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Research article

Prioritized aggregation operators for Schweizer-Sklar multi-attribute decision-making for complex spherical fuzzy information in mobile e-tourism applications

Khawlah Alhulwah[1], Muhammad Azeem[2,*], Mehwish Sarfraz[3], Nasreen Almohanna[1] and Ali Ahmad[4]

[1] Department of Mathematics and Statistics, Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh 11564, Saudi Arabia
[2] Department of Mathematics, Riphah International University, Lahore 54000, Pakistan
[3] Independent Researcher, Lahore 54000, Pakistan
[4] Department of Computer Science, College of Engineering and Computer Science, Jazan University, Jazan, Saudi Arabia

* **Correspondence:** Email: azeemali7009@gmail.com; Tel: +923005067913.

Abstract: Complex spherical fuzzy sets (CSFSs) are a theory that addresses confusing and unreliable information in real-life decision-making contexts by integrating elements of two theories: spherical fuzzy sets (SFSs) and complex fuzzy sets (CFSs). CSFSs are classified into three categories, represented by polar coordinates: membership, nonmember, and abstention. These grades are located on a complex plane within a unit disc. It is necessary for the total squares representing the real components of the grades for abstinence, membership, and non-membership to not surpass a certain interval. Several aspects of CSFS and the corresponding operational laws were examined in this work. The key components of this article were based on CSFs, including complex spherical fuzzy Schweizer-Sklar prioritized aggregation (CSFSSPA), complex spherical fuzzy Schweizer-Sklar weighted prioritized aggregation (CSFSSWPA), complex spherical fuzzy Schweizer-Sklar prioritized geometry (CSFSSPG), and complex spherical fuzzy Schweizer-Sklar prioritized weighted geometry (CSFSSWPG). Additionally, the suggested operators' specific instances were examined. The main outcome of this work includes new aggregation techniques for CSFS information, based on t-conorm and t-norm from Schweizer-Sklar (SS). The basic characteristics of the operators were established by this study. We looked at a numerical example centered on efficient mobile e-tourism selection to show the effectiveness and viability of the recommended approaches. Additionally, we carried out a thorough comparative analysis to assess the outcomes of the suggested aggregation approaches in comparison to the current methods. Last, we offer an overview of the planned study and talk about potential directions for the future.

Keywords: prioritized aggregation operators; Schweizer-Sklar; multi-attribute decision-making; complex spherical fuzzy information; mobile e-tourism applications
Mathematics Subject Classification: 05C72, 68R10

Abbreviations	
CSFSSPA	complex spherical fuzzy Schweizer-Sklar prioritized aggregation
CSFSSWPA	the complex spherical fuzzy Schweizer-Sklar weighted prioritized aggregation
CSFSSPG	complex spherical fuzzy Schweizer-Sklar prioritized geometry
CSFSSWPG	complex spherical fuzzy Schweizer-Sklar weighted prioritized geometry
SSPA	Schweizer-Sklar prioritized aggregation
CSFSs	complex spherical fuzzy sets
SFSs	spherical fuzzy sets
CFSs	complex fuzzy sets
PAOs	prioritized aggregation operators
SS	Schweizer-Sklar
MADM	multi-attribute decision-making
MG	membership great
AG	abstinence great
NMG	non-membership great
DMs	decision-makers

1. Introduction

By providing consumers with an easy way to access travel-related information and services straight from their mobile devices, mobile e-tourism applications have revolutionized the travel and tourism sector. These apps improve users' overall travel experiences by offering features like location-based recommendations, virtual tours, real-time booking, and customized itineraries. The prioritized aggregation operators (PAOs) combined with complex spherical fuzzy information improve the accuracy and efficiency of decision-making in these applications. This method makes it possible to handle user preference ambiguity and uncertainty more effectively, which results in more individualized recommendations and higher user satisfaction. These apps can also help travel agencies better understand consumers' trends and needs, and that will help them modify their products and services and stay competitive.

Zadeh [1] clarified the uncertainty and imprecision that occur throughout the evaluation process using fuzzy sets (FSs) as an example. For instance, researchers looked at intuitionistic FSs (IFSs) with supporting and non-supporting grades with the restriction that the total of the two cannot be greater than a unit interval Atanassov [2]. However, in terms of choosing the total of supporting and non-supporting grades that do not surpass a unit interval, the standards of an IFS for a decision-maker are too strict. Yager [3] looked into the Pythagorean fuzzy set (PyFS) to solve these issues, with the caveat that the sum of the squares of both must not exceed a unit interval. The picture fuzzy sets (PFSs) were later introduced by Cuong and Kreinovich [4]. PFSs are grades for membership, abstention, and non-membership; all grades cannot exceed a set unit interval. Mahmood et al.'s theory of SFS [5] was applied to resolve these problems; however, a unit interval must be maintained in the sum of squares for all classes. Ullah et al. developed T-spherical fuzzy sets (T-SFSs) [6] by extending squares with q-powers. These sets, which have several applications in diverse disciplines [7–11], contain the sum of the q-powers of positive, abstinence, and negative grades, corresponding to $[0,1]$. Consequently, the complex FS (CFS) was invented by Ramot et al. [12] and has been applied in various ways [13–15].

1.1. Literature review

Furthermore, the notion of complex IFSs (CIFSs) was established by Alkouri and Salleh [16] to provide a decision-maker with many possibilities. A unit disc in a complex plane's complex numbers representing the supporting and non-supporting grades make up CIFSs. Adding the real and imaginary parts of two classes that are more than one unit apart is a limitation of CIFSs.

The grades of imaginary and real components, whose sum surpasses a unit interval, may be provided by a decision maker, nevertheless. Yin et al. [17] introduced the concept of the MADM with Pythagorean fuzzy information. To address this type of problem Ullah et al. [18] proposed the theory of complex PFSs (CPFSs). PFS concept was based on the SS aggregation operator by Hussain et al. [19]. Rahman et al. [20–22] used the concept of a complex polytypic fuzzy model. The SS operator concept was developed by Wei et al. using an entropy-combined solution approach. Liu et al. [23] used the q-rung orthopair as the basis for the SS aggregation operator's notion based on MADM. Considering the idea of a complicated spherical fuzzy Aczel-Alsina aggregation operator, Hussain et al. [24] and Sarfraz et al. [25], in light of the Aczel-Alsina aggregation operator-based MAGDM approach, developed and gave an application to authenticate the developed method. A prioritized aggregation operator is a concept introduced by Ullah et al. [26] for complex IFS. The idea of MAGDM based on SS TN and TCM, Hussain et al. [27] and Sarfraz et al. [28–30] introduced the concepts of SSPAOs with the application of recycled water and different parameters. Hussain et al. [31] introduced the theory of spherical fuzzy Sugeno-weber aggregation operator and Asif et al. [32] used the concept of the Pythagorean fuzzy set and its application in terms of MADM.

The flowchart in Figure 1 shows how to apply prioritized aggregation operators for multi-attribute decision-making in mobile e-tourism applications using Schweizer-Sklar t-norms and t-conorms with an optional feedback loop to fine-tune user preferences, it delineates crucial phases, ranging from data collection to decision-making.

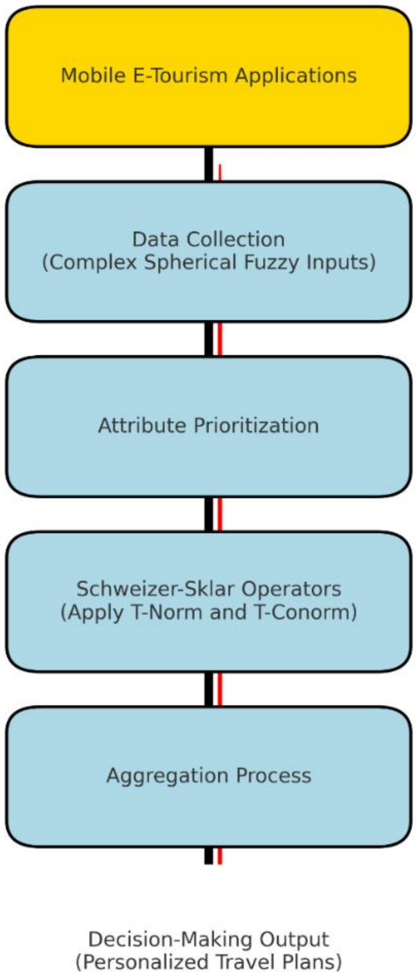

Figure 1. Depiction of a flowchart of the methodology of the application.

Worndl et al. [33] used the MADM to extend the use of mobile e-tourism on a spherical basis. Considering the idea of e-tourism for a system utilizing data management, Hamid et al. [34], and Mohammed and others [35], worked on it and developed different methodologies. The decision-making strategy for e-tourism based on a spherical rough notion. Leung et al. [36] extended the theory based on the idea of e-tourism to smart tourism using communication and information advancements. By considering the idea of e-tourism, Utomo et al. [37] implemented the business intelligent topics. According to Krishnan et al. [38], the interval type 2 fuzzy with zero weights is applied for e-tourism. E-tourism services are expanded by Vdovenko et al. [39] through the improvement of the smart room system. E-tourism was developed by Alamoodi et al. [40,41] using a neutrosophic fuzzy environment as its foundation. Qurashi et al. [42] introduced the concept of rough substructures with relations of fuzzy overlaps. Wang et al. [43] used the Fermatean cubic fuzzy method in the energy sector.

1.2. Identifying the research gap

- Limited research has been done on applying Schweizer-Sklar aggregation operators (SSAOs) to complex spherical fuzzy information in decision-making scenarios, despite the fact that many studies have been conducted on MADM techniques using conventional fuzzy sets.

- The uncertainty and complexity present in real-world applications, like mobile e-tourism, where user preferences are highly variable and data is frequently ambiguous, are frequently too much for current approaches to handle.
- Furthermore, prioritizing criteria is crucial when different factors have differing degrees of importance in decision-making, but most researchers concentrate on standard aggregation without taking this into account.

1.3. Motivation for studying

The CSFS environment system has many benefits, but it can occasionally be unable to handle unpredictable data about human opinions. For example, if data is membership grade (MG), abstinence grade (AG), and non-membership grade (NMG), among the other three components, the environments, IFS, PyFSs, and PFS that are discussed cannot handle such nature of the data. The decision-maker takes into account robust environments such as SFS and CSFS to get past this circumstance. The method that has been developed offers the benefit of handling incomplete data without compromising the degree of weight. Creating a class of mathematical techniques under the CSFS information system is our main objective. The following list includes some important benefits of the derived theory:

a) Our suggested research methodologies enable us to more accurately express uncertainty and complexity under the CSFS information system during the MADM problem (see Figures 1 and 2).
b) We can also assess any object's given information without the need for any external weight vectors, which the decision-maker assigns to the attributes or characteristics.

Figure 2. Flowchart illustrating applications for mobile e-tourism.

1.4. Contribution

The previously mentioned study suggests that the real world is complex when it comes to PAOs used in MADM. To discover the best alternative in MADM, the information needs to be handled more conveniently. Moreover, CSFSs manage environment ambiguity better than IFSs, PyFSs, q-ROFSs, and PFSs. The T-conorm and the SS T-norm have not yet been applied to CSFSs. These considerations motivate us to investigate the uses of SSPAOs in MADM, having first formalized their concept in the construction of CSFSs. This article is organized as follows. We derive the prioritized aggregation operator, often known as the SS theory, for CSFSs.

(1) For derived work, we deduce qualities of idempotency, monotonicity, and boundedness.
(2) Drawing from the ideas put forward, we introduce a MADM method.
(3) We present some valuable characteristics of the above-present operators and pioneer a MADM tool to handle the unreliable and vague type of information with the help of CSFSSPA, CSFSSWPA, CSFSSPG, and CSF.
(4) We provide a numerical example of a travel app that compares the suggested effort to certain previous research.
(5) Compared to more conventional approaches like technique for order of preference by similarity to ideal solution (TOPSIS) or VIseKriterijumska Optimizacija I Kompromisno Resenje (VIKOR), the suggested method gives decision-makers the ability to rank criteria, offering a more accurate and flexible assessment of e-tourism options.

1.5. Organization of the study

The following profile was used to compute this paper: In Section 2, we looked at the specific historical statistics on CSFS and its SS functioning laws. Furthermore, we used optimistic data to develop the PAO idea. In Section 3, we discussed the creation, beneficial traits, and results of CSFSSPA, CSFSSWPA, CSFSSPG, and CSFSSWPG operators. We cover several important characteristics of the previously described operators in Section 4. We are the first to use the CSFSSPA, CSFSSWPA, CSFSSPG, and CSFSSWPG operators to handle unclear and untrustworthy sorts of information. In Section 5, we demonstrate that the effort generated is obviously better than the other workers by contrasting novel and tested methods to raise the value and capacity of the evaluated operators. We conclude in Section 6 with a few insightful remarks. A geometric depiction of the obtained theory is provided.

2. Preliminaries

To create some ideal and useful theories, we must update some knowledge about CSFSs and their SS operational regulations. Additionally, we update the theory of positive integer-based PAOs, and the universal set is denoted by $\overline{\overline{\mho}}$.

Definition 2.1. [44] *It is found that a CSFS* $\overline{\overline{\emptyset}}$ *in a universal set* $\overline{\overline{\mho}}$*:*

$$\overline{\overline{\emptyset}} = \left\{\left(\overline{\overline{\mathfrak{A}}}(\mathcal{X})exp^{2\pi i\left(\overline{\overline{u}}(\mathcal{X})\right)}, \overline{\overline{\mathscr{f}}}(\mathcal{X})exp^{2\pi i\left(\overline{\overline{v}}(\mathcal{X})\right)}, \overline{\overline{\ell}}(\mathcal{X})exp^{2\pi i\left(\overline{\overline{k}}(\mathcal{X})\right)}\right) : \mathcal{X} \in \overline{\overline{\mho}}\right\}.$$

In the following cases: $\overline{\overline{\mathfrak{A}}}(\mathcal{X}) : \overline{\overline{\emptyset}} \to [0,1]$, $\overline{\overline{\mathscr{f}}}(\mathcal{X}) : \overline{\overline{\emptyset}} \to [0,1]$, $\overline{\overline{\ell}}(\mathcal{X}) : \overline{\overline{\emptyset}} \to [0,1]$ encompassing the requirement: for the element $\mathcal{X}$ in $\overline{\overline{\emptyset}}'$, $0 \le \overline{\overline{\mathfrak{A}}}^2(\mathcal{X}) + \overline{\overline{\mathscr{f}}}^2(\mathcal{X}) + \overline{\overline{\ell}}^2(\mathcal{X}) \le 1$. The membership great (MG), abstinence great (AG), and non-membership great (NMG) of the element X in the set $\overline{\overline{\emptyset}}$ are represented by the numbers $\overline{\overline{\mathfrak{A}}}^2(\mathcal{X})'$, $\overline{\overline{\mathscr{f}}}^2(\mathcal{X})$ and $\overline{\overline{\ell}}^2(\mathcal{X})$, respectively. Additionally, the following is how we explained the importance of refusal information $\overline{\overline{\delta_\mu}}(\mathcal{X}) = 1 - \Big(\overline{\overline{\mathfrak{A}}}^2(\mathcal{X}) +$

$\bar{\bar{\mathfrak{f}}}^2(\mathcal{X})+\bar{\bar{\ell}}^2(\mathcal{X})\big)$. Last, the mention of the CSF number's representation (CSFs) is made by $\bar{\bar{\emptyset}}=\left(\overline{\overline{\mathfrak{A}_\tau}},\bar{\bar{\mathfrak{f}}}_\tau,\bar{\bar{\ell}}_\tau\right)$, $\tau=1,2,\dots,\alpha$.

Definition 2.2. [45] *Based on defined CSFVs* $\overline{\overline{\emptyset_\tau}}=\left(\overline{\overline{\mathfrak{A}_\tau^2}}exp^{2\pi i\left(\overline{\overline{u_\tau^2}}\right)},\overline{\overline{\mathfrak{f}_\tau^2}}exp^{2\pi i\left(\overline{\overline{v_\tau^2}}\right)},\overline{\overline{\ell_\tau^2}}exp^{2\pi i\left(\overline{\overline{k_\tau^2}}\right)}\right)$, $\tau=1,2,\dots,\alpha$. *The accuracy function and score are derived, as follows:*

$$\wedge_s\overline{\overline{\emptyset_\tau}}=\frac{\left(2+\overline{\overline{\mathfrak{A}_\tau^2}}-\overline{\overline{\mathfrak{f}_\tau^2}}-\overline{\overline{\ell_\tau^2}}\right)}{3},$$
$$\wedge_a\overline{\overline{\emptyset_\tau}}=\frac{\left(2+\overline{\overline{\mathfrak{A}_\tau^2}}+\overline{\overline{\mathfrak{f}_\tau^2}}+\overline{\overline{\ell_\tau^2}}\right)}{3}.$$

See that $\wedge_s\overline{\overline{\emptyset_\tau}}\in[-1,1]$ *and* $\wedge_a\overline{\overline{\emptyset_\tau}}\in[0,1]$ *for all CSFVs* $\overline{\overline{\emptyset_\tau}}$.

To further justify the information, we have derived some limitations such as when we consider $\overline{\overline{\emptyset_1}}>\overline{\overline{\emptyset_2}}$, we get $\wedge_s\overline{\overline{\emptyset_1}}>\wedge_s\overline{\overline{\emptyset_2}}$; if we consider $\overline{\overline{\emptyset_1}}<\overline{\overline{\emptyset_2}}$, then we get $\wedge_s\overline{\overline{\emptyset_1}}<\wedge_s\overline{\overline{\emptyset_2}}$; but if we get $\wedge_s\overline{\overline{\emptyset_1}}=\wedge_s\overline{\overline{\emptyset_2}}$, then we will follow the idea such as if we consider $\overline{\overline{\emptyset_1}}>\overline{\overline{\emptyset_2}}$, then we get $\wedge_a\overline{\overline{\emptyset_1}}>\wedge_a\overline{\overline{\emptyset_2}}$; we will consider $\overline{\overline{\emptyset_1}}<\overline{\overline{\emptyset_2}}$ when we get $\wedge_a\overline{\overline{\emptyset_1}}<\wedge_a\overline{\overline{\emptyset_2}}$ and which is the property of CSFVs.

Definition 2.3. [45] *We obtain the concept of PAOS based on the positive integers* $\overline{\overline{\emptyset_\tau}}=\tau=1,2,\dots,\alpha$ *such as*

$$PA=\left(\overline{\overline{\emptyset_1}},\overline{\overline{\emptyset_2}},\dots,\overline{\overline{\emptyset_\alpha}}\right)=\beta_1\overline{\overline{\emptyset_1}}\oplus\beta_2\overline{\overline{\emptyset_2}}\oplus\dots.\oplus\beta_\alpha\overline{\overline{\emptyset_\alpha}}=\oplus_{\tau=1}^{\alpha}\beta_\tau\overline{\overline{\emptyset_\tau}}.$$

Noticed that $\beta_\tau=\frac{\beta_\tau}{\sum_{\tau=1}^{\alpha}\beta_\tau}$, where $\beta_1=1$ and $\beta_\tau=\oplus_\kappa^{\tau-1}\wedge_s\left(\overline{\overline{\emptyset_\kappa}}\right)$, $\kappa=1,2,\dots,\alpha$.

Definition 2.4. [44] *For any CSFV* $\overline{\overline{\emptyset_\tau}}=\left(\overline{\overline{\mathfrak{A}_\tau^2}}exp^{2\pi i\left(\overline{\overline{u_\tau^2}}\right)},\overline{\overline{\mathfrak{f}_\tau^2}}exp^{2\pi i\left(\overline{\overline{v_\tau^2}}\right)},\overline{\overline{\ell_\tau^2}}exp^{2\pi i\left(\overline{\overline{k_\tau^2}}\right)}\right)$, $\tau=1,2,\dots,\alpha$, *some necessary operations of the Schweizer-Sklar tools are expressed as*

$$\overline{\overline{\emptyset_1}}\oplus\overline{\overline{\emptyset_2}}=\begin{pmatrix}\sqrt[2]{\left(1-\left(\left(1-\overline{\overline{\mathfrak{A}_1^2}}\right)^{\bar{\bar{\zeta}}}+\left(1-\overline{\overline{\mathfrak{A}_2^2}}\right)^{\bar{\bar{\zeta}}}\right)-1\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\,exp^{2\pi i\left(\sqrt[2]{\left(1-\left(\left(1-\overline{\overline{u_1^2}}\right)^{\bar{\bar{\zeta}}}+\left(1-\overline{\overline{u_2^2}}\right)^{\bar{\bar{\zeta}}}\right)-1\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\right)}\\ \sqrt[2]{\left(\left(\overline{\overline{\mathfrak{f}_1^2}}\right)^{\bar{\bar{\zeta}}}+\left(\overline{\overline{\mathfrak{f}_2^2}}\right)^{\bar{\bar{\zeta}}}-1\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\,exp^{2\pi i\left(\sqrt[2]{\left(\left(\overline{\overline{v_1^2}}\right)^{\bar{\bar{\zeta}}}+\left(\overline{\overline{v_2^2}}\right)^{\bar{\bar{\zeta}}}-1\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\right)}\\ \sqrt[2]{\left(\left(\overline{\overline{\ell_1^2}}\right)^{\bar{\bar{\zeta}}}+\left(\overline{\overline{\ell_2^2}}\right)^{\bar{\bar{\zeta}}}-1\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\,exp^{2\pi i\left(\sqrt[2]{\left(\left(\overline{\overline{k_1^2}}\right)^{\bar{\bar{\zeta}}}+\left(\overline{\overline{k_2^2}}\right)^{\bar{\bar{\zeta}}}-1\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\right)}\end{pmatrix},$$

$$\overline{\overline{\emptyset_1}} \otimes \overline{\overline{\emptyset_2}} = \begin{pmatrix} \sqrt[2]{\left(\left(\overline{\overline{\mathscr{f}_1^2}}\right)^{\overline{\overline{\zeta}}} + \left(\overline{\overline{\mathscr{f}_2^2}}\right)^{\overline{\overline{\zeta}}} - 1\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\, exp^{2\pi i\left(\sqrt[2]{\left(\left(\overline{\overline{\mathscr{u}_1^2}}\right)^{\overline{\overline{\zeta}}} + \left(\overline{\overline{\mathscr{u}_2^2}}\right)^{\overline{\overline{\zeta}}} - 1\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\right)} \\ \sqrt[2]{\left(1 - \left(\left(1 - \overline{\overline{\mathfrak{A}_1^2}}\right)^{\overline{\overline{\zeta}}} + \left(1 - \overline{\overline{\mathfrak{A}_2^2}}\right)^{\overline{\overline{\zeta}}}\right) - 1\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\, exp^{2\pi i\left(\sqrt[2]{\left(1 - \left(\left(1 - \overline{\overline{\mathscr{v}_1^2}}\right)^{\overline{\overline{\zeta}}} + \left(1 - \overline{\overline{\mathscr{v}_2^2}}\right)^{\overline{\overline{\zeta}}}\right) - 1\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\right)} \\ \sqrt[2]{\left(1 - \left(\left(1 - \overline{\overline{\ell_1^2}}\right)^{\overline{\overline{\zeta}}} + \left(1 - \overline{\overline{\ell_2^2}}\right)^{\overline{\overline{\zeta}}}\right) - 1\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\, exp^{2\pi i\left(\sqrt[2]{\left(1 - \left(\left(1 - \overline{\overline{\mathscr{k}_1^2}}\right)^{\overline{\overline{\zeta}}} + \left(1 - \overline{\overline{\mathscr{k}_2^2}}\right)^{\overline{\overline{\zeta}}}\right) - 1\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\right)} \end{pmatrix},$$

$$\Psi\overline{\overline{\emptyset_1}} = \begin{pmatrix} \sqrt[2]{\left(1 - \left(\Psi\left(1 - \overline{\overline{\mathfrak{A}_1^2}}\right)^{\overline{\overline{\zeta}}} - (\Psi - 1)\right)\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\, exp^{2\pi i\left(\sqrt[2]{\left(1 - \left(\Psi\left(1 - \overline{\overline{\mathscr{u}_1^2}}\right)^{\overline{\overline{\zeta}}} - (\Psi - 1)\right)\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\right)} \\ \sqrt[2]{\left(\Psi\left(1 - \overline{\overline{\mathscr{f}_1^2}}\right)^{\overline{\overline{\zeta}}} - (\Psi - 1)\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\, exp^{2\pi i\left(\sqrt[2]{\left(\Psi\left(1 - \overline{\overline{\mathscr{v}_1^2}}\right)^{\overline{\overline{\zeta}}} - (\Psi - 1)\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\right)} \\ \sqrt[2]{\left(\Psi\left(1 - \overline{\overline{\ell_1^2}}\right)^{\overline{\overline{\zeta}}} - (\Psi - 1)\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\, exp^{2\pi i\left(\sqrt[2]{\left(\Psi\left(1 - \overline{\overline{\mathscr{k}_1^2}}\right)^{\overline{\overline{\zeta}}} - (\Psi - 1)\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\right)} \end{pmatrix},$$

$$\overline{\overline{\emptyset_1}}^{\Psi} = \begin{pmatrix} \sqrt[2]{\left(\Psi\left(1 - \overline{\overline{\mathfrak{A}_1^2}}\right)^{\overline{\overline{\zeta}}} - (\Psi - 1)\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\, exp^{2\pi i\left(\sqrt[2]{\left(\Psi\left(1 - \overline{\overline{\mathscr{u}_1^2}}\right)^{\overline{\overline{\zeta}}} - (\Psi - 1)\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\right)} \\ \sqrt[2]{1 - \left(\Psi\left(1 - \overline{\overline{\mathscr{f}_1^2}}\right)^{\overline{\overline{\zeta}}} - (\Psi - 1)\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\, exp^{2\pi i\left(\sqrt[2]{1 - \left(\Psi\left(1 - \overline{\overline{\mathscr{v}_1^2}}\right)^{\overline{\overline{\zeta}}} - (\Psi - 1)\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\right)} \\ \sqrt[2]{\left(1 - \left(\Psi\left(1 - \overline{\overline{\ell_1^2}}\right)^{\overline{\overline{\zeta}}} - (\Psi - 1)\right)\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\, exp^{2\pi i\left(\sqrt[2]{\left(1 - \left(\Psi\left(1 - \overline{\overline{\mathscr{k}_1^2}}\right)^{\overline{\overline{\zeta}}} - (\Psi - 1)\right)\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\right)} \end{pmatrix}.$$

3. Schweizer-Sklar PAOs for data on CSF

The main contribution of this section is the derivation of the operators CSFSSPA, CSFSSWPA, CSFSSPG, and CSFSSWPG, along with the identification of their practical properties and results.

Definition 3.1. *For any CSFVs* $\overline{\overline{\emptyset_\tau}} = \left(\overline{\overline{\mathfrak{A}_\tau^2}} exp^{2\pi i\left(\overline{\overline{u_\tau^2}}\right)}, \overline{\overline{\mathfrak{f}_\tau^2}} exp^{2\pi i\left(\overline{\overline{v_\tau^2}}\right)}, \overline{\overline{\ell_\tau^2}} exp^{2\pi i\left(\overline{\overline{k_\tau^2}}\right)}\right)$, $\tau = 2, 3, \ldots, \alpha$, *the notion CSFSSPA operator is expressed as follows:*

$$CSFSSPA\left(\overline{\overline{\emptyset_1}}, \overline{\overline{\emptyset_2}}, \ldots, \overline{\overline{\emptyset_\alpha}}\right) = \beta_1 \overline{\overline{\emptyset_1}} \oplus \beta_2 \overline{\overline{\emptyset_2}} \oplus \ldots \oplus \beta_\alpha \overline{\overline{\emptyset_\alpha}} = \oplus_{\tau=1}^{\alpha} \beta_\tau \overline{\overline{\emptyset_\tau}}. \quad (2.1)$$

Noticed that $\beta_\tau = \frac{\beta_\tau}{\sum_{\tau=1}^{\alpha} \beta_\tau}$, where $\beta_1 = 1$ and $\beta_\tau = \oplus_\kappa^{\tau-1} \wedge_s \left(\overline{\overline{\emptyset_\kappa}}\right)$, $\kappa = 2, 3, \ldots, \alpha$.

Theorem 3.1. *For any CSFVs* $\overline{\overline{\emptyset_\tau}} = \left(\overline{\overline{\mathfrak{A}_\tau^2}} exp^{2\pi i\left(\overline{\overline{u_\tau^2}}\right)}, \overline{\overline{\mathfrak{f}_\tau^2}} exp^{2\pi i\left(\overline{\overline{v_\tau^2}}\right)}, \overline{\overline{\ell_\tau^2}} exp^{2\pi i\left(\overline{\overline{k_\tau^2}}\right)}\right)$, $\tau = 1, 2, \ldots, \alpha$, *the integrated values of the CSFSSPA operator are CSFV such as*

$$CSFSSPA\left(\overline{\overline{\emptyset_1}}, \overline{\overline{\emptyset_2}}, \ldots, \overline{\overline{\emptyset_\alpha}}\right)$$

$$= \begin{pmatrix} \sqrt[2]{1 - \left(\sum_{\tau=1}^{\alpha} \beta_\tau \left(1 - \overline{\overline{\mathfrak{A}_\tau^2}}\right)^{\bar{\bar{\zeta}}} - \sum_{\tau=1}^{\alpha} \beta_\tau + 1\right)^{\frac{1}{\bar{\bar{\zeta}}}}} exp^{2\pi i\left(\sqrt[2]{1 - \left(\sum_{\tau=1}^{\alpha} \beta_\tau \left(1 - \overline{\overline{u_\tau^2}}\right)^{\bar{\bar{\zeta}}} - \sum_{\tau=1}^{\alpha} \beta_\tau + 1\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\right)} \\ \sqrt[2]{\left(\sum_{\tau=1}^{\alpha} \beta_j \left(\overline{\overline{\mathfrak{f}_\tau^2}}\right)^{\bar{\bar{\zeta}}} - \sum_{\tau=1}^{\alpha} \beta_\tau + 1\right)^{\frac{1}{\bar{\bar{\zeta}}}}} exp^{2\pi i\left(\sqrt[2]{\left(\sum_{\tau=1}^{\alpha} \beta_j \left(\overline{\overline{v_\tau^2}}\right)^{\bar{\bar{\zeta}}} - \sum_{\tau=1}^{\alpha} \beta_\tau + 1\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\right)} \\ \sqrt[2]{\left(\sum_{\tau=1}^{\alpha} \beta_j \left(\overline{\overline{\ell_\tau^2}}\right)^{\bar{\bar{\zeta}}} - \sum_{\tau=1}^{\alpha} \beta_\tau + 1\right)^{\frac{1}{\bar{\bar{\zeta}}}}} exp^{2\pi i\left(\sqrt[2]{\left(\sum_{\tau=1}^{\alpha} \beta_j \left(\overline{\overline{k_\tau^2}}\right)^{\bar{\bar{\zeta}}} - \sum_{\tau=1}^{\alpha} \beta_\tau + 1\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\right)} \end{pmatrix}. \quad (2.2)$$

Proof. Deriving Eq (2.2) is a very challenging task for scholars; to overcome this challenge, we use a popular and effective method of calculation known as "induction method". Therefore, we consider $\alpha = 2$ to properly evidence Eq (2.2) for this, as in

$$\beta_1 \overline{\overline{\emptyset_1}} = \begin{pmatrix} \sqrt[2]{1 - \left(\beta_1 \left(1 - \overline{\overline{\mathfrak{A}_1^2}}\right)^{\bar{\bar{\zeta}}} - (\beta_1 - 1)\right)^{\frac{1}{\bar{\bar{\zeta}}}}} exp^{2\pi i\left(\sqrt[2]{1 - \left(\beta_1 \left(1 - \overline{\overline{u_1^2}}\right)^{\bar{\bar{\zeta}}} - (\beta_1 - 1)\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\right)} \\ \sqrt[2]{\left(\beta_1 \left(\overline{\overline{\mathfrak{f}_1^2}}\right)^{\bar{\bar{\zeta}}} - (\beta_1 - 1)\right)^{\frac{1}{\bar{\bar{\zeta}}}}} exp^{2\pi i\left(\sqrt[2]{\left(\beta_1 \left(\overline{\overline{v_1^2}}\right)^{\bar{\bar{\zeta}}} - (\beta_1 - 1)\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\right)} \\ \sqrt[2]{\left(\beta_1 \left(\overline{\overline{\ell_1^2}}\right)^{\bar{\bar{\zeta}}} - (\beta_1 - 1)\right)^{\frac{1}{\bar{\bar{\zeta}}}}} exp^{2\pi i\left(\sqrt[2]{\left(\beta_1 \left(\overline{\overline{k_1^2}}\right)^{\bar{\bar{\zeta}}} - (\beta_1 - 1)\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\right)} \end{pmatrix},$$

$$\beta_2\overline{\overline{\emptyset_2}} = \begin{pmatrix} \sqrt[2]{1-\left(\beta_2\left(1-\overline{\overline{\mathfrak{A}_2^2}}\right)^{\bar{\bar{\zeta}}}-(\beta_2-1)\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\, exp^{2\pi i\left(\sqrt[2]{1-\left(\beta_2\left(1-\overline{\overline{u_2^2}}\right)^{\bar{\bar{\zeta}}}-(\beta_2-1)\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\right)} \\ \sqrt[2]{\left(\beta_2\left(\overline{\overline{\mathfrak{f}_2^2}}\right)^{\bar{\bar{\zeta}}}-(\beta_2-1)\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\, exp^{2\pi i\left(\sqrt[2]{\left(\beta_2\left(\overline{\overline{v_2^2}}\right)^{\bar{\bar{\zeta}}}-(\beta_2-1)\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\right)} \\ \sqrt[2]{\left(\beta_2\left(\overline{\overline{\ell_2^2}}\right)^{\bar{\bar{\zeta}}}-(\beta_2-1)\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\, exp^{2\pi i\left(\sqrt[2]{\left(\beta_2\left(\overline{\overline{k_2^2}}\right)^{\bar{\bar{\zeta}}}-(\beta_2-1)\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\right)} \end{pmatrix}.$$

To further justify Eq (2.2), we consider

$CSFSSPA\left(\overline{\overline{\emptyset_1}}, \overline{\overline{\emptyset_2}}\right) = \beta_1\overline{\overline{\emptyset_1}} \oplus \beta_2\overline{\overline{\emptyset_2}}$

$$= \begin{pmatrix} \begin{pmatrix} \sqrt[2]{1-\left(\beta_1\left(1-\overline{\overline{\mathfrak{A}_1^2}}\right)^{\bar{\bar{\zeta}}}-(\beta_1-1)\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\, exp^{2\pi i\left(\sqrt[2]{1-\left(\beta_1\left(1-\overline{\overline{u_1^2}}\right)^{\bar{\bar{\zeta}}}-(\beta_1-1)\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\right)} \\ \sqrt[2]{\left(\beta_1\left(\overline{\overline{\mathfrak{f}_1^2}}\right)^{\bar{\bar{\zeta}}}-(\beta_1-1)\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\, exp^{2\pi i\left(\sqrt[2]{\left(\beta_1\left(\overline{\overline{v_1^2}}\right)^{\bar{\bar{\zeta}}}-(\beta_1-1)\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\right)} \\ \sqrt[2]{\left(\beta_1\left(\overline{\overline{\ell_1^2}}\right)^{\bar{\bar{\zeta}}}-(\beta_1-1)\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\, exp^{2\pi i\left(\sqrt[2]{\left(\beta_1\left(\overline{\overline{k_1^2}}\right)^{\bar{\bar{\zeta}}}-(\beta_1-1)\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\right)} \end{pmatrix} \oplus \\ \begin{pmatrix} \sqrt[2]{1-\left(\beta_2\left(1-\overline{\overline{\mathfrak{A}_2^2}}\right)^{\bar{\bar{\zeta}}}-(\beta_2-1)\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\, exp^{2\pi i\left(\sqrt[2]{1-\left(\beta_2\left(1-\overline{\overline{u_2^2}}\right)^{\bar{\bar{\zeta}}}-(\beta_2-1)\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\right)} \\ \sqrt[2]{\left(\beta_2\left(\overline{\overline{\mathfrak{f}_2^2}}\right)^{\bar{\bar{\zeta}}}-(\beta_2-1)\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\, exp^{2\pi i\left(\sqrt[2]{\left(\beta_2\left(\overline{\overline{v_2^2}}\right)^{\bar{\bar{\zeta}}}-(\beta_2-1)\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\right)} \\ \sqrt[2]{\left(\beta_2\left(\overline{\overline{\ell_2^2}}\right)^{\bar{\bar{\zeta}}}-(\beta_2-1)\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\, exp^{2\pi i\left(\sqrt[2]{\left(\beta_2\left(\overline{\overline{k_2^2}}\right)^{\bar{\bar{\zeta}}}-(\beta_2-1)\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\right)} \end{pmatrix} \end{pmatrix}$$

$$
= \begin{pmatrix}
\sqrt[2]{\left(1-\begin{pmatrix}\left(1-\left(1-\left(\beta_1\left(1-\overline{\overline{\mathfrak{A}_1^2}}\right)^{\bar{\bar{\zeta}}}-(\beta_1-1)\right)^{\frac{1}{\bar{\bar{\zeta}}}}\right)\right)^{\bar{\bar{\zeta}}}+\\ \left(1-\left(1-\left(\beta_2\left(1-\overline{\overline{\mathfrak{A}_2^2}}\right)^{\bar{\bar{\zeta}}}-(\beta_2-1)\right)^{\frac{1}{\bar{\bar{\zeta}}}}\right)\right)^{\bar{\bar{\zeta}}}\end{pmatrix}-1\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\; exp^{2\pi i\left(\sqrt[2]{\left(1-\begin{pmatrix}\left(1-\left(1-\left(\beta_1\left(1-\overline{\overline{u_1^2}}\right)^{\bar{\bar{\zeta}}}-(\beta_1-1)\right)^{\frac{1}{\bar{\bar{\zeta}}}}\right)\right)^{\bar{\bar{\zeta}}}+\\ \left(1-\left(1-\left(\beta_2\left(1-\overline{\overline{u_2^2}}\right)^{\bar{\bar{\zeta}}}-(\beta_2-1)\right)^{\frac{1}{\bar{\bar{\zeta}}}}\right)\right)^{\bar{\bar{\zeta}}}\end{pmatrix}-1\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\right)} \\
\sqrt[2]{\left(\left(\left(\beta_1\left(\overline{\overline{f_1^2}}\right)^{\bar{\bar{\zeta}}}-(\beta_1-1)\right)^{\frac{1}{\bar{\bar{\zeta}}}}\right)^{\bar{\bar{\zeta}}}+\left(\left(\beta_2\left(\left(\overline{\overline{f_1^2}}\right)^2\right)^{\bar{\bar{\zeta}}}-(\beta_2-1)\right)^{\frac{1}{\bar{\bar{\zeta}}}}\right)^{\bar{\bar{\zeta}}}-1\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\; exp^{2\pi i\left(\sqrt[2]{\begin{pmatrix}\left(\left(\beta_1\left(\overline{\overline{v_1^2}}\right)^{\bar{\bar{\zeta}}}-(\beta_1-1)\right)^{\frac{1}{\bar{\bar{\zeta}}}}\right)^{\bar{\bar{\zeta}}}+\\ \left(\left(\beta_2\left(\left(\overline{\overline{v_1^2}}\right)^2\right)^{\bar{\bar{\zeta}}}-(\beta_2-1)\right)^{\frac{1}{\bar{\bar{\zeta}}}}\right)^{\bar{\bar{\zeta}}}-1\end{pmatrix}^{\frac{1}{\bar{\bar{\zeta}}}}}\right)} \\
\sqrt[2]{\left(\left(\left(\beta_1\left(\overline{\overline{\ell_1^2}}\right)^{\bar{\bar{\zeta}}}-(\beta_1-1)\right)^{\frac{1}{\bar{\bar{\zeta}}}}\right)^{\bar{\bar{\zeta}}}+\left(\left(\beta_2\left(\overline{\overline{\ell_2^2}}\right)^{\bar{\bar{\zeta}}}-(\beta_2-1)\right)^{\frac{1}{\bar{\bar{\zeta}}}}\right)^{\bar{\bar{\zeta}}}-1\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\; exp^{2\pi i\left(\sqrt[2]{\begin{pmatrix}\left(\left(\beta_1\left(\overline{\overline{k_1^2}}\right)^{\bar{\bar{\zeta}}}-(\beta_1-1)\right)^{\frac{1}{\bar{\bar{\zeta}}}}\right)^{\bar{\bar{\zeta}}}+\\ \left(\left(\beta_2\left(\overline{\overline{k_2^2}}\right)^{\bar{\bar{\zeta}}}-(\beta_2-1)\right)^{\frac{1}{\bar{\bar{\zeta}}}}\right)^{\bar{\bar{\zeta}}}-1\end{pmatrix}^{\frac{1}{\bar{\bar{\zeta}}}}}\right)}
\end{pmatrix}
$$

$$
= \begin{pmatrix}
\sqrt[2]{\left(1-\left(\left(1-1+\left(\beta_1\left(1-\overline{\overline{\mathfrak{A}_1^2}}\right)^{\bar{\bar{\zeta}}}-(\beta_1-1)\right)^{\frac{1}{\bar{\bar{\zeta}}}}\right)^{\bar{\bar{\zeta}}}+\left(1-1+\left(\beta_2\left(1-\overline{\overline{\mathfrak{A}_2^2}}\right)^{\bar{\bar{\zeta}}}-(\beta_2-1)\right)^{\frac{1}{\bar{\bar{\zeta}}}}\right)^{\bar{\bar{\zeta}}}\right)-1\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\; exp^{2\pi i\left(\sqrt[2]{\left(1-\begin{pmatrix}\left(1-1+\left(\beta_1\left(1-\overline{\overline{u_1^2}}\right)^{\bar{\bar{\zeta}}}-(\beta_1-1)\right)^{\frac{1}{\bar{\bar{\zeta}}}}\right)^{\bar{\bar{\zeta}}}+\\ \left(1-1+\left(\beta_2\left(1-\overline{\overline{u_2^2}}\right)^{\bar{\bar{\zeta}}}-(\beta_2-1)\right)^{\frac{1}{\bar{\bar{\zeta}}}}\right)^{\bar{\bar{\zeta}}}\end{pmatrix}-1\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\right)} \\
\sqrt[2]{\left(\left(\beta_1\left(\overline{\overline{f_1^2}}\right)^{\bar{\bar{\zeta}}}-(\beta_1-1)\right)+\left(\beta_2\left(\overline{\overline{f_2^2}}\right)^{\bar{\bar{\zeta}}}-(\beta_2-1)\right)-1\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\; exp^{2\pi i\left(\sqrt[2]{\left(\left(\beta_1\left(\overline{\overline{v_1^2}}\right)^{\bar{\bar{\zeta}}}-(\beta_1-1)\right)+\left(\beta_2\left(\overline{\overline{v_2^2}}\right)^{\bar{\bar{\zeta}}}-(\beta_2-1)\right)-1\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\right)} \\
\sqrt[2]{\left(\left(\beta_1\left(\overline{\overline{\ell_1^2}}\right)^{\bar{\bar{\zeta}}}-(\beta_1-1)\right)+\left(\beta_2\left(\overline{\overline{\ell_2^2}}\right)^{\bar{\bar{\zeta}}}-(\beta_2-1)\right)-1\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\; exp^{2\pi i\left(\sqrt[2]{\left(\left(\beta_1\left(\overline{\overline{k_1^2}}\right)^{\bar{\bar{\zeta}}}-(\beta_1-1)\right)+\left(\beta_2\left(\overline{\overline{k_2^2}}\right)^{\bar{\bar{\zeta}}}-(\beta_2-1)\right)-1\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\right)}
\end{pmatrix}
$$

$$
=\begin{pmatrix}
\sqrt[2]{\left(1-\left(\begin{array}{c}\left(\left(\beta_1\left(1-\overline{\overline{\mathfrak{A}_1^2}}\right)^{\overline{\overline{\zeta}}}-(\beta_1-1)\right)^{\frac{1}{\overline{\overline{\zeta}}}}\right)^{\overline{\overline{\zeta}}}+\\ \left(\left(\beta_2\left(1-\overline{\overline{\mathfrak{A}_2^2}}\right)^{\overline{\overline{\zeta}}}-(\beta_2-1)\right)^{\frac{1}{\overline{\overline{\zeta}}}}\right)^{\overline{\overline{\zeta}}}\end{array}\right)-1\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\, exp^{2\pi i\left(\sqrt[2]{\left(1-\left(\begin{array}{c}\left(\left(\beta_1\left(1-\overline{\overline{u_1^2}}\right)^{\overline{\overline{\zeta}}}-(\beta_1-1)\right)^{\frac{1}{\overline{\overline{\zeta}}}}\right)^{\overline{\overline{\zeta}}}+\\ \left(\left(\beta_2\left(1-\overline{\overline{u_2^2}}\right)^{\overline{\overline{\zeta}}}-(\beta_2-1)\right)^{\frac{1}{\overline{\overline{\zeta}}}}\right)^{\overline{\overline{\zeta}}}\end{array}\right)-1\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\right)} \\
\sqrt[2]{\left(\beta_1\left(\left(\overline{\overline{\mathfrak{f}_1^2}}\right)\right)^{\overline{\overline{\zeta}}}-\beta_1+1+\beta_2\left(\left(\overline{\overline{\mathfrak{f}_2^2}}\right)\right)^{\overline{\overline{\zeta}}}-\beta_2+1-1\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\, exp^{2\pi i\left(\sqrt[2]{\left(\beta_1\left(\left(\overline{\overline{v_1^2}}\right)\right)^{\overline{\overline{\zeta}}}-\beta_1+1+\beta_2\left(\left(\overline{\overline{v_2^2}}\right)\right)^{\overline{\overline{\zeta}}}-\beta_2+1-1\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\right)} \\
\sqrt[2]{\left(\beta_1\left(\overline{\overline{\ell_1^2}}\right)^{\overline{\overline{\zeta}}}-\beta_1+1+\beta_2\left(\overline{\overline{\ell_2^2}}\right)^{\overline{\overline{\zeta}}}-\beta_2+1-1\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\, exp^{2\pi i\left(\sqrt[2]{\left(\beta_1\left(\overline{\overline{\mathfrak{k}_1^2}}\right)^{\overline{\overline{\zeta}}}-\beta_1+1+\beta_2\left(\overline{\overline{\mathfrak{k}_2^2}}\right)^{\overline{\overline{\zeta}}}-\beta_2+1-1\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\right)}
\end{pmatrix}
$$

$$
=\begin{pmatrix}
\sqrt[2]{\left(1-\left(\begin{array}{c}\left(\beta_1\left(1-\overline{\overline{\mathfrak{A}_1^2}}\right)^{\overline{\overline{\zeta}}}-(\beta_1-1)\right)+\\ \left(\beta_2\left(1-\overline{\overline{\mathfrak{A}_2^2}}\right)^{\overline{\overline{\zeta}}}-(\beta_2-1)\right)\end{array}\right)-1\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\, exp^{2\pi i\left(\sqrt[2]{\left(1-\left(\begin{array}{c}\left(\beta_1\left(1-\overline{\overline{u_1^2}}\right)^{\overline{\overline{\zeta}}}-(\beta_1-1)\right)+\\ \left(\beta_2\left(1-\overline{\overline{u_2^2}}\right)^{\overline{\overline{\zeta}}}-(\beta_2-1)\right)\end{array}\right)-1\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\right)} \\
\sqrt[2]{\left(\beta_1\left(\overline{\overline{\mathfrak{f}_1^2}}\right)^{\overline{\overline{\zeta}}}-\beta_1+1+\beta_2\left(\overline{\overline{\mathfrak{f}_2^2}}\right)^{\overline{\overline{\zeta}}}-\beta_2+1-1\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\, exp^{2\pi i\left(\sqrt[2]{\left(\beta_1\left(\overline{\overline{v_1^2}}\right)^{\overline{\overline{\zeta}}}-\beta_1+1+\beta_2\left(\overline{\overline{v_2^2}}\right)^{\overline{\overline{\zeta}}}-\beta_2+1-1\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\right)} \\
\sqrt[2]{\left(\beta_1\left(\overline{\overline{\ell_1^2}}\right)^{\overline{\overline{\zeta}}}-\beta_1+1+\beta_2\left(\overline{\overline{\ell_2^2}}\right)^{\overline{\overline{\zeta}}}-\beta_2+1-1\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\, exp^{2\pi i\left(\sqrt[2]{\left(\beta_1\left(\overline{\overline{\mathfrak{k}_1^2}}\right)^{\overline{\overline{\zeta}}}-\beta_1+1+\beta_2\left(\overline{\overline{\mathfrak{k}_2^2}}\right)^{\overline{\overline{\zeta}}}-\beta_2+1-1\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\right)}
\end{pmatrix}
$$

$$
=\begin{pmatrix}
\sqrt[2]{\left(1-\left(\beta_1\left(1-\overline{\overline{\mathfrak{A}_1^2}}\right)^{\overline{\overline{\zeta}}}+\beta_2\left(1-\overline{\overline{\mathfrak{A}_2^2}}\right)^{\overline{\overline{\zeta}}}-(\beta_1-1)-(\beta_2-1)\right)-1\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\, exp^{2\pi i\left(\sqrt[2]{\left(1-\left(\beta_1\left(1-\overline{\overline{u_1^2}}\right)^{\overline{\overline{\zeta}}}+\beta_2\left(1-\overline{\overline{u_2^2}}\right)^{\overline{\overline{\zeta}}}-(\beta_1-1)-(\beta_2-1)\right)-1\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\right)} \\
\sqrt[2]{\left(\beta_1\left(\overline{\overline{\mathfrak{f}_1^2}}\right)^{\overline{\overline{\zeta}}}+\beta_2\left(\overline{\overline{\mathfrak{f}_2^2}}\right)^{\overline{\overline{\zeta}}}-\beta_1-\beta_2+1\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\, exp^{2\pi i\left(\sqrt[2]{\left(\beta_1\left(\overline{\overline{v_1^2}}\right)^{\overline{\overline{\zeta}}}+\beta_2\left(\overline{\overline{v_2^2}}\right)^{\overline{\overline{\zeta}}}-\beta_1-\beta_2+1\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\right)} \\
\sqrt[2]{\left(\beta_1\left(\overline{\overline{\ell_1^2}}\right)^{\overline{\overline{\zeta}}}+\beta_2\left(\overline{\overline{\ell_2^2}}\right)^{\overline{\overline{\zeta}}}-\beta_1-\beta_2+1\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\, exp^{2\pi i\left(\sqrt[2]{\left(\beta_1\left(\overline{\overline{\mathfrak{k}_1^2}}\right)^{\overline{\overline{\zeta}}}+\beta_2\left(\overline{\overline{\mathfrak{k}_2^2}}\right)^{\overline{\overline{\zeta}}}-\beta_1-\beta_2+1\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\right)}
\end{pmatrix}
$$

$$
= \begin{pmatrix}
\sqrt[2]{\left(1-\left(\beta_1\left(1-\overline{\overline{\mathfrak{A}_1^2}}\right)^{\bar{\bar{\zeta}}}+\beta_2\left(1-\overline{\overline{\mathfrak{A}_2^2}}\right)^{\bar{\bar{\zeta}}}-\beta_1+1-\beta_2+1\right)-1\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\, exp^{2\pi i\left(\sqrt[2]{\left(1-\left(\beta_1\left(1-\overline{\overline{\mathcal{u}_1^2}}\right)^{\bar{\bar{\zeta}}}+\beta_2\left(1-\overline{\overline{\mathfrak{A}_2^2}}\right)^{\bar{\bar{\zeta}}}-\beta_1+1-\beta_2+1\right)-1\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\right)} \\
\sqrt[2]{\left(\beta_1\left(\overline{\overline{\mathcal{f}_1^2}}\right)^{\bar{\bar{\zeta}}}+\beta_2\left(\overline{\overline{\mathcal{f}_1^2}}\right)^{\bar{\bar{\zeta}}}-\beta_1-\beta_2+1\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\, exp^{2\pi i\left(\sqrt[2]{\left(\beta_1\left(\overline{\overline{\mathcal{v}_1^2}}\right)^{\bar{\bar{\zeta}}}+\beta_2\left(\overline{\overline{\mathcal{v}_1^2}}\right)^{\bar{\bar{\zeta}}}-\beta_1-\beta_2+1\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\right)} \\
\sqrt[2]{\left(\beta_1\left(\overline{\overline{\ell_1^2}}\right)^{\bar{\bar{\zeta}}}+\beta_2\left(\overline{\overline{\ell_2^2}}\right)^{\bar{\bar{\zeta}}}-\beta_1-\beta_2+1\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\, exp^{2\pi i\left(\sqrt[2]{\left(\beta_1\left(\overline{\overline{\mathcal{k}_1^2}}\right)^{\bar{\bar{\zeta}}}+\beta_2\left(\overline{\overline{\mathcal{k}_2^2}}\right)^{\bar{\bar{\zeta}}}-\beta_1-\beta_2+1\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\right)}
\end{pmatrix}
$$

$$
= \begin{pmatrix}
\sqrt[2]{\left(1-\left(\beta_1\left(1-\overline{\overline{\mathfrak{A}_1^2}}\right)^{\bar{\bar{\zeta}}}+\beta_2\left(1-\overline{\overline{\mathfrak{A}_2^2}}\right)^{\bar{\bar{\zeta}}}-\beta_1-\beta_2+1\right)\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\, exp^{2\pi i\left(\sqrt[2]{\left(1-\left(\beta_1\left(1-\overline{\overline{\mathcal{u}_1^2}}\right)^{\bar{\bar{\zeta}}}+\beta_2\left(1-\overline{\overline{\mathcal{u}_2^2}}\right)^{\bar{\bar{\zeta}}}-\beta_1-\beta_2+1\right)\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\right)} \\
\sqrt[2]{\left(\beta_1\left(\overline{\overline{\mathcal{f}_1^2}}\right)^{\bar{\bar{\zeta}}}+\beta_2\left(\overline{\overline{\mathcal{f}_1^2}}\right)^{\bar{\bar{\zeta}}}-\beta_1-\beta_2+1\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\, exp^{2\pi i\left(\sqrt[2]{\left(\beta_1\left(\overline{\overline{\mathcal{v}_1^2}}\right)^{\bar{\bar{\zeta}}}+\beta_2\left(\overline{\overline{\mathcal{v}_1^2}}\right)^{\bar{\bar{\zeta}}}-\beta_1-\beta_2+1\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\right)} \\
\sqrt[2]{\left(\beta_1\left(\overline{\overline{\ell_1^2}}\right)^{\bar{\bar{\zeta}}}+\beta_2\left(\overline{\overline{\ell_2^2}}\right)^{\bar{\bar{\zeta}}}-\beta_1-\beta_2+1\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\, exp^{2\pi i\left(\sqrt[2]{\left(\beta_1\left(\overline{\overline{\mathcal{k}_1^2}}\right)^{\bar{\bar{\zeta}}}+\beta_2\left(\overline{\overline{\mathcal{k}_2^2}}\right)^{\bar{\bar{\zeta}}}-\beta_1-\beta_2+1\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\right)}
\end{pmatrix}
$$

$$
= \begin{pmatrix}
\sqrt[2]{1-\left(\sum_{\tau=1}^{2}\beta_\tau\left(1-\overline{\overline{\mathfrak{A}_\tau^2}}\right)^{\bar{\bar{\zeta}}}-\sum_{\tau=1}^{2}\beta_\tau+1\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\, exp^{2\pi i\left(\sqrt[2]{1-\left(\sum_{\tau=1}^{2}\beta_\tau\left(1-\overline{\overline{\mathcal{u}_\tau^2}}\right)^{\bar{\bar{\zeta}}}-\sum_{\tau=1}^{2}\beta_\tau+1\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\right)} \\
\sqrt[2]{\left(\sum_{\tau=1}^{2}\beta_\tau\left(\overline{\overline{\mathcal{f}_\tau^2}}\right)^{\bar{\bar{\zeta}}}-\sum_{\tau=1}^{2}\beta_\tau+1\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\, exp^{2\pi i\left(\sqrt[2]{\left(\sum_{\tau=1}^{2}\beta_\tau\left(\overline{\overline{\mathcal{v}_\tau^2}}\right)^{\bar{\bar{\zeta}}}-\sum_{\tau=1}^{2}\beta_\tau+1\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\right)} \\
\sqrt[2]{\left(\sum_{\tau=1}^{2}\beta_\tau\left(\overline{\overline{\ell_\tau^2}}\right)^{\bar{\bar{\zeta}}}-\sum_{\tau=1}^{2}\beta_\tau+1\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\, exp^{2\pi i\left(\sqrt[2]{\left(\sum_{\tau=1}^{2}\beta_\tau\left(\overline{\overline{\mathcal{k}_\tau^2}}\right)^{\bar{\bar{\zeta}}}-\sum_{\tau=1}^{2}\beta_\tau+1\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\right)}
\end{pmatrix}.
$$

We found that it holds for $\alpha = 2$, and further investigation, we found that it also holds for $\alpha = \kappa$ as follows:

$$CSFSSPA\left(\overline{\overline{\emptyset_1}},\overline{\overline{\emptyset_2}},\ldots,\overline{\overline{\emptyset_\alpha}}\right)$$

$$=\begin{pmatrix} \sqrt[2]{1-\left(\sum_{\tau=1}^{\kappa}\beta_\tau\left(1-\overline{\overline{\mathfrak{A}_\tau^2}}\right)^{\bar{\bar{\zeta}}}-\sum_{\tau=1}^{\kappa}\beta_\tau+1\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\,exp^{2\pi i\left(\sqrt[2]{1-\left(\sum_{\tau=1}^{\kappa}\beta_\tau\left(1-\overline{\overline{u_\tau^2}}\right)^{\bar{\bar{\zeta}}}-\sum_{\tau=1}^{\kappa}\beta_\tau+1\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\right)} \\ \sqrt[2]{\left(\sum_{\tau=1}^{\kappa}\beta_\tau\left(\overline{\overline{\mathfrak{f}_\tau^2}}\right)^{\bar{\bar{\zeta}}}-\sum_{\tau=1}^{\kappa}\beta_\tau+1\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\,exp^{2\pi i\left(\sqrt[2]{\left(\sum_{\tau=1}^{\kappa}\beta_\tau\left(\overline{\overline{v_\tau^2}}\right)^{\bar{\bar{\zeta}}}-\sum_{\tau=1}^{\kappa}\beta_\tau+1\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\right)} \\ \sqrt[2]{\left(\sum_{\tau=1}^{\kappa}\beta_\tau\left(\overline{\overline{\ell_\tau^2}}\right)^{\bar{\bar{\zeta}}}-\sum_{\tau=1}^{\kappa}\beta_\tau+1\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\,exp^{2\pi i\left(\sqrt[2]{\left(\sum_{\tau=1}^{\kappa}\beta_\tau\left(\overline{\overline{\hbar_\tau^2}}\right)^{\bar{\bar{\zeta}}}-\sum_{\tau=1}^{\kappa}\beta_\tau+1\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\right)} \end{pmatrix}.$$

Then, we prove it for $\alpha=\kappa+1$ such as

$$CSFSSPA\left(\overline{\overline{\emptyset_1}},\overline{\overline{\emptyset_2}},\ldots,\overline{\overline{\emptyset_\alpha}}\right)$$

$$=\beta_1\overline{\overline{\emptyset_1}}\oplus\beta_2\overline{\overline{\emptyset_2}}\oplus\ldots\oplus\beta_\kappa\overline{\overline{\emptyset_\kappa}}\oplus\beta_{\kappa+1}\overline{\overline{\emptyset_{\kappa+1}}}$$

$$=\oplus_{\tau=1}^{\kappa}\beta_\tau\overline{\overline{\emptyset_\tau}}\oplus\beta_{\kappa+1}\overline{\overline{\emptyset_{\kappa+1}}}$$

$$=\left(\begin{pmatrix} \sqrt[2]{1-\left(\sum_{\tau=1}^{\kappa}\beta_\tau\left(1-\overline{\overline{\mathfrak{A}_\tau^2}}\right)^{\bar{\bar{\zeta}}}-\sum_{\tau=1}^{K}\beta_\tau+1\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\,exp^{2\pi i\left(\sqrt[2]{1-\left(\sum_{\tau=1}^{\kappa}\beta_\tau\left(1-\overline{\overline{u_\tau^2}}\right)^{\bar{\bar{\zeta}}}-\sum_{\tau=1}^{K}\beta_\tau+1\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\right)} \\ \sqrt[2]{\left(\sum_{\tau=1}^{K}\beta_\tau\left(\overline{\overline{\mathfrak{f}_\tau^2}}\right)^{\bar{\bar{\zeta}}}-\sum_{\tau=1}^{K}\beta_\tau+1\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\,exp^{2\pi i\left(\sqrt[2]{\left(\sum_{\tau=1}^{K}\beta_\tau\left(\overline{\overline{v_\tau^2}}\right)^{\bar{\bar{\zeta}}}-\sum_{\tau=1}^{K}\beta_\tau+1\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\right)} \\ \sqrt[2]{\left(\sum_{\tau=1}^{K}\beta_\tau\left(\overline{\overline{\ell_\tau^2}}\right)^{\bar{\bar{\zeta}}}-\sum_{\tau=1}^{K}\beta_\tau+1\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\,exp^{2\pi i\left(\sqrt[2]{\left(\sum_{\tau=1}^{K}\beta_\tau\left(\overline{\overline{\hbar_\tau^2}}\right)^{\bar{\bar{\zeta}}}-\sum_{\tau=1}^{K}\beta_\tau+1\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\right)} \end{pmatrix}\right.$$

$$\left.\oplus\begin{pmatrix} \sqrt[2]{1-\left(\beta_{K+1}\left(1-\overline{\overline{\mathfrak{A}_{\kappa+1}^2}}\right)^{\bar{\bar{\zeta}}}-(\beta_{K+1}-1)\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\,exp^{2\pi i\left(\sqrt[2]{1-\left(\beta_{K+1}\left(1-\overline{\overline{u_{\kappa+1}^2}}\right)^{\bar{\bar{\zeta}}}-(\beta_{K+1}-1)\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\right)} \\ \sqrt[2]{\left(\beta_{K+1}\left(\overline{\overline{\mathfrak{f}_{K+1}^2}}\right)^{\bar{\bar{\zeta}}}-(\beta_{K+1}-1)\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\,exp^{2\pi i\left(\sqrt[2]{\left(\beta_{K+1}\left(\overline{\overline{v_{K+1}^2}}\right)^{\bar{\bar{\zeta}}}-(\beta_{K+1}-1)\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\right)} \\ \sqrt[2]{\left(\beta_{K+1}\left(\overline{\overline{\ell_{K+1}^2}}\right)^{\bar{\bar{\zeta}}}-(\beta_{K+1}-1)\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\,exp^{2\pi i\left(\sqrt[2]{\left(\beta_{K+1}\left(\overline{\overline{\hbar_{K+1}^2}}\right)^{\bar{\bar{\zeta}}}-(\beta_{K+1}-1)\right)^{\frac{1}{\bar{\bar{\zeta}}}}}\right)} \end{pmatrix}\right)$$

$$= \begin{pmatrix} \sqrt[2]{1-\left(\sum_{\tau=1}^{\kappa+1}\beta_\tau\left(1-\overline{\overline{\mathfrak{A}_\tau^2}}\right)^{\overline{\overline{\zeta}}}-\sum_{\tau=1}^{\kappa+1}\beta_\tau+1\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\, exp^{2\pi i\left(\sqrt[2]{1-\left(\sum_{\tau=1}^{\kappa+1}\beta_\tau\left(1-\overline{\overline{u_\tau^2}}\right)^{\overline{\overline{\zeta}}}-\sum_{\tau=1}^{\kappa+1}\beta_\tau+1\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\right)} \\ \sqrt[2]{\left(\sum_{\tau=1}^{\kappa+1}\beta_\tau\left(\overline{\overline{\mathfrak{f}_\tau^2}}\right)^{\overline{\overline{\zeta}}}-\sum_{\tau=1}^{\kappa+1}\beta_\tau+1\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\, exp^{2\pi i\left(\sqrt[2]{\left(\sum_{\tau=1}^{\kappa+1}\beta_\tau\left(\overline{\overline{v_\tau^2}}\right)^{\overline{\overline{\zeta}}}-\sum_{\tau=1}^{\kappa+1}\beta_\tau+1\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\right)} \\ \sqrt[2]{\left(\sum_{\tau=1}^{\kappa+1}\beta_\tau\left(\overline{\overline{\ell_\tau^2}}\right)^{\overline{\overline{\zeta}}}-\sum_{\tau=1}^{\kappa+1}\beta_\tau+1\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\, exp^{2\pi i\left(\sqrt[2]{\left(\sum_{\tau=1}^{\kappa+1}\beta_\tau\left(\overline{\overline{k_\tau^2}}\right)^{\overline{\overline{\zeta}}}-\sum_{\tau=1}^{\kappa+1}\beta_\tau+1\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\right)} \end{pmatrix}.$$

Here, we were successful in our evaluation of the targeted data.

We also assessed the information in Eq (2.2) for idempotency, monotonicity, and boundedness.

Theorem 3.2. *The following properties are found to hold successfully based on defined CSFVs* $\overline{\overline{\emptyset_\tau}} = \left(\overline{\overline{\mathfrak{A}_\tau^2}} exp^{2\pi i\left(\overline{\overline{u_\tau^2}}\right)}, \overline{\overline{\mathfrak{f}_\tau^2}} exp^{2\pi i\left(\overline{\overline{v_\tau^2}}\right)}, \overline{\overline{\ell_\tau^2}} exp^{2\pi i\left(\overline{\overline{k_\tau^2}}\right)}\right)',\ \tau = 1, 2, \dots, \alpha.$

For any CSFVs $\overline{\overline{\emptyset}} = \overline{\overline{\emptyset_\tau}}$, *that is*

$$\overline{\overline{\emptyset_\tau}} = CSFSSPA\left(\overline{\overline{\emptyset_{\tau_1}}}, \overline{\overline{\emptyset_{\tau_2}}}, \dots, \overline{\overline{\emptyset_{\tau_\alpha}}}\right) = \overline{\overline{\emptyset}}.$$

Proof. If $\overline{\overline{\emptyset}} = \overline{\overline{\emptyset_\tau}} = \left(\overline{\overline{\mathfrak{A}_\tau^2}} exp^{2\pi i\left(\overline{\overline{u_\tau^2}}\right)}, \overline{\overline{\mathfrak{f}_\tau^2}} exp^{2\pi i\left(\overline{\overline{v_\tau^2}}\right)}, \overline{\overline{\ell_\tau^2}} exp^{2\pi i\left(\overline{\overline{k_\tau^2}}\right)}\right)$, then using Eq (2.2), we have

$$CSFSSPA\left(\overline{\overline{\emptyset_{\tau_1}}}, \overline{\overline{\emptyset_{\tau_2}}}, \dots, \overline{\overline{\emptyset_{\tau_\alpha}}}\right)$$

$$= \begin{pmatrix} \sqrt[2]{1-\left(\sum_{\tau=1}^{\alpha}\beta_\tau\left(1-\overline{\overline{\mathfrak{A}^2}}\right)^{\overline{\overline{\zeta}}}-\sum_{\tau=1}^{\alpha}\beta_\tau+1\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\, exp^{2\pi i\left(\sqrt[2]{1-\left(\sum_{\tau=1}^{\alpha}\beta_\tau\left(1-\overline{\overline{u^2}}\right)^{\overline{\overline{\zeta}}}-\sum_{\tau=1}^{\alpha}\beta_\tau+1\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\right)} \\ \sqrt[2]{\left(\sum_{\tau=1}^{\alpha}\beta_\tau\left(\overline{\overline{\mathfrak{f}^2}}\right)^{\overline{\overline{\zeta}}}-\sum_{\tau=1}^{\alpha}\beta_\tau+1\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\, exp^{2\pi i\left(\sqrt[2]{\left(\sum_{\tau=1}^{\alpha}\beta_\tau\left(\overline{\overline{v^2}}\right)^{\overline{\overline{\zeta}}}-\sum_{\tau=1}^{\alpha}\beta_\tau+1\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\right)} \\ \sqrt[2]{\left(\sum_{\tau=1}^{\alpha}\beta_\tau\left(\overline{\overline{\ell^2}}\right)^{\overline{\overline{\zeta}}}-\sum_{\tau=1}^{\alpha}\beta_\tau+1\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\, exp^{2\pi i\left(\sqrt[2]{\left(\sum_{\tau=1}^{\alpha}\beta_\tau\left(\overline{\overline{k^2}}\right)^{\overline{\overline{\zeta}}}-\sum_{\tau=1}^{\alpha}\beta_\tau+1\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\right)} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt[2]{1-\left(\left(1-\overline{\overline{\mathfrak{A}^2}}\right)^{\overline{\overline{\zeta}}}-1+1\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\, exp^{2\pi i\left(\sqrt[2]{1-\left(\left(1-\overline{\overline{u^2}}\right)^{\overline{\overline{\zeta}}}-1+1\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\right)} \\ \sqrt[2]{\left(\left(\overline{\overline{\mathscr{f}^2}}\right)^{\overline{\overline{\zeta}}}-1+1\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\, exp^{2\pi i\left(\sqrt[2]{\left(\left(\overline{\overline{v^2}}\right)^{\overline{\overline{\zeta}}}-1+1\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\right)} \\ \sqrt[2]{\left(\left(\overline{\overline{\ell^2}}\right)^{\overline{\overline{\zeta}}}-1+1\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\, exp^{2\pi i\left(\sqrt[2]{\left(\left(\overline{\overline{k^2}}\right)^{\overline{\overline{\zeta}}}-1+1\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\right)} \end{pmatrix}, \quad \sum_{\tau=1}^{\alpha}\beta_\tau = 1$$

$$= \begin{pmatrix} \sqrt[2]{1-\left(\left(1-\overline{\overline{\mathfrak{A}^2}}\right)^{\overline{\overline{\zeta}}}\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\, exp^{2\pi i\left(\sqrt[2]{1-\left(\left(1-\overline{\overline{u^2}}\right)^{\overline{\overline{\zeta}}}\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\right)} \\ \sqrt[2]{\left(\left(\overline{\overline{\mathscr{f}^2}}\right)^{\overline{\overline{\zeta}}}\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\, exp^{2\pi i\left(\sqrt[2]{\left(\left(\overline{\overline{v^2}}\right)^{\overline{\overline{\zeta}}}\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\right)} \\ \sqrt[2]{\left(\left(\overline{\overline{\ell^2}}\right)^{\overline{\overline{\zeta}}}\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\, exp^{2\pi i\left(\sqrt[2]{\left(\left(\overline{\overline{k^2}}\right)^{\overline{\overline{\zeta}}}\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\right)} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt[2]{1-\left(1-\overline{\overline{\mathfrak{A}^2}}\right)}\, exp^{2\pi i\left(\sqrt[2]{1-\left(1-\overline{\overline{u^2}}\right)}\right)} \\ \sqrt[2]{\left(\overline{\overline{\mathscr{f}^2}}\right)}\, exp^{2\pi i\left(\sqrt[2]{\left(\overline{\overline{v^2}}\right)}\right)} \\ \sqrt[2]{\left(\overline{\overline{\ell^2}}\right)}\, exp^{2\pi i\left(\sqrt[2]{\left(\overline{\overline{k^2}}\right)}\right)} \end{pmatrix} = \left(\overline{\overline{\mathfrak{A}_\tau}}, \overline{\overline{\mathscr{f}}}_\tau, \overline{\overline{\ell}}_\tau\right) = \overline{\overline{\emptyset}}.$$

If $\overline{\overline{\emptyset_\tau}} = \left(\overline{\overline{\mathfrak{A}^2_\tau}}, \overline{\overline{\mathscr{f}^2_\tau}}, \overline{\overline{\ell^2_\tau}}\right) \le \overline{\overline{\emptyset'_\tau}} = \left(\overline{\overline{(\mathfrak{A}^2_\tau)'}}, \overline{\overline{(\mathscr{f}^2_\tau)'}}, \overline{\overline{(\ell^2_\tau)'}}\right)$, that is $\overline{\overline{\mathfrak{A}^2_\tau}} \le \overline{\overline{(\mathfrak{A}^2_\tau)'}}$, $\overline{\overline{\mathscr{f}^2_\tau}} \ge \overline{\overline{(\mathscr{f}^2_\tau)'}}$ and $\overline{\overline{\ell^2_\tau}} \ge \overline{\overline{(\ell^2_\tau)'}}$, then

$$\overline{\overline{\mathfrak{A}^2_\tau}} \le \overline{\overline{(\mathfrak{A}^2_\tau)'}} \Rightarrow 1-\overline{\overline{\mathfrak{A}^2_\tau}} \le 1-\overline{\overline{(\mathfrak{A}^2_\tau)'}} \Rightarrow \left(1-\overline{\overline{\mathfrak{A}^2_\tau}}\right)^{\overline{\overline{\zeta}}} \ge \left(1-\overline{\overline{(\mathfrak{A}^2_\tau)'}}\right)^{\overline{\overline{\zeta}}}$$
$$\Rightarrow \sum_{\tau=1}^{\alpha}\beta_\tau\left(1-\overline{\overline{\mathfrak{A}^2_\tau}}\right)^{\overline{\overline{\zeta}}} \ge \sum_{\tau=1}^{\alpha}\beta_\tau\left(1-\overline{\overline{(\mathfrak{A}^2_\tau)'}}\right)^{\overline{\overline{\zeta}}}$$

$$= \sum_{\tau=1}^{\alpha} \beta_\tau \left(1 - \overline{\overline{\mathfrak{A}_\tau^2}}\right)^{\overline{\overline{\zeta}}} - \sum_{\tau=1}^{\alpha} \beta_\tau + 1 \geq \sum_{\tau=1}^{\alpha} \beta_\tau \left(1 - \overline{\overline{(\mathfrak{A}_\tau^2)'}}\right)^{\overline{\overline{\zeta}}} - \sum_{\tau=1}^{\alpha} \beta_\tau + 1$$

$$= \left(\sum_{\tau=1}^{\alpha} \beta_\tau \left(1 - \overline{\overline{\mathfrak{A}_\tau^2}}\right)^{\overline{\overline{\zeta}}} - \sum_{\tau=1}^{\alpha} \beta_\tau + 1\right)^{\frac{1}{\overline{\overline{\zeta}}}} \geq \left(\sum_{\tau=1}^{\alpha} \beta_\tau \left(1 - \overline{\overline{(\mathfrak{A}_\tau^2)'}}\right)^{\overline{\overline{\zeta}}} - \sum_{\tau=1}^{\alpha} \beta_\tau + 1\right)^{\frac{1}{\overline{\overline{\zeta}}}}$$

$$= \left(\sum_{\tau=1}^{\alpha} \beta_\tau \left(1 - \overline{\overline{\mathfrak{A}_\tau^2}}\right)^{\overline{\overline{\zeta}}} - \sum_{\tau=1}^{\alpha} \beta_\tau + 1\right)^{\frac{1}{\overline{\overline{\zeta}}}} \leq \left(\sum_{\tau=1}^{\alpha} \beta_\tau \left(1 - \overline{\overline{(\mathfrak{A}_\tau^2)'}}\right)^{\overline{\overline{\zeta}}} - \sum_{\tau=1}^{\alpha} \beta_\tau + 1\right)^{\frac{1}{\overline{\overline{\zeta}}}}.$$

Theorem 3.3. *For any two CSFVs* $\overline{\overline{\emptyset_\tau}} = \left(\overline{\overline{\mathfrak{A}_\tau}}, \overline{\overline{\mathfrak{f}_\tau}}, \overline{\overline{\ell_\tau}}\right) \leq \overline{\overline{\emptyset'_\tau}} = \left(\overline{\overline{\mathfrak{A}'_\tau}}, \overline{\overline{\mathfrak{f}'_\tau}}, \overline{\overline{\ell'_\tau}}\right)$*, that is* $\overline{\overline{\mathfrak{A}_\tau}} \leq \overline{\overline{\mathfrak{A}'_\tau}}$*,* $\overline{\overline{\mathfrak{f}_\tau}} \geq \overline{\overline{\mathfrak{f}'_\tau}}$ *and* $\overline{\overline{\ell_\tau}} \geq \overline{\overline{\ell'_\tau}}$*, then* $CSFSSPA\left(\overline{\overline{\emptyset_1}}{}'\overline{\overline{\emptyset_2}}{}' \ldots {}'\overline{\overline{\emptyset_\alpha}}\right) \leq CSFSSPA\left(\overline{\overline{\emptyset'_1}}, \overline{\overline{\emptyset'_2}}, \ldots, \overline{\overline{\emptyset'_\alpha}}\right)$.

Proof. If $\overline{\overline{\mathfrak{f}_\tau}} \geq \overline{\overline{\mathfrak{f}'_\tau}}$, then

$$\overline{\overline{\mathfrak{f}_\tau^2}} \geq \overline{\overline{(\mathfrak{f}_\tau^2)'}} \Rightarrow \overline{\overline{\mathfrak{f}_\tau^2}}^{\overline{\overline{\zeta}}} \geq \overline{\overline{(\mathfrak{f}_\tau^2)'}}^{\overline{\overline{\zeta}}} \Rightarrow \sum_{\tau=1}^{\alpha} \beta_\tau \overline{\overline{\mathfrak{f}_\tau^2}}^{\overline{\overline{\zeta}}} \geq \sum_{\tau=1}^{\alpha} \beta_\tau \overline{\overline{(\mathfrak{f}_\tau^2)'}}^{\overline{\overline{\zeta}}}$$

$$\Rightarrow \sum_{\tau=1}^{\alpha} \beta_\tau \overline{\overline{\mathfrak{f}_\tau^2}}^{\overline{\overline{\zeta}}} - \sum_{\tau=1}^{\alpha} \beta_\tau + 1 \geq \sum_{\tau=1}^{\alpha} \beta_\tau \overline{\overline{(\mathfrak{f}_\tau^2)'}}^{\overline{\overline{\zeta}}} - \sum_{\tau=1}^{\alpha} \beta_\tau + 1$$

$$\Rightarrow \left(\sum_{\tau=1}^{\alpha} \beta_\tau \overline{\overline{\mathfrak{f}_\tau^2}}^{\overline{\overline{\zeta}}} - \sum_{\tau=1}^{\alpha} \beta_\tau + 1\right)^{\frac{1}{\overline{\overline{\zeta}}}} \geq \left(\sum_{\tau=1}^{\alpha} \beta_\tau \overline{\overline{(\mathfrak{f}_\tau^2)'}}^{\overline{\overline{\zeta}}} - \sum_{\tau=1}^{\alpha} \beta_\tau + 1\right)^{\frac{1}{\overline{\overline{\zeta}}}}.$$

Further, if $\overline{\overline{\ell_\tau^2}} \geq \overline{\overline{(\ell_\tau^2)'}}$, then

$$\overline{\overline{\ell_\tau^2}} \geq \overline{\overline{(\ell_\tau^2)'}} \Rightarrow \overline{\overline{\ell_\tau^2}}^{\overline{\overline{\zeta}}} \geq \overline{\overline{(\ell_\tau^2)'}}^{\overline{\overline{\zeta}}} \Rightarrow \sum_{\tau=1}^{\alpha} \beta_\tau \overline{\overline{\ell_\tau^2}}^{\overline{\overline{\zeta}}} \geq \sum_{\tau=1}^{\alpha} \beta_\tau \overline{\overline{(\ell_\tau^2)'}}^{\overline{\overline{\zeta}}}$$

$$\Rightarrow \sum_{\tau=1}^{\alpha} \beta_\tau \overline{\overline{\ell_\tau^2}}^{\overline{\overline{\zeta}}} - \sum_{\tau=1}^{\alpha} \beta_\tau + 1 \geq \sum_{\tau=1}^{\alpha} \beta_\tau \overline{\overline{(\ell_\tau^2)'}}^{\overline{\overline{\zeta}}} - \sum_{\tau=1}^{\alpha} \beta_\tau + 1$$

$$\Rightarrow \left(\sum_{\tau=1}^{\alpha} \beta_\tau \overline{\overline{\ell_\tau^2}}^{\overline{\overline{\zeta}}} - \sum_{\tau=1}^{\alpha} \beta_\tau + 1\right)^{\frac{1}{\overline{\overline{\zeta}}}} \geq \left(\sum_{\tau=1}^{\alpha} \beta_\tau \overline{\overline{(\ell_\tau^2)'}}^{\overline{\overline{\zeta}}} - \sum_{\tau=1}^{\alpha} \beta_\tau + 1\right)^{\frac{1}{\overline{\overline{\zeta}}}}.$$

Theorem 3.4. *For any CSFVs* $\emptyset_\tau^- = \left\{min\overline{\overline{\mathfrak{A}_\tau}}, max\overline{\overline{\mathfrak{f}_\tau}}, max\overline{\overline{\ell_\tau}}\right\}$ *and* $\emptyset_\tau^+ = \left\{max\overline{\overline{\mathfrak{A}_\tau}}, min\overline{\overline{\mathfrak{f}_\tau}}, min\overline{\overline{\ell_\tau}}\right\}$*, then we have* $\emptyset_\tau^- = CSFSSPA\left(\overline{\overline{\emptyset_{\tau_1}}}, \overline{\overline{\emptyset_{\tau_2}}}, \ldots, \overline{\overline{\emptyset_{\tau_\alpha}}}\right) = \emptyset_\tau^+$.

$$CSFSSPA\left(\overline{\overline{\emptyset_{\tau_1}}}{}'\overline{\overline{\emptyset_{\tau_2}}}{}' \ldots {}'\overline{\overline{\emptyset_{\tau_\alpha}}}\right) \leq CSFSSPA\left(\overline{\overline{\emptyset'_{\tau_1}}}{}'\overline{\overline{\emptyset'_{\tau_2}}}{}' \ldots {}'\overline{\overline{\emptyset'_{\tau_\alpha}}}\right).$$

Proof. Since the theory in ideas Theorems 3.2 and 3.3, we have

$$CSFSSPA\left(\overline{\overline{\emptyset_1}}, \overline{\overline{\emptyset_2}}, \ldots, \overline{\overline{\emptyset_\alpha}}\right) \geq CSFSSPA\left(\overline{\overline{\emptyset_1^-}}, \overline{\overline{\emptyset_2^-}}, \ldots, \overline{\overline{\emptyset_\alpha^-}}\right) = \overline{\overline{\emptyset_\tau^-}},$$

$$CSFSSPA\left(\overline{\overline{\emptyset_1}}, \overline{\overline{\emptyset_2}}, \ldots, \overline{\overline{\emptyset_\alpha}}\right) \leq CSFSSPA\left(\overline{\overline{\emptyset_1^+}}, \overline{\overline{\emptyset_2^+}}, \ldots, \overline{\overline{\emptyset_\alpha^+}}\right) = \overline{\overline{\emptyset_\tau^+}}.$$

Then,

$$\overline{\overline{\emptyset_{\tau}^{-}}} \leq CSFSSPA\left(\overline{\overline{\emptyset_1}}, \overline{\overline{\emptyset_2}}, \dots, \overline{\overline{\emptyset_\alpha}}\right) \leq \overline{\overline{\emptyset_{\tau}^{+}}}.$$

Definition 3.2. *For any CSFVs* $\overline{\overline{\emptyset_\tau}} = \left(\overline{\overline{\mathfrak{A}_\tau^2}} exp^{2\pi i\left(\overline{\overline{u_\tau^2}}\right)}, \overline{\overline{\mathfrak{f}_\tau^2}} exp^{2\pi i\left(\overline{\overline{v_\tau^2}}\right)}, \overline{\overline{\ell_\tau^2}} exp^{2\pi i\left(\overline{\overline{k_\tau^2}}\right)}\right)$, $\tau = 1, 2, \dots, \alpha$, *the following is how the CSFSSWPA operator is expressed as*

$$CSFSSWPA\left(\overline{\overline{\emptyset_1}}, \overline{\overline{\emptyset_2}}, \dots, \overline{\overline{\emptyset_\alpha}}\right) = \beta_1 \overline{\overline{\emptyset_1}} \oplus \beta_2 \overline{\overline{\emptyset_2}} \oplus \dots \oplus \beta_\alpha \overline{\overline{\emptyset_\alpha}} = \oplus_{\tau=1}^{\alpha} \beta_\tau \overline{\overline{\emptyset_\tau}}.$$

Noticed that $\beta_\tau = \frac{\varpi_\tau \beta_\tau}{\sum_{\tau=1}^{\alpha} \varpi_\tau \beta_\tau}$, where $\beta_1 = 1$ and $\beta_\tau = \oplus_{\kappa}^{\tau-1} \wedge_s \left(\overline{\overline{\emptyset_\kappa}}\right)$, when $\kappa = 2, 3, \dots, \alpha$. Additionally, the weight vector's representation is expressed as follows: $\varpi_\tau \in [0, 1]$ with $\sum_{\tau=1}^{\alpha} \varpi_\tau = 1$.

Theorem 3.5. *For any CSFVs* $\overline{\overline{\emptyset_\tau}} = \left(\overline{\overline{\mathfrak{A}_\tau^2}} exp^{2\pi i\left(\overline{\overline{u_\tau^2}}\right)}, \overline{\overline{\mathfrak{f}_\tau^2}} exp^{2\pi i\left(\overline{\overline{v_\tau^2}}\right)}, \overline{\overline{\ell_\tau^2}} exp^{2\pi i\left(\overline{\overline{k_\tau^2}}\right)}\right)$, $\tau = 1, 2, \dots, \alpha$, *then the integrated values of the CSFSSWPA operator are CSFV such as*

$$CSFSSWPA\left(\overline{\overline{\emptyset_1}}, \overline{\overline{\emptyset_2}}, \dots, \overline{\overline{\emptyset_\alpha}}\right)$$

$$= \left(\begin{array}{c} \sqrt[2]{1 - \left(\sum_{\tau=1}^{\alpha} \beta_\tau \left(1 - \overline{\overline{\mathfrak{A}_\tau^2}}\right)^{\overline{\overline{\zeta}}} - \sum_{\tau=1}^{\alpha} \beta_\tau + 1\right)^{\frac{1}{\overline{\overline{\zeta}}}}} exp^{2\pi i \left(\sqrt[2]{1 - \left(\sum_{\tau=1}^{\alpha} \beta_\tau \left(1 - \overline{\overline{u_\tau^2}}\right)^{\overline{\overline{\zeta}}} - \sum_{\tau=1}^{\alpha} \beta_\tau + 1\right)^{\frac{1}{\overline{\overline{\zeta}}}}} \right)} \\ \sqrt[2]{\left(\sum_{\tau=1}^{\alpha} \beta_j \left(\overline{\overline{\mathfrak{f}_\tau^2}}\right)^{\overline{\overline{\zeta}}} - \sum_{\tau=1}^{\alpha} \beta_\tau + 1\right)^{\frac{1}{\overline{\overline{\zeta}}}}} exp^{2\pi i \left(\sqrt[2]{\left(\sum_{\tau=1}^{\alpha} \beta_j \left(\overline{\overline{v_\tau^2}}\right)^{\overline{\overline{\zeta}}} - \sum_{\tau=1}^{\alpha} \beta_\tau + 1\right)^{\frac{1}{\overline{\overline{\zeta}}}}} \right)} \\ \sqrt[2]{\left(\sum_{\tau=1}^{\alpha} \beta_j \left(\overline{\overline{\ell_\tau^2}}\right)^{\overline{\overline{\zeta}}} - \sum_{\tau=1}^{\alpha} \beta_\tau + 1\right)^{\frac{1}{\overline{\overline{\zeta}}}}} exp^{2\pi i \left(\sqrt[2]{\left(\sum_{\tau=1}^{\alpha} \beta_j \left(\overline{\overline{k_\tau^2}}\right)^{\overline{\overline{\zeta}}} - \sum_{\tau=1}^{\alpha} \beta_\tau + 1\right)^{\frac{1}{\overline{\overline{\zeta}}}}} \right)} \end{array} \right). \quad (3.1)$$

Theorem 3.1's proof and Eq (3.1)'s proof are equivalent universally. We also measured the evidence in Eq (3.1) for idempotency, monotonicity, and boundedness.

Theorem 3.6. *The following properties hold successfully based on defined CSFVs* $\overline{\overline{\emptyset_\tau}} = \left(\overline{\overline{\mathfrak{A}_\tau^2}} exp^{2\pi i\left(\overline{\overline{u_\tau^2}}\right)}, \overline{\overline{\mathfrak{f}_\tau^2}} exp^{2\pi i\left(\overline{\overline{v_\tau^2}}\right)}, \overline{\overline{\ell_\tau^2}} exp^{2\pi i\left(\overline{\overline{k_\tau^2}}\right)}\right)$, $\tau = 1, 2, \dots, \alpha$. *For any CSFVs* $\overline{\overline{\emptyset}} = \overline{\overline{\emptyset_\tau}}$, *that is*

$$\overline{\overline{\emptyset_\tau}} = CSFSSWPA\left(\overline{\overline{\emptyset_{\tau_1}}}, \overline{\overline{\emptyset_{\tau_2}}}, \dots, \overline{\overline{\emptyset_{\tau_\alpha}}}\right) = \overline{\overline{\emptyset}}.$$

Proof. Theorem 3.6's proof is like the proof of Theorem 3.2.

Theorem 3.7. *For any two CSFVs* $\overline{\overline{\emptyset_\tau}} = \left(\overline{\overline{\mathfrak{A}_\tau^2}}, \overline{\overline{\mathfrak{f}_\tau^2}}, \overline{\overline{\ell_\tau^2}}\right) \leq \overline{\overline{\emptyset_\tau'}} = \left(\overline{\overline{(\mathfrak{A}_\tau^2)'}}, \overline{\overline{(\mathfrak{f}_\tau^2)'}}, \overline{\overline{(\ell_\tau^2)'}}\right)$, *that is* $\overline{\overline{\mathfrak{A}_\tau^2}} \leq \overline{\overline{(\mathfrak{A}_\tau^2)'}}$, $\overline{\overline{\mathfrak{f}_\tau^2}} \geq \overline{\overline{(\mathfrak{f}_\tau^2)'}}$ *and* $\overline{\overline{\ell_\tau^2}} \geq \overline{\overline{(\ell_\tau^2)'}}$, *then* $CSFSSWPA\left(\overline{\overline{\emptyset_1}}, \overline{\overline{\emptyset_2}}, \dots, \overline{\overline{\emptyset_\alpha}}\right) \leq CSFSSWPA\left(\overline{\overline{\emptyset_1'}}, \overline{\overline{\emptyset_2'}}, \dots, \overline{\overline{\emptyset_\alpha'}}\right)$.

Proof. Theorem 3.7's proof is like the proof of Theorem 3.3.

Theorem 3.8. *For any CSFVs* $\emptyset_\tau^- = \left\{min\overline{\overline{\mathfrak{A}_\tau^2}}, max\overline{\overline{\mathfrak{f}_\tau^2}}, max\overline{\overline{\ell_\tau^2}}\right\}$ *and* $\emptyset_\tau^+ = \left\{max\overline{\overline{\mathfrak{A}_\tau^2}}, min\overline{\overline{\mathfrak{f}_\tau^2}}, min\overline{\overline{\ell_\tau^2}}\right\}$, *then we have*

$$\emptyset_\tau^- = CSFSSWPA\left(\overline{\overline{\emptyset_{\tau_1}}}, \overline{\overline{\emptyset_{\tau_2}}}, \dots, \overline{\overline{\emptyset_{\tau_\alpha}}}\right) = \emptyset_\tau^+.$$

Proof. Theorem 3.8's proof is like the proof of Theorem 3.4.

Definition 3.3. *For any CSFVs* $\overline{\overline{\emptyset_\tau}} = \left(\overline{\overline{\mathfrak{A}_\tau^2}} exp^{2\pi i\left(\overline{\overline{u_\tau^2}}\right)}, \overline{\overline{\mathfrak{f}_\tau^2}} exp^{2\pi i\left(\overline{\overline{v_\tau^2}}\right)}, \overline{\overline{\ell_\tau^2}} exp^{2\pi i\left(\overline{\overline{k_\tau^2}}\right)}\right)$, $\tau = 2, 3, \dots, \alpha$, *then the notion of the CSFSSPG operator is expressed as follows:*

$$CSFSSPG\left(\overline{\overline{\emptyset_1}}, \overline{\overline{\emptyset_2}}, \dots, \overline{\overline{\emptyset_\alpha}}\right) = \overline{\overline{\emptyset_1^{\beta_1}}} \otimes \overline{\overline{\emptyset_2^{\beta_2}}} \dots \overline{\overline{\emptyset_\alpha^{\beta_\alpha}}} = \oplus_{\tau=1}^{\alpha} \overline{\overline{\emptyset_\tau^{\beta_\tau}}}. \quad (3.4)$$

Noticed that $\beta_\tau = \frac{\beta_\tau}{\sum_{\tau=1}^{\alpha} \beta_\tau}$, where $\beta_1 = 1$ and $\beta_\tau = \oplus_\kappa^{\tau-1} \wedge_s\left(\overline{\overline{\emptyset_\kappa}}\right)$, $\kappa = 1, 2, \dots, \alpha$.

Theorem 3.9. *For any CSFVs* $\overline{\overline{\emptyset_\tau}} = \left(\overline{\overline{\mathfrak{A}_\tau^2}} exp^{2\pi i\left(\overline{\overline{u_\tau^2}}\right)}, \overline{\overline{\mathfrak{f}_\tau^2}} exp^{2\pi i\left(\overline{\overline{v_\tau^2}}\right)}, \overline{\overline{\ell_\tau^2}} exp^{2\pi i\left(\overline{\overline{k_\tau^2}}\right)}\right)$, $\tau = 1, 2, \dots, \alpha$, *then the integrated values of the CSFSSPG operator still a CSFV such as*

$$CSFSSPG\left(\overline{\overline{\emptyset_1}}, \overline{\overline{\emptyset_2}}, \dots, \overline{\overline{\emptyset_\alpha}}\right)$$

$$= \begin{pmatrix} \sqrt[2]{\left(\sum_{\tau=1}^{\alpha} \beta_j \left(\overline{\overline{\mathfrak{f}_\tau^2}}\right)^{\overline{\overline{\zeta}}} - \sum_{\tau=1}^{\alpha} \beta_\tau + 1\right)^{\frac{1}{\overline{\overline{\zeta}}}}} exp^{2\pi i\left(\sqrt[2]{\left(\sum_{\tau=1}^{\alpha} \beta_j \left(\overline{\overline{u_\tau^2}}\right)^{\overline{\overline{\zeta}}} - \sum_{\tau=1}^{\alpha} \beta_\tau + 1\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\right)} \\ \sqrt[2]{1 - \left(\sum_{\tau=1}^{\alpha} \beta_j \left(1 - \overline{\overline{\ell_\tau^2}}\right)^{\overline{\overline{\zeta}}} - \sum_{\tau=1}^{\alpha} \beta_\tau + 1\right)^{\frac{1}{\overline{\overline{\zeta}}}}} exp^{2\pi i\left(\sqrt[2]{1 - \left(\sum_{\tau=1}^{\alpha} \beta_j \left(1 - \overline{\overline{v_\tau^2}}\right)^{\overline{\overline{\zeta}}} - \sum_{\tau=1}^{\alpha} \beta_\tau + 1\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\right)} \\ \sqrt[2]{1 - \left(\sum_{\tau=1}^{\alpha} \beta_\tau \left(1 - \overline{\overline{\mathfrak{A}_\tau^2}}\right)^{\overline{\overline{\zeta}}} - \sum_{\tau=1}^{\alpha} \beta_\tau + 1\right)^{\frac{1}{\overline{\overline{\zeta}}}}} exp^{2\pi i\left(\sqrt[2]{1 - \left(\sum_{\tau=1}^{\alpha} \beta_\tau \left(1 - \overline{\overline{k_\tau^2}}\right)^{\overline{\overline{\zeta}}} - \sum_{\tau=1}^{\alpha} \beta_\tau + 1\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\right)} \end{pmatrix}. \quad (3.5)$$

Proof. The proof can be done on the similar steps of the proof of Theorem 3.1.

Theorem 3.10. *The following properties are found to hold successfully based on defined CSFVs* $\overline{\overline{\emptyset_\tau}} = \left(\overline{\overline{\mathfrak{A}_\tau^2}} exp^{2\pi i\left(\overline{\overline{u_\tau^2}}\right)}, \overline{\overline{\mathfrak{f}_\tau^2}} exp^{2\pi i\left(\overline{\overline{v_\tau^2}}\right)}, \overline{\overline{\ell_\tau^2}} exp^{2\pi i\left(\overline{\overline{k_\tau^2}}\right)}\right)$, $\tau = 1, 2, \dots, \alpha$, *for any CSFVs* $\overline{\overline{\emptyset}} = \overline{\overline{\emptyset_\tau}}$, *that is*

$CSFSSPG\left(\overline{\overline{\emptyset_{\tau_1}}}, \overline{\overline{\emptyset_{\tau_2}}}, \dots, \overline{\overline{\emptyset_{\tau_\alpha}}}\right) = \overline{\overline{\emptyset}}$.

Proof. The proof can be done on the similar steps of the proof of Theorem 3.2.

Theorem 3.11. *For any two CSFVs* $\overline{\overline{\emptyset_\tau}} = \left(\overline{\overline{\mathfrak{A}_\tau^2}}, \overline{\overline{\mathfrak{f}_\tau^2}}, \overline{\overline{\ell_\tau^2}}\right) \leq \overline{\overline{\emptyset'_\tau}} = \left(\overline{\overline{(\mathfrak{A}_\tau^2)'}}, \overline{\overline{(\mathfrak{f}_\tau^2)'}}, \overline{\overline{(\ell_\tau^2)'}}\right)$, *that is* $\overline{\overline{\mathfrak{A}_\tau^2}} \leq \overline{\overline{(\mathfrak{A}_\tau^2)'}}$, $\overline{\overline{\mathfrak{f}_\tau^2}} \geq \overline{\overline{(\mathfrak{f}_\tau^2)'}}$ *and* $\overline{\overline{\ell_\tau^2}} \geq \overline{\overline{(\ell_\tau^2)'}}$, *then* $CSFSSPG\left(\overline{\overline{\emptyset_1}}, \overline{\overline{\emptyset_2}}, \dots, \overline{\overline{\emptyset_\alpha}}\right) \leq CSFSSPG\left(\overline{\overline{\emptyset'_1}}, \overline{\overline{\emptyset'_2}}, \dots, \overline{\overline{\emptyset'_\alpha}}\right)$.

Proof. The proof can be done on the similar steps of the proof of Theorem 3.2.

Theorem 3.12. *For any CSFVs* $\emptyset_\tau^- = \left\{min\overline{\overline{\mathfrak{A}_\tau^2}}, max\overline{\overline{\mathfrak{f}_\tau^2}}, max\overline{\overline{\ell_\tau^2}}\right\}$ *and* $\emptyset_\tau^+ = \left\{max\overline{\overline{\mathfrak{A}_\tau^2}}, min\overline{\overline{\mathfrak{f}_\tau^2}}, min\overline{\overline{\ell_\tau^2}}\right\}$, *then*

$$\emptyset_\tau^- = CSFSSPG\left(\overline{\overline{\emptyset_{\tau_1}}}, \overline{\overline{\emptyset_{\tau_2}}}, \dots, \overline{\overline{\emptyset_{\tau_\alpha}}}\right) = \emptyset_\tau^+.$$

Proof. The proof can be done on the similar steps of the proof of Theorem 3.4.

Definition 3.3. *For any CSFVs* $\overline{\overline{\emptyset_\tau}} = \left(\overline{\overline{\mathfrak{A}_\tau^2}} exp^{2\pi i\left(\overline{\overline{u_\tau^2}}\right)}, \overline{\overline{\mathfrak{f}_\tau^2}} exp^{2\pi i\left(\overline{\overline{v_\tau^2}}\right)}, \overline{\overline{\ell_\tau^2}} exp^{2\pi i\left(\overline{\overline{k_\tau^2}}\right)}\right)$, $\tau = 1, 2, \dots, \alpha$, *then the notion of the CSFSSWPG operator is expressed as follows:*

$$CSFSSWPG\left(\overline{\overline{\emptyset_1}}, \overline{\overline{\emptyset_2}}, \dots, \overline{\overline{\emptyset_\alpha}}\right) = \overline{\overline{\emptyset_1^{\beta_1}}} \otimes \overline{\overline{\emptyset_2^{\beta_2}}} \otimes \dots \otimes \overline{\overline{\emptyset_\alpha^{\beta_\alpha}}} = \otimes_{\tau=1}^{\alpha} \overline{\overline{\emptyset_\tau^{\beta_\tau}}}.$$

Noticed that $\beta_\tau = \frac{\varpi_\tau \beta_\tau}{\sum_{\tau=1}^{\alpha} \varpi_\tau \beta_\tau}$, where $\beta_1 = 1$ and $\beta_\tau = \oplus_\kappa^{\tau-1} \wedge_s\left(\overline{\overline{\emptyset_\kappa}}\right)$, $\kappa = 2, 3, \dots, \alpha$. Additionally, the illustration of the weight vector is measured by $\varpi_\tau \in [0, 1]$ through $\sum_{\tau=1}^{\alpha} \varpi_\tau = 1$.

Theorem 3.13. *For CSFVs* $\overline{\overline{\emptyset_\tau}} = \left(\overline{\overline{\mathfrak{A}_\tau^2}} exp^{2\pi i\left(\overline{\overline{u_\tau^2}}\right)}, \overline{\overline{\mathfrak{f}_\tau^2}} exp^{2\pi i\left(\overline{\overline{v_\tau^2}}\right)}, \overline{\overline{\ell_\tau^2}} exp^{2\pi i\left(\overline{\overline{k_\tau^2}}\right)}\right)$, $\tau = 1, 2, \dots, \alpha$, *then the integrated values of the CSFSSWPA operator still a CSFV such as*

$$CSFSSWPG\left(\overline{\overline{\emptyset_1}}, \overline{\overline{\emptyset_2}}, \dots, \overline{\overline{\emptyset_\alpha}}\right)$$

$$= \left(\begin{array}{c} \sqrt[2]{\left(\sum_{\tau=1}^{\alpha} \beta_j \left(\overline{\overline{\mathfrak{A}_\tau^2}}\right)^{\overline{\overline{\zeta}}} - \sum_{\tau=1}^{\alpha} \beta_\tau + 1\right)^{\frac{1}{\overline{\overline{\zeta}}}}} exp^{2\pi i\left(\sqrt[2]{\left(\sum_{\tau=1}^{\alpha} \beta_j \left(\overline{\overline{u_\tau^2}}\right)^{\overline{\overline{\zeta}}} - \sum_{\tau=1}^{\alpha} \beta_\tau + 1\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\right)} \\ \sqrt[2]{1 - \left(\sum_{\tau=1}^{\alpha} \beta_j \left(1 - \overline{\overline{\mathfrak{f}_\tau^2}}\right)^{\overline{\overline{\zeta}}} - \sum_{\tau=1}^{\alpha} \beta_\tau + 1\right)^{\frac{1}{\overline{\overline{\zeta}}}}} exp^{2\pi i\left(\sqrt[2]{1 - \left(\sum_{\tau=1}^{\alpha} \beta_j \left(1 - \overline{\overline{v_\tau^2}}\right)^{\overline{\overline{\zeta}}} - \sum_{\tau=1}^{\alpha} \beta_\tau + 1\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\right)} \\ \sqrt[2]{1 - \left(\sum_{\tau=1}^{\alpha} \beta_\tau \left(1 - \overline{\overline{\ell_\tau^2}}\right)^{\overline{\overline{\zeta}}} - \sum_{\tau=1}^{\alpha} \beta_\tau + 1\right)^{\frac{1}{\overline{\overline{\zeta}}}}} exp^{2\pi i\left(\sqrt[2]{1 - \left(\sum_{\tau=1}^{\alpha} \beta_\tau \left(1 - \overline{\overline{k_\tau^2}}\right)^{\overline{\overline{\zeta}}} - \sum_{\tau=1}^{\alpha} \beta_\tau + 1\right)^{\frac{1}{\overline{\overline{\zeta}}}}}\right)} \end{array}\right). \quad (3.6)$$

Proof. The proof can be done on the similar steps of the proof of Theorem 3.1.

Theorem 3.14. *Based on defined CSFVs* $\overline{\overline{\emptyset_\tau}} = \left(\overline{\overline{\mathfrak{A}_\tau^2}} exp^{2\pi i\left(\overline{\overline{u_\tau^2}}\right)}, \overline{\overline{\mathfrak{f}_\tau^2}} exp^{2\pi i\left(\overline{\overline{v_\tau^2}}\right)}, \overline{\overline{\ell_\tau^2}} exp^{2\pi i\left(\overline{\overline{k_\tau^2}}\right)}\right)$, $\tau = 1, 2, \dots, \alpha$, *we determine that the resulting properties hold efficaciously.*

For any CSFVs $\overline{\overline{\emptyset}} = \overline{\overline{\emptyset_\tau}}$, *that is* $CSFSSWPG\left(\overline{\overline{\emptyset_{\tau_1}}}, \overline{\overline{\emptyset_{\tau_2}}}, \dots, \overline{\overline{\emptyset_{\tau_\alpha}}}\right) = \overline{\overline{\emptyset}}$.

*Proof***.** The proof can be done on the similar steps of the proof of Theorem 3.2.

Theorem 3.15. *For any two CSFVs* $\overline{\overline{\emptyset_\tau}} = \left(\overline{\overline{\mathfrak{A}_\tau^2}}, \overline{\overline{\mathfrak{f}_\tau^2}}, \overline{\overline{\ell_\tau^2}}\right) \leq \overline{\overline{\emptyset'_\tau}} = \left(\overline{\overline{(\mathfrak{A}_\tau^2)'}}, \overline{\overline{(\mathfrak{f}_\tau^2)'}}, \overline{\overline{(\ell_\tau^2)'}}\right)$, *that is* $\overline{\overline{\mathfrak{A}_\tau^2}} \leq \overline{\overline{(\mathfrak{A}_\tau^2)'}}$, $\overline{\overline{\mathfrak{f}_\tau^2}} \geq \overline{\overline{(\mathfrak{f}_\tau^2)'}}$ *and* $\overline{\overline{\ell_\tau^2}} \geq \overline{\overline{(\ell_\tau^2)'}}$, *then* $CSFSSWPG\left(\overline{\overline{\emptyset_1}}, \overline{\overline{\emptyset_2}}, \dots, \overline{\overline{\emptyset_\alpha}}\right) \leq CSFSSWPG\left(\overline{\overline{\emptyset'_1}}, \overline{\overline{\emptyset'_2}}, \dots, \overline{\overline{\emptyset'_\alpha}}\right)$.

Proof. The proof can be done on the similar steps of the proof of Theorem 3.2.

Theorem 3.16. *For any CSFVs* $\emptyset_\tau^- = \left\{min\overline{\overline{\mathfrak{A}_\tau^2}}, max\overline{\overline{\mathfrak{f}_\tau^2}}, max\overline{\overline{\ell_\tau^2}}\right\}$ *and* $\emptyset_\tau^+ = \left\{max\overline{\overline{\mathfrak{A}_\tau^2}}, min\overline{\overline{\mathfrak{f}_\tau^2}}, min\overline{\overline{\ell_\tau^2}}\right\}$, *then we have* $\emptyset_\tau^- = CSFSSWPG\left(\overline{\overline{\emptyset_{\tau_1}}}{}'\overline{\overline{\emptyset_{\tau_2}}}{}' \dots {}'\overline{\overline{\emptyset_{\tau_\alpha}}}\right) = \emptyset_\tau^+$.

Proof. The proof can be done on the similar steps of the proof of Theorem 3.4.

4. MADM Techniques for CSF data

The CSFSSWPA and CSFSSWPG operators are to be used in this situation to help us construct a MADM problem using CSF data. By calculating a decision matrix with their values as CSF numbers, we can accomplish this. A set of data is described as a personal of alternatives $\{\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_m\}$ and a personal of attributes $\{\overline{\overline{\Lambda_1}}, \overline{\overline{\Lambda_2}}, \dots, \overline{\overline{\Lambda_\alpha}}\}$. To unceasingly explain the process, we describe a weight vector $\varpi_\tau \in [0, 1]$ through $\sum_{\tau=1}^{\alpha} \varpi_\tau = 1$. We also allocate an SF quantity to every attribute in each option and effort to represent it in a locked medium M=$\left[\overline{\overline{\emptyset_{\iota\tau}}}\right]_{m\times\alpha}$. The CSF quantity can be expressed in the following form: $\overline{\overline{\emptyset_\tau}} = \overline{\overline{\mathfrak{A}_\tau^2}} exp^{2\pi i\left(\overline{\overline{u_\tau^2}}\right)}, \overline{\overline{\mathfrak{f}_\tau^2}} exp^{2\pi i\left(\overline{\overline{v_\tau^2}}\right)}, \overline{\overline{\ell_\tau^2}} exp^{2\pi i\left(\overline{\overline{k_\tau^2}}\right)}$, $\tau = 1, 2, \dots, \alpha$. In this case, the symbols $\overline{\overline{\mathfrak{A}}}^2(\mathcal{X})$ MG, $\overline{\overline{\mathfrak{f}}}^2(\mathcal{X})$ AG and NMG convey the information MG with the age-old and valuable properties: $0 \leq \left(\overline{\overline{\mathfrak{A}}}^2(\mathcal{X}) + \overline{\overline{\mathfrak{f}}}^2(\mathcal{X}) + \overline{\overline{\ell}}^2(\mathcal{X})\right) \leq 1$ and $0 \leq \left(\overline{\overline{u_\tau^2}}(\mathcal{X}) + \overline{\overline{v_\tau^2}}(\mathcal{X}) + \overline{\overline{k_\tau^2}}(\mathcal{X})\right) \leq 1$. Additionally, we have discussed the importance of rejection information, that is expressed as $\overline{\overline{\delta_\mu}}(\mathcal{X}) = \left(1 - \left(\overline{\overline{\mathfrak{A}}}^2(\mathcal{X}) + \overline{\overline{\mathfrak{f}}}^2(\mathcal{X}) + \overline{\overline{\ell}}^2(\mathcal{X})\right)\right) exp^{i2\pi\left(1 - \left(\overline{\overline{u_\tau^2}}(\mathcal{X}) + \overline{\overline{v_\tau^2}}(\mathcal{X}) + \overline{\overline{k_\tau^2}}(\mathcal{X})\right)\right)}$. We have demonstrated a technique whose highest assessment technique is lower:

Step 1. After collecting the CSF data, we have tried to evaluate it. Normalization is necessary if cost-type values are present in the data. The ideas listed below can assist this:

$$\text{Decision Matrix} = \begin{cases} \left(\overline{\overline{\mathfrak{A}_\tau^2}} exp^{2\pi i\left(\overline{\overline{u_\tau^2}}\right)}, \overline{\overline{\mathfrak{f}_\tau^2}} exp^{2\pi i\left(\overline{\overline{v_\tau^2}}\right)}, \overline{\overline{\ell_\tau^2}} exp^{2\pi i\left(\overline{\overline{k_\tau^2}}\right)}\right), & for\ benefit\ kinds, \\ \left(\overline{\overline{\mathfrak{A}_\tau^2}} exp^{2\pi i\left(\overline{\overline{u_\tau^2}}\right)}, \overline{\overline{\mathfrak{f}_\tau^2}} exp^{2\pi i\left(\overline{\overline{v_\tau^2}}\right)}, \overline{\overline{\ell_\tau^2}} exp^{2\pi i\left(\overline{\overline{k_\tau^2}}\right)}\right), & for\ cost\ kinds. \end{cases}$$

When assistance forms are included in the attention data, there is no need to normalize the environment.

Step 2. We object to calculate the assessment of $\beta_\tau = \frac{\beta_\tau}{\sum_{\tau=1}^{\alpha} \beta_\tau}$, where $\beta_1 = 1$ and $\beta_\tau =$

$\oplus_{\kappa=1}^{\tau-1} \wedge_k \left(\overline{\overline{\emptyset_\kappa}}\right)'$, $\kappa = 2, 3, \dots, \alpha$.

Step 3. To gather all the data, we want to calculate the CSF numbers using the operators CSFSSWPA and CSFSSWPG.

Step 4. We use information to look at the data's accuracy and score.

Step 5. We eventually outgrew each preference and attempted to identify the most advantageous ideal among the preferences.

Additionally, by offering a real-world example to back up the previously described process, we hope to increase the value of the derived information.

4.1. Example

In this section, we select mobile applications to play a crucial role in enhancing the e-tourism experience by providing users with convenient access to information, bookings, and personalized recommendations. In this example, we provide some features of mobile applications for e-tourism.

Some features of mobile applications

Destination discovery: Interactive map with points of interest, landmarks, and attractions. Detailed information about each destination, including historical background, cultural significance, and local tips.

Trip planning: Itinerary for users to plan their trips by selecting attractions, activities, and accommodations. Real-time availability and pricing for hotels, flights, and activities.

Augmented reality (AR) navigation: AR feature for easy navigation in unfamiliar locations. Point-and-view functionality to identify landmarks and get information in real-time through the phone’s camera.

Personalized recommendations: AI-driven recommendations based on user preferences, previous travel history, and ratings. Suggestions for local restaurants, events, and hidden gems.

Booking integration: Seamless integration with popular booking platforms for flights, hotels, and activities. Exclusive discounts and offers for in-app bookings.

Language translation: Built-in language translator for overcoming language barriers. Translate text and spoken words for better communication with locals.

Offline access: Downloadable maps, itineraries, and essential information for offline access during travel. Access to emergency information, including local emergency numbers and embassy contacts.

Social integration: Shareable itineraries and experiences on social media. Connect with friends and fellow travelers for collaborative trip planning.

User reviews and ratings: In-app reviews and ratings for destinations, accommodations, and activities. User-generated content to help others make informed decisions.

Safety alerts: Real-time safety alerts and notifications for potential risks or emergencies in the chosen destination. Emergency contact information and guidance on local safety protocols.

Feedback and support: User feedback feature for continuous improvement. 24/7 customer support for any issues or queries.

Mobile application for tourism aims to provide a comprehensive and user-friendly platform for travelers, making their e-tourism experience enjoyable, stress-free, and tailored to their preferences.

Now, we consist of alternatives Λ_τ $(\tau = 1, 2, \dots, 4)$: $\overline{\overline{\Lambda_1}}$ Travel traverse, $\overline{\overline{\Lambda_2}}$ Wander wave,

$\overline{\overline{\Lambda_3}}$ Roam rover and $\overline{\overline{\Lambda_4}}$ Explore express. The processes have four attributes $\mathcal{L}_\tau$ $(\tau = 1, 2 \dots 4)$: $\overline{\overline{\mathcal{L}_1}}$ Swift voyage; $\overline{\overline{\mathcal{L}_2}}$ Adventura ease; $\overline{\overline{\mathcal{L}_3}}$ Discover ease and $\overline{\overline{\mathcal{L}_4}}$ Guided globe and $\varpi_\tau = (0.30, 0.35, 0.10, 0.25)$ are the weight vector and $\overline{\overline{\zeta}} = 2$.

These names blend alliteration for a catchy sound and attributes that convey the essence of mobile applications in the e-tourism sector.

Step 1. We attempt to evaluate the CSF decision matrix by gathering examples, such as those in Table 1. If the data included cost-type values, then normalization is required. The following concepts can help with this:

$$DM = \begin{cases} \left(\overline{\overline{\mathfrak{A}_\tau^2}} exp^{2\pi i\left(\overline{\overline{u_\tau^2}}\right)}, \overline{\overline{\mathfrak{f}_\tau^2}} exp^{2\pi i\left(\overline{\overline{v_\tau^2}}\right)}, \overline{\overline{\ell_\tau^2}} exp^{2\pi i\left(\overline{\overline{\hbar_\tau^2}}\right)}\right), & for\ benefit\ kinds, \\ \left(\overline{\overline{\mathfrak{A}_\tau^2}} exp^{2\pi i\left(\overline{\overline{u_\tau^2}}\right)}, \overline{\overline{\mathfrak{f}_\tau^2}} exp^{2\pi i\left(\overline{\overline{v_\tau^2}}\right)}, \overline{\overline{\ell_\tau^2}} exp^{2\pi i\left(\overline{\overline{\hbar_\tau^2}}\right)}\right), & for\ cost\ kinds. \end{cases}$$

Table 1. Alternative and attribute.

	$\overline{\overline{\Lambda_1}}$	$\overline{\overline{\Lambda_2}}$
$\mathcal{L}_1$	$\left(0.3exp^{i2\pi(0.2)}, 0.3exp^{i2\pi(0.4)}, 0.5exp^{i2\pi(0.7)}\right)$	$\left(0.4exp^{i2\pi(0.4)}, 0.4exp^{i2\pi(0.3)}, 0.5exp^{i2\pi(0.7)}\right)$
$\mathcal{L}_2$	$\left(0.2exp^{i2\pi(0.4)}, 0.5exp^{i2\pi(0.4)}, 0.4exp^{i2\pi(0.5)}\right)$	$\left(0.3exp^{i2\pi(0.5)}, 0.6exp^{i2\pi(0.4)}, 0.4exp^{i2\pi(0.5)}\right)$
$\mathcal{L}_3$	$\left(0.5exp^{i2\pi(0.5)}, 0.4exp^{i2\pi(0.6)}, 0.3exp^{i2\pi(0.3)}\right)$	$\left(0.3exp^{i2\pi(0.3)}, 0.4exp^{i2\pi(0.5)}, 0.5exp^{i2\pi(0.4)}\right)$
$\mathcal{L}_4$	$\left(0.4exp^{i2\pi(0.6)}, 0.5exp^{i2\pi(0.5)}, 0.5exp^{i2\pi(0.4)}\right)$	$\left(0.6exp^{i2\pi(0.5)}, 0.3exp^{i2\pi(0.4)}, 0.5exp^{i2\pi(0.4)}\right)$
	$\overline{\overline{\Lambda_3}}$	$\overline{\overline{\Lambda_4}}$
$\mathcal{L}_1$	$\left(0.3exp^{i2\pi(0.4)}, 0.4exp^{i2\pi(0.5)}, 0.5exp^{i2\pi(0.2)}\right)$	$\left(0.4exp^{i2\pi(0.3)}, 0.5exp^{i2\pi(0.6)}, 0.3exp^{i2\pi(0.5)}\right)$
$\mathcal{L}_2$	$\left(0.4exp^{i2\pi(0.3)}, 0.6exp^{i2\pi(0.4)}, 0.3exp^{i2\pi(0.4)}\right)$	$\left(0.4exp^{i2\pi(0.5)}, 0.6exp^{i2\pi(0.4)}, 0.4exp^{i2\pi(0.3)}\right)$
$\mathcal{L}_3$	$\left(0.7exp^{i2\pi(0.5)}, 0.1exp^{i2\pi(0.6)}, 0.3exp^{i2\pi(0.5)}\right)$	$\left(0.5exp^{i2\pi(0.6)}, 0.5exp^{i2\pi(0.4)}, 0.1exp^{i2\pi(0.2)}\right)$
$\mathcal{L}_4$	$\left(0.5exp^{i2\pi(0.4)}, 0.2exp^{i2\pi(0.5)}, 0.4exp^{i2\pi(0.3)}\right)$	$\left(0.6exp^{i2\pi(0.4)}, 0.4exp^{i2\pi(0.6)}, 0.4exp^{i2\pi(0.4)}\right)$

Step 2. Our goal is to assess the worth of $\beta_\tau = \frac{\beta_\tau}{\sum_{\tau=1}^{\alpha} \beta_\tau}$, where $\beta_1 = 1$ and $\beta_\tau = \oplus_{\kappa=1}^{\tau-1} \wedge_s\left(\overline{\overline{\emptyset_\kappa}}\right)$, $\kappa = 2, 3, \dots, \alpha$. The score values are presented in Table 2.

Table 2. Score matrix.

	$CSFSSWPA$ and $CSFSSWPG$
$\mathcal{L}_1$	$\left(0.5833exp^{i2\pi(0.4633)}, 0.5833exp^{i2\pi(0.5267)}, ,0.5600exp^{i2\pi(0.6233)}, 0.6067exp^{i2\pi(0.4933)}\right)$
$\mathcal{L}_2$	$\left(0.5433exp^{i2\pi(0.5833)}, 0.5233exp^{i2\pi(0.6133)}, 0.5700exp^{i2\pi(0.5900)}, 0.5467exp^{i2\pi(0.6667)}\right)$
$\mathcal{L}_3$	$\left(0.667exp^{i2\pi(0.6000)}, 0.5600exp^{i2\pi(0.5600)}, 0.7967exp^{i2\pi(0.5467)}, 0.6633exp^{i2\pi(0.7200)}\right)$
$\mathcal{L}_4$	$\left(0.5533exp^{i2\pi(0.6500)}, 0.6733exp^{i2\pi(0.6433)}, 0.6833exp^{i2\pi(0.6067)}, 0.6800exp^{i2\pi(0.5467)}\right)$

Step 3. Our goal is to determine the CSF numbers using the operators CSFSSWPA and CSFSSWPG to compile the collective information in Table 3.

Table 3. Weight matrix.

	$CSFSSWPA$ and $CSFSSWPG$
$\mathcal{L}_1$	$\left(0.5121exp^{i2\pi(0.5719)}, 0.3485exp^{i2\pi(0.3091)}, ,0.0581exp^{i2\pi(0.0465)}, 0.0813exp^{i2\pi(0.0725)}\right)$
$\mathcal{L}_2$	$\left(0.5366exp^{i2\pi(0.5061)}, 0.3401exp^{i2\pi(0.3445)}, 0.0509exp^{i2\pi(0.0604)}, 0.0725exp^{i2\pi(0.0890)}\right)$
$\mathcal{L}_3$	$\left(0.4651exp^{i2\pi(0.5089)}, 0.3617exp^{i2\pi(0.3562)}, 0.0579exp^{i2\pi(0.0570)}, 0.1153exp^{i2\pi(0.0779)}\right)$
$\mathcal{L}_4$	$\left(0.5046exp^{i2\pi(0.4741)}, 0.3257exp^{i2\pi(0.3595)}, 0.0627exp^{i2\pi(0.0661)}, 0.1070exp^{i2\pi(0.1002)}\right)$

Step 4. We use information to look at the data's accuracy and score in Table 4.

Table 4. Aggregated information matrix.

	$CSFSSWPA$	$CSFSSWPG$
$\mathcal{L}_1$	$\left(0.3456exp^{i2\pi(0.2908)}, 0.3744exp^{i2\pi(0.4142)}, 0.4909exp^{i2\pi(0.6818)}\right)$	$\left(0.3535exp^{i2\pi(0.3209)}, 0.3607exp^{i2\pi(0.3927)}, 0.4855exp^{i2\pi(0.6639)}\right)$
$\mathcal{L}_2$	$\left(0.2663exp^{i2\pi(0.4395)}, 0.5531exp^{i2\pi(0.4000)}, 0.3965exp^{i2\pi(0.4852)}\right)$	$\left(0.2869exp^{i2\pi(0.4487)}, 0.5467exp^{i2\pi(0.4000)}, 0.3953exp^{i2\pi(0.4783)}\right)$
$\mathcal{L}_3$	$\left(0.4465exp^{i2\pi(0.4436)}, 0.4104exp^{i2\pi(0.5589)}, 0.4048exp^{i2\pi(0.3607)}\right)$	$\left(0.4800exp^{i2\pi(0.4689)}, 0.4005exp^{i2\pi(0.5496)}, 0.3664exp^{i2\pi(0.3446)}\right)$
$\mathcal{L}_4$	$\left(0.4960exp^{i2\pi(0.5322)}, 0.4386exp^{i2\pi(0.4864)}, 0.4870exp^{i2\pi(0.3954)}\right)$	$\left(0.5197exp^{i2\pi(0.5449)}, 0.4138exp^{i2\pi(0.4757)}, 0.4837exp^{i2\pi(0.3939)}\right)$

Step 5. Finally, we outgrew each preference and tried to determine which of the favorites the most advantageous ideal was in Table 5.

Table 5. Final result.

	$CSFSSWPA$	$CSFSSWPG$
$\mathcal{L}_1$	0.3955	0.4224
$\mathcal{L}_2$	0.4685	0.4799
$\mathcal{L}_3$	0.5404	0.5783
$\mathcal{L}_4$	0.5689	0.5934

Considering score values, the best solution for a mobile application for tourism is explore express $\mathcal{L}_4$. $\mathcal{L}_4$ is the high score value in Table 6, and Figure 3 represents the graphical result of CSFSSWPA and CSFSSWPG.

Table 6. Ranking order representation.

Aggregation operators	Ranking and ordering
$CSFSSWPA$	$\mathcal{L}_4 > \mathcal{L}_3 > \mathcal{L}_2 > \mathcal{L}_1$
$CSFSSWPG$	$\mathcal{L}_4 > \mathcal{L}_3 > \mathcal{L}_2 > \mathcal{L}_1$

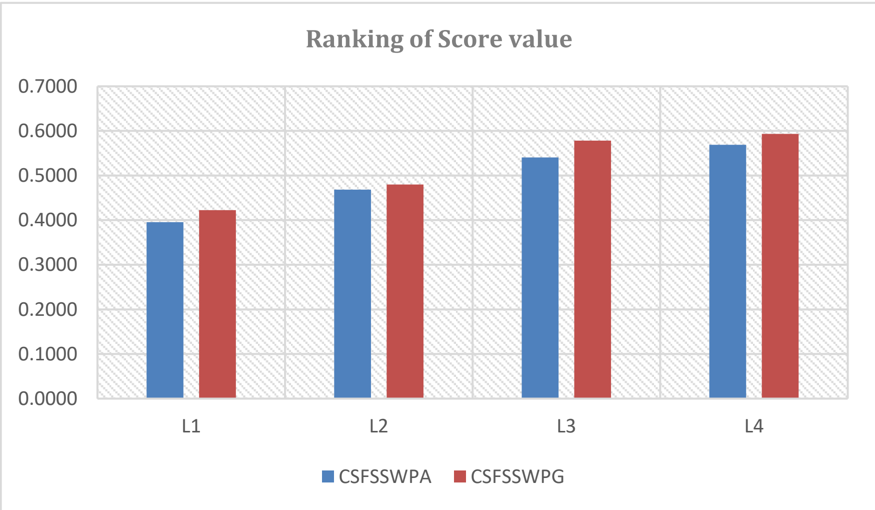

Figure 3. Sensitivity analysis of CSFSSWPA and CSFSSWPG operator.

4.2. Influence study

Since the value of parameters $\bar{\bar{\rho}} = -5$ is fixed in Table 6, we change the value of $\bar{\bar{\rho}}$ in this part and compare the result in different parametric. We found that the result of CSFSSPWA in Table 6 has the same result with different parameters in Table 7.

The CSFSSPWA result in Table 7 has the same result with different parameters.

Table 7. Different parametric of CSFSSPWA.

Parametric values	Ranking and ordering
$\rho = -1$	$\mathcal{L}_4 > \mathcal{L}_3 > \mathcal{L}_2 > \mathcal{L}_1$
$\rho = -5$	$\mathcal{L}_4 > \mathcal{L}_3 > \mathcal{L}_2 > \mathcal{L}_1$
$\rho = -20$	$\mathcal{L}_4 > \mathcal{L}_3 > \mathcal{L}_2 > \mathcal{L}_1$
$\rho = -35$	$\mathcal{L}_4 > \mathcal{L}_3 > \mathcal{L}_2 > \mathcal{L}_1$
$\rho = -70$	$\mathcal{L}_4 > \mathcal{L}_3 > \mathcal{L}_2 > \mathcal{L}_1$
$\rho = -100$	$\mathcal{L}_4 > \mathcal{L}_3 > \mathcal{L}_2 > \mathcal{L}_1$

Figure 4 gives the same result with different parameters.

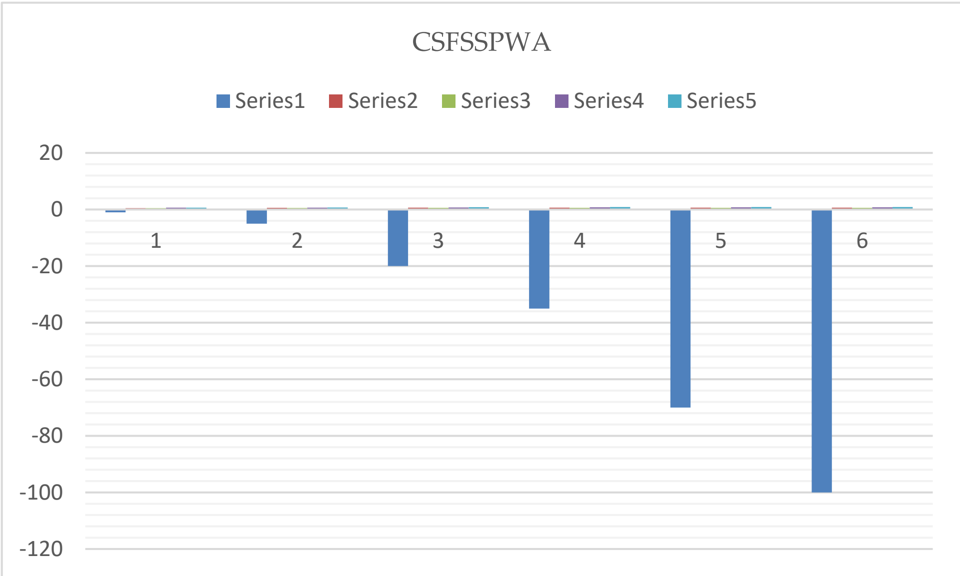

Figure 4. Sensitivity analysis of CSFSSPWA operator.

In Table 8, CSFSSPWG results have same different parameters give the same result.

Table 8. Different parametric of CSFSSPWG.

Parametric values	Ranking and ordering
$\rho = -1$	$\mathcal{L}_4 > \mathcal{L}_3 > \mathcal{L}_2 > \mathcal{L}_1$
$\rho = -5$	$\mathcal{L}_4 > \mathcal{L}_3 > \mathcal{L}_2 > \mathcal{L}_1$
$\rho = -20$	$\mathcal{L}_4 > \mathcal{L}_3 > \mathcal{L}_2 > \mathcal{L}_1$
$\rho = -35$	$\mathcal{L}_4 > \mathcal{L}_3 > \mathcal{L}_2 > \mathcal{L}_1$
$\rho = -70$	$\mathcal{L}_4 > \mathcal{L}_3 > \mathcal{L}_2 > \mathcal{L}_1$
$\rho = -100$	$\mathcal{L}_4 > \mathcal{L}_3 > \mathcal{L}_2 > \mathcal{L}_1$

Figure 5 provides the same result with different parameters. In a similar manner, we vary the SS triangular norm's parametric values to see the results of the CSFSSPWA and CSFSSPWG operators. When we raise the SS triangular norms' parametric value in the CSFSSPWA and CSFSSPWG operators, the results of the investigation start to get same, and the score value ranking stays the same:

$\mathcal{L}_4 > \mathcal{L}_3 > \mathcal{L}_2 > \mathcal{L}_1$. Tables 7 and 8 contain a list of all the examined outcomes for the CSFSSPWA and CSFSSPWG operators.

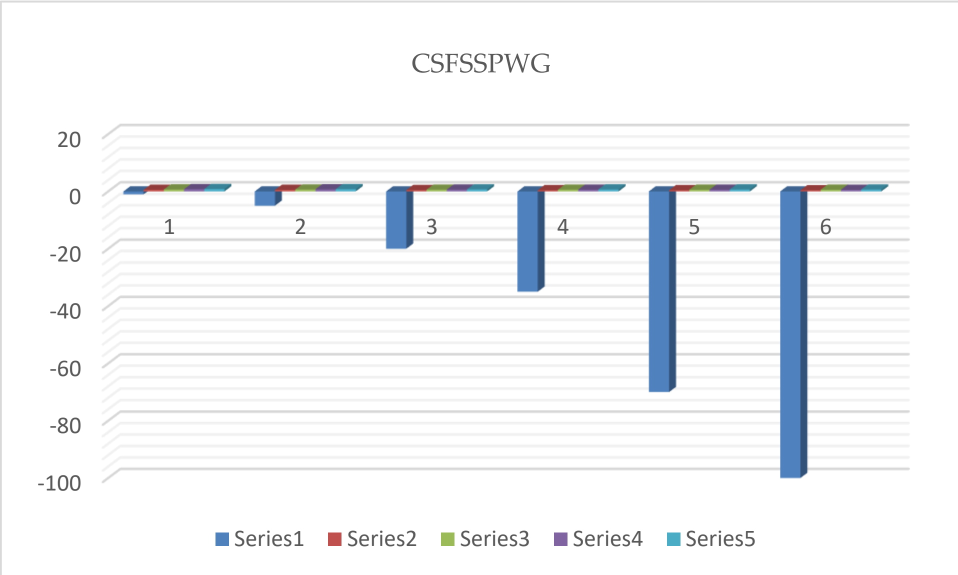

Figure 5. Sensitivity analysis of CSFSSPWG operator.

4.3. Comparative study

The suggested operators, CSFSSPWA and CSFSSWPG, are compared to several well-known aggregation techniques in this section. We assess how well they handle complex spherical fuzzy information, which is essential for decision-making in mobile e-tourism applications, using the data shown in Table 1. The purpose of the comparison is to evaluate each operator's precision and efficacy in handling ambiguous and uncertain data. We determine which operator is best suited to provide dependable decision support in mobile e-tourism scenarios by comparing their outcomes with other approaches. This analysis demonstrates how the suggested operators outperform earlier techniques, providing increased robustness and precision in multi-attribute decision-making: Worndl et al. [33] developed the application for E-tourism, Hamid et al. [34] aggregated an operator based data management system using smart tourism, Sarfraz et al. [33] aggregation operators based on prioritized Aczel-Alsina aggregation operator for (IF) and Ullah et al. [35] aggregated operators based on a complex IF and its application in decision-making problem and tried to contrast with operators that have been diagnosed. Table 9 presents the comparative analysis based on the data presented in Table 1.

Table 9. Comparison information.

Method	Ranking information
CSFSSWPA	$\mathcal{L}_4 > \mathcal{L}_3 > \mathcal{L}_2 > \mathcal{L}_1$
CSFSSWPG	$\mathcal{L}_4 > \mathcal{L}_3 > \mathcal{L}_2 > \mathcal{L}_1$
Worndl et al. [37]	$\mathcal{L}_4 > \mathcal{L}_3 > \mathcal{L}_2 > \mathcal{L}_1$
Hamid et al. [38]	$\mathcal{L}_4 > \mathcal{L}_3 > \mathcal{L}_2 > \mathcal{L}_1$
Sarfraz et al. [33]	$\mathcal{L}_4 > \mathcal{L}_3 > \mathcal{L}_2 > \mathcal{L}_1$
Ullah et al. [35]	$\mathcal{L}_4 > \mathcal{L}_3 > \mathcal{L}_2 > \mathcal{L}_1$

Upon closer inspection, we found that the prevailing operators and the derived operators both produce the same advantageous ideal $\mathcal{L}_4$. Thus, our deduced operators from the CSFS theory represent a novel and validated concept for handling difficult and inconsistent data in real-world applications.

5. Conclusions

For the arrangement with uncertain information in real-world decision-making situations, CSFSs typically combine elements of CFSs and SFSs. We study the operational laws of CSFSs with certain features in this research. In addition, we construct four aggregation operators, which we refer to as the complex spherical fuzzy Schweizer-Sklar prioritized aggregation (CSFSSPA), the complex spherical fuzzy Schweizer-Sklar weighted prioritized aggregation (CSFSSWPA), the complex spherical fuzzy Schweizer-Sklar prioritized geometry (CSFSSPG), and the complex spherical fuzzy Schweizer-Sklar weighted prioritized geometry (CSFSSWPG). These operators are based on CSFSs and Schweizer-Sklar prioritized aggregation (SSPA) operators. Additionally, we provide the idempotent, monotonicity, and boundedness features for the operators CSFSSPA, CSFSSWPA, CSFSSPG, and CSFWBM. We examine these operators' behavior in more detail by varying the value of the relevant parameter, $\overline{\overline{\zeta}}$. The CSFSSWPA and CSFSSWPG operators were taken into consideration to address a MADM problem.

5.1. Future directions

The suggested methodology may be expanded in future studies to include other decision-making frameworks, such as group decision-making scenarios, in which several experts participate in the evaluation of alternatives. Furthermore, investigating hybrid models that combine the Schweizer-Sklar technique with additional fuzzy or non-fuzzy techniques may enhance decision accuracy even more.

5.2. Study limitations

Dependency on expert input: To define criteria and prioritization levels, the suggested model mainly depends on the opinions and input of experts. These experts' subjective biases may affect how accurate the results are.

Computational complexity: The handling of complex spherical fuzzy information and the nature of Schweizer-Sklar operators can make computations time-consuming, particularly when working with large datasets or numerous options.

Restricted application scope: Although the study concentrates on mobile e-tourism applications, the results may need to be modified for use in other sectors with distinct data structures or decision-making standards.

Sensitivity to parameter selection: Depending on how the parameters (such as prioritization weights and Schweizer-Sklar parameters) are chosen, the prioritized aggregation process's efficacy may change. Inaccurate parameter values may result in less-than-ideal choices.

Comparative analysis with limited methods: While TOPSIS, VIKOR, and Multi-MOORA are compared with the suggested method in the study, other new MADM approaches might not have been considered, which could have limited the evaluation of its effectiveness.

Author contributions

Khawlah Alhulwah: Conceptualization, Data curation, Formal analysis; Muhammad Azeem: Resources, Funding acquisition; Mehwish Sarfraz: Software, Visualization, Investigation; Nasreen Almohanna: Validation, Methodology; Ali Ahmad: Supervision, Project administration; Khawlah Alhulwah， Muhammad Azeem，Mehwish Sarfraz，Nasreen Almohanna and Ali Ahmad: Writing–original draft, Writing–review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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