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Research article

Solution of linear correlated fuzzy differential equations in the linear correlated fuzzy spaces

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Abstract: Linear correlated fuzzy differential equations (LCFDEs) are a valuable approach to handling physical problems, optimizations problems, linear programming problems etc. with uncertainty. But, LCFDEs employed on spaces with symmetric basic fuzzy numbers often exhibit multiple solutions due to the extension process. This abundance of solutions poses challenges in the existing literature's solution methods for LCFDEs. These limitations have led to reduced applicability of LCFDEs in dealing with such types of problems. Therefore, in the current study, we focus on establishing existence and uniqueness results for LCFDEs. Moreover, we will discuss solutions in the canonical form of LCFDEs in the space of symmetric basic fuzzy number which is currently absent in the literature. To enhance the practicality of our work, we provide examples and plots to illustrate our findings.

Keywords: Non-increasing diameter; non-decreasing diameter; extend system; uncertainty; nodal point; symmetric and non-symmetric numbers; linear correlation; nodal point
Mathematics Subject Classification: 26E50, 35R13

Table 1. Nomenclature tabular column.

Name	Symbole
Space of fuzzy numbers	R_F
LC-spaces with non-symmetric BF-number A	$R^n_{F(A)}$
LC-spaces with symmetric BF-number A	$R^s_{F(A)}$
Length or diameter of Fuzzy number μ	$len(B)$
LC-fuzzy numbers of Non-symmetric A	$\Psi_A(p, r)$
LC-fuzzy numbers of symmetric A	$\widetilde{\Psi}_A([(p, s)]_{\equiv_A})$
Metric on LC-space of non-symmetric A	d_{Ψ_A}
Metric on LC-space of symmetric A	$d_{\widetilde{\Psi}_A}$
Addition operator of F-numbers	$\oplus_A$
Operator of scalar multiplication	$\odot_A$
LC-Fuzzy difference operator	$\boxminus_A$
Equivalence relation	$\equiv_A$
Equivalence class	$[(p, s)]_{\equiv_A}$
Quotient in R^2	$R^2/\equiv_A$

1. Introduction

Fuzzy numbers (F-numbers) and fuzzy operations were introduced by Zadeh [1] and Zadeh's extension principle respectively. Fuzzy set theory is a strong mathematical tool to deal with the complexity generally arising from uncertainty in real-life scenarios. The fuzzy concept is important for the optimization of problems, like the robot routing problem, to optimize path length and energy consumption [2]. Fuzzy numbers are also important in the banking industry in data envelopment analysis [3] and linear programming problems. The fuzzy concept is also used in medical resource allocation [4] and medical health resource allocation evaluation in public health emergencies [5]. Hesitant fuzzy linguistic entropy [6] and probabilistic double hierarchy linguistic term set [7] are important in decision making. The decision making model for operating systems and human-computer interaction are studied by [8] and [9] respectively. The arithmetic operations of F-numbers indicate they are non-interactive due to the operation of random variables. Therefore, Carlsson et al. [10] introduce interactive F-numbers. The interactivities were based on some fuzzy joint distribution functions. The operations of interactive F-numbers were obtained by the generalized extension principle. To relax the need of a joint distribution function, linear correlated fuzzy numbers (LCF-numbers) were introduced by Barros and Pedro [11]. Esmi et al. [12] define an operator from a two dimension Euclidean space to the space of LCF-numbers. This operator is a bijection if the basic fuzzy numbers (BF-numbers) of LCF-numbers are non-symmetric and therefore operations in LCF-numbers are defined from the linear isomorphism. But, the operator is not bijection in the case of symmetric BF-number, therefore operations in LCF-numbers are not possible to be defined by this process. Therefore, Shen [13] define an equivalence relation in R^2 if the BF-number is symmetric and uses canonical representation to define the bijection from an equivalent class to the space of LCF-numbers. The operations in the LC-space of symmetric BF-number was obtained by linear isomorphism. Just like the modification in F-numbers and operations, fuzzy differentiability has also been a focus for researchers. Consequently some fuzzy differentiations were introduced in the fuzzy calculus. The most prominent fuzzy differentiations are H-differentiability [14], generalized differentiability [15], gH-differentiability [16], interactive differentiability [11], Fŕechet differentiability [12] and LC-differentiability [13]. To discuss linear correlated fuzzy differential equations (LCFDEs) in the LC-spaces of both non-symmetric and symmetric BF-numbers, Shen [17] discussed the calculus of LCF-numbers and [18] studied first order FDEs with LC-differentiability. The main importance of FDEs is to dealt the uncertainty in the solutions of problems. A problem is dealt with in a model if the solution is unique. But, a unique solution of any differential equation, and especially fore LCFDEs with symmetric BF-numbers, is not possible. The results of [19] ensure a unique solution of the problem discussed in [18] by reducing the solution region to the nearest extension point. Moreover, in the existing literature, the solutions methods of LCFDEs have some difficulties. These drawbacks reduce the usability of LCFDEs to deal with such types of problems.

In the current work, we will study the existence and uniqueness results of LCFDEs to ensure a unique solution in the LC-spaces of non-symmetric and symmetric BF-numbers. This study will point out the cause due to which the LCFDEs do not have a unique solution and the solution extended to a new system at the nodal points. In the existing literature of LCFDEs in the LC-spaces of symmetric basic fuzzy number $R^s_{F(A)}$ the LCFDEs are taken with the canonical form but the solution is not in the canonical form, therefore in this study we will discuss the canonical form of solution of LCFDEs in the space $R^s_{F(A)}$. Moreover, in this study we will discuss the importance of the conical form of solution. The main cause of the extension of LCFDEs and the non-availability of a unique canonical form of solution is the form of LCFDEs. If first order LCFDEs have the following form then solution does not extend, therefore one of the basic difficulties of existence of a unique solution is solved

$$\begin{cases} z^{'}(t) = g(t, z(t)), \ t \in I, \\ z(0) = z_0, \end{cases} \tag{1.1}$$

where $g : I \times R_{F(A)} \to R_{F(A)}$ is a continuous LC fuzzy number valued function. The first order LCFDE discussed in [18] do not have a unique solution due to the extension of the solution to a new system at the nodal points, but the solution of Eq (1.1) does not extend to new system at the nodal points even when the extension conditions holds. At the nodal points of solution of Eq (1.1) alternately the non-increasing and non-decreasing diameter will change and no new solution of extend system will exist. In this study we will discuss the concept of the solution of the canonical form having a pair of solutions with non-increasing and non-decreasing diameter in the LC-spaces of symmetric BF-numbers. These are the main contribution of this work. Moreover, Eq (1.1) will produce a form similar to the first order FDEs discussed in [18], which have unique solution. To show the validity of this manuscript, examples and their $2D$ and $3D$ fuzzy plots of solutions are also provided.

2. Preliminaries

Now, we provide the mathematical background used in the current work.

2.1. The space of fuzzy numbers

The upper semi continuous and fuzzy convex mapping $B : R \to [0, 1]$ is an F-number if B is normal at some $t_0 \in R$ and the closure of $\{t \in R, B(t) > 0\}$ is compact. The set R_F of all F-numbers is called the space of fuzzy numbers [20].

Let $B \in R_F$ be a fuzzy number. Then, the set $\{t \in R \mid B(t) \geq \alpha\}$ with $\alpha \in [0,1]$ is called the $\alpha-$level set of the F-number. If $\underline{B}(t)$ and $\overline{B}(t)$ are the lower and upper bound of the α-level set, then $B = [\underline{B}(t), \overline{B}(t)]$, see [21]. In particular, the triple $B_1 = (t_1; t_2; t_3)$ and quadruple $B_2 = (t_1; t_2; t_3; t_4)$, represent triangular and trapezoidal F-numbers, respectively, where $t_1 \leq t_2 \leq t_3 \leq t_4$. Their respective $\alpha-$level sets are given by

$$[B_1]_\alpha = [t_1 + (t_2 - t_1)\alpha, t_3 - (t_3 - t_2)\alpha],$$

$$[B_2]_\alpha = [t_1 + (t_2 - t_1)\alpha, t_4 - (t_4 - t_3)\alpha].$$

The set $B_0 = \{t \in R \mid B(t) \geq 0\}$ is the support of F-number B. Meanwhile $len(B) = \overline{B}_0 - \underline{B}_0$ is the length of the support, known as the diameter of B.

The fuzzy number B is a symmetric F-number with respect to t_0, if for unique $t_0 \in R$, $B(t + t_0) = B(t_0 - t)$ for all $t \in R$; otherwise, B is non-symmetric.

2.2. The space of linear correlated fuzzy numbers

Let $\Psi_A : R^2 \to R_F$ be an operator such that, for all $A \in R_F$, each pair (p, s) is associated with an F-number $\Psi_A(p, s)$. If there exist $(p, s) \in R^2$ such that $B = \Psi_A(p, s) = pA + s$, then B is called an $A-$linear correlated fuzzy number(LCF-number). The set $\{\Psi_A(p, s) \mid \forall\, (p, s) \in R^2\}$ which consists of all $A-$linear correlated fuzzy numbers is called the space of A-linear correlated fuzzy numbers $R_{F(A)}$. If the basic F-number A is non-symmetric, then the LC-space is denoted by $R^n_{F(A)}$, but in the case of symmetric basic F-number A the LC-space is denoted by $R^s_{F(A)}$. Moreover, $[\Psi_A(p, s)]_\alpha = \{pt+s \in R \mid t \in [A]_\alpha\} = p[A]_\alpha+s$ is the α-level set. Clearly, if A is a non-symmetric F-number then Ψ_A is one-to-one and onto, therefore $(R^n_{F(A)}, \oplus_A, \odot_A)$ is a linear space where $\oplus_A$ and $\odot_A$ are defined as $B_1 \oplus_A B_2 = \Psi_A({\Psi_A}^{-1}(B_1) + {\Psi_A}^{-1}(B_2))$ and $\beta \odot_A B = \Psi_A(\beta {\Psi_A}^{-1}(B))$. But, if $A \in R_F \backslash R$ is symmetric, then $\Psi_A : R^2 \to R_{F(A)}$ is not one-to-one because $\Psi_A(p, s) = \Psi_A(-p, 2py + s)$, where y is a symmetric point. Now, we need to define an equivalence relation. Let, for any $(p, s), (q, r) \in R^2$, the equivalence relation be defined as $(p, s) \equiv_A (q, r)$ if and only if $(p, s) = (q, r)$ or $(p, s) = (-q, 2qy + r)$. With equivalence relation $\equiv_A$, the quotient in R^2 is defined as $R^2/\equiv_A = \{[(p, s)]_{\equiv_A} \mid (p, s) \in R^2\}$, where $[(p, s)]_{\equiv_A} = \{(p, s), (-p, 2py + s)\}$ is the equivalence class. The operations $\oplus_A$ and $\odot_A$ in $R^2/\equiv_A$ are defined as $[(p, s)]_{\equiv_A} \oplus_A [(q, r)]_{\equiv_A} = [(p + q, s + r)]_{\equiv_A}$ and

$$\beta \odot_A [(p, s)]_{\equiv_A} = \begin{cases} [(\beta p, \beta s)]_{\equiv_A}, \beta \geq 0 \\ [(-\beta p, 2\beta py + \beta s)]_{\equiv_A}, \beta < 0. \end{cases}$$

If $\widetilde{\Psi}_A([(p, s)]_{\equiv_A}) = \widetilde{p}A + \widetilde{s}$, then $\widetilde{\Psi}_A$ is a bijection from $R^2/\equiv_A$ to $\widetilde{R}_{F(A)}$. The difference $\ominus_A$ is defined as from the operation of $\oplus_A$ and $\odot_A$ in $R^2/\equiv_A$ $A_1 \ominus_A A_2 = A_1 \oplus_A (-1) \odot_A A_2$. From Proposition 3.5 in [17], $C \ominus_A C \neq 0$ if $\in R_F \backslash R$ is symmetric. To remove the above drawback, the linear correlated fuzzy difference was introduced in [13].

2.3. Linear correlated fuzzy difference

The LC-difference in the spaces of non-symmetric and symmetric basic fuzzy numbers $R^n_{F(A)}$ and $R^s_{F(A)}$, respectively, is defined as follows:

$A_1 \boxminus_A A_2 = \Psi_A(p_1, s_1) \boxminus_A \Psi_A(p_2, s_2) = \Psi_A(p_1 - p_2, s_1 - s_2) = (p_1 - p_2)A + s_1 - s_2$ for all $A_1, A_2 \in R^n_{F(A)}$.

$A_1 \boxminus_A A_2 = \widetilde{\Psi}_A([(p_1, s_1)]_{\equiv_A}) \boxminus_A \widetilde{\Psi}_A([(p_2, s_2)]_{\equiv_A}) = \widetilde{\Psi}_A([(p_1, s_1)]_{\equiv_A} \boxminus_A [(p_2, s_2)]_{\equiv_A})$ for all $A_1, A_2 \in R^s_{F(A)}$

$$\text{where, } \widetilde{\Psi}_A([(p_1, s_1)]_{\equiv_A} \boxminus_A [(p_2, s_2)]_{\equiv_A}) = \begin{cases} (p_1 - p_2)A + s_1 - s_2, p_1 \geq p_2, \\ (p_2 - p_1)A + 2(p_1 - p_2)y + s_1 - s_2, p_1 < p_2. \end{cases}$$

2.4. The metric in the space of linear correlated fuzzy numbers

The metric d_{Ψ_A} in the space $R^n_{F(A)}$ and $d_{\widetilde{\Psi}_A}$ in $R^s_{F(A)}$ with LC-difference is defined as follows:
$d_{\Psi_A}(A_1, A_2) = \|A_1 \boxminus_A A_2\|_{\Psi_A}$ for all $A_1, A_2 \in R^n_{F(A)}$ where the norm is defined as $\|C\|_{\Psi_A} = \|{\Psi_A}^{-1}(C)\|_\infty$.
$d_{\widetilde{\Psi}_A}(A_1, A_2) = \|A_1 \boxminus_A A_2\|_{\widetilde{\Psi}_A} = \|[(p_1, s_1)]_{\equiv_A} \boxminus_A [(p_2, s_2)]_{\equiv_A}\|_\infty$ for all $A_1, A_2 \in R^s_{F(A)}$, and
$[(p_1, s_1)]_{\equiv_A}, [(p_2, s_2)]_{\equiv_A} \in R^2/\equiv_A$,
where the norm is defined as $\|C\|_{\widetilde{\Psi}_A} = \|\widetilde{\Psi}^{-1}_A(C)\|_\infty = \|[(p, s)]_{\equiv_A}\|_\infty = \max\{\|(p, s)\|_\infty, (-p, 2py + s)\|_\infty\}$.

2.5. Differentiability of linear correlated fuzzy functions

Let $A \in R_F$ and the differentiable function $g : I \to R_{F(A)}$ be continuous. Then, g is left (right) LC-differentiable on $t_0 \in I$ if, in the sense of metrics d_{Ψ_A} or $d_{\widetilde{\Psi}_A}$, the following limit exists [13]:

$$\lim_{t \to t_0^- (t \to t_0^+)} \frac{1}{t - t_0} \odot_A (g(t) \boxminus_A g(t_0)),$$

if g is left and right differentiable on $t_0 \in I$, denoted by g'_-, g'_+ respectively, where $g'_+ = g'_-$. Then, g is LC-differentiable on $t_0 \in I$.

Moreover, if $A \in R_F$ is non-symmetric and $g(t) = p(t)A + s(t)$ such that $p(t), s(t)$ are differentiable g is LC-differentiable and $g'(t) = p'(t)A + s'(t)$.

But, if $A \in R_F \backslash R$ is symmetric with a symmetric point y, and the canonical form of $g(t) = \widetilde{p}(t)A + \widetilde{s}(t)$, then $g(t)$ is LC-differentiable if $\widetilde{p'}_-(t) = \widetilde{p'}_+(t)$, $\widetilde{s'}_-(t) = \widetilde{s'}_+(t)$ or $\widetilde{p'}_-(t) = -\widetilde{p'}_+(t)$, $\widetilde{s'}_-(t) = 2\widetilde{p'}_+(t)y + \widetilde{s'}_+(t)$, where $\widetilde{p'}_+, \widetilde{s'}_+$ and $\widetilde{p'}_-, \widetilde{s'}_-$ denote the right and left derivative of $\widetilde{p}, \widetilde{s}$, respectively.

Definition 2.1. *Let problem (1.1) have a solution $z(t)$ in the space of continuous fuzzy functions, $C(I, R_F)$. Then, there exists a vector (p, s) such that $z(t) = p(t)A + s(t)$ and $z(t)$ satisfies Eq (1.1). Also, for $z_0 \in C(I, R_F)$ there exist s_0, p_0 such that $z_0 = p_0A + s_0$.*

3. LCFDEs of non-symmetric basic fuzzy numbers

Real problems with uncertainty are dealt with fuzzy models. A fuzzy model in the spaces $R^n_{F(A)}$ of non-symmetric basic fuzzy numbers deal with the problem in a simple and easy way. A model deals a problem properly if it has a unique solution. Therefore, the current study is concerned with the existence and uniqueness of solutions of fuzzy models of LCFDEs. Moreover, we will discuss the solutions of fuzzy models of LCFDEs in the LC-spaces $R^n_{F(A)}$.

In this work, we will discuss the existence result and solution of Eq (1.1) in the LC-space $R^n_{F(A)}$. If $g(t, z(t)) = a(t)z(t) + b(t)$, where $a, b : I \to R$ are continuous functions, then problem (1.1) will have the form

$$\begin{cases} z'(t) = a(t)z(t) + b(t), \\ z(t_0) = p_0A + s_0. \end{cases} \tag{3.1}$$

Eq (3.1) can be easily expressed in the following equivalent systems of equations

$$\begin{cases} p'(t) = a(t)p(t), \\ s'(t) = a(t)s(t) + b(t), \\ p(t_0) = p_0, s(t_0) = s_0. \end{cases}$$

From this, the following solutions were obtained easily.

$$\begin{cases} p(t) = p_0 e^{\int_{t_0}^{t} a(w)dw}, \\ s(t) = e^{\int_{t_0}^{t} a(w)dw}\Big\{s_0 + \int_{t_0}^{t} b(w)e^{-\int_{t_0}^{t} a(w)dw} dw\Big\}. \end{cases} \tag{3.2}$$

Hence,

$$z(t) \quad = \quad p_0 e^{\int_{t_0}^{t} a(w)dw} A \ + \ e^{\int_{t_0}^{t} a(w)dw}\Big\{s_0 \ + \ \int_{t_0}^{t} b(w)e^{-\int_{t_0}^{t} a(w)dw} dw\Big\}. \tag{3.3}$$

Now, we have to state and prove the existence and uniqueness result of problem (1.1) if BF-numbers are non-symmetric.

Theorem 3.1. *Let $g : [t_0, t_0 + \delta] \times U_{F(A)} \to R_{F(A)}$ with $\delta > 0$ be continuous a LCF-number valued function, where $U_{F(A)} = \{z \in R_{F(A)} \mid ||z(t) \boxminus z_0||_{\Psi_A} \leq \rho\}$ and $M = sup\{|| \ g(t, z(t)) \ ||_{\Psi_A} \mid t \in [t_0, t_0+\delta]$ and $z \in U_{F(A)}\}$. If for $z_1, z_2 \in U_{F(A)}$ and $t \in [t_0, t_0 + \delta]$, the following Lipschitz condition holds,*

$$||g(t, z_1(t) \boxminus_A g(t, z_2(t))||_{\Psi_A} \leq k||z_1(t) \boxminus_A z_2(t)||_{\Psi_A}$$

then Eq (1.1) has a unique solution $z(t)$ in the LC-space defined on $t \in [t_0, t_0 + \tau]$ and $\tau = \min\{\delta, \frac{\rho}{M}, \frac{1}{\alpha k}\}$, where $\alpha > 1$.

Proof. Suppose $D = \{C([t_0, t_0 + \tau], U_{F(A)})\}$ such that $T : D \to D$ is defined by

$$\begin{cases} T(z(t)) = z_0 \oplus_A \displaystyle\int_{t_0}^{t} g(s, z(w))dw \\ \qquad T(z_0) = z_0. \end{cases}$$

Since T is well defined on the complete metric space D, we also have

$$\begin{aligned} || \ T(z(t)) \boxminus z_0 \ ||_{\Psi_A} \ &\leq \ \int_{t_0}^{t} || \ g(t, z(t)) \ ||_{\Psi_A}, \\ &\leq \ \tau M < \rho. \end{aligned}$$

To complete the proof we need to show a contraction of T in D.

$$||T(z_1(t) \boxminus_A T(z_2(t))||_{\Psi_A} \leq \int_{t_0}^{t} ||g(y, z_1(t)) \boxminus_A g(z_2(t))||_{\Psi_A} dw \leq k\tau||z_1(t) \boxminus_A z_2(t)||_{\Psi_A}.$$

If $\tau < \frac{1}{k}$, then T is contraction on D. Therefore, there exists a unique solution $z(t) = p(t)A + r(t)$ defined on $t \in [t_0, t_0 + \tau]$ of problem Eq (1.1). □

Example 3.2. *For a non-symmetric BF-number* $A = (-1;\ 0;\ 2)$, *consider the following FDEs*

$$\begin{cases} z^{'}(t) = -\lambda z(t), \\ z(0) = A + 1. \end{cases} \tag{3.4}$$

In integral form, Eq (3.4) can be written as

$$\begin{cases} z(t) = z_0 \boxminus_A \lambda \int_{t_0}^{t} z(w)dw, \\ \quad z(0) = A + 1. \end{cases} \tag{3.5}$$

Now, one can easily show the following condition:

$$\begin{aligned} \|g(y, z_1(t) \boxminus_A g(t, z_1(t))\| &\leq \lambda k \|z_1(t) \boxminus_A z_1(t)\| \\ &= \lambda k \|z_1(t) \boxminus_A z_1(t)\| \\ &= \frac{0.693k}{T_{\frac{1}{2}}} \|z_1(t) \boxminus_A z_1(t)\|. \end{aligned}$$

Here, $T_{\frac{1}{2}}$ is the half life of a radioactive sample. Clearly, for $k < \frac{0.693}{T_{\frac{1}{2}}}$, the condition of Theorem 3.1 holds, therefore the FDEs (3.4) have a unique solution. One can easily find the solution

$$z(t) = z_0 e^{-\lambda t} = e^{-\lambda t} A + e^{-\lambda t} \tag{3.6}$$

The Figure 1 shows $2D$ and $3D$ fuzzy plots of the solution (3.6) of Eq (3.4) with non-symmetric basic fuzzy number $A = (-1;\ 0;\ 2)$, and $\lambda = 0.5$.

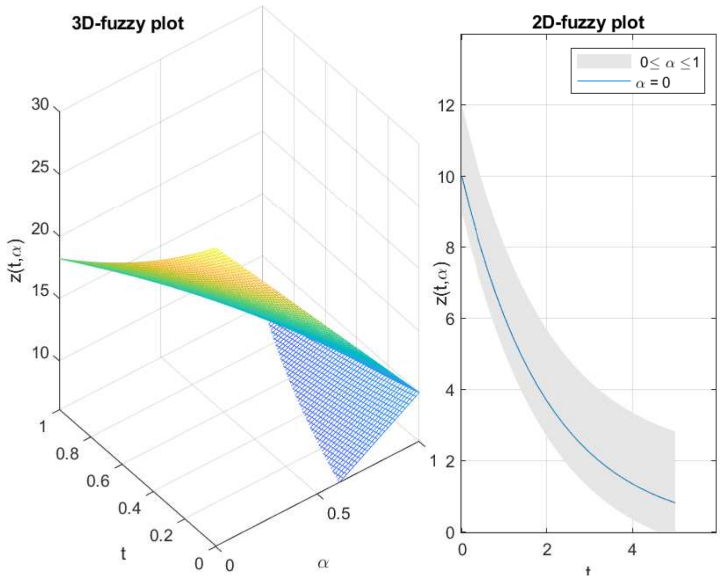

Figure 1. 3D-fuzzy plot and 2D-fuzzy plot of Example 3.2 with $\lambda = 0.5$.

Example 3.3. *For a non-symmetric BF-number* $A = (-1;\ 0;\ 2)$, *consider the FDEs*

$$\begin{cases} z'(t) = \dfrac{1}{t+1} \odot_A z(t) \boxminus_A t + 1, \\ z(0) = A + 1. \end{cases} \tag{3.7}$$

One can easily show the following condition:

$$\|g(t, z_1(t) \boxminus_A g(t, z_1(t))\| \leq \sup |\frac{1}{t+1}| \odot_A \|z_1(t) \boxminus_A z_1(t)\|.$$

Hence, the condition of Theorem 3.1 holds for all $t \in (0, \infty)$, therefore the FDEs (3.7) have a unique solution.

Now, for the solution of Eq (3.7), we need to solve the following equations

$$\begin{cases} p'(t) = \dfrac{1}{t+1} p(t), \\ s'(t) = \dfrac{1}{t+1} s(t) - (t+1). \end{cases}$$

One can get the solution

$$z(t) = (t+1)A + 1 - t^2. \tag{3.8}$$

The Figure 2 shows $2D$ and $3D$ fuzzy plots of the solution (3.8) of Eq (3.7) with non-symmetric basic fuzzy number $A = (-1;\ 0;\ 2)$.

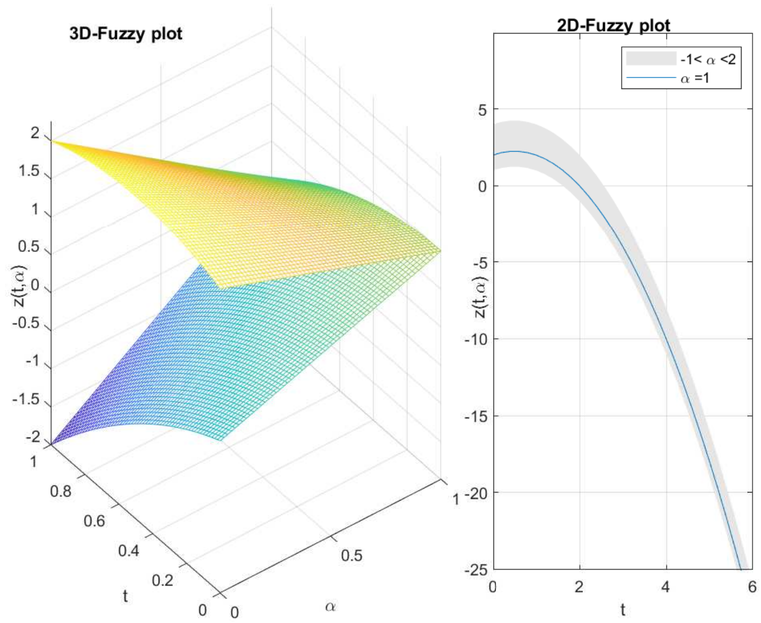

Figure 2. 3D-fuzzy plot and 2D-fuzzy plot of the solution of Example 3.3.

Example 3.4. *For a non-symmetric BF-number* $A = (-1;\ 0;\ 2)$,

$$\begin{cases} z'(t) = -\cos(t) \odot_A z(t) \oplus_A \dfrac{\sin(4t)}{4}, \\ z(0) = A - 1. \end{cases} \tag{3.9}$$

Clearly, the conditions of Theorem 3.1 hold for all $t \in (0, \infty)$, therefore the FDEs (3.9) have a unique solution. Now, for the solution of Eq (3.9), we need to solve the following equations:

$$\begin{cases} p'(t) = -\cos(t)p(t), \\ s'(t) = -\cos(t)s(t) \oplus_A \dfrac{\sin(4t)}{4}, \\ p(0) = 1, \ s(0) = -1. \end{cases}$$

This produces the following solutions:

$$\begin{cases} p(t) = e^{-\sin t}, \\ s(t) = -12e^{-\sin t} + 3\cos 2t - \dfrac{\sin(3t)}{2} + \dfrac{25\sin(t)}{2} + 14. \end{cases} \tag{3.10}$$

Hence, the required solution of FDE (3.9) is

$$z(t) = e^{-\sin t}A - 12e^{-\sin t} + 3\cos 2t - \frac{\sin(3t)}{2} + \frac{25\sin(t)}{2} + 14. \tag{3.11}$$

The Figure 3 shows $2D$ and $3D$ fuzzy plots of the solution (3.11) of Eq (3.9) with non-symmetric basic fuzzy number $A = (-1;\ 0;\ 2)$.

Now, Problem (1.1) can also have the following form, similar to the main problem discussed in paper [18]:

$$\begin{cases} z'(t) = \dfrac{\frac{c(t)}{p(t)} - b(t)}{a(t)} \odot_A z(t) + \dfrac{d(t)}{a(t)}, \\ z(t_0) = p_0 A + s_0. \end{cases}$$

$$\begin{cases} a(t) \odot_A z'(t) \oplus_A b(t) \odot_A z(t) = \dfrac{c(t)}{p(t)}A + \dfrac{c(t)s(t)}{p(t)} + d(t), \\ z(t_0) = p_0 A + s_0. \end{cases} \tag{3.12}$$

Now, we discuss the solutions of Problem (3.12) and for this we rewrite (3.12) as

$$\begin{cases} z'(t) \oplus_A \dfrac{b(t)}{a(t)} \odot_A z(t) = \dfrac{c(t)}{a(t)p(t)}A + \dfrac{c(t)s(t)}{a(t)p(t)} + \dfrac{d(t)}{a(t)}, \\ z(t_0) = p_0 A + s_0. \end{cases} \tag{3.13}$$

Since Eq (3.13) can be easily expressed in the following equivalent systems of equations:

$$\begin{cases} p(t)p'(t) + \dfrac{b(t)}{a(t)}p^2(t) = \dfrac{c(t)}{a(t)}, \\ s'(t) + \dfrac{b(t)}{a(t)}s(t) = \dfrac{c(t)s(t)}{a(t)p(t)} + \dfrac{d(t)}{a(t)}. \end{cases} \tag{3.14}$$

Let $v = p^2(t)$, $\frac{v'}{2} = p(t)p'(t)$ and, for the simplicity of this study, take the positive root of v so $p(t) \geq 0$. Therefore, Eq (4.27) produces

$$\begin{cases} v'(t) + \dfrac{2b(t)}{a(t)}v(t) = \dfrac{2c(t)}{a(t)}, \text{if } v' \geq 0 \\ s'(t) + \dfrac{b(t)}{a(t)}s(t) = \dfrac{c(t)s(t)}{a(t)p(t)} + \dfrac{d(t)}{a(t)}. \end{cases} \tag{3.15}$$

By solving Eq (3.15), the following solutions and obtained:

$$\begin{cases} p(t) = \sqrt{v(t)} = \sqrt{e^{-\int_{t_0}^{t}\frac{2b(w)}{a(w)}dw}\left\{(p_0)^2 + 2\displaystyle\int_{t_0}^{t}\left\{\frac{c(w)}{a(w)}e^{\int_{t_0}^{t}\frac{2b(w)}{a(w)}dw}\right\}dw\right\}}, \\ s(t) = e^{-\int_{t_0}^{t}(\frac{b(w)}{a(w)} - \frac{c(w)}{a(w)p(t)})dw}\left\{(s_0) + \displaystyle\int_{t_0}^{t}\left\{\frac{d(w)}{a(w)}e^{\int_{t_0}^{t}(\frac{b(w)}{a(w)} - \frac{c(w)}{a(w)p(t)})dw}\right\}dw\right\}. \end{cases} \tag{3.16}$$

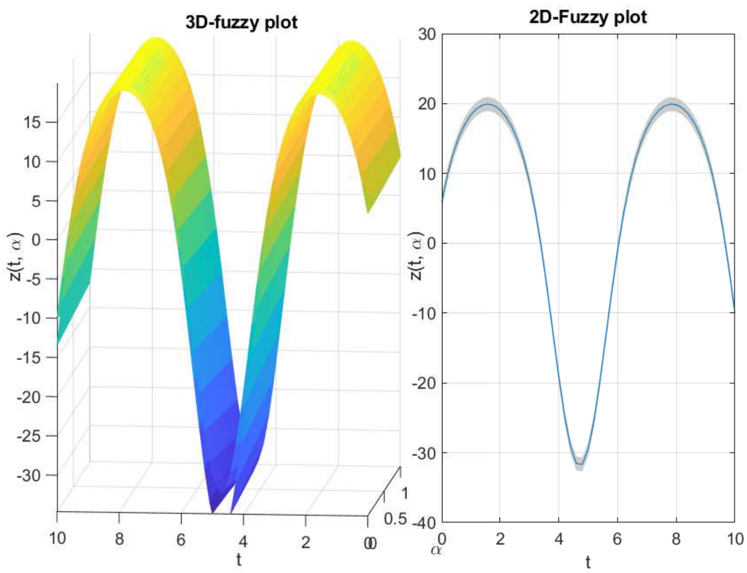

Figure 3. 3D-fuzzy plot and 2D-fuzzy plot of the solution of Example 3.4.

Example 3.5. *For a non-symmetric BF-number* $A = (-1;\ 0;\ 2)$,

$$\begin{cases} z'(t) \oplus_A \dfrac{1}{t} \odot_A z(t) = \dfrac{2t}{p(t)}A + \dfrac{2ts(t)}{p(t)} - 2t, \\ z(1) = A + 1. \end{cases} \tag{3.17}$$

The conditions of Theorem 3.1 hold easily, therefore Eq (3.17) has a unique solution. Using Eq (3.16) to obtain the solution of Eq (3.17) in the interval $(0, \infty)$,

$$z(t) = tA + \frac{2t^2 + 2t + 1 - 3e^{2(t-1)}}{2t}. \tag{3.18}$$

The Figure 4 shows $2D$ and $3D$ fuzzy plots of the solution (3.18) of Eq (3.17) with non-symmetric basic fuzzy number $A = (-1;\ 0;\ 2)$.

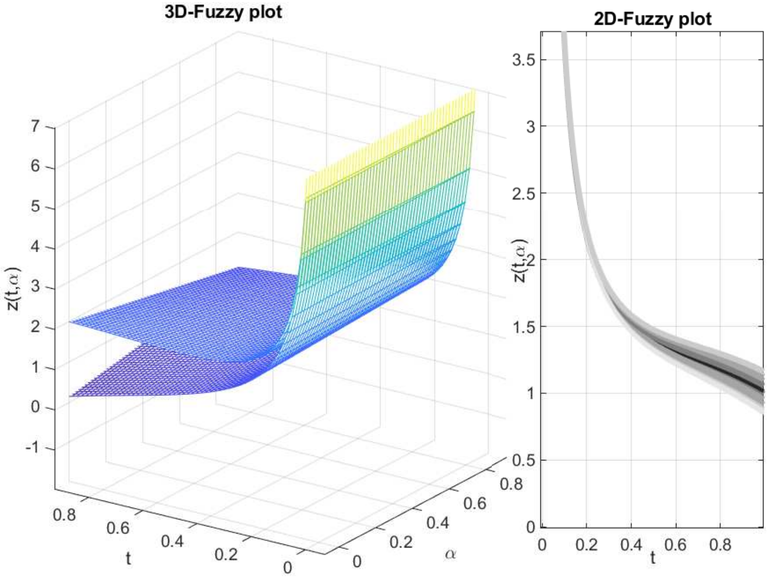

Figure 4. 3D-fuzzy plot and 2D-fuzzy plot of the solution of Example 3.5.

4. LCFDEs in the spaces of symmetric basic fuzzy numbers

Physical and biological models of linear correlated fuzzy differential equations (LCFDEs) easily deal with problems with uncertainty. A model can deal a problem better if it has a unique solution, but LCFDEs in the space $R^s_{F(A)}$ do not have unique solutions. Therefore, this study is concerned with the existence of unique solution of LCFDEs in the space $R^s_{F(A)}$. Moreover, LCFDEs in the space $R^s_{F(A)}$ are based on the canonical form, but the solutions discussed in the existing literature are not in the canonical form. Therefore, in the current work we will discuss the solution of LCFDEs in the canonical form. Let $A \in R_F \backslash R$ be a symmetric F-number. Then $\Psi_A : R^2 \to R_{F(A)}$ is not one-to-one because $\Psi_A(p, s) = \Psi_A(-p, 2px + s)$, where x is a symmetric point, but $\Psi_A : R^2 / \equiv A \to R^s_{F(A)}$ is a bijection and has the canonical form $\widetilde{\Psi}_A([\widetilde{p}(t), \widetilde{s}(t)]_{\equiv A}) = \widetilde{p}(t)A + \widetilde{s}(t)$, where $[\widetilde{p}(t), \widetilde{s}(t)]_{\equiv A} = \{(p(t), s(t)), (-p(t), 2p(t)x + s(t))\}$ and $z'(t) = \widetilde{\Psi}_A([|\widetilde{p'}|, \widetilde{s'}(t)]_{\equiv A}) = |\widetilde{p'}(t)|A + \widetilde{s'}(t)$ can be expressed as

$$z'(t) = |\widetilde{p'}(t)|A + \widetilde{s'}(t) = \begin{cases} p'(t)A + s'(t) & \text{if } p'(t) \geq 0, \\ -p'(t) + 2p'(t)x + s'(t), & \text{if } p'(t) < 0. \end{cases}$$

$$z(t) = \widetilde{p}(t)A + \widetilde{s}(t) = \begin{cases} p(t) + s(t) \text{ if } p(t) \geq 0, \\ -p(t) + 2p(t)x + s(t), \text{ if } p(t) < 0. \end{cases}$$

Therefore, Problem (1.1) has the following form:

$$z'(t) = \widetilde{\Psi}_A([|\widetilde{p'}|, \widetilde{s'}(t)]_{\equiv A}) = \begin{cases} g(t, \widetilde{\Psi}_A([\widetilde{p}(t), \widetilde{s}(t)]_{\equiv A})) \\ z(t_0) = z_0 = \widetilde{p}_0 A + \widetilde{s}. \end{cases} \tag{4.1}$$

Let $g(t,z(t)) = a(t)z(t) + b(t)$, where $a,b : I \to R$ are continuous functions. Then, Problem (1.1) has the form:

$$\begin{cases} z'(t) = a(t)z(t) + b(t), \\ z(t_0) = \widetilde{p}_0 A + \widetilde{s}_0. \end{cases} \tag{4.2}$$

Equation (4.2) can be expressed in the canonical form as

$$\begin{aligned} \widetilde{\Psi}_A([\widetilde{p'}(t), \widetilde{s'}(t)]_{\equiv A}) &= \widetilde{\Psi}_A([a(t)\widetilde{p}(t), a(t)\widetilde{s}(t)]_{\equiv A})) + b(t) \ \text{ if } \ a(t) \geq 0, \\ \widetilde{\Psi}_A([-\widetilde{p'}(t), \widetilde{s'}(t) + 2\widetilde{p'}(t)x]_{\equiv A}) &= \widetilde{\Psi}_A([-a(t)\widetilde{p}(t), a(t)\widetilde{s}(t) + 2a(t)\widetilde{p}(t)]_{\equiv A})) + b(t) \ \text{ if } \ a(t) < 0. \end{aligned} \tag{4.3}$$

Equation (4.3) produces the following systems of equations:

$$\begin{cases} p'(t) = a(t)p(t), \text{if } a(t) \geq 0, \\ s'(t) = a(t)s(t) + b(t) \\ -p'(t) = a(t)p(t), \text{if } a(t) \geq 0, \\ s'(t) + 2p'(t)x = a(t)s(t) + b(t), \end{cases} \quad \text{and} \quad \begin{cases} p'(t) = -a(t)p(t), \text{if } a(t) < 0, \\ s'(t) = a(t)s(t) + 2p(t)x + b(t) \\ -p'(t) = -a(t)p(t), \text{if } a(t) < 0, \\ s'(t) + 2p'(t)x = a(t)s(t) + 2p(t)x + b(t). \end{cases} \tag{4.4}$$

By solving Eq (4.4), the following solutions are obtained for $t \in I$.
Case (i). If $p'(t) \geq 0$ and $a(t) \geq 0$, the following solution is obtained for $t \in I_0 \subseteq I$.

$$\begin{cases} p(t) = p_0 e^{\int_{t_0}^{t} a(w)dw}, \\ s(t) = e^{\int_{t_0}^{t} a(w)dw} \left\{ s_0 + \displaystyle\int_{t_0}^{t} \{ b(w) e^{-\int_{t_0}^{t} a(w)dw} \} dw \right\}. \end{cases}$$

$$z(t) \quad = \quad p_0 e^{\int_{t_0}^{t} a(w)dw} A \ \oplus_A \ e^{\int_{t_0}^{t} a(w)dw} \{ s_0 \ + \ \int_{t_0}^{t} \{ b(w) e^{-\int_{t_0}^{t} a(w)dw} \} dw \}. \tag{4.5}$$

Case (ii). If $p'(t) < 0$ and $a(t) \geq 0$, the following solution is obtained for $t \in I_0 \subseteq I$.

$$\begin{cases} p(t) = p_0 e^{-\int_{t_0}^{t} a(w)dw}, \\ s(t) = e^{\int_{t_0}^{t} a(w)dw} \left\{ s_0 + \displaystyle\int_{t_0}^{t} \left\{ (b(w) - 2a(w)p'(w)x) e^{-\int_{t_0}^{t} a(w)dw} \right\} dw \right\}. \end{cases}$$

$$z(t) \quad = \quad p_0 e^{-\int_{t_0}^{t} a(w)dw} A \ \oplus_A \ e^{\int_{t_0}^{t} a(w)dw} \left\{ s_0 \ + \ \int_{t_0}^{t} \left\{ (b(w) \ - \ 2a(w)p'(w)x) e^{-\int_{t_0}^{t} a(w)dw} \right\} dw \right\}. \tag{4.6}$$

Case (iii). If $p'(t) \geq 0$ and $a(t) < 0$, the following solution is obtained for $t \in I_0 \subseteq I$.

$$\begin{cases} p(t) = p_0 e^{-\int_{t_0}^{t} a(w)dw}, \\ s(t) = e^{\int_{t_0}^{t} a(w)dw} \left\{ s_0 + \displaystyle\int_{t_0}^{t} \left\{ (2p(w)x + b(w)) e^{-\int_{t_0}^{t} a(w)dw} \right\} dw \right\}. \end{cases}$$

$$\begin{cases} p(t) = p_0 e^{-\int_{t_0}^{t} a(w)dw}, \\ s(t) = e^{\int_{t_0}^{t} a(w)dw}\Big\{s_0 + \int_{t_0}^{t}\Big\{(2a(w)p(w)x + b(w))e^{-\int_{t_0}^{t} a(w)dw}\Big\}dw\Big\}. \end{cases}$$

$$z(t) = q_0 e^{-\int_{t_0}^{t} a(w)dw} A \oplus_A e^{\int_{t_0}^{t} a(w)dw}\Big\{s_0 + \int_{t_0}^{t}\Big\{(2a(w)p(w)x + b(w))e^{-\int_{t_0}^{t} a(w)dw}\Big\}dw\Big\}. \quad (4.7)$$

Case (iv). If $p'(t) < 0$, and $a(t) < 0$, the following solution is obtained for $t \in I_0 \subseteq I$.

$$\begin{cases} p(t) = p_0 e^{\int_{t_0}^{t} a(w)dw}, \\ s(t) = e^{\int_{t_0}^{t} a(w)dw}\Big\{s_0 + \int_{t_0}^{t}\Big\{(2a(w)p(w)x - 2\widetilde{p'}(w)x + b(w))e^{-\int_{t_0}^{t} a(w)dw}\Big\}dw\Big\}. \end{cases}$$

$$\begin{cases} p(t) = p_0 e^{\int_{t_0}^{t} a(w)dw}, \\ s(t) = e^{\int_{t_0}^{t} a(w)dw}\Big\{s_0 + \int_{t_0}^{t} b(w)e^{-\int_{t_0}^{t} a(w)dw}dw\Big\}. \end{cases}$$

$$z(t) = p_0 e^{\int_{t_0}^{t} a(w)dw} A \oplus_A e^{\int_{t_0}^{t} a(w)dw}\Big\{s_0 + \int_{t_0}^{t} b(w)e^{-\int_{t_0}^{t} a(w)dw}dw\Big\}. \quad (4.8)$$

Now, we discuss the existence and uniqueness results for first order linear correlated FDEs of symmetric BF-number $A \in R_F \backslash R$ in the space of LCF-numbers $R^s_{F(A)}$.

Theorem 4.1. *Let* $g : I \times U^s_{F(A)} \to R^s_{F(A)}$ *be a continuous LCF-number valued function, where* $I = \{t \in R \mid |t - t_0| \leq \delta\}$ *and* $U^s_{F(A)} = \{\widetilde{\Psi}_A([p(t), s(t)]_{\equiv A}) = z(t) \in R^s_{F(A)} \mid \|z(t) \boxminus z_0\|_{\widetilde{\Psi}_A} \leq \rho\}$*. If, for* $z_1, z_2 \in U^s_{F(A)}$ *and* $t \in I$ *the Lipschitz condition*

$$\|g(t, z_1(t)) \boxminus_A g(t, z_2(t))\|_{\widetilde{\Psi}_A} \leq k\|z_1(t) \boxminus_A z_2(t)\|_{\widetilde{\Psi}_A},$$

holds, then Eq (4.1) has a unique pair of LC-differentiable solutions representing the unique canonical form of solutions in $R^s_{F(A)}$ in the interval

$$I_0 = \{t \in R \mid |t - t_0| \leq \tau\},$$

where
$\tau = \min\{\delta, \frac{\rho}{M}, \frac{1}{\alpha k}\}$, with $\alpha > 1$ and $M = sup\{\| g(t, z(t)) \|_{\widetilde{\Psi}_A} \mid t \in I$ and $z \in U^s_{F(A)}\}$.

Proof. If the BF-number $A \in R_F \backslash R$ is symmetric with symmetric point x, then Eq (1.1) will have the following form:

$$z'(t) = \begin{cases} \widetilde{\Psi}_A([\widetilde{p'}(t), \widetilde{s'}(t)]_{\equiv A}) \\ \widetilde{\Psi}_A([-\widetilde{p'}(t), \widetilde{s'}(t) + 2\widetilde{p'}(t)x]_{\equiv A}) \end{cases} = \begin{cases} g(t, \widetilde{\Psi}_A([\widetilde{p}(t), \widetilde{s}(t)]_{\equiv A})) \\ z(t_0) = z_0 = \widetilde{p}_0 A + \widetilde{s}. \end{cases}$$

In the integral form, the above equation can be written as

$$z(t) \boxminus z_0 = \int_{t_0}^{t} g(s, z(t))dw,$$

Suppose $D = \{C([t_0, t_0 + \tau], \widetilde{U}_{F(A)})\}$ such that $T : D \to D$ is defined by

$$\begin{cases} T(z(t)) = z_0 \oplus_A \displaystyle\int_{t_0}^{t} g(s, z(w))dw \\ \quad T(z_0) = z_0. \end{cases} \tag{4.9}$$

Since T is well defined in the complete metric space D, we also have

$$\begin{aligned} \| T(z(t)) \boxminus z_0 \|_{\widetilde{\Psi}_A} &\leq \int_{t_0}^{t} \| g(t, z(t)) \|_{\widetilde{\Psi}_A} \\ &\leq \tau M < \rho. \end{aligned}$$

To complete the proof we need to show the contraction of T in D.

$$\begin{aligned} \|T(z_1(t)) \boxminus_A T(z_2(t))\|_{\widetilde{\Psi}_A} &\leq \int_{t_0}^{t} \|g(y, z_1(t)) \boxminus_A g(z_2(t))\|_{\widetilde{\Psi}_A} dw \\ &\leq k\tau \|z_1(t) \boxminus_A z_2(t)\|_{\widetilde{\Psi}_A}. \end{aligned}$$

If $\tau < \frac{1}{k}$, then T is a contraction on D. Therefore, there exist a unique pair of LC-differentiable solutions representing the unique canonical form of solution defined on $t \in I_0$ of the Problem (4.1). □

Example 4.2. *For a symmetric BF-number* $A = (-1;\ 0;\ 1)$, *with symmetric point* 0, *consider the following FDEs:*

$$\begin{cases} z'(t) = \dfrac{1}{1-t} \odot_A z(t) \oplus_A 2t - \dfrac{t^2}{1-t}, \\ z(0) = A. \end{cases} \tag{4.10}$$

Clearly, the conditions of Theorem 4.1 hold, therefore a unique pair of solutions exist in the interval $I_0 = (-\infty, 1)$. If $p'(t) \geq 0$, then the canonical form of $\widetilde{s'}(t) = s'(t)$. Therefore, $z(t) = \frac{1}{1-t}A + t^2$ for all $t \in (-\infty, 1)$ is a solution with non-decreasing diameter in $I_0 = (-\infty, 1]$.

Moreover, if $p'(t) < 0$, the canonical form of $\widetilde{s'}(t) = 2p'(t)x + s'(t)$. Therefore, $z(t) = (1-t)A + t^2$ for all $t \in (-\infty, 1)$ is a solution with non-increasing diameter $I_0 = (-\infty, 1]$.

Hence, the following unique solution pair represents the unique solution in the canonical form:

$$z(t) = \widetilde{p}(t)A + \widetilde{s}(t) = \begin{cases} \dfrac{1}{1-t}A + t^2, \text{ for all } t \in (-\infty, 1), \\ (1-t)A + t^2, \text{ for all } t \in (-\infty, 1). \end{cases} \tag{4.11}$$

The Figures 5 and 6 shows $2D$ and $3D$ fuzzy plots of the solution (4.11) of Eq (4.10) with non-decreasing and non-increasing diameter respectively.

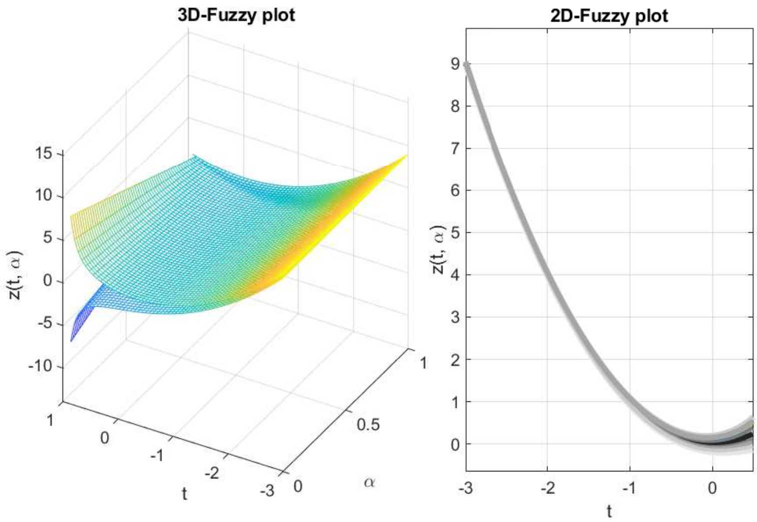

Figure 5. 2D and 3D plots of the solution of Example 4.2 with non-decreasing diameter in I_0.

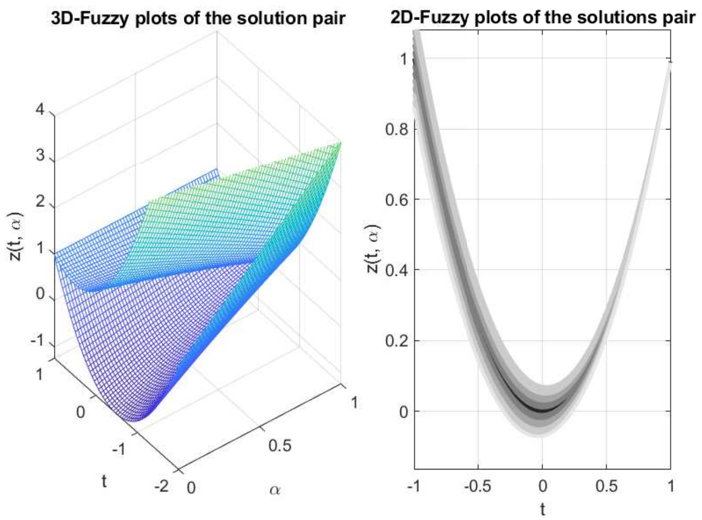

Figure 6. 2D and 3D plots of the solution of Example 4.2 with non-increasing diameter in I_0.

Moreover, the solution $z(t) = (1-t)A + t^2$ is also the solution of the following FDEs discussed in Example (3) of paper [18]:

$$\begin{cases} z^{'}(t) = A + 2t, \\ z(0) = A. \end{cases} \tag{4.12}$$

But, Eq (4.12) has many solutions and Eq (4.10) has a unique solution in the canonical form. Therefore, Eq (4.12) is not a proper form of first order FDE if the BF-number is symmetric. This shows the importance of existence theory, and the form of Eq (1.1) to deal with the physical model of FDEs.

Moreover, Eq (4.10) is discontinuous at $t = 1$, therefore its solution lies in the interval $(-\infty, 1)$. Also, Example (3) of paper [18] is extended in Example (4), but the conditions of Theorem 4.1 prevent such type of extension of Eq (4.10), and it has a unique solutions in the canonical form.

Example 4.3. *For a symmetric BF-number* $A = (0;\ 1;\ 2)$ *with symmetric point* 1, *consider the following FDEs:*

$$\begin{cases} z^{'}(t) = \dfrac{1}{t} \odot_A z(t) + t^2, \\ \quad z(1) = A. \end{cases} \tag{4.13}$$

Clearly, the conditions of Theorem 4.1 hold, therefore a unique pair of solutions in the canonical form exist in the interval $I_0 = (0, \infty)$. If $p'(t) \geq 0$, then $\widetilde{s'}(t) = s'(t)$. Therefore, $z(t) = tA + \frac{t^3-t}{2}$ for all $t \in (0, \infty)$ is a solution with non-decreasing diameter $(0, \infty)$.

Moreover, if $p'(t) < 0$, then $\widetilde{s'}(t) = 2p'(t)x + s'(t)$. Therefore, $z(t) = \frac{1}{t}A + \frac{t^4+t^2-2}{2t}$ for all $t \in [0, \infty)$ is a solution with non-increasing diameter in $(0, \infty)$. Hence, the following is the required unique solution pair which represents the unique solution in the canonical form.

$$z(t) = \widetilde{p}(t)A + \widetilde{s}(t) = \begin{cases} tA + \dfrac{t^3 - t}{2}, \text{ for all } t \in (0, \infty), \\ \dfrac{1}{t}A + \dfrac{t^4 + t^2 - 2}{2t}, \text{ for all } t \in (0, \infty). \end{cases} \tag{4.14}$$

This example also shows that neither solutions with non-decreasing nor non-increasing diameter can be extended. Therefore, Theorem 4.1 ensures the existence of unique solution in the canonical form.

The Figures 7 and 8 shows $2D$ and $3D$ fuzzy plots of the solution (4.14) of Eq (4.13) with non-decreasing and non-increasing diameter respectively.

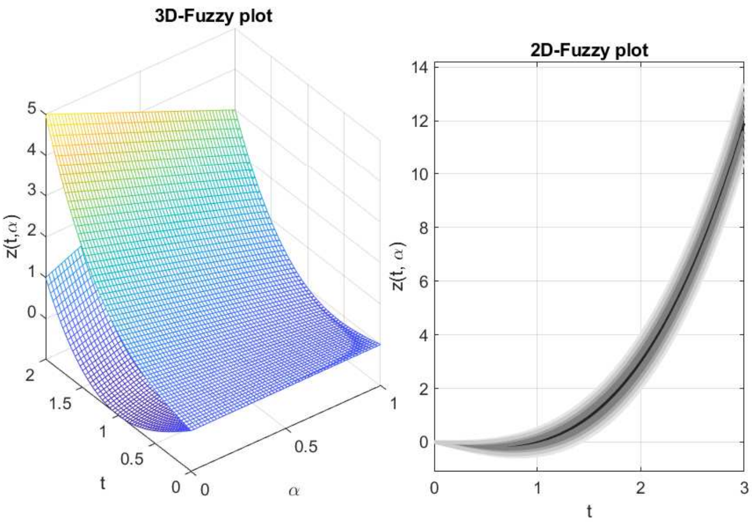

Figure 7. 2D and 3D plots of the solution of Example 4.3 with non-decreasing diameter in I_0.

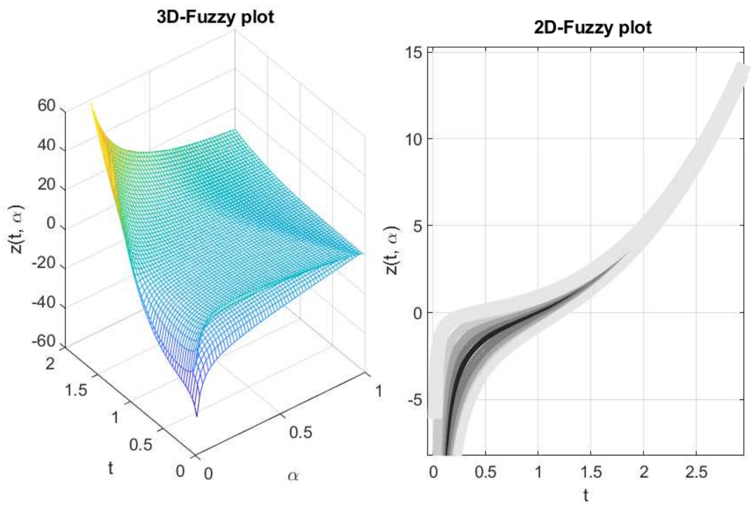

Figure 8. 2D and 3D plots of the solution of Example 4.3 with non-increasing diameter in I_0.

Example 4.4. *For a symmetric BF-number* $A = (0; 1; 2)$ *with symmetric point* 1, *we consider the following FDEs:*

$$\begin{cases} z^{'}(t) = (t^3 - 3t^2 + 2t) \odot_A z(t), \\ z(0) = A + 1. \end{cases} \tag{4.15}$$

Since $a(t) < 0$ in the interval $(-\infty, 0)$, then for $p'(t) \geq 0$ the following solution with non-decreasing diameter in $(-\infty, 0)$ is obtained:

$$z(t) = \widetilde{p}(t)A + \widetilde{s}(t) = Ae^{-(\frac{t^4}{4}-t^3+t^2)} + 2e^{\frac{t^4}{4}-t^3+t^2} - e^{-(\frac{t^4}{4}-t^3+t^2)}. \tag{4.16}$$

Also, if $a(t) < 0$ in the interval $(-\infty, 0)$, then for $p'(t) < 0$ the following solution with non-increasing diameter in $(-\infty, 0)$ is obtained:

$$z(t) = \widetilde{p}(t)A + \widetilde{s}(t) = Ae^{(\frac{t^4}{4}-t^3+t^2)} + e^{\frac{t^4}{4}-t^3+t^2}. \tag{4.17}$$

Now, if $a(t) \geq 0$ in the interval $[0, 1]$, then for $p'(t) \geq 0$ the following solution with non-decreasing diameter in $[0, 1]$ is obtained:

$$z(t) = \widetilde{p}(t)A + \widetilde{s}(t) = Ae^{(\frac{t^4}{4}-t^3+t^2)} + e^{\frac{t^4}{4}-t^3+t^2}. \tag{4.18}$$

Also, if $a(t) \geq 0$ in the interval $[0, 1]$, then for $p'(t) < 0$ the following solution in $[0, 1]$ is obtained:

$$z(t) = \widetilde{p}(t)A + \widetilde{s}(t) = Ae^{-(\frac{t^4}{4}-t^3+t^2)} + 2e^{\frac{t^4}{4}-t^3+t^2} - e^{-(\frac{t^4}{4}-t^3+t^2)}. \tag{4.19}$$

Moreover, if $a(t) < 0$ in the interval $(1, 2)$, then for $p'(t) \geq 0$ the following solution in $(1, 2)$ is obtained:

$$z(t) = \widetilde{p}(t)A + \widetilde{s}(t) = Ae^{-(\frac{t^4}{4}-t^3+t^2)} + 2e^{\frac{t^4}{4}-t^3+t^2} - e^{-(\frac{t^4}{4}-t^3+t^2)}. \tag{4.20}$$

Also, if $a(t) < 0$ in the interval $(1, 2)$, then for $p'(t) < 0$ the following solution in $(1, 2)$ is obtained:

$$z(t) = \widetilde{p}(t)A + \widetilde{s}(t) = Ae^{(\frac{t^4}{4}-t^3+t^2)} + e^{\frac{t^4}{4}-t^3+t^2}. \tag{4.21}$$

Now, if $a(t) \geq 0$ in the interval $[2, \infty)$, then for $p'(t) \geq 0$ the following solution with non-decreasing diameter in $[2, \infty)$ is obtained:

$$z(t) = \widetilde{p}(t)A + \widetilde{s}(t) = Ae^{(\frac{t^4}{4}-t^3+t^2)} + e^{\frac{t^4}{4}-t^3+t^2}. \tag{4.22}$$

Also, if $a(t) \geq 0$ in the interval $[2, \infty)$, then for $p'(t) < 0$ the following solution with non-increasing diameter in $[2, \infty)$ is obtained:

$$z(t) = \widetilde{p}(t)A + \widetilde{s}(t) = Ae^{-(\frac{t^4}{4}-t^3+t^2)} + 2e^{\frac{t^4}{4}-t^3+t^2} - e^{-(\frac{t^4}{4}-t^3+t^2)}. \tag{4.23}$$

Thus, the following unique solution pair of Eq (4.15) exists in the subintervals $(-\infty, 0)$, $[0, 1]$, $(1, 2)$ and $[2, \infty)$ of $I = (-\infty, \infty)$ represents a unique canonical solution.

$$z(t) = \widetilde{p}(t)A + \widetilde{s}(t) = \begin{cases} Ae^{(\frac{t^4}{4}-t^3+t^2)} + e^{(\frac{t^4}{4}-t^3+t^2)}, \\ Ae^{-(\frac{t^4}{4}-t^3+t^2)} + 2e^{\frac{t^4}{4}-t^3+t^2} - e^{-(\frac{t^4}{4}-t^3+t^2)}. \end{cases} \tag{4.24}$$

Moreover, the solution $z(t)$ alternately changes the non-increasing and non-decreasing diameter in the subinterval with nodal points $0, 1, 2$. Also, the solution pair is LC-differentiable on the nodal points $\{0, 1, 2\}$.

The Figures 9 and 10 shows $2D$ and $3D$ fuzzy plots of the solution (4.24) of Eq (4.15) with non-increasing and non-decreasing diameter respectively.

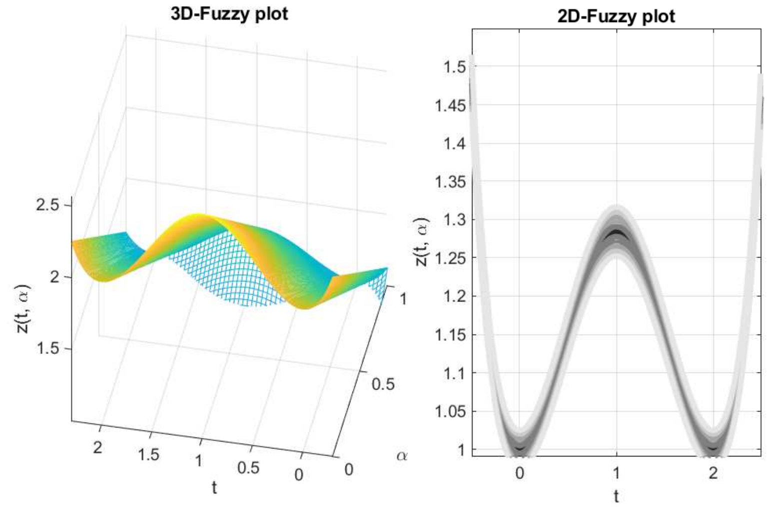

Figure 9. 2D and 3D plots of the solution of Example 4.4 with non-increasing diameter in I.

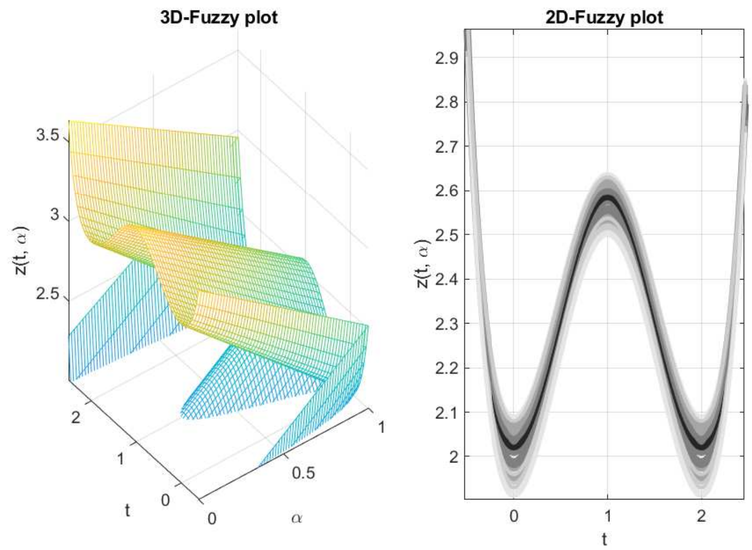

Figure 10. 2D and 3D plots of the solution of Example 4.4 with non-decreasing diameter in I.

Remark 4.5. Problem (1.1) can also have the following form with canonical representation in the case of symmetric BF-number A:

$$\begin{cases} a(t) \odot_A z^{'}(t) \oplus_A b(t) \odot_A z(t) = \dfrac{c(t)}{\widetilde{p}(t)}A + \dfrac{c(t)\widetilde{s}(t)}{\widetilde{p}(t)} + d(t), \\ z(t_0) = \widetilde{p}_0 A + \widetilde{s}_0. \end{cases} \tag{4.25}$$

This form is similar to the main problem discussed in paper [18], but this can not be extended by the extension procedure in Remark (7) of [18]. Because the solutions in the interval $[t_0, t_1]$ and $[t_1, t_2]$ can be extended to $[t_0, t_2]$ if $p(t_1) = 0$, Problem (6.3) cannot be extended and may have a unique pair of solutions representing the unique canonical solution if the condition of Theorem 4.1 holds. Note that the extension produces many solutions of the problem for examples 4 to 8 in [18] do not have unique canonical solutions.

Now, we discuss the solutions to Problem (6.3)and for this we rewrite (6.3) as

$$\begin{cases} z^{'}(t) \oplus_A \dfrac{b(t)}{a(t)} \odot_A z(t) = \dfrac{c(t)}{a(t)\widetilde{p}(t)}A + \dfrac{c(t)\widetilde{s}(t)}{a(t)\widetilde{p}(t)} + \dfrac{d(t)}{a(t)}, \\ z(t_0) = \widetilde{p}_0 A + \widetilde{s}_0. \end{cases} \tag{4.26}$$

Eq (4.26) can be easily expressed in the following equivalent systems of equations:

$$\begin{cases} \widetilde{p}(t)\widetilde{p}'(t) + \dfrac{b(t)}{a(t)}\widetilde{p}^2(t) = \dfrac{c(t)}{a(t)}, \\ \widetilde{s}'(t) + \dfrac{b(t)}{a(t)}\widetilde{s}(t) = \dfrac{c(t)\widetilde{s}(t)}{a(t)\widetilde{p}(t)} + \dfrac{d(t)}{a(t)}, \end{cases} \tag{4.27}$$

Let $\widetilde{v} = \widetilde{p}^2(t)$, $\frac{\widetilde{p}'}{2} = \widetilde{p}(t)\widetilde{p}'(t)$ and, for the simplicity of this study, take the positive root of $\widetilde{v}$, so $\widetilde{p}(t) \geq 0$, therefore Eq (4.27) produces

$$\begin{cases} \widetilde{v}'(t) + \dfrac{2b(t)}{a(t)}\widetilde{v}(t) = \dfrac{2c(t)}{a(t)}, \text{if } \widetilde{v}' \geq 0 \\ \widetilde{s}'(t) + \dfrac{b(t)}{a(t)}\widetilde{s}(t) = \dfrac{c(t)\widetilde{s}(t)}{a(t)\widetilde{p}(t)} + \dfrac{d(t)}{a(t)} \end{cases} \begin{cases} \widetilde{v}'(t) - \dfrac{2b(t)}{a(t)}\widetilde{v}(t) = -\dfrac{2c(t)}{a(t)}, \text{if } \widetilde{p}' < 0, \\ \widetilde{s}'(t) + 2\widetilde{p}'(t)x + \dfrac{b(t)}{a(t)}\widetilde{s}(t) = \dfrac{c(t)\widetilde{v}(t)}{a(t)\widetilde{p}(t)} + \dfrac{d(t)}{a(t)} \end{cases} \tag{4.28}$$

By solving Eq (4.28), the following solutions are obtained for $t \in I$.

Case (i). If $\widetilde{p}(t)\widetilde{p}'(t) = \widetilde{v}'(t) \geq 0$, the non-decreasing solution is obtained for $t \in I$.

$$\begin{cases} \widetilde{p}(t) = \sqrt{\widetilde{v}(t)} = \sqrt{e^{-\int_{t_0}^{t} \frac{2b(w)}{a(w)}dw}\left\{(\widetilde{p}_0)^2 + 2\displaystyle\int_{t_0}^{t}\left\{\dfrac{c(w)}{a(w)}e^{\int_{t_0}^{t}\frac{2b(w)}{a(w)}dw}\right\}dw\right\}}, \\ \widetilde{s}(t) = e^{-\int_{t_0}^{t}\left(\frac{b(w)}{a(w)} - \frac{c(w)}{a(w)\widetilde{p}(t)}\right)dw}\left\{\widetilde{s}_0 + \displaystyle\int_{t_0}^{t}\left\{\dfrac{d(w)}{a(w)}e^{\int_{t_0}^{t}\left(\frac{b(w)}{a(w)} - \frac{c(w)}{a(w)\widetilde{p}(t)}\right)dw}\right\}dw\right\}. \end{cases} \tag{4.29}$$

Case (ii). If $\widetilde{p}(t)\widetilde{p}'(t) = \widetilde{v}'(t) < 0$, the non-increasing solution is obtained for $t \in I$.

$$\begin{cases} \widetilde{p}(t) = \sqrt{\widetilde{v}(t)} = \sqrt{e^{\int_{t_0}^{t}\frac{2b(w)}{a(w)}dw}\left\{(\widetilde{p}_0)^2 - 2\displaystyle\int_{t_0}^{t}\left\{\dfrac{c(w)}{a(w)}e^{-\int_{t_0}^{t}\frac{2b(w)}{a(w)}dw}\right\}dw\right\}} \\ \widetilde{s}(t) = e^{-\int_{t_0}^{t}(\frac{b(w)}{a(w)} - \frac{c(w)}{a(w)\widetilde{p}(t)})dw}\left\{\widetilde{s}_0 + \displaystyle\int_{t_0}^{t}\left\{(\dfrac{d(w)}{a(w)} - 2\widetilde{p}(w)\widetilde{p}'(w)x)e^{\int_{t_0}^{t}\left(\frac{b(w)}{a(w)} - \frac{c(w)}{a(w)\widetilde{p}(w)}\right)dw}\right\}dw\right\}. \end{cases} \tag{4.30}$$

The solutions of Problem (6.3) for both cases have the canonical form

$$z(t) = \widetilde{p}(t)A + \widetilde{s}(t).$$

Example 4.6. *For a symmetric BF-number* $A = (-1;\ 0;\ 1)$ *with symmetric point* 0, *consider the following FDEs:*

$$\begin{cases} z^{'}(t) \oplus_A \dfrac{1}{2t} \odot_A z(t) = \dfrac{3t}{2\widetilde{p}(t)}A + \dfrac{3t\widetilde{s}(t)}{2\widetilde{p}(t)}, \\ z(1) = A + 1. \end{cases} \tag{4.31}$$

The conditions of Theorem 4.1 hold easily and Eq (4.15) has a unique pair of solutions representing the unique canonical solution. Using Eqs (4.29) and (4.30) to obtain the canonical form of solutions of Eq (4.15), the non-decreasing solution in the interval $I = (0, \infty)$ is $z(t) = tA + \frac{1}{\sqrt{t}}e^{\frac{3}{2}(t-1)}$.

The non-increasing solution in the interval $(0, \frac{4}{3}]$

$$z(t) = \sqrt{4t - 3t^2}A + \frac{1}{\sqrt{t}}e^{\arcsin(\frac{t-2}{2}) - \frac{\sqrt{4t-3t^2}}{2} + \frac{\pi+3}{6}}. \tag{4.32}$$

The Figure 11 shows $2D$ and $3D$ fuzzy plots of the solution (4.32) of Eq (4.31) with non-increasing and non-decreasing diameter.

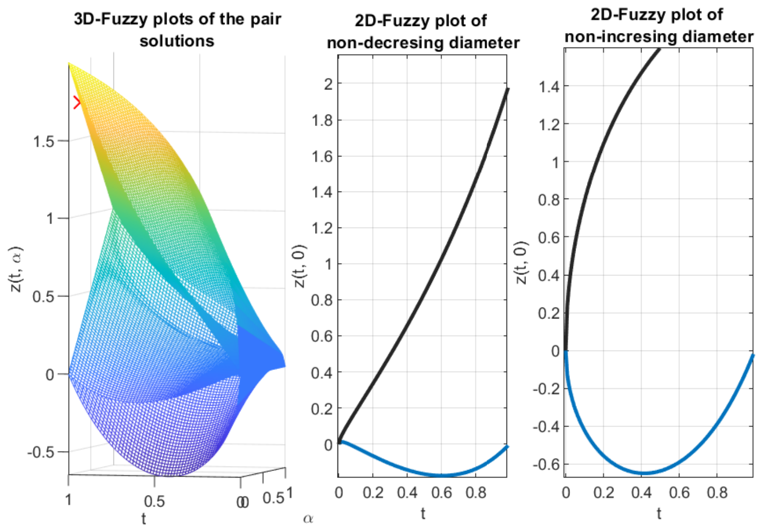

Figure 11. 2D and 3D plots of the solutions of Example 4.6 with non-decreasing and non-increasing diameter in I.

5. Practical example

Example 5.1. *The following is the discharging LCFDE of an electric capacitor of capacitance C connected in series with an electric resistor of resistance R,*

$$\begin{cases} Q^{'}(t) = -\dfrac{1}{RC}Q(t), \\ Q(t_0) = \widetilde{q}_0 A + \widetilde{r}_0. \end{cases} \tag{5.1}$$

Eq (5.1) is equivalent to the following systems of equations

$$\begin{cases} q^{'}(t) = -\dfrac{1}{RC}q(t), \text{for } q' \geq 0, \\ r^{'}(t) = -\dfrac{1}{RC}r(t). \end{cases} \text{And} \begin{cases} q'(t) = \dfrac{1}{RC}q(t), \text{for } q' < 0, \\ r'(t) = -\dfrac{1}{RC}r(t) - 2\dfrac{1}{RC}q(t)x. \end{cases} \tag{5.2}$$

The solution with non-decreasing diameter can be easily obtained as

$$Q(t) = q_0 e^{-\frac{t}{RC}} A + r_0 e^{-\frac{t}{RC}} \tag{5.3}$$

The solution with non-increasing diameter can be obtained from case (vi) or from integration by parts as

$$Q(t) = q_0 e^{\frac{t}{RC}} A + r_0 e^{-\frac{t}{RC}} + q_0 x\{e^{-\frac{t}{RC}} - e^{\frac{t}{RC}}\}. \tag{5.4}$$

This solution pair of non-decreasing and non-increasing diameter represents the unique solution in the canonical form.

$$Q(t) = \begin{cases} q_0 e^{-\frac{t}{RC}} A + r_0 e^{-\frac{t}{RC}}, \\ q_0 e^{\frac{t}{RC}} A + r_0 e^{-\frac{t}{RC}} + q_0 x\{e^{-\frac{t}{RC}} - e^{\frac{t}{RC}}\}. \end{cases}$$

The uncertainty in the discharging process with non-increasing diameter of Eq (5.3) is commonly used, but the non-decreasing diameter Eq (5.4) is new in the literature due to the symmetric behavior of variation in the space $R^s_{F(A)}$. The uncertainty in the discharging process with non-decreasing diameter is more suitable because the rate of discharging of the capacitor gradually reduces, and fully discharging of a capacitor takes infinite time, therefore the uncertainty is non-decreasing.

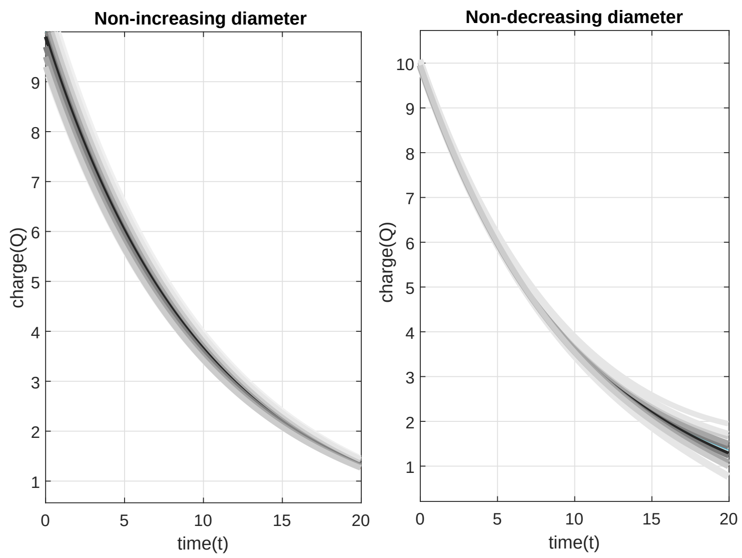

Figure 12. 2D plots of non-decreasing and non-increasing diameter of solutions of Example 5.1 if $C = 2F, R = 5\mho$ and the maximum charge stored by capacitor is $10C$.

6. Comparative analysis of this work with existing literature

The solutions of the linear correlated fuzzy differential equations in the space $R^s_{F(A}$ discussed in paper [18] see examples 4 to 8 are extended to new systems, due to which they have many solutions. In this study, we point out the cause of the extension of a system to a new system. The main cause of extension of a system to a new system is the form of LCFDEs discussed in paper [18]. If first order LCFDEs are in the form of Eq (1.1), then the condition to extend a system given in paper [18] hold only at the points where the function $g(t, z(t))$ is discontinuous or it is not included in the domain of solutions or the solution of the extended system and the original problem are same. Lemma 6.1 illustrates the first two possibilities, and Lemma 6.2 the last possibility.

Lemma 6.1. *If $g(t, z(t))$ and $a(t)$ of Problem (4.2) is not defined on $\eta \in I$, then*

(i) $\widetilde{p}(\eta) = 0$, if $\widetilde{p'}(t)$ is constant.
(ii) $\widetilde{p}(\eta)$ is undefined if $\widetilde{p'}(t)$ is not constant.

Proof. Eq (4.2) produces the following pair of equations:

$$\begin{cases} \widetilde{p'}(t) = a(t)\widetilde{p}(t), \text{for } \widetilde{p'} \geq 0, \\ \widetilde{s'}_(t) = a(t)\widetilde{s}(t) + b(t) \end{cases} \text{and } \begin{cases} -\widetilde{p'}(t) = a(t)\widetilde{p}(t), \text{for } \widetilde{p'} < 0, \\ 2\widetilde{p'}(t)x + \widetilde{s'}(t) = a(t)\widetilde{s}(t) + b(t). \end{cases} \tag{6.1}$$

If $\widetilde{p'}(t)$ is constant for all $t \in I$ (i.e., $\widetilde{p'}(t) = k$)), then Eq (6.1) produces $\widetilde{p}(t) = \frac{k}{a(t)}$. Now, $a(\eta)$ is infinite on $\eta \in I$, therefore $\widetilde{p}(\eta) = 0$.

Also, if $\widetilde{p'}(t)$ is not constant and $a(\eta)$ is infinite, then $\widetilde{p'}(\eta)$ also infinite. Now, Eq (6.1) produces

$$\widetilde{p}(t) = \frac{\widetilde{p'}(t)}{a(t)},$$

therefore

$$\widetilde{p}(\eta) = \frac{\widetilde{p'}(\eta)}{a(\eta)}$$

is undefined. □

In Example 4.2, functions $g(1, z(1))$ and $a(1)$ are not defined at $t = 1$ and $\widetilde{p'}(t)$ is constant in the case with $\widetilde{p'}(t) \geq 0$ and $\widetilde{p}(1) = 0$. But, $\widetilde{p'}(t)$ is not constant in the case with $\widetilde{p'}(t) < 0$ and $\widetilde{p}(1)$ is undefined. Similarly, in Example 4.3, functions $g(0, z(0))$ and $a(0)$ are not defined at $t = 0$ and $\widetilde{p'}(t)$ is constant in the case with $\widetilde{p'}(t) \geq 0$ and $\widetilde{p}(0) = 0$. But, $\widetilde{p'}(t)$ is not constant in the case with $\widetilde{p'}(t) < 0$, and $\widetilde{p}(0)$ is undefined.

Lemma 6.2. *The canonical solutions of Problem (4.2) form nodes on all solutions of $a(t) = 0$. Moreover, each solution $z(t)$ of the canonical solution is LC-differentiable on these nodal points.*

Proof. Let $\eta \in I$ be a solution of $a(t) = 0$. Then, each solution from the canonical solution of Problem (4.2) form nodes on the $\eta \in I$, because if Problem (4.2) has a solution with $\widetilde{p'}(t) \geq 0$ in the interval $I_0 = [t_0, t_0 + \eta] \subset I$, then in the interval $I_1 = (t_0 + \eta, s) \subset I$ the solution has $\widetilde{p'}(t) < 0$ therefore a node forms on the $\eta \in I$. Similarly, if Problem (4.2) has a solution with $\widetilde{p'}(t) < 0$ in the interval $I_0 = [t_0, t_0 + \eta] \subset I$, then in the interval $I_1 = (t_0 + \eta, s) \subset I$, the solution has $\widetilde{p'}(t) \geq 0$ and node forms on the $\eta \in I$. Moreover, $\widetilde{p}(\eta) = \widetilde{p}_0$ and $\widetilde{s}(\eta) = \widetilde{s}_0$. Similarly, on all the solutions $\eta \in I$ of $a(t) = 0$, the solution pair of Problem (4.2) form nodes in the subintervals of I.

Now, we show that both solutions of the canonical solution $z(t)$ are LC-differentiable on the nodal points. Each solution of the canonical solution has non-decreasing diameter on the one side of nodal points $\eta \in I$, but have a non-increasing diameter on the opposite side. Therefore, for each $\eta \in I$, the following condition holds:

$$\begin{cases} \widetilde{p'}_-(\eta) = \widetilde{p'}_+(\eta), \\ \widetilde{s'}_-(\eta) = \widetilde{s'}_+(\eta) + 2\widetilde{p'}_+(\eta)x. \end{cases} \tag{6.2}$$

Hence, both solutions of the canonical solution $z(t)$ are LC-differentiable on the nodal points $\eta \in I$. □

In Example 4.4, the system can be extended to a new system by taking the initial value at points 1 and 2, but the extended system has solutions similar to the solutions of Eq (4.15). Therefore, it has a unique solution of canonical form and these extensions are meaningless.

Moreover, the existence and uniqueness results ensure the existence of a unique solution, but according to Theorem 4.1, first order FDEs in the LC-space $R^s_{F(A)}$, produce a unique pair of solutions representing a unique solution of canonical form because the space is symmetric therefore two types of symmetric variations are produced simultaneously. One solution has a non-decreasing diameter, but the second solution of the pair has a non-increasing diameter, and this pair of solutions representing the unique solution of canonical form. Therefore, the solutions of LCFDEs in the space $R^s_{F(A)}$ are in the canonical form.

In this study we discuss the LCFDEs (1.1) due to previously discussed difficulties. Moreover, the following form of LCFDEs (1.1) is similar to the main problem discussed in paper [18], but this problem can not be extended by the extension procedure in Remark (7) of [18]. Because the solutions in the interval $[t_0, t_1]$ and $[t_1, t_2]$ can be extended to $[t_0, t_2]$ if $p(t_1) = 0$, the following form can not be extended and may have a unique solution of the conical form if the condition of Theorem 4.1 holds.

$$\begin{cases} a(t) \odot_A z'(t) \oplus_A b(t) \odot_A z(t) = \dfrac{c(t)}{\widetilde{p}(t)}A + \dfrac{c(t)\widetilde{s}(t)}{\widetilde{p}(t)} + d(t), \\ z(t_0) = \widetilde{p}_0 A + \widetilde{s}_0. \end{cases} \tag{6.3}$$

This study shows that both solutions of the unique canonical form of solution changed alternately the non-decreasing and non-increasing diameter at the nodal points in the overall domain of the solution.

7. Advantages and limitations of this study

Physical problems, optimization problems, linear programming problems etc. with uncertainty can be easily dealt with using LCFDEs, but LCFDEs often have many solutions and sometimes do not have solutions as in Examples 4 to 8 of [18]. A problem is modeled correctly if it has a unique solution. Therefore, this study is concerned with the existence and uniqueness of solutions of first order LCFDEs. LCFDEs satisfying the conditions of Theorems 3.1 or 4.1 must have a unique solution. In the existing literature of LCFDEs in the space $R^s_{F(A)}$, LCFDEs are taken in the canonical form of initial value and function, but the solution is not in the canonical form, therefore the question arises of why the solution does not have canonical form. Therefore, this study provides the concept of the canonical form of the solution of LCFDEs in the space $R^s_{F(A)}$. This achievement of solution in the canonical form is not only response to the said question, but it is a useful concept to have a solution with both non-decreasing and non-increasing diameter. The uncertainty in the discharging process discussed in the Example 5.1 elaborates the importance of the said concept. This work removes the difficulties in the solution process due to the sign of the coefficient function and extend the domain of the solution.

This work has two main limitations: the uncertainty of real life problems exists in the space of linear correlated fuzzy numbers, and fuzzy differential equations must be in the form of Eq (1.1).

8. Conclusion and future direction

In this work, we have discussed the existence and uniqueness conditions of solutions of linear correlated fuzzy differential equations (LCFDEs) in the LC-spaces of both symmetric and non-symmetric BF-numbers. In the existing literature [18], first order LCFDEs in LC-spaces of symmetric BF-numbers mostly extend to new systems and produce many continuous and differentiable solutions. From Example 3 to 7 of [18], all have many solutions due to extensions. In this work, we point out the causes of extensions of first order LCFDEs. If first order LCFDEs are in the form of Eq (1.1), they does not extend, or they do extend but the extended system and initial system have the same solutions. To support this work, we provide Examples 4.2 and 4.3 of non-extend systems, and Example 4.4 of the system which extends but the extended system and initial system have the same solutions. Moreover, Eq (1.1) can also produce first order LCFDEs like [18] which do not extend and have the unique solution discussed in Examples 3.5 and 4.6. The second problem in first order LCFDEs is the existence of solutions such as the Example 8 of [18], do not have any solution. Therefore, we obtained the conditions for the existence and uniqueness of the solutions in Theorems 3.1 and 4.1. First order LCFDEs satisfying these conditions must have unique solutions. We provide examples of the usability and authenticity of the established results. This work is applicable to all real life problems where uncertainty lies in the spaces of linear correlated fuzzy numbers, because $R \subseteq R_{F(A)} \subset R_F$. The existence of solutions of second order LCFDEs in LC-spaces and stability like [22] etc. are also interesting topics of future study. The LCFDEs with fractional order like [23–25] are also interesting. This study can also extend to the work of [26].

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of Interest

The authors declare that there is no conflict of interest.

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