8,0.9\},\{0.2,0.5\}\rangle$	$\langle\{0.6,0.7,0.8\},\{0.7\}\rangle$	$\langle\{0.4,0.7\},\{0.6,0.7\}\rangle$

Step 1. Normalization of the decision matrix is unnecessary because each attribute belongs to the benefit type. Then, calculate the exponential entropy $(E(\alpha_{ij}))_{5\times 4}$ of each attribute.

$$(E(\alpha_{ij}))_{5\times 4} = \begin{pmatrix} 0.9425 & 0.8936 & 0.8647 & 0.8394 \\ 0.8647 & 0.8912 & 0.7563 & 0.8141 \\ 0.7742 & 0.8647 & 0.8113 & 0.8647 \\ 0.8971 & 0.8263 & 0.8552 & 0.8552 \\ 0.7852 & 0.6682 & 0.8330 & 0.9041 \end{pmatrix}$$

Then, $E_1 = 0.8527$, $E_2 = 0.8288$, $E_3 = 0.8241$ and $E_4 = 0.8555$.

Calculate the attribute weight $\omega = (\omega_1, \omega_2, \omega_3, \omega_4)^T$ by the Fermatean hesitant fuzzy exponential entropy hybrid weighting method, then $\omega = (0.2305,0.2680,0.2753,0.2262)^T$.

Step 2. Compute the scoring function and accuracy function of the attribute values in Table 1, and then the cardinals of each set are supplemented from a pessimistic perspective and the positive and negative ideal solutions A^{+} and A^{-} are determined, as shown in Table 4. For example, the smallest values in the membership degree set of P_{11} and P_{12} are 0.4 and 0.6, respectively. Therefore, when completing the cardinal number of each set, 0.4 and 0.6 are, respectively, added to the corresponding membership degrees so that the number of elements in all membership sets are three. Since the number of elements in the membership set of P_{11} and P_{12} is the maximum of two, there is no need to complete the cardinal number of the non-membership set.

Table 4. The normalized processed Fermatean hesitant fuzzy decision matrix.

	e_1	e_2	e_3	e_4
P_1	⟨{0.4,0.4,0.6},{0.3,0.5}⟩	⟨{0.6,0.6,0.8},{0.4,0.5}⟩	⟨{0.7,0.7,0.8},{0.4,0.5}⟩	⟨{0.5,0.7,0.8},{0.3,0.3}⟩
P_2	⟨{0.4,0.4,0.5},{0.2,0.7}⟩	⟨{0.3,0.4,0.5},{0.7,0.7}⟩	⟨{0.3,0.3,0.4},{0.7,0.9}⟩	⟨{0.6,0.6,0.9},{0.4,0.6}⟩
P_3	⟨{0.3,0.5,0.7},{0.8,0.8}⟩	⟨{0.7,0.7,0.8},{0.4,0.5}⟩	⟨{0.7,0.8,0.9},{0.5,0.5}⟩	⟨{0.5,0.5,0.8},{0.6,0.7}⟩
P_4	⟨{0.5,0.7,0.8},{0.4,0.4}⟩	⟨{0.3,0.3,0.7},{0.6,0.8}⟩	⟨{0.6,0.6,0.7},{0.6,0.8}⟩	⟨{0.6,0.6,0.7},{0.6,0.8}⟩
P_5	⟨{0.7,0.7,0.9},{0.4,0.6}⟩	⟨{0.8,0.8,0.9},{0.2,0.5}⟩	⟨{0.6,0.7,0.8},{0.7,0.7}⟩	⟨{0.4,0.4,0.7},{0.6,0.7}⟩
A^{+}	⟨{0.7,0.7,0.9},{0.4,0.6}⟩	⟨{0.8,0.8,0.9},{0.2,0.5}⟩	⟨{0.7,0.8,0.9},{0.5,0.5}⟩	⟨{0.6,0.6,0.9},{0.4,0.6}⟩
A^{-}	⟨{0.3,0.5,0.7},{0.8,0.8}⟩	⟨{0.3,0.4,0.5},{0.7,0.7}⟩	⟨{0.3,0.3,0.4},{0.7,0.9}⟩	⟨{0.6,0.6,0.7},{0.6,0.8}⟩

Step 3. According to Definition 12, the distances between each attribute value α_{ij} and α_j^{+}, α_{ij} and α_j^{-} can be calculated respectively. The results are as follows:

$$(d(\alpha_{ij},\alpha_j^{+}))_{5\times4}=\begin{pmatrix}0.4405 & 0.2630 & 0.1702 & 0.2353\\ 0.4424 & 0.5238 & 0.5107 & 0.0000\\ 0.3846 & 0.1890 & 0.0000 & 0.1678\\ 0.2300 & 0.1440 & 0.2845 & 0.2306\\ 0.0000 & 0.0000 & 0.2123 & 0.2495\end{pmatrix},$$

$$(d(\alpha_{ij},\alpha_j^{-}))_{5\times4}=\begin{pmatrix}0.5083 & 0.2997 & 0.4550 & 0.3548\\ 0.4467 & 0.0000 & 0.0000 & 0.2078\\ 0.0000 & 0.3348 & 0.5601 & 0.1113\\ 0.4471 & 0.2086 & 0.3120 & 0.0000\\ 0.3846 & 0.5238 & 0.3340 & 0.1560\end{pmatrix}.$$

Step 4. Use normal distribution weighting to compute the position weighting vector of D_{GFHFHWD}, which is given as $\omega=(0.155,0.345,0.345,0.155)$. Then the distances between each attribute and the solutions A^{+} and A^{-} are calculated when parameter $\lambda=2$.

First, arrange $4\omega_j d(\alpha_{ij},\alpha_1)$ in descending order to calculate $D_{\text{GFHFHWD}}(P_1,A^{+})$, which is shown as follows:

$$D_{\mathrm{GFHFHWD}}(P_1, A^+) = 0.2722.$$

Similarly, it is easy to get

$$D_{\mathrm{GFHFHWD}}(P_2, A^+) = 0.4639$$
$$D_{\mathrm{GFHFHWD}}(P_3, A^+) = 0.2040$$
$$D_{\mathrm{GFHFHWD}}(P_4, A^+) = 0.3020$$
$$D_{\mathrm{GFHFHWD}}(P_5, A^+) = 0.1614$$
$$D_{\mathrm{GFHFHWD}}(P_1, A^-) = 0.4077$$
$$D_{\mathrm{GFHFHWD}}(P_2, A^-) = 0.2032$$
$$D_{\mathrm{GFHFHWD}}(P_3, A^-) = 0.3127$$
$$D_{\mathrm{GFHFHWD}}(P_4, A^-) = 0.2577$$
$$D_{\mathrm{GFHFHWD}}(P_5, A^-) = 0.3778$$

Step 5. Calculate the closeness $\delta_i$of each alternative, then $\delta_1 = 0.5997$, $\delta_2 = 0.3047$, $\delta_3 = 0.6052$, $\delta_4 = 0.4604$, $\delta_5 = 0.7006$. The ranking result is listed below:

$$P_5 \succ P_3 \succ P_1 \succ P_4 \succ P_2.$$

4.2. Parameter analysis

As shown in Table 5, the ranking will change when the parameter λ selects different values. That is, the value of parameter λ in $\mathrm{D_{GFHFHWD}}$ has a significant impact on the decision results. When $\lambda < 2$, the ranking is $\mathrm{P}_5 \succ \mathrm{P}_1 \succ \mathrm{P}_3 \succ \mathrm{P}_4 \succ \mathrm{P}_2$, while when $\lambda \geq 2$, the result is $\mathrm{P}_5 \succ \mathrm{P}_3 \succ \mathrm{P}_1 \succ \mathrm{P}_4 \succ \mathrm{P}_2$. With the increase of λ, the closeness of P_1 and P_2 decreases to varying degrees, whereas the closeness of P_3, P_4 and P_5 increases gradually. Particularly, when $\lambda \geq 2$, the sorting results tend to be stable and the distinction between different alternatives becomes smaller with the increase of λ. In addition, as can be seen from Table 3, the alternative P_2 always scores the lowest, which is consistent with the fact that the attribute values of P_2 in Table 1 are relatively small in all cases. Therefore, the decision approach based on the distance $\mathrm{D_{GFHFHWD}}$ is reasonable and stable, and it can sufficiently satisfy the multiple purposes and demands of different decision-makers.

Table 5. The ranking results of λ with different values.

	The closeness of all alternatives	The ranking results
$\lambda = 0.1$	(0.6063,0.0028,0.5524,0.1517,0.9993)	$P_5 \succ P_1 \succ P_3 \succ P_4 \succ P_2$
$\lambda = 0.2$	(0.6061,0.0368,0.5565,0.2929,0.9797)	$P_5 \succ P_1 \succ P_3 \succ P_4 \succ P_2$
$\lambda = 0.5$	(0.6052,0.1572,0.5683,0.4057,0.8601)	$P_5 \succ P_1 \succ P_3 \succ P_4 \succ P_2$
$\lambda = 1$	(0.6035,0.2436,0.5847,0.4443,0.7596)	$P_5 \succ P_1 \succ P_3 \succ P_4 \succ P_2$
$\lambda = 2$	(0.5997,0.3047,0.6052,0.4604,0.7006)	$P_5 \succ P_3 \succ P_1 \succ P_4 \succ P_2$
$\lambda = 4$	(0.5908,0.3514,0.6160,0.4636,0.6791)	$P_5 \succ P_3 \succ P_1 \succ P_4 \succ P_2$
$\lambda = 6$	(0.5822,0.3739,0.6154,0.4635,0.6792)	$P_5 \succ P_3 \succ P_1 \succ P_4 \succ P_2$
$\lambda = 10$	(0.5700,0.3941,0.6137,0.4639,0.6872)	$P_5 \succ P_3 \succ P_1 \succ P_4 \succ P_2$

The graphically comparison of ranking results for different values of λ is given in Figure 1.

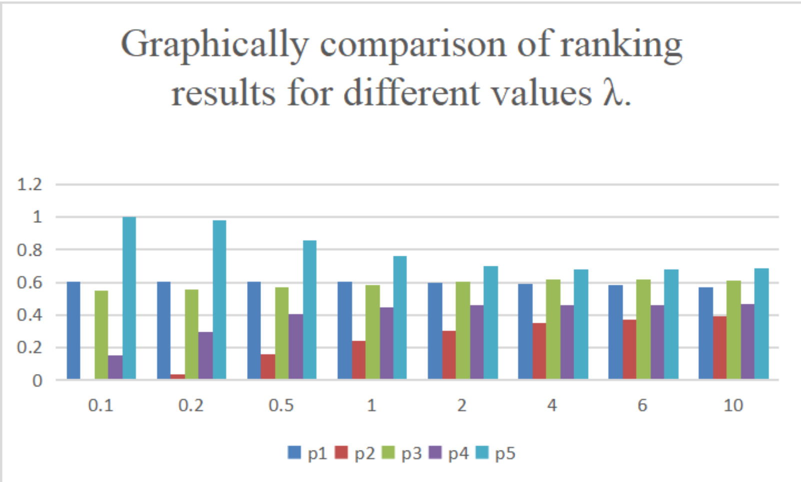

Figure 1. Graphically comparison of ranking results for different values λ.

4.3. Comparative analysis

In this section, we contrast and analyze the approach employed in this article with several Fermatean fuzzy DM algorithms that have recently been published, as stated in Table 6, in order to demonstrate its viability.

Yang et al. [53] proposed Fermatean fuzzy integrated weighted average distance (FFIWAD) measure and presented the FFIWAD-TOPSIS method to assess the green low-carbon port evaluation of five major ports in China. Kirişci [54] introduced new cosine similarity and Euclidean distance measures for Fermatean fuzzy sets, and used the suggested FFDMT-DM$^{\omega}_{FFS}$ and FFDMT-DM$_{FFS}$ to compute the separation of each alternative between positive ideal and negative ideal solutions. Mishra et al. [40] proposed a MADM method based on FHFS and the modified VIKOR method that can overcome the drawbacks of the existing MADM methods. The authorss of [25] looked at the Euclidean distance between two FFSs and developed a Fermatean fuzzy TOPSIS (FF-TOPSIS) approach to select a site for a house based on this information. Hamacher operations and FFS were merged by Hadi et al. [29] to create the Fermatean fuzzy Hamacher hybrid weight averaging (FFHHWA) operator. Literature [26] introduced two new weighted aggregation operators: Fermatean fuzzy weighted average (FFWA) and Fermatean fuzzy weighted geometric (FFWG) operators, and proposed a MADM method with Fermatean fuzzy imformation based on these operators.

Table 6. A comparison of the existing Fermatean fuzzy decision methods.

	The closeness of all alternatives	The ranking results
*D*GFHFHWD	(0.5997,0.3047,0.6052,0.4604,0.7006)	$P_5 \succ P_3 \succ P_1 \succ P_4 \succ P_2$
FFIWAD-TOPSIS [53]	(0.6070,0.2953,0.6129,0.4047,0.7671)	$P_5 \succ P_3 \succ P_1 \succ P_4 \succ P_2$
FFDMT-DMωFFS [54]	(0.3991,0.6832,0.3894,0.5952,0.2734)	$P_5 \succ P_3 \succ P_1 \succ P_4 \succ P_2$
FFDMT-DMFFS [54]	(0.3961,0.6592,0.4104,0.5909,0.2769)	$P_5 \succ P_1 \succ P_3 \succ P_4 \succ P_2$
FHF-VIKOR [40]	(0.5008,0.5090,0.4899,0.5117.0.4233)	$P_5 \succ P_3 \succ P_1 \succ P_2 \succ P_4$
FF-TOPSIS [25]	(-1.4576,-2.7835,-1.1789,-2.1386,-0.4433)	$P_5 \succ P_3 \succ P_1 \succ P_4 \succ P_2$
FFHHWA-MADM [29]	(0.3643,-0.5475,0.4235,0.3387,0.4569)	$P_5 \succ P_3 \succ P_1 \succ P_4 \succ P_2$
FFWA-MADM [26]	(0.2215,-0.1353,0.1143,0.0200,0.2226)	$P_5 \succ P_1 \succ P_3 \succ P_4 \succ P_2$
FFWG-MADM [26]	(0.2146,-0.1293,0.1151,-0.0066,0.2253)	$P_5 \succ P_1 \succ P_3 \succ P_4 \succ P_2$

The graphically comparison of the rankings of different Fermatean fuzzy decision methods is shown in Figure 2.

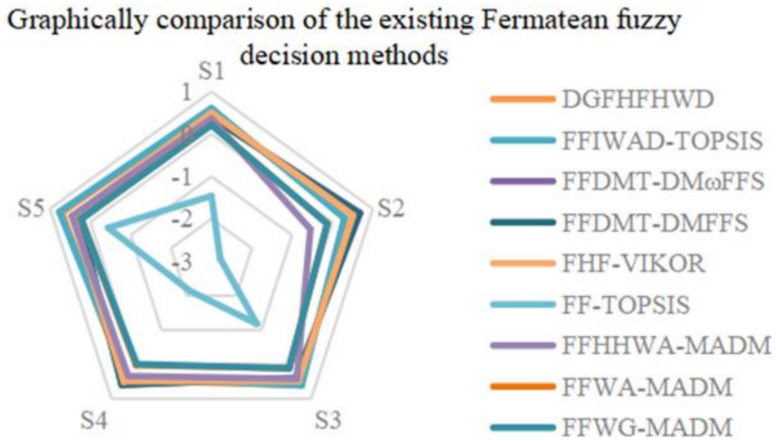

Figure 2. Graphically comparison of the exisrting Fermatean fuzzy decision methods.

The attribute weights for FFIWAD-TOPSIS, FFDMT-DM$^{\omega}{}_{FFS}$, FFDMT-DM$_{FFS}$, FHF-VIKOR, FF-TOPSIS, FFHHWA-MADM, FFWA-MADM and FFWG-MADM approaches are given in advance. To reduce the differences between the methods, all the above methods uniformly use the entropy-revised G1 combination weighting to process the weight. In the following, the ranking orders of the alternatives obtained by different MADM approaches are demonstrated, as shown in Table 6. From Table 6, it can be seen that although the sorting results obtained by different methods are slightly different, the optimal scheme remains unchanged and the overall ranking of the schemes is relatively stable. In particular, in FFDMT-DM$^{\omega}{}_{FFS}$ and FFDMT-DM$_{FFS}$ methods, the smaller the score value of the alternative, the better the scheme. Then, P_5 with the smallest scoring result is still the optimal scheme.

The MADM approaches presented in [25] and [29] cannot distinguish the ranking order of the alternatives sometimes when the membership or non-membership values in the calculation process become zero. The FFDMT-DM^{ω}_{FFS} and FFDMT-DM_{FFS} cannot indicate the sorting order of options well when the membership or non-membership values in the calculation process become zero. However, when the data volume is large, FFDMT-DM^{ω}_{FFS} and FFDMT-DM_{FFS} are difficult to get perfect results. The sorting of FFDMT-DM^{ω}_{FFS} and FFDMT-DM_{FFS} proposed in the literature [54] are different to some extent. Among them, the second-best scheme of FFDMT-DM^{ω}_{FFS} is P_3, while the second-best scheme of FFDMT-DM_{FFS} is P_1. This is because both methods have a small λ, and when the parameter value is larger, the results obtained by the two decision-making methods are more similar. Specifically, the FF-TOPSIS, FFIWAD-TOPSIS, FFDMT-DM^{ω}_{FFS}, FFDMT-DM_{FFS} and FHF-VIKOR MADM approaches are consistent with the basic framework of the proposed method. These methods use TOPSIS or VIKOR and different distance methods to obtain the closeness of each alternative. In conclusion, the worst choice of the above four methods is the same, which is consistent with the proposed D_{GFHFHWD} method. However, the FHF-VIKOR MADM method has the disadvantage of "ranking inversion or irregular ranking", so the worst choice becomes P_4. Compared with the above methods, the MADM method based on D_{GFHFHWD} in this paper focuses on the importance of each attribute information and its location and fully considers the hesitancy of evaluators, so both subjective and objective weight information of attributes are considered. In conclusion, the best alternative of the above aggregation methods remains the same, which proves the feasibility and rationality of the presented decision-making method in this paper.

The decision-making methods in references [25,26,29,53,54] are based on FFS. Compared with FHFS, these methods fail to consider the hesitancy of the evaluators. FFS has a limited ability to deal with complex decision-making problems and can only use single values to express the degree of agreement and disagreement of evaluators. However, in many real-world decision situations, the evaluators are often hesitant between multiple values, and it is difficult for them to express their preferences with single values. The MADM method in this paper uses FHFNs to represent the evaluation information, which reflects the hesitancy of the evaluators between these numbers. To make these methods suitable for Fermatean hesitant fuzzy environment, we add the number of elements in the membership and non-membership sets from a pessimistic perspective, which fully reflects the hesitancy of the evaluators during the decision-making process. By doing so, the evaluators can express agreement and disagreement with multiple possible values. Therefore, when considering complex unwillingness, the MADM method based on HFFS is superior to that based on FFS.

5. Conclusions

For MADM problems with unknown attribute weights and FHFNs, we define several new distance measures between different FHFSs, including the weighted distance measure (D_{FHFWD}), the ordered weighted distance measure (D_{FHFOWD}) and the hybrid weighted distance measure (D_{FHFHWD}). Then, a new generalized Fermatean hesitant fuzzy hybrid weighted distance measure, which is represented as D_{GFHFHWD}, is constructed and its related properties and special forms are discussed. This new distance measure is obtained according to the exponential entropy hybrid weighting method. It can effectively make full use of the existing evaluation information, solve the allocation problem of subjective and objective weight and eliminate the influence of unduly large or unduly small deviations on the decision results. On this basis, a modified Fermatean hesitant fuzzy TOPSIS method based on D_{GFHFHWD} is proposed, and a combined attribute weighting method combining exponential entropy and normal distribution weighting is constructed. Finally, the feasibility and rationality of the proposed MADM method are illustrated by a numerical example, and the influence of parameter λ on the ranking results is discussed as well. Through comparative analysis, it is easy to see that as different values are selected, the sorting results also change, indicating that the values of λ has a significant impact on the decision results. When $\lambda \geq 2$, the sorting results tend to be stable, and the distinction between different alternatives descends with the increase of λ.

The MADM method proposed in this paper effectively combines the TOPSIS method and the HWD measure under the Fermatean hesitant fuzzy environment. Not only the importance of the attributes is considered, but also the importance of the location of the attributes is reflected, so as to obtain a more scientific and reasonable decision result. Moreover, we further analyze the influence of the change of parameter λ in D_{GFHFHWD} on the closeness and decision result, helping experts choose the appropriate parameter according to their needs and preferences. In conclusion, the developed distance measures and decision method have broad application prospects and important theoretical value. They can enrich the theoretical system of FHFS, provide new methods and ideas for many DM problems and meet the increasingly complex needs of decision- makers in real life. However, in view of the unequal number of the membership and non-membership degrees in different FHFNs, this paper selects and adds the smallest value from a pessimistic perspective, which may increase the subjectivity during the decision process, change the original information and lead to DM.

In the future, we may show the applicability of the suggested MADM method for FHFSs in more fields, such as green supplier selection, new product investment, project evaluation and so on. The developed distance measure of FFSs can not only effectively improve the defects of the existing TOPSIS method, but also can be extended to various classical decision methods and generations of FFSs, which has certain promotion value, including VIKOR, TODIM, preference ranking organisation method for enrichment evaluations (PROMETHEE) [72], complex FHFSs [73] and so on. Furthermore, how to define the Fermatean hesitant fuzzy distance measures of dimension reduction [74], which can reflect the original information and solve the shortcoming of the traditional data completion scheme which needs to artificially add the maximum or minimum membership value, is a problem worthy of further research and discussion.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest.

References

1. Y. Seo, S. Kim, O. Kisi, V. P. Singh, Daily water level forecasting using wavelet decomposition and artificial intelligence techniques, *J. Hydrol.*, **520** (2015), 224–243. https://doi.org/10.1016/j.jhydrol.2014.11.050
2. G. Wei, M. Lu, Pythagorean fuzzy power aggregation operators in multiple attribute decision making, *Int. J. Intell. Syst.*, **33** (2018), 169–186. https://doi.org/10.1002/int.21946
3. O. Castillo, L. Amador-Angulo, J. R. Castro, M. Garcia-Valdez, A comparative study of type-1 fuzzy logic systems, interval type-2 fuzzy logic systems and generalized type-2 fuzzy logic systems in control problems, *Inform. Sci.*, **354** (2016), 257–274. https://doi.org/10.1016/j.ins.2016.03.026
4. G. Wei, Y. Wei, Similarity measures of Pythagorean fuzzy sets based on the cosine function and their applications, *Int. J. Intell. Syst.*, **33** (2018), 634–652. https://doi.org/10.1002/int.21965
5. L. Sahoo, Some score functions on Fermatean fuzzy sets and its application to bride selection based on TOPSIS method, *Int. J. Fuzzy Syst. Appl.*, **10** (2021), 18–29. https://doi.org/10.4018/IJFSA.2021070102
6. H. T. X. Chi, F. Y. Vincent, Ranking generalized fuzzy numbers based on centroid and rank index, *Appl. Soft Comput.*, **68** (2018), 283–292. https://doi.org/10.1016/j.asoc.2018.03.050
7. M. S. Kuo, G. S. Liang, W. C. Huang, Extensions of the multicriteria analysis with pairwise comparison under a fuzzy environment, *Int. J. Approx. Reason.*, **43** (2006), 268–285. https://doi.org/10.1016/j.ijar.2006.04.006

8. J. Pan, S. Rahman, Multiattribute utility analysis with imprecise information: An enhanced decision support technique for the evaluation of electric generation expansion strategies, *Electr. Pow. Syst. Res.*, **46** (1998), 101–109. https://doi.org/10.1016/S0378-7796(98)00022-4
9. S. Y. Chou, Y. H. Chang, C. Y. Shen, A fuzzy simple additive weighting system under group decision-making for facility location selection with objective/subjective attributes, *Eur. J. Oper. Res.*, **189** (2008), 132–145. https://doi.org/10.1016/j.ejor.2007.05.006
10. C. L. Hwang, K. Yoon, Methods for multiple attribute decision making, In: *Multiple Attribute Decision Making*, Berlin, Heidelberg: Springer, 1981, 58–191. https://doi.org/10.1007/978-3-642-48318-9_3
11. G. Nalcaci, A. Özmen, G. W. Weber, Long-term load forecasting: models based on MARS, ANN and LR methods, *Cent. Eur. J. Oper. Res.*, **27** (2019), 1033–1049. https://doi.org/10.1007/s10100-018-0531-1
12. M. A. Ahmadi, M. Ebadi, A. Shokrollahi, S. M. J. Majidi, Evolving artificial neural network and imperialist competitive algorithm for prediction oil flow rate of the reservoir, *Appl. Soft Comput.*, **13** (2013), 1085–1098. https://doi.org/10.1016/j.asoc.2012.10.009
13. A. F. Hayes, A. K. Montoya, A tutorial on testing, visualizing, and probing an interaction involving a multicategorical variable in linear regression analysis, *Commun. Methods. Meas.*, **11** (2017), 1–30. https://doi.org/10.1080/19312458.2016.1271116
14. L. A. Zadeh, Fuzzy sets, *Inf. Control*, **8** (1965), 338–353. https://doi.org/10.1016/S0019-9958(65)90241-X
15. K. T. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Sets Syst.*, **20** (1986), 87–96. https://dl.acm.org/doi/10.5555/1708507.1708520
16. Z. Xu, Intuitionistic fuzzy aggregation operator, *IEEE T. Fuzzy Syst.*, **15** (2007), 1179–1187. https://doi.org/10.1109/TFUZZ.2006.890678
17. H. Garg, A new generalized improved score function of interval-valued intuitionistic fuzzy sets and applications in expert systems, *Appl. Soft Comput.*, **38** (2016), 988–999. https://doi.org/10.1016/j.asoc.2015.10.040
18. F. Feng, H. Fujita, M. I. Ali, R. R. Yager, X. Liu, Another view on generalized intuitionistic fuzzy soft sets and related multiattribute decision making methods, *IEEE T. Fuzzy Syst.*, **27** (2019), 474–488. https://doi.org/10.1109/TFUZZ.2018.2860967
19. R. R. Yager, A. M. Abbasov, Pythagorean membership grades, complex numbers, and decision making, *Int. J. Intell. Syst.*, **28** (2013), 436–452. https://doi.org/10.1002/int.21584
20. Z. Li, G. Wei, M. Lu, Pythagorean fuzzy hamy mean operators in multiple attribute group decision making and their application to supplier selection, *Symmetry*, **10** (2018), 505. https://doi.org/10.3390/sym10100505
21. K. Naeem, M. Riaz, X. Peng, D. Afzal, Pythagorean fuzzy soft MCGDM methods based on TOPSIS, VIKOR and aggregation operators, *J. Intell. Fuzzy Syst.*, **37** (2019), 6937–6957. https://doi.org/10.3233/JIFS-190905
22. M. Akram, M. Ramzan, M. Deveci, Linguistic Pythagorean fuzzy CRITIC-EDAS method for multiple-attribute group decision analysis, *Eng. Appl. Artif. Intell.*, **119** (2023), 105777. https://doi.org/10.1016/j.engappai.2022.105777
23. S. Singh, A. H. Ganie, On some correlation coefficients in Pythagorean fuzzy environment with applications, *Int. J. Intell. Syst.*, **35** (2020), 682–717. https://doi.org/10.1002/int.22222

24. R. Verma, J. M. Merigó, On generalized similarity measures for Pythagorean fuzzy sets and their applications to multiple attribute decision-making, *Int. J. Intell. Syst.*, **34** (2019), 2556–2583. https://doi.org/10.1002/int.22160
25. T. Senapati, R. R. Yager, Fermatean fuzzy sets, *J. Ambient Intell. Hum. Comput.*, **11** (2020), 663–674. https://doi.org/10.1007/s12652-019-01377-0
26. T. Senapati, R. R. Yager, Fermatean fuzzy weighted averaging/geometric operators and its application in multi-criteria decision-making methods, *Eng. Appl. Artif. Intell.*, **85** (2019), 112–121. https://doi.org/10.1016/j.engappai.2019.05.012
27. Y. Pan, S. Z. Zeng, W. Chen, J. Gu, Service quality evaluation of crowdsourcing logistics platform based on Fermatean fuzzy TODIM and regret theory, *Eng. Appl. Artif. Intell.*, **123** (2023), 106385. https://doi.org/10.1016/j.engappai.2023.106385
28. A. H. Ganie, Multicriteria decision-making based on distance measures and knowledge measures of Fermatean fuzzy sets, *Granul. Comput.*, **7** (2022), 979–998. https://doi.org/10.1007/s41066-021-00309-8
29. Z. Deng, J. Wang, New distance measure for Fermatean fuzzy sets and its application, *Int. J. Intell. Syst.*, **37** (2022), 1903–1930. https://doi.org/10.1002/int.22760
30. C. Xu, J. Shen, Multi-criteria decision making and pattern recognition based on similarity measures for Fermatean fuzzy sets, *J. Intell. Fuzzy Syst.*, **41** (2021), 5847–5863. https://doi.org/10.3233/JIFS-201557
31. L. Sahoo, A new score function based Fermatean fuzzy transportation problem, *Results Control Optim.*, **4** (2022), 100040. https://doi.org/10.1016/j.rico.2021.100040
32. L. Zhou, S. Wan, J. Dong, A Fermatean fuzzy ELECTRE method for multi-criteria group decision-making, *Informatica*, **33** (2022), 181–224. https://doi.org/10.15388/21-INFOR463
33. V. Torra, Hesitant fuzzy sets, *Int. J. Intell. Syst.*, **25** (2010), 529–539. https://doi.org/10.1002/int.20418
34. J. Peng, J. Wang, X. Wu, Novel multi-criteria decision-making approaches based on hesitant fuzzy sets and prospect theory, *Int. J. Inf. Tech. Decis.*, **15** (2016), 621–643. https://doi.org/10.1142/S0219622016500152
35. J. Peng, J. Wang, X. Wu, H. Zhang, X. Chen, The fuzzy cross-entropy for intuitionistic hesitant fuzzy sets and their application in multi-criteria decision-making, *Int. J. Syst. Sci.*, **46** (2015), 2335–2350. https://doi.org/10.1080/00207721.2014.993744
36. B. Farhadinia, Information measures for hesitant fuzzy sets and interval-valued hesitant fuzzy sets, *Inform. Sci.*, **240** (2013), 129–144. https://doi.org/10.1016/j.ins.2013.03.034
37. M. Khan, S. Abdullah, A. Ali, F. Amin, F. Hussain, Pythagorean hesitant fuzzy Choquet integral aggregation operators and their application to multi-attribute decision-making, *Soft Comput.*, **23** (2019), 251–267. https://doi.org/10.1007/s00500-018-3592-0
38. A. Hussain, M. I. Ali, T. Mahmood, Hesitant q-rung orthopair fuzzy aggregation operators with their applications in multi-criteria decision making, *Iran. J. Fuzzy Syst.*, **17** (2020), 117–134. https://doi.org/10.22111/IJFS.2020.5353
39. H. M. A. Farid, M. Riaz, B. Almohsin, D. Marinkovic, Optimizing filtration technology for contamination control in gas processing plants using hesitant q-rung orthopair fuzzy information aggregation, *Soft Comput.*, 2023. https://doi.org/10.1007/s00500-023-08588-w

40. A. R. Mishra, S. M. Chen, P. Rani, Multiattribute decision making based on Fermatean hesitant fuzzy sets and modified VIKOR method, *Inform. Sci.*, **607** (2022), 1532–1549. https://doi.org/10.1016/j.ins.2022.06.037
41. M. Kirişci, Fermatean Hesitant Fuzzy Sets for Multiple Criteria Decision-Making with Applications, *Fuzzy Inform. Eng.*, **15** (2023), 100–127. https://doi.org/10.26599/fie.2023.9270011
42. A. Khan, M. Aslam, Q. Iqbal, Cyclone disaster assessment based on Fermatean hesitant fuzzy information and extended TOPSIS method, *J. Intell. Fuzzy Syst.*, **44** (2023), 10633–10660. https://doi.org/10.3233/JIFS-222144
43. H. Lai, H. Liao, Y. Long, E. K. Zavadskas, A Hesitant Fermatean Fuzzy CoCoSo Method for Group Decision-Making and an Application to Blockchain Platform Evaluation, *Int. J. Fuzzy Syst.*, **24** (2022), 2643–2661. https://doi.org/10.1007/s40815-022-01319-7
44. Y. Wang, X. Ma, H. Qin, H. Sun, W. Wei, Hesitant Fermatean fuzzy Bonferroni mean operators for multi-attribute decision-making, *Complex Intell. Syst.*, 2023. https://doi.org/10.1007/s40747-023-01203-3
45. C. Y. Ruan, X. J. Chen, L. N. Han, Fermatean Hesitant Fuzzy Prioritized Heronian Mean Operator and Its Application in Multi-Attribute Decision Making, *Comput. Mater. Con.*, **75** (2023), 3204–3222. https://doi.org/10.32604/cmc.2023.035480
46. L. Sha, Y. Shao, Fermatean Hesitant Fuzzy Choquet Integral Aggregation Operators, *IEEE Access*, **11** (2023), 38548–38562. https://doi.org/10.1109/ACCESS.2023.3267512
47. A. R. Mishra, P. Liu, P. Rani, COPRAS method based on interval-valued hesitant Fermatean fuzzy sets and its application in selecting desalination technology, *Appl. Soft Comput.*, **119** (2022), 108570. https://doi.org/10.1016/j.asoc.2022.108570
48. I. Demir, Novel correlation coefficients for interval-valued Fermatean hesitant fuzzy sets with pattern recognition application, *Turkish J. Math.*, **47** (2023), 213–233. https://doi.org/10.55730/1300-0098.3355
49. W. Zeng, H. Cui, Y. Liu, Q. Yin, Z. Xu, Novel distance measure between intuitionistic fuzzy sets and its application in pattern recognition, *Iran. J. Fuzzy Syst.*, **19** (2022), 127–137. https://doi.org/10.22111/IJFS.2022.6947
50. Y. Washizawa, S. Hotta, Mahalanobis distance on extended Grassmann manifolds for variational pattern analysis, *IEEE T. Neur. Net. Learn. Syst.*, **25** (2014), 1980–1990. https://doi.org/10.1109/TNNLS.2014.2301178
51. H. Kamacı, F. Marinkovic, S. Petchimuthu, M. Riaz, S. Ashraf, Novel distance-measures-based extended TOPSIS method under linguistic linear Diophantine fuzzy information, *Symmetry*, **14** (2022), 2140. https://doi.org/10.3390/sym14102140
52. D. Liu, Y. Liu, L. Wang, Distance measure for Fermatean fuzzy linguistic term sets based on linguistic scale function: An illustration of the TODIM and TOPSIS methods, *Int. J. Intell. Syst.*, **34** (2019), 2807–2834. https://doi.org/10.1002/int.22162
53. S. Yang, Y. Pan, S. Zeng, Decision making framework based Fermatean fuzzy integrated weighted distance and TOPSIS for green low-carbon port evaluation, *Eng. Appl. Artif. Intell.*, **114** (2022), 105048. https://doi.org/10.1016/j.engappai.2022.105048
54. M. Kirişci, New cosine similarity and distance measures for Fermatean fuzzy sets and TOPSIS approach, *Knowl. Inf. Syst.*, **65** (2023), 855–868. https://doi.org/10.1007/s10115-022-01776-4

55. A. H. Ganie, Multicriteria decision-making based on distance measures and knowledge measures of Fermatean fuzzy sets, *Granul. Comput.*, **7** (2022), 979–998. https://doi.org/10.1007/s41066-021-00309-8
56. Z. Deng, J. Wang, New distance measure for Fermatean fuzzy sets and its application, *Int. J. Intell. Syst.*, **37** (2022), 1903–1930. https://doi.org/10.1002/int.22760
57. S. Zeng, J. Gu, X. Peng, Low-carbon cities comprehensive evaluation method based on Fermatean fuzzy hybrid distance measure and TOPSIS, *Artif. Intell. Rev.*, **56** (2023), 8591–8607. https://doi.org/10.1007/s10462-022-10387-y
58. Z. Xu, J. Chen, Ordered weighted distance measure, *J. Syst. Sci. Syst. Eng.*, **17** (2008), 432–445. https://doi.org/10.1007/s11518-008-5084-8
59. L. Zhou, J. Wu, H. Chen, Linguistic continuous ordered weighted distance measure and its application to multiple attributes group decision making, *Appl. Soft Comput.*, **25** (2014), 266–276. https://doi.org/10.1016/j.asoc.2014.09.027
60. S. Zeng, J. M. Merigo, D. Palacios-Marques, H. Jin, F. Gu, Intuitionistic fuzzy induced ordered weighted averaging distance operator and its application to decision making, *J. Intell. Fuzzy Syst.*, **32** (2017), 11–22. https://doi.org/10.3233/JIFS-141219
61. Y. Qin, Y. Liu, Z. Hong, Multicriteria decision making method based on generalized Pythagorean fuzzy ordered weighted distance measures, *J. Intell. Fuzzy Syst.*, **33** (2017), 3665–3675. https://doi.org/10.3233/JIFS-17506
62. Z. Xu, M. Xia, Distance and similarity measures for hesitant fuzzy sets, *Inform. Sci.*, **181** (2011), 2128–2138. https://doi.org/10.1016/j.ins.2011.01.028
63. H. Liu, J. You, X. You, Evaluating the risk of healthcare failure modes using interval 2-tuple hybrid weighted distance measure, *Comput. Ind. Eng.*, **78** (2014), 249–258. https://doi.org/10.1016/j.cie.2014.07.018
64. S. Zeng, J. Chen, X. Li, A hybrid method for Pythagorean fuzzy multiple-criteria decision making, *Int. J. Inf. Tech. Decis.*, **15** (2016), 403–422. https://doi.org/10.1142/S0219622016500012
65. Q. Ding, Y. Wang, M. Goh, TODIM dynamic emergency decision-making method based on hybrid weighted distance under probabilistic hesitant fuzzy information, *Int. J. Fuzzy Syst.*, **23** (2021), 474–491. https://doi.org/10.1007/s40815-020-00978-8
66. S. Zeng, J. Gu, X. Peng, Low-carbon cities comprehensive evaluation method based on Fermatean fuzzy hybrid distance measure and TOPSIS, *Artif. Intell. Rev.*, **56** (2023), 8591–8607. https://doi.org/10.1007/s10462-022-10387-y
67. Z. Xu, An overview of methods for determining OWA weights, *Int. J. Intell. Syst.*, **20** (2005), 843–865. https://doi.org/10.1002/int.20097
68. X. Gou, Z. Xu, H. Liao, Hesitant fuzzy linguistic entropy and cross-entropy measures and alternative queuing method for multiple criteria decision making, *Inform. Sci.*, **388** (2017), 225–246. https://doi.org/10.1016/j.ins.2017.01.033
69. J. Rezaei, T. Nispeling, J. Sarkis, L. Tavasszy, A supplier selection life cycle approach integrating traditional and environmental criteria using the best worst method, *J. Clean. Prod.*, **135** (2016), 577–588. https://doi.org/10.1016/j.jclepro.2016.06.125
70. K. Krishnamoorthy, M. Lee, Improved tests for the equality of normal coefficients of variation, *Comput. Stat.*, **29** (2014), 215–232. https://doi.org/10.1007/s00180-013-0445-2

71. S. Zeng, Y. Hu, C. Llopis-Albert, Stakeholder-inclusive multi-criteria development of smart cities, *J. Bus. Res.*, **154** (2023), 113281. https://doi.org/10.1016/j.jbusres.2022.08.045
72. B. Sennaroglu, G. V. Celebi, A military airport location selection by AHP integrated PROMETHEE and VIKOR methods, *Transport. Res. D-Tr. E.*, **59** (2018), 160–173. https://doi.org/10.1016/j.trd.2017.12.022
73. M. S. A. Khan, F. Anjum, I. Ullah, T. Senapati, S. Moslem, Priority Degrees and Distance Measures of Complex Hesitant Fuzzy Sets With Application to Multi-Criteria Decision Making, *IEEE Access*, **11** (2023), 13647–13666. https://doi.org/10.1109/ACCESS.2022.3232371
74. X. Sha, C. Yin, Z. Xu, Weighted hesitant fuzzy Lance distance measure of dimension reduction based on exponential entropy and its application, *Control. Decis.*, **35** (2020), 728–734. https://doi.org/10.13195/j.kzyjc.2018.0910

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