AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(12): 33800–33817.
DOI: 10.3934/math.20241612
Received: 02 September 2024
Revised: 31 October 2024
Accepted: 12 November 2024
Published: 28 November 2024

Research article

Set-valued mappings and best proximity points: A study in $\mathcal{F}$-metric spaces

Amer Hassan Albargi*

Department of Mathematics, Faculty of Sciences, King Abdulaziz University, P. O. Box 80203, Jeddah 21589, Saudi Arabia

* **Correspondence:** Email: aalbarqi@kau.edu.sa.

Abstract: This paper introduces the concept of set-valued almost Υ-contractions in $\mathcal{F}$-metric spaces, aiming to obtain the best proximity point results for set-valued mappings. The newly proposed idea of set-valued almost Υ-contractions includes various contractive conditions like set-valued almost contractions, set-valued Υ-contractions, and traditional Υ-contractions. Consequently, the results presented here extend and unify numerous established works in this domain. To illustrate the practical significance of the theoretical findings, a specific example is provided.

Keywords: best proximity points; set-valued almost Υ-contractions; $\mathcal{F}$-metric spaces; fixed points
Mathematics Subject Classification: 46S40, 47H10, 54H25

1. Introduction

The foundational idea of the metric space (MS), autonomously conceived by a French mathematician Fréchet [1], serves as the cornerstone for fixed point theory. The concept of b-metric spaces (b-MSs) was introduced by I. A. Bakhtin in 1989. However, it was Czerwik [2] in 1993 who formally defined and studied b-MSs in more detail. By replacing the traditional triangle inequality with a more relaxed "b-metric inequality", Bakhtin introduced a generalized MS capable of accommodating a wider range of distance structures. Leveraging this groundwork, Jleli et al. [3] subsequently expanded the MS concept by introducing $\mathcal{F}$-metric spaces ($\mathcal{F}$-MSs), a flexible structure incorporating both b-metric and standard MS. This innovative approach has proven fruitful in fixed point theory and holds significant potential for future advancements in functional analysis and topology.

Capturing the notion of distance between points, this concept allows fixed point theorems to set forth conditions under which specific mappings within MS. The first result in this theory, the eminent Banach contraction principle (BCP) [4], was begun by Stefan Banach in 1922. Wardowski [5] introduced a novel contraction concept, termed the Υ-contraction, alongside a corresponding fixed point (FP) theorem. This groundbreaking work expanded the boundaries of the BCP. By generalizing the contraction mapping framework, Wardowski's contributions have significantly advanced the field and enriched its applicability across diverse mathematical domains. The study of FP theory for set-valued mappings has been a focal point in nonlinear analysis since Nadler [6] initiated the concept. He gave the idea of set-valued contraction and established an FP theorem for set-valued mappings. Building upon this foundation, researchers have delved into various extensions and generalizations. For instance, Berinde et al. [7] extended the result of Nadler [6] by introducing set-valued almost contraction in the foundation of complete MSs. Afterwards, Altune et al. [8, 9] gave the thoughts of set-valued Υ-contraction and set-valued almost Υ-contraction to obtain FP results for set-valued mappings. Ali et al. [10] introduced a generalized set-valued Υ-contraction in the situation of b-MSs and obtained FPs of set-valued mappings.

The BCP has been generalized in different ways, as in [11]. A primary focus of research has been extending BCP to non-self-mappings $S : \mathcal{U} \to \mathcal{V}$, where $(\mathcal{U}, \mathcal{V})$ is a couple of subsets of a MS (Ω, d). Unlike standard contraction mappings, these mappings do not necessarily possess FPs. Instead, the objective is to identify points $\zeta^* \in \mathcal{U}$ whenever $d(\zeta^*, S\zeta^*) = dist(\mathcal{U}, \mathcal{V})$, where $dist(\mathcal{U}, \mathcal{V}) = \inf\{d(\zeta, \mathfrak{z}) : \zeta \in \mathcal{U}, \mathfrak{z} \in \mathcal{V}\}$. These points are termed the best proximity points (BPPs) of mapping T. Fan [12] introduced a foundational theorem on best approximations in 1969. Building upon this work, Sadiq Basha [13] established necessary and sufficient conditions for identifying points that achieve the minimum possible distance between two sets for specific types of mappings. In this context, Omidvari et al. [14] and Şahina [15] investigated BPP results for non-self mappings satisfying Υ-contractions. Abkar et al. [16] presented existence theorems for BPPs of set-valued contraction in MSs. This work represents a significant extension of Nadler's classical FP theorem [6] to the setting of set-valued mappings. Subsequently, Debnath [17] discussed the existence of a BPP result for set-valued Υ-contractions. Under the umbrella of MS topology, Patel et al. [18] established BPP results for set-valued almost Υ-contractions. De La Sen et al. [19] studied optimal fuzzy best proximity coincidence points for cyclic contractions in non-Archimedean fuzzy metric spaces. Lateef's [20] recent work pushed the boundaries within $\mathcal{F}$-MS, establishing BPP results for generalized contractions and coupled BPPs under an arbitrary binary relation. The work of Albargi et al. [21] focuses on demonstrating the existence of BPPs for Υ-contractions in the setting of $\mathcal{F}$-MS. For a more in-depth discussion of these concepts, see [22–27].

The current investigation focuses on the concept of set-valued almost Υ-contractions in the background of $\mathcal{F}$-MS and to demonstrate BPPs for the aforementioned contractions. By generalizing existing contractive conditions, our research unifies and extends upon seminal works by Banach [4], Wardowski [5], Nadler [6], Berinde et al. [7], Altun et al. [8, 9], Abkar et al. [16], Debnath [17] and Patel et al. [18] in the background of classical MS. Moreover, our results encompass and generalize key theorems of Jleli et al. [3], Asif et al. [22], Lateef [20], Albargi et al. [21] and Işık et al. [26] in the framework of $\mathcal{F}$-MS as a consequence. The practical utility of our theorems is illustrated through a concrete example.

2. Preliminaries

Czerwik [2] introduced the idea of b-MSs by altering the triangle inequality inherent in standard MSs.

For all $\zeta, \nu, \mathfrak{z} \in \Omega$ and for some $b \geq 1$,

$$d(\zeta, \mathfrak{z}) \leq b\left[d(\zeta, \nu) + d(\nu, \mathfrak{z})\right].$$

Jleli et al. [3] unveiled a captivating generalization of MS, known as $\mathcal{F}$-MS.

Let Ξ be the collection of functions $f : (0, +\infty) \to \mathbb{R}$ such that

($\mathfrak{F}_1$) for all $t_1, t_2 \in (0, +\infty)$ such that $t_1 < t_2 \Longrightarrow f(t_1) < f(t_2)$,
($\mathfrak{F}_2$) for $\{t_n\} \subseteq (0, +\infty)$, $\lim_{n\to\infty} t_n = 0$ and $\lim_{n\to\infty} f(t_n) = -\infty$ are equivalent.

Definition 1. [3] Let $\Omega \neq \emptyset$ and $d : \Omega\times\Omega \to [0, +\infty)$ be a distance function such that

(i) $d(\zeta, \mathfrak{z}) = 0 \Longleftrightarrow \zeta = \mathfrak{z}$,
(ii) $d(\zeta, \mathfrak{z}) = d(\mathfrak{z}, \zeta)$,
(iii) for every $(u_\iota)_{\iota=1}^{p} \subset \Omega$ with $(u_1, u_p) = (\zeta, \mathfrak{z})$, we have

$$d(\zeta, \mathfrak{z}) > 0 \Rightarrow f(d(\zeta, \mathfrak{z})) \leq f\left(\sum_{\iota=1}^{p-1} d(u_\iota, u_{\iota+1})\right) + \beta,$$

for all $(\zeta, \mathfrak{z}) \in \Omega\times\Omega$ and for $p \in \mathbb{N}$ with $p \geq 2$. If there exists a combination $(f,\beta) \in \Xi \times [0, +\infty)$ satisfying the aforementioned properties, then d is referred to as an $\mathcal{F}$- metric on Ω, and the combination (Ω, d) is termed an $\mathcal{F}$-MS.

In the setting of MS, Stefan Banach [4] gave the concept of contraction mapping in this way.

Definition 2. [4] Let (Ω, d) be a MS. A mapping $S : \Omega \to \Omega$ is said to be a contraction mapping if there exists some constant $\lambda \in [0, 1)$ such that

$$d(S\zeta, S\mathfrak{z}) \leq \lambda d(\zeta, \mathfrak{z})$$

for all $\zeta, \mathfrak{z} \in \Omega$.

Theorem 1. *[4] Let (Ω, d) be a complete MS and $S : \Omega \to \Omega$ be a contraction mapping, then S has a unique FP.*

Wardowski [5] introduced a novel contraction concept, known as Υ-contractions, which expanded upon the traditional contraction mapping framework. This innovative approach led to the development of innovative FP results within the context of complete MSs.

Definition 3. Denote by Ψ the collection of all continuous functions Υ from the positive real numbers to the real numbers satisfying the following conditions:

(Υ_1) for all $t_1, t_2 \in (0, +\infty)$ such that $t_1 < t_2 \Longrightarrow \Upsilon(t_1) < \Upsilon(t_2)$,
(Υ_2) for $\{t_n\} \subseteq (0, +\infty)$, $\lim_{n\to\infty} t_n = 0$ if and only if $\lim_{n\to\infty} \Upsilon(t_n) = -\infty$,
(Υ_3) there exists $k \in (0, 1)$ such that $\lim_{t\to 0^+} t^k \Upsilon(t) = 0$.

A self mapping $S : \Omega \to \Omega$ is termed as an Υ-contraction if there is a function $\Upsilon \in \Psi$ fulfilling (Υ_1)–(Υ_3) and a positive constant $\mathfrak{w} > 0$ such that

$$d(S\zeta, S\mathfrak{z}) > 0 \Longrightarrow \mathfrak{w} + \Upsilon(d(S\zeta, S\mathfrak{z})) \leq \Upsilon(d(\zeta, \mathfrak{z}))$$

for all $\zeta, \mathfrak{z} \in \Omega$.

Theorem 2. *[5] If (Ω, d) is a complete MS and $S : \Omega \to \Omega$ be a mapping fulfilling the criteria of an Υ-contraction, then S has a unique FP in Ω.*

The subsequent discussion employs the following notation. Let $\mathcal{K}(\Omega)$ denote the set of all non-empty compact subsets of Ω and $\mathcal{CB}(\Omega)$ the set of all non-empty, closed, and bounded subsets of Ω. Given $\mathcal{U}, \mathcal{V} \in \mathcal{CB}(\Omega)$, define

$$H(\mathcal{U}, \mathcal{V}) = \max\left\{\sup_{\zeta\in\mathcal{U}} D(\zeta, \mathcal{V}), \sup_{\mathfrak{z}\in\mathcal{V}} D(\mathfrak{z}, \mathcal{U})\right\},$$

where $D(\zeta, \mathcal{V}) = \inf_{\mathfrak{z}\in\mathcal{V}} d(\zeta, \mathfrak{z})$. The H is called Pompeiu-Hausdorff metric with respect to d.

Nadler [6] pioneered the concept of set-valued contraction in MSs in such a manner.

Definition 4. [6] Let (Ω, d) be a MS. A mapping $S : \Omega \to CB(\Omega)$ is termed a set-valued contraction if there exists a constant $\lambda \in [0, 1)$ satisfying

$$H(S\zeta, S\mathfrak{z}) \leq \lambda d(\zeta, \mathfrak{z})$$

for all $\zeta, \mathfrak{z} \in \Omega$.

Theorem 3. *[6] Let (Ω, d) be a complete MS and $S : \Omega \to CB(\Omega)$ be a set-valued contraction, then S has a FP.*

Berinde et al. [7] gave the idea of set-valued almost contraction in MSs in this fashion.

Definition 5. [7] Let (Ω, d) be a MS. A mapping $S : \Omega \to CB(\Omega)$ is said to be a set-valued almost contraction if there exist some constants $\lambda \in (0, 1)$ and $L \geq 0$ such that

$$H(S\zeta, S\mathfrak{z}) \leq \lambda d(\zeta, \mathfrak{z}) + LD(\mathfrak{z}, S\zeta),$$

for all $\zeta, \mathfrak{z} \in \Omega$.

Theorem 4. *[7] Let (Ω, d) be a complete MS and $S : \Omega \to CB(\Omega)$ be a set-valued almost contraction, then S possesses a FP.*

Later on, Altun et al. [8] added the following condition (Υ_4) with (Υ_1)–(Υ_3) and defined the concept of set-valued Υ-contraction in this way.

(Υ_4) $\Upsilon(\inf \mathcal{U}) = \inf \Upsilon(\mathcal{U})$ for all $\mathcal{U} \subset \mathbb{R}^+$.

Let Φ represent the collection of all functions Υ pleasing the axioms (Υ_1)–(Υ_4). Clearly, Φ is a subset of Ψ. Examples of functions within Φ include

- $\Upsilon_1(t) = \ln t$, for positive t,
- $\Upsilon_2(t) = t + \ln t$, for positive t,
- $\Upsilon_3(t) = -\frac{1}{\sqrt{t}}$, for positive t.

If we define $\Upsilon_4(t) = \ln t$ for $t \leq 1$ and $\Upsilon_4(t) = t$ for $t > 1$, then $\Upsilon_4 \in \Psi \backslash \Phi$.

Definition 6. [8] Consider a MS (Ω, d) and $S : \Omega \to \mathcal{CB}(\Omega)$. We define S as a set-valued Υ-contraction if there exists a positive number $\mathfrak{w} > 0$ and a function $\Upsilon \in \Phi$ such that for all $\zeta, \mathfrak{z} \in \Omega$,

$$H(S\zeta, S\mathfrak{z}) > 0 \Rightarrow \mathfrak{w} + \Upsilon(H(S\zeta, S\mathfrak{z})) \leq \Upsilon(d(\zeta, \mathfrak{z})).$$

Theorem 5. *[8] Let (Ω, d) be a complete MS and $S : \Omega \to C\mathcal{B}(\Omega)$ be a set-valued Υ-contraction, then S has a FP.*

Altun et al. [9] also introduced the notion of set-valued almost Υ-contraction as follows:

Definition 7. [9] Let (Ω, d) be a MS. A mapping $S : \Omega \to C\mathcal{B}(\Omega)$ is termed a set-valued almost Υ-contraction if there exist some constants $\mathfrak{w} > 0$, $L \geq 0$ and a function $\Upsilon \in \Phi$ such that for all $\zeta, \mathfrak{z} \in \Omega$,

$$H(S\zeta, S\mathfrak{z}) > 0 \Rightarrow \mathfrak{w} + \Upsilon(H(S\zeta, S\mathfrak{z})) \leq \Upsilon(d(\zeta, \mathfrak{z}) + LD(\mathfrak{z}, S\zeta)).$$

Theorem 6. *[9] Let (Ω, d) be a complete MS and $S : \Omega \to C\mathcal{B}(\Omega)$ be a set-valued almost Υ-contraction, then S has a FP.*

Consistent with the approach of Sadiq Basha [13], consider $\mathcal{U}$ and $\mathcal{V}$ as non-empty subsets of the MS Ω. We introduce the following subsets of $\mathcal{U}$ and $\mathcal{V}$:

$$\begin{aligned} \mathcal{U}_0 &= \{\zeta \in \mathcal{U} : \text{there exists } \mathfrak{z} \in \mathcal{V} \text{ such that } d(\zeta, \mathfrak{z}) = dist(\mathcal{U}, \mathcal{V})\}, \\ \mathcal{V}_0 &= \{\mathfrak{z} \in \mathcal{V} : \text{there exists } \zeta \in \mathcal{U} \text{ such that } d(\zeta, \mathfrak{z}) = dist(\mathcal{U}, \mathcal{V})\}. \end{aligned}$$

Definition 8. [13] Let $\mathcal{U}$ and $\mathcal{V}$ are non-empty subsets of MS Ω. If

$$\left.\begin{array}{l} d(\zeta_1, \mathfrak{z}_1) = dist(\mathcal{U}, \mathcal{V}) \\ d(\zeta_2, \mathfrak{z}_2) = dist(\mathcal{U}, \mathcal{V}) \end{array}\right\} \Rightarrow d(\zeta_1, \zeta_2) = d(\mathfrak{z}_1, \mathfrak{z}_2)$$

for all $\zeta_1, \zeta_2 \in \mathcal{U}$ and $\mathfrak{z}_1, \mathfrak{z}_2 \in \mathcal{V}$, then the combination $(\mathcal{U}, \mathcal{V})$ is said to satisfy the P-property. Additionally, $(\mathcal{U}, \mathcal{V})$ is said to satisfy the weak P-property if for all $\zeta_1, \zeta_2 \in \mathcal{U}$ and $\mathfrak{z}_1, \mathfrak{z}_2 \in \mathcal{V}$, the following holds

$$\left.\begin{array}{l} d(\zeta_1, \mathfrak{z}_1) = dist(\mathcal{U}, \mathcal{V}) \\ d(\zeta_2, \mathfrak{z}_2) = dist(\mathcal{U}, \mathcal{V}) \end{array}\right\} \Rightarrow d(\zeta_1, \zeta_2) \leq d(\mathfrak{z}_1, \mathfrak{z}_2).$$

It is well known that in a MS, every pair $(\mathcal{U}, \mathcal{V})$ satisfy P-property also satisfy the weak P-property.

Abkar et al. [16] presented an elegant study of the following result concerning pairs of sets satisfying the P-property.

Theorem 7. *[16] Given two non-empty, closed subsets $\mathcal{U}$ and $\mathcal{V}$ embedded in the complete MS (Ω, d) such that $\mathcal{U}_0 \neq \emptyset$ and the combination $(\mathcal{U}, \mathcal{V})$ satisfies P-property. Given a set-valued contraction $S : \mathcal{U} \to C\mathcal{B}(\mathcal{V})$ with the property that $S\zeta \subseteq \mathcal{V}_0$ for all $\zeta \in \mathcal{U}_0$. Then S has a BPP.*

The subsequent theorem establishes the existence of a BPP for set-valued Υ-contractions, as explored by Debnath [17].

Theorem 8. *[17] Given two non-empty, closed subsets $\mathcal{U}$ and $\mathcal{V}$ embedded in the complete MS (Ω, d) such that $\mathcal{U}_0 \neq \emptyset$ and the combination $(\mathcal{U}, \mathcal{V})$ satisfies weak P-property. Suppose $S : \mathcal{U} \to C\mathcal{B}(\mathcal{V})$ be a set-valued Υ-contraction such that $S\zeta \subseteq \mathcal{V}_0$ for all $\zeta \in \mathcal{U}_0$. Then S has a BPP.*

Patel et al. [18] established BPP theorems for set-valued almost Υ-contractions in the context of MSs.

Theorem 9. *[18] Given two non-empty, closed subsets $\mathcal{U}$ and $\mathcal{V}$ embedded in the complete MS (Ω, d) such that $\mathcal{U}_0 \neq \emptyset$ and the combination $(\mathcal{U}, \mathcal{V})$ satisfies weak P-property. Assume that $S : \mathcal{U} \to C\mathcal{B}(\mathcal{V})$ be a set-valued almost Υ-contraction such that $S\zeta \subseteq \mathcal{V}_0, \forall\ \zeta \in \mathcal{U}_0$. Then S has a BPP.*

Lemma 1. *[26] Let (Ω, d) be $\mathcal{F}$-MS and $\mathcal{U}, \mathcal{V} \in C\mathcal{B}(\Omega)$. Then*

(1) $D(\zeta, \mathcal{V}) \leq d(\zeta, \mathfrak{z})$ for any $\mathfrak{z} \in \mathcal{V}$ and $\zeta \in \mathcal{U}$,
(2) $D(\zeta, \mathcal{V}) \leq H(\mathcal{U}, \mathcal{V})$ for any $\zeta \in \mathcal{U}$,
(3) if $\mathcal{V}$ is compact, thenfor any $\zeta \in \mathcal{U}$, there exists a point $\mathfrak{z} \in \mathcal{V}$ such that $d(\zeta, \mathfrak{z}) \leq H(\mathcal{U}, \mathcal{V})$.

In the context of $\mathcal{F}$-MSs, Lateef [20] established BPP result for contraction mappings.

Theorem 10. *[20] Given two non-empty, closed subsets $\mathcal{U}$ and $\mathcal{V}$ embedded in the $\mathcal{F}$-complete $\mathcal{F}$-MS (Ω, d) such that $\mathcal{U}_0 \neq \emptyset$ and the combination $(\mathcal{U}, \mathcal{V})$ satisfies P-property. Suppose $S : \mathcal{U} \to \mathcal{V}$ be a contraction mapping such that $S(\mathcal{U}_0) \subseteq \mathcal{V}_0$. Then S has a BPP.*

Albargi et al. [21] demonstrated the existence of BPPs for Υ-contractions within the setting of $\mathcal{F}$-MSs.

Theorem 11. *[21] Given two non-empty, closed subsets $\mathcal{U}$ and $\mathcal{V}$ embedded in the $\mathcal{F}$-complete $\mathcal{F}$-MS (Ω, d) such that $\mathcal{U}_0 \neq \emptyset$ and the combination $(\mathcal{U}, \mathcal{V})$ satisfies P-property. Suppose $S : \mathcal{U} \to \mathcal{V}$ be an Υ-contraction such that $S(\mathcal{U}_0) \subseteq \mathcal{V}_0$. Then S has a BPP.*

3. Main results

Definition 9. Given non-empty closed sets $\mathcal{U}$ and $\mathcal{V}$ in an $\mathcal{F}$-MS (Ω, d). A set-valued mapping $S : \mathcal{U} \to 2^{\mathcal{V}}$ is said to be set-valued almost Υ-contraction if there exists a function $\Upsilon \in \Psi/\Phi$ and the positive numbers $\mathfrak{w} > 0$ and $L \geq 0$ such that

$$H(S\zeta, S\mathfrak{z}) > 0 \Rightarrow \mathfrak{w} + \Upsilon(H(S\zeta, S\mathfrak{z})) \leq \Upsilon(d(\zeta, \mathfrak{z}) + L(D(\mathfrak{z}, S\zeta) - dist(\mathcal{U}, \mathcal{V}))) \tag{3.1}$$

for all $\zeta, \mathfrak{z} \in \mathcal{U}$.

If $\Upsilon(t) = \ln t$ for positive t and $\mathcal{U} = \mathcal{V}$, then every set-valued almost contraction is a set-valued almost Υ-contraction.

Theorem 12. *Given two non-empty, closed subsets $\mathcal{U}$ and $\mathcal{V}$ embedded in the $\mathcal{F}$-complete $\mathcal{F}$-MS (Ω, d) such that $\mathcal{U}_0 \neq \emptyset$ and the combination $(\mathcal{U}, \mathcal{V})$ satisfies weak P-property. Assume that $S : \mathcal{U} \to \mathcal{K}(\mathcal{V})$ be a set-valued almost Υ-contraction such that $S\zeta \subseteq \mathcal{V}_0, \forall \zeta \in \mathcal{U}_0$. Then S has a BPP in $\mathcal{U}$.*

Proof. Let $\epsilon > 0$ be fixed and $(f, \beta) \in \mathcal{F} \times [0, +\infty)$ be such that the condition (*iii*) of Definition 1 is satisfied. By $(\mathfrak{F}_2)$, there exists $\delta > 0$ such that

$$0 < t < \delta \Longrightarrow f(t) < f(\varepsilon) - \beta. \tag{3.2}$$

Let ζ_0 be an arbitrary element of $\mathcal{U}_0$ and $\mathfrak{z}_0 \in S\zeta_0 \subseteq \mathcal{V}_0$. Using the definition of $\mathcal{V}_0$, we can choose $\zeta_1 \in \mathcal{U}_0$ such that $d(\zeta_1, \mathfrak{z}_0) = dist(\mathcal{U}, \mathcal{V})$. If $\mathfrak{z}_0 \in S\zeta_1$, then □

$$dist(\mathcal{U},\mathcal{V}) \le D(\zeta_1, S\zeta_1) \le d(\zeta_1, \mathfrak{z}_0) = dist(\mathcal{U},\mathcal{V}).$$

Hence $D(\zeta_1, S\zeta_1) = dist(\mathcal{U},\mathcal{V})$, so ζ_1 is a BPP of S. Now assume that $\mathfrak{z}_0 \notin S\zeta_1$. By the hypothesis that $S\zeta_1$ is compact, we can find an element $\mathfrak{z}_1$ in $S\zeta_1$ such that

$$0 < d(\mathfrak{z}_0, \mathfrak{z}_1) \le H(S\zeta_0, S\zeta_1).$$

Combining condition (Υ_1) with the previous inequality yields

$$\begin{aligned}\Upsilon(d(\mathfrak{z}_0, \mathfrak{z}_1)) &\le \Upsilon(H(S\zeta_0, S\zeta_1))\\ &\le \Upsilon(d(\zeta_0, \zeta_1) + L(D(\zeta_1, S\zeta_0) - dist(\mathcal{U},\mathcal{V}))) - \mathfrak{w}\\ &= \Upsilon(d(\zeta_0, \zeta_1)) - \mathfrak{w}. \end{aligned} \tag{3.3}$$

Since $\mathfrak{z}_1 \in S\zeta_1 \subseteq \mathcal{V}_0$, there exists $\zeta_2 \in \mathcal{U}_0$ such that

$$d(\zeta_2, \mathfrak{z}_1) = dist(\mathcal{U},\mathcal{V}).$$

Therefore, by using weak P-property, we have

$$d(\zeta_1, \zeta_2) \le d(\mathfrak{z}_0, \mathfrak{z}_1). \tag{3.4}$$

Form the inequalities (3.3) and (3.4), we obtain

$$\Upsilon(d(\zeta_1, \zeta_2)) \le \Upsilon(d(\mathfrak{z}_0, \mathfrak{z}_1)) \le \Upsilon(d(\zeta_0, \zeta_1)) - \mathfrak{w}. \tag{3.5}$$

If $\mathfrak{z}_1 \in S\zeta_2$, then

$$dist(\mathcal{U},\mathcal{V}) \le D(\zeta_2, S\zeta_2) \le d(\zeta_2, \mathfrak{z}_1) = dist(\mathcal{U},\mathcal{V}).$$

Hence $D(\zeta_2, S\zeta_2) = dist(\mathcal{U},\mathcal{V})$, so ζ_2 is a BPP of S. On the other hand, if $\mathfrak{z}_1 \notin S\zeta_2$, then by the assumption that $S\zeta_2$ is compact, there exists an element $\mathfrak{z}_2$ in $S\zeta_2$ such that

$$0 < d(\mathfrak{z}_1, \mathfrak{z}_2) \le H(S\zeta_1, S\zeta_2).$$

By (Υ_1), we have

$$\begin{aligned}\Upsilon(d(\mathfrak{z}_1, \mathfrak{z}_2)) &\le \Upsilon(H(S\zeta_1, S\zeta_2))\\ &\le \Upsilon(d(\zeta_1, \zeta_2) + L(D(\zeta_2, S\zeta_1) - dist(\mathcal{U},\mathcal{V}))) - \mathfrak{w}\\ &= \Upsilon(d(\zeta_1, \zeta_2)) - \mathfrak{w}\\ &\le \Upsilon(d(\zeta_0, \zeta_1)) - 2\mathfrak{w}. \end{aligned} \tag{3.6}$$

Since $\mathfrak{z}_2 \in S\zeta_2 \subseteq \mathcal{V}_0$, there exists $\zeta_3 \in \mathcal{U}_0$ such that

$$d(\zeta_3, \mathfrak{z}_2) = dist(\mathcal{U},\mathcal{V}).$$

Consequently, applying the weak P-property, we obtain

$$d(\zeta_2, \zeta_3) \le d(\mathfrak{z}_1, \mathfrak{z}_2). \tag{3.7}$$

Form the inequalities (3.6) and (3.7), we obtain

$$\Upsilon(d(\zeta_2,\zeta_3)) \le \Upsilon(d(\mathfrak{z}_1,\mathfrak{z}_2)) \le \Upsilon(d(\zeta_0,\zeta_1)) - 2\mathfrak{w}. \tag{3.8}$$

Proceeding inductively, we construct two sequences $\{\zeta_n\}$ and $\{\mathfrak{z}_n\}$ in $\mathcal{U}_0$ and $\mathcal{V}_0$, respectively, which satisfy $\mathfrak{z}_n \in S\zeta_n \subseteq \mathcal{V}_0$ and $d(\zeta_{n+1},\mathfrak{z}_n) = dist(\mathcal{U},\mathcal{V}), \forall\, n \in \mathbb{N}$. Also,

$$\Upsilon(d(\zeta_n,\zeta_{n+1})) \le \Upsilon(d(\mathfrak{z}_{n-1},\mathfrak{z}_n)) \le \Upsilon(d(\zeta_0,\zeta_1)) - n\mathfrak{w}. \tag{3.9}$$

Then $d(\zeta_n,\zeta_{n+1}) > 0, \forall\, n \in \mathbb{N}$. By evaluating the limit as $n \to \infty$, we obtain $\lim_{n\to\infty} \Upsilon(d(\zeta_n,\zeta_{n+1})) = -\infty$. Applying (Υ_2), we obtain

$$\lim_{n\to\infty} d(\zeta_n,\zeta_{n+1}) = 0.$$

Forthwith, using (Υ_3), there exist $k \in (0,1)$ such that

$$\lim_{n\to\infty} d(\zeta_n,\zeta_{n+1})^k\, \Upsilon(d(\zeta_n,\zeta_{n+1})) = 0. \tag{3.10}$$

From the inequality (3.9), for each $n \in \mathbb{N}$, we have

$$d(\zeta_n,\zeta_{n+1})^k\, \Upsilon(d(\zeta_n,\zeta_{n+1})) - d(\zeta_n,\zeta_{n+1})^k\, \Upsilon(d(\zeta_0,\zeta_1)) \le -nd(\zeta_n,\zeta_{n+1})^k\, \mathfrak{w} \le 0. \tag{3.11}$$

Taking $n \to \infty$ and using (3.10), we have

$$\lim_{n\to\infty} nd(\zeta_n,\zeta_{n+1})^k = 0.$$

Therefore, there exists a positive integer n_1 such that $nd(\zeta_n,\zeta_{n+1})^k \le 1$ for every $n \ge n_1$. This implies that

$$d(\zeta_n,\zeta_{n+1}) \le \frac{1}{n^{\frac{1}{k}}}, \tag{3.12}$$

which yields

$$\sum_{i=n}^{m-1} d(\zeta_i,\zeta_{i+1}) \le \sum_{i=n}^{m-1} \frac{1}{i^{1/k}}$$

for $m > n \ge n_1$. Since $\sum_{i=1}^{\infty} \frac{1}{i^{1/k}}$ is a convergent series; there exists $n_2 \in \mathbb{N}$ such that

$$0 < \sum_{i=n}^{m-1} \frac{1}{i^{1/k}} < \sum_{i=1}^{\infty} \frac{1}{i^{1/k}} < \delta, \quad m > n \ge n_2. \tag{3.13}$$

Hence, by inequalities (3.13) and ($\mathfrak{F}_1$), we have

$$f\left(\sum_{i=n}^{m-1} d(\zeta_i,\zeta_{i+1})\right) \le f\left(\sum_{i=1}^{\infty} \frac{1}{i^{1/k}}\right) < f(\varepsilon) - \beta \tag{3.14}$$

for $m > n \ge n_2$. Employing the condition (*iii*) of Definition 1 and the inequality (3.14), we obtain

$$d(\zeta_n,\zeta_m) > 0,\ m > n > n_2 \implies f(d(\zeta_n,\zeta_m)) \le f\left(\sum_{i=n}^{m-1} d(\zeta_i,\zeta_{i+1})\right) + \beta < f(\epsilon).$$

Condition ($\mathfrak{F}_1$) implies that the distance among consecutive terms, $d(\zeta_n, \zeta_m) < \epsilon$, for $m > n > n_2$. Consequently, the sequence $\{\zeta_n\}$ is Cauchy within $\mathcal{U}_0$, a subset of $\mathcal{U}$. Given the completeness of (Ω, d) and the closedness of $\mathcal{U}$, there exists an element $\zeta^* \in \mathcal{U}$ such that the sequence $\{\zeta_n\}$ is convergent to ζ^*, that is,

$$\lim_{n\to\infty} d(\zeta_n, \zeta^*) = 0. \tag{3.15}$$

Repeating the above process, using the inequality (3.9), standard proof techniques establish that the sequence $\{\mathfrak{z}_n\}$ is Cauchy within $\mathcal{V}$. The completeness of $\mathcal{V}$, guarantees the existence of an element $\mathfrak{z}^* \in \mathcal{V}$ such that the sequence $\{\mathfrak{z}_n\}$ converges to $\mathfrak{z}^*$. Using the fact, $d(\zeta_{n+1}, \mathfrak{z}_n) = dist(\mathcal{U}, \mathcal{V})$, $\forall\, n \in \mathbb{N}$, we have

$$\lim_{n\to\infty} d(\zeta_{n+1}, \mathfrak{z}_n) = d(\zeta^*, \mathfrak{z}^*) = dist(\mathcal{U}, \mathcal{V}).$$

By (3.1), we have

$$\begin{aligned} \Upsilon(H(S\zeta_n, S\zeta^*)) &\leq \Upsilon(d(\zeta_n, \zeta^*) + L(D(\zeta^*, S\zeta_n) - dist(\mathcal{U}, \mathcal{V}))) - \mathfrak{w} \\ &< \Upsilon(d(\zeta_n, \zeta^*) + L(D(\zeta^*, S\zeta_n) - dist(\mathcal{U}, \mathcal{V}))), \end{aligned}$$

which implies by (Υ_1), we have

$$H(S\zeta_n, S\zeta^*) \leq d(\zeta_n, \zeta^*) + L(D(\zeta^*, S\zeta_n) - dist(\mathcal{U}, \mathcal{V})) - \mathfrak{w}. \tag{3.16}$$

Now by ($\mathfrak{F}_1$) and (3.16), we obtain

$$\begin{aligned} f(dist(\mathcal{U}, \mathcal{V})) \leq f(D(\zeta^*, S\zeta^*)) &\leq f(D(\zeta^*, S\zeta_n) + H(S\zeta_n, S\zeta^*)) \\ &\leq f(D(\zeta^*, S\zeta_n) + d(\zeta_n, \zeta^*) + L(D(\zeta^*, S\zeta_n) - dist(\mathcal{U}, \mathcal{V}))) \\ &\leq f((1+L)d(\zeta^*, \mathfrak{z}_n) + d(\zeta_n, \zeta^*) - Ldist(\mathcal{U}, \mathcal{V})). \end{aligned}$$

Taking limit $n \to \infty$, we obtain $D(\zeta^*, S\zeta^*) = dist(\mathcal{U}, \mathcal{V})$. Consequently, ζ^* is a BPP of S.

As a corollary to Theorem 12, we have the following result.

Corollary 1. *Given two non-empty, closed subsets $\mathcal{U}$ and $\mathcal{V}$ embedded in the $\mathcal{F}$-complete $\mathcal{F}$-MS (Ω, d) such that $\mathcal{U}_0 \neq \emptyset$ and the combination $(\mathcal{U}, \mathcal{V})$ satisfies P-property. Assume that $S : \mathcal{U} \to \mathcal{K}(\mathcal{V})$ be a set-valued almost Υ-contraction such that $S\zeta \subseteq \mathcal{V}_0, \forall \zeta \in \mathcal{U}_0$, then S has a BPP in $\mathcal{U}$.*

Corollary 2. *Given two non-empty, closed subsets $\mathcal{U}$ and $\mathcal{V}$ embedded in the $\mathcal{F}$-complete $\mathcal{F}$-MS (Ω, d) such that $\mathcal{U}_0 \neq \emptyset$ and the combination $(\mathcal{U}, \mathcal{V})$ satisfies weak P-property. Assume that $S : \mathcal{U} \to \mathcal{K}(\mathcal{V})$ be a set-valued Υ-contraction such that $S\zeta \subseteq \mathcal{V}_0, \forall \zeta \in \mathcal{U}_0$. Then S has a BPP in $\mathcal{U}$.*

Proof. Take $L = 0$ in the Theorem 12. □

A direct corollary of the preceding theorem is the following:

Corollary 3. *Given two non-empty, closed subsets $\mathcal{U}$ and $\mathcal{V}$ embedded in the $\mathcal{F}$-complete $\mathcal{F}$-MS (Ω, d) such that the set $\mathcal{U}_0$ is non-empty. If the combination $(\mathcal{U}, \mathcal{V})$ satisfies weak P-property (P-property) and $S : \mathcal{U} \to \mathcal{K}(\mathcal{V})$ is a set-valued almost contractions, that is*

$$H(S\zeta, S\mathfrak{z}) \leq \lambda d(\zeta, \mathfrak{z}) + L(D(\mathfrak{z}, S\zeta) - dist(\mathcal{U}, \mathcal{V})) \tag{3.17}$$

for some $\lambda \in [0, 1), L \geq 0$ and for all $\zeta, \mathfrak{z} \in \mathcal{U}$ provided $S\zeta \subseteq \mathcal{V}_0$ for all $\zeta \in \mathcal{U}_0$, then S admits a BPP in $\mathcal{U}$.

Proof. Considering $\Upsilon(t) = \ln t$, for $t > 0$, every contraction fulfilling (3.17), is a set-valued almost Υ-contraction. Therefore, in light of Theorem 12 (Corollary 1), S has a BPP in $\mathcal{U}$. □

Corollary 4. *Given two non-empty, closed subsets $\mathcal{U}$ and $\mathcal{V}$ embedded in the $\mathcal{F}$-complete $\mathcal{F}$-MS (Ω, d) such that the set $\mathcal{U}_0$ is non-empty. If the combination $(\mathcal{U}, \mathcal{V})$ satisfies weak P-property (P-property) and $S : \mathcal{U} \to \mathcal{K}(\mathcal{V})$ is a set-valued contraction, that is*

$$H(S\zeta, S\mathfrak{z}) \leq \lambda d(\zeta, \mathfrak{z})$$

for some $\lambda \in [0, 1)$ and for all $\zeta, \mathfrak{z} \in \mathcal{U}$ provided $S\zeta \subseteq \mathcal{V}_0$ for all $\zeta \in \mathcal{U}_0$, then S admits a BPP in $\mathcal{U}$.

Proof. Take $L = 0$ in the Corollary 3. □

Example 1. Let $\Omega = \mathbb{R}$ and $d(\zeta, \mathfrak{z}) = |\zeta - \mathfrak{z}|$ for all $\zeta, \mathfrak{z} \in \Omega$, then (Ω, d) is a complete $\mathcal{F}$-MS with $f(t) = \ln t$, for $t > 0$ and $\beta = \ln 1$. Given two closed subsets of Ω, $\mathcal{U} = [2, 3]$ and $\mathcal{V} = [-1, 0]$, it is evident that $dist(\mathcal{U}, \mathcal{V}) = d(2, 0) = 2$. Moreover, $\mathcal{U}_0 = \{2\}$ and $\mathcal{V}_0 = \{0\}$. Thus, the combination $(\mathcal{U}, \mathcal{V})$ adheres to the P-property and weak P-property. Define $S : \mathcal{U} \to \mathcal{CB}(\mathcal{V})$ as follows:

$$S\zeta = \begin{cases} \left\{-\frac{3}{4}\right\} & \text{if } \zeta \in \left(\frac{5}{2}, 3\right] \\ \{-1\} & \text{if } \zeta = \frac{5}{2} \\ \{0\} & \text{otherwise.} \end{cases}$$

Clearly, $S(\mathcal{U}_0) \subseteq \mathcal{V}_0$ and S is not the set-valued Υ-contraction. Define

$$\Upsilon(t) = \begin{cases} \ln t & \text{if } t \leq 1 \\ t & \text{otherwise.} \end{cases}$$

Subsequently, we demonstrate that the condition, for all $\zeta, \mathfrak{z} \in \mathcal{U}$ with $H(S\zeta, S\mathfrak{z}) > 0$, implies that

$$\mathfrak{w} + \Upsilon(H(S\zeta, S\mathfrak{z})) \leq \Upsilon(d(\zeta, \mathfrak{z}) + L(D(\mathfrak{z}, S\zeta) - dist(\mathcal{U}, \mathcal{V}))).$$

We have the following cases:

Case 1. Let $\zeta \in \left(\frac{5}{2}, 3\right]$ and $\mathfrak{z} = \frac{5}{2}$, then for $\mathfrak{w} = 1 + \ln 4, L \geq \frac{4}{5}$, we have

$$\mathfrak{w} + \Upsilon\left(\frac{1}{4}\right) \leq \left|\zeta - \frac{5}{2}\right| + L\left(\frac{5}{4}\right) = \Upsilon\left(\left|\zeta - \frac{5}{2}\right| + L\left(\frac{13}{4} - 2\right)\right).$$

Case 2. Let $\mathfrak{z} \in \left(\frac{5}{2}, 3\right]$ and $\zeta = \frac{5}{2}$, then for $\mathfrak{w} = 1 + \ln 4, L \geq \frac{2}{3}$, we have

$$\mathfrak{w} + \Upsilon\left(\frac{1}{4}\right) \leq \left|\mathfrak{z} - \frac{5}{2}\right| + L\left(\frac{3}{2}\right) = \Upsilon\left(\left|\mathfrak{z} - \frac{5}{2}\right| + L\left(\frac{7}{2} - 2\right)\right).$$

Case 3. Let $\zeta \in \left(\frac{5}{2}, 3\right]$ and $\mathfrak{z} \in \left[2, \frac{5}{2}\right)$, then for $\mathfrak{w} = 1 + \ln \frac{4}{3}, L \geq \frac{4}{3}$, we have

$$\mathfrak{w}+\Upsilon\left(\frac{3}{4}\right)\leq|\zeta-\mathfrak{z}|+L\left(\frac{3}{4}\right)=\Upsilon\left(|\zeta-\mathfrak{z}|+L\left(\frac{11}{4}-2\right)\right).$$

Case 4. Let $\mathfrak{z}\in\left(\frac{5}{2},3\right]$ and $\zeta\in\left[2,\frac{5}{2}\right)$, then for $\mathfrak{w}=1+\ln\frac{4}{3}, L\geq 2$, we have

$$\mathfrak{w}+\Upsilon\left(\frac{3}{4}\right)\leq|\zeta-\mathfrak{z}|+L(\frac{1}{2})=\Upsilon(|\zeta-\mathfrak{z}|+L(\frac{5}{2}-2)).$$

Case 5. Let $\zeta=\frac{5}{2}$ and $\mathfrak{z}\in\left[2,\frac{5}{2}\right)$ then, for $\mathfrak{w}=1, L\geq 1$, we have

$$\mathfrak{w}+\Upsilon(1)\leq\left|\mathfrak{z}-\frac{5}{2}\right|+L(1)=\Upsilon\left(\left|\mathfrak{z}-\frac{5}{2}\right|+L(3-2)\right).$$

Case 6. Let $\mathfrak{z}=\frac{5}{2}$ and $\zeta\in\left[2,\frac{5}{2}\right)$ then, for $\mathfrak{w}=1, L\geq 2$, we have

$$\mathfrak{w}+\Upsilon(1)\leq\left|\zeta-\frac{5}{2}\right|+L\left(\frac{1}{2}\right)=\Upsilon\left(\left|\zeta-\frac{5}{2}\right|+L\left(\frac{5}{2}-2\right)\right).$$

Hence, the mapping S fulfills the conditions of Theorem 12 and Corollary 1 for all $\zeta,\mathfrak{z}$ in $\mathcal{U}$, where $\mathfrak{w}$ lies within the interval $(0,1+\ln 4]$ and $L\geq 2$. Consequently, S possesses a BPP in $\mathcal{U}$, which is equal to 2. Notably, the mapping S exhibits discontinuous behavior.

By incorporating condition $(\mathfrak{F}_4)$ into Theorem 12, the compactness requirement of $\mathcal{K}(\mathcal{V})$ can be relaxed to the closedness and boundedness of $\mathcal{CB}(\mathcal{V})$. The following theorem explores this generalization.

Theorem 13. *Given two non-empty, closed subsets $\mathcal{U}$ and $\mathcal{V}$ embedded in the $\mathcal{F}$-complete $\mathcal{F}$-MS (Ω,d). If the combination $(\mathcal{U},\mathcal{V})$ satisfies weak P-property and $S:\mathcal{U}\to\mathcal{CB}(\mathcal{V})$ is a set-valued almost Υ-contraction with Υ belonging to Φ in such a way that $S\zeta\subseteq\mathcal{V}_0$, $\forall\,\zeta\in\mathcal{U}_0$, then S admits a BPP in $\mathcal{U}$.*

Proof. Let $\epsilon>0$ be fixed and $(f,\beta)\in\mathcal{F}\times[0,+\infty)$ be such that (iii) of Definition 1 is satisfied. By $(\mathfrak{F}_2)$, there exists $\delta>0$ such that

$$0<t<\delta\Longrightarrow f(t)<f(\varepsilon)-\beta. \tag{3.18}$$

Let $\zeta_0\in\mathcal{U}_0$ and $\mathfrak{z}_0\in S\zeta_0\subseteq\mathcal{V}_0$. Using the definition of $\mathcal{V}_0$, we can choose $\zeta_1\in\mathcal{U}_0$ such that $d(\zeta_1,\mathfrak{z}_0)=dist(\mathcal{U},\mathcal{V})$. If $\mathfrak{z}_0\in S\zeta_1$, then □

$$dist(\mathcal{U},\mathcal{V})\leq D(\zeta_1,S\zeta_1)\leq d(\zeta_1,\mathfrak{z}_0)=dist(\mathcal{U},\mathcal{V}).$$

Hence $D(\zeta_1,S\zeta_1)=dist(\mathcal{U},\mathcal{V})$, so ζ_1 is a BPP of S. Now assume that $\mathfrak{z}_0\notin S\zeta_1$. Since $S\zeta_1$ is closed, it follows that

$$0<D(\mathfrak{z}_0,S\zeta_1)\leq H(S\zeta_0,S\zeta_1).$$

By (Υ_1), we conclude that

$$\begin{aligned}\Upsilon(D(\mathfrak{z}_0,S\zeta_1))&\leq\Upsilon(H(S\zeta_0,S\zeta_1))\\&\leq\Upsilon(d(\zeta_0,\zeta_1)+L(D(\zeta_1,S\zeta_0)-dist(\mathcal{U},\mathcal{V})))-\mathfrak{w}\end{aligned}$$

$$= \Upsilon\left(d\left(\zeta_0, \zeta_1\right)\right) - \mathfrak{w}. \tag{3.19}$$

Note that $D\left(\mathfrak{z}_0, S\zeta_1\right) > 0$, so, by (Υ_4), we can write

$$\Upsilon\left(D\left(\mathfrak{z}_0, S\zeta_1\right)\right) = \inf_{\mathfrak{z}\in S\zeta_1} \Upsilon\left(D\left(\mathfrak{z}_0, \mathfrak{z}\right)\right).$$

Consequently, considering the aforementioned property and inequality (3.19), there exists an element $\mathfrak{z}_2 \in S\zeta_1$ such that

$$\Upsilon\left(d\left(\mathfrak{z}_0, \mathfrak{z}_1\right)\right) \leq \Upsilon\left(d\left(\zeta_0, \zeta_1\right)\right) - \mathfrak{w}. \tag{3.20}$$

Since $\mathfrak{z}_1 \in S\zeta_1 \subseteq \mathcal{V}_0$, there exists $\zeta_2 \in \mathcal{U}_0$ such that

$$d\left(\zeta_2, \mathfrak{z}_1\right) = dist(\mathcal{U}, \mathcal{V}).$$

Therefore, by using weak P-property, we have

$$d\left(\zeta_1, \zeta_2\right) \leq d\left(\mathfrak{z}_0, \mathfrak{z}_1\right). \tag{3.21}$$

Form the inequalities (3.20) and (3.21), we obtain

$$\Upsilon\left(d\left(\zeta_1, \zeta_2\right)\right) \leq \Upsilon\left(d\left(\mathfrak{z}_0, \mathfrak{z}_1\right)\right) \leq \Upsilon\left(d\left(\zeta_0, \zeta_1\right)\right) - \mathfrak{w}. \tag{3.22}$$

If $\mathfrak{z}_1 \in S\zeta_2$, then

$$dist(\mathcal{U}, \mathcal{V}) \leq D\left(\zeta_2, S\zeta_2\right) \leq d\left(\zeta_2, \mathfrak{z}_1\right) = dist(\mathcal{U}, \mathcal{V}).$$

Hence $D\left(\zeta_2, S\zeta_2\right) = dist(\mathcal{U}, \mathcal{V})$, so ζ_2 is a BPP of S. On the other hand, if $\mathfrak{z}_1 \notin S\zeta_2$. Since $S\zeta_2$ is closed, we obtain

$$0 < D\left(\mathfrak{z}_1, S\zeta_2\right) \leq H\left(S\zeta_1, S\zeta_2\right).$$

By (Υ_1), we have

$$\begin{aligned}
\Upsilon\left(D\left(\mathfrak{z}_1, S\zeta_2\right)\right) &\leq \Upsilon\left(H\left(S\zeta_1, S\zeta_2\right)\right) \\
&\leq \Upsilon\left(d\left(\zeta_1, \zeta_2\right) + L\left(D\left(\zeta_2, S\zeta_1\right) - dist(\mathcal{U}, \mathcal{V})\right)\right) - \mathfrak{w} \\
&= \Upsilon\left(d\left(\zeta_1, \zeta_2\right)\right) - \mathfrak{w} \\
&\leq \Upsilon\left(d\left(\zeta_0, \zeta_1\right)\right) - 2\mathfrak{w}.
\end{aligned} \tag{3.23}$$

Since $D\left(\mathfrak{z}_1, S\zeta_2\right) > 0$. Applying condition (Υ_4) to inequality (3.23), we can find $\mathfrak{z}_2 \in S\zeta_2$ such that

$$\Upsilon\left(d\left(\mathfrak{z}_1, \mathfrak{z}_2\right)\right) \leq \Upsilon\left(d\left(\zeta_0, \zeta_1\right)\right) - 2\mathfrak{w}. \tag{3.24}$$

Since $\mathfrak{z}_2 \in S\zeta_2 \subseteq \mathcal{V}_0$, there exists $\zeta_3 \in \mathcal{U}_0$ such that

$$d\left(\zeta_3, \mathfrak{z}_2\right) = dist(\mathcal{U}, \mathcal{V}).$$

By virtue of the weak P-property, it follows that

$$d(\zeta_2, \zeta_3) \leq d(\mathfrak{z}_1, \mathfrak{z}_2). \tag{3.25}$$

Form the inequalities (3.24) and (3.25), we obtain

$$\Upsilon(d(\zeta_2, \zeta_3)) \leq \Upsilon(d(\mathfrak{z}_1, \mathfrak{z}_2)) \leq \Upsilon(d(\zeta_0, \zeta_1)) - 2\mathfrak{w}. \tag{3.26}$$

Through successive steps, we generate sequences $\{\zeta_n\}$ and $\{\mathfrak{z}_n\}$ in $\mathcal{U}_0$ and $\mathcal{V}_0$, respectively, which satisfies $\mathfrak{z}_n \in S\zeta_n \subseteq \mathcal{V}_0$ and $d(\zeta_{n+1}, \mathfrak{z}_n) = dist(\mathcal{U}, \mathcal{V})$, $\forall\, n \in \mathbb{N}$. In addition

$$\Upsilon(d(\zeta_n, \zeta_{n+1})) \leq \Upsilon(d(\mathfrak{z}_{n-1}, \mathfrak{z}_n)) \leq \Upsilon(d(\zeta_0, \zeta_1)) - n\mathfrak{w}. \tag{3.27}$$

Then $d(\zeta_n, \zeta_{n+1}) > 0, \forall\, n \in \mathbb{N}$. By evaluating the limit as $n \to \infty$, we obtain $\lim_{n\to\infty} \Upsilon(d(\zeta_n, \zeta_{n+1})) = -\infty$. Applying (Υ_2), we obtain

$$\lim_{n\to\infty} d(\zeta_n, \zeta_{n+1}) = 0.$$

Now, using (Υ_3), there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} d(\zeta_n, \zeta_{n+1})^k\, \Upsilon(d(\zeta_n, \zeta_{n+1})) = 0. \tag{3.28}$$

From the inequality (3.27), for each $n \in \mathbb{N}$, we have

$$d(\zeta_n, \zeta_{n+1})^k\, \Upsilon(d(\zeta_n, \zeta_{n+1})) - d(\zeta_n, \zeta_{n+1})^k\, \Upsilon(d(\zeta_0, \zeta_1)) \leq -nd(\zeta_n, \zeta_{n+1})^k\, \mathfrak{w} \leq 0. \tag{3.29}$$

Taking $n \to \infty$ and using (3.28), we have

$$\lim_{n\to\infty} nd(\zeta_n, \zeta_{n+1})^k = 0.$$

Therefore, there exists a positive number n_1 provided that $nd(\zeta_n, \zeta_{n+1})^k \leq 1$ for every $n \geq n_1$. This implies for $n \geq n_1$,

$$d(\zeta_n, \zeta_{n+1}) \leq \frac{1}{n^{\frac{1}{k}}}, \tag{3.30}$$

which yields

$$\sum_{i=n}^{m-1} d(\zeta_i, \zeta_{i+1}) \leq \sum_{i=n}^{m-1} \frac{1}{i^{1/k}}$$

for $n < m$. Since $\sum_{i=1}^{\infty} \frac{1}{i^{1/k}}$ is a convergent series, then there exists $n_2 \in \mathbb{N}$ such that

$$0 < \sum_{i=n}^{m-1} \frac{1}{i^{1/k}} < \sum_{i=1}^{\infty} \frac{1}{i^{1/k}} < \delta, \quad m > n \geq n_2. \tag{3.31}$$

Hence, by inequality (3.31) and $(\mathfrak{F}_1)$, we have

$$f\left(\sum_{i=n}^{m-1} d(\zeta_i, \zeta_{i+1})\right) \leq f\left(\sum_{i=1}^{\infty} \frac{1}{i^{1/k}}\right) < f(\varepsilon) - \beta \tag{3.32}$$

for $m > n \geq n_2$. Employing the condition (*iii*) of Definition 1 and the inequality (3.32), we have

$$d(\zeta_n, \zeta_m) > 0,\ m > n > n_2 \implies f(d(\zeta_n, \zeta_m)) \leq f\left(\sum_{i=n}^{m-1} d(\zeta_i, \zeta_{i+1})\right) + \beta < f(\epsilon).$$

Condition $(\mathfrak{F}_1)$ ensures that the distance between any two distinct terms of the sequence $\{\zeta_n\}$ is smaller than ϵ for sufficiently large indices m and n and $d(\zeta_n, \zeta_m) < \epsilon$, for $m > n > n_2$. Consequently, $\{\zeta_n\}$ forms a Cauchy sequence within $\mathcal{U}_0$, which is a subset of $\mathcal{U}$. Given the completeness of (Ω, d) and the closedness of $\mathcal{U}$, there exists an element $\zeta^* \in \mathcal{U}$ such that $\{\zeta_n\}$ is convergent to ζ^*, that is,

$$\lim_{n\to\infty} d(\zeta_n, \zeta^*) = 0. \tag{3.33}$$

By iteratively applying the procedure outlined above and utilizing inequality (3.28), the sequence $\{\mathfrak{z}_n\}$ can be shown to be Cauchy within the $\mathcal{V}$ and by closeness of $\mathcal{V}$, $\exists \mathfrak{z}^* \in \mathcal{V}$ such that $\lim_{n\to\infty} \mathfrak{z}_n = \mathfrak{z}^*$. Given that the $d(\zeta_{n+1}, \mathfrak{z}_n) = dist(\mathcal{U}, \mathcal{V})$, $\forall\, n \in \mathbb{N}$, we have

$$\lim_{n\to\infty} d(\zeta_{n+1}, \mathfrak{z}_n) = d(\zeta^*, \mathfrak{z}^*) = dist(\mathcal{U}, \mathcal{V}).$$

By following the same procedure as in the proof of Theorem 12, we can conclude that ζ^* is a BPP of S.

Corollary 5. *Given two non-empty, closed subsets $\mathcal{U}$ and $\mathcal{V}$ embedded in the $\mathcal{F}$-complete $\mathcal{F}$-MS (Ω, d). If the combination $(\mathcal{U}, \mathcal{V})$ satisfies P-property and $S : \mathcal{U} \to \mathcal{CB}(\mathcal{V})$ is a set-valued almost Υ-contraction with Υ belonging to Φ in such a way that $S\zeta \subseteq \mathcal{V}_0$ for all $\zeta \in \mathcal{U}_0$, then S admits a BPP in $\mathcal{U}$.*

Corollary 6. *Given two non-empty, closed subsets $\mathcal{U}$ and $\mathcal{V}$ embedded in the $\mathcal{F}$-complete $\mathcal{F}$-MS (Ω, d) such that $\mathcal{U}_0 \neq \emptyset$ and the pair $(\mathcal{U}, \mathcal{V})$ satisfies weak P-property. Assume that $S : \mathcal{U} \to \mathcal{CB}(\mathcal{V})$ be a set-valued Υ-contraction such that $S\zeta \subseteq \mathcal{V}_0$ for all $\zeta \in \mathcal{U}_0$. Then S has a BPP in $\mathcal{U}$.*

Proof. Take $L = 0$ in Theorem 13. □

A special case of the preceding theorem yields the following result:

Corollary 7. *Given two non-empty, closed subsets $\mathcal{U}$ and $\mathcal{V}$ embedded in the $\mathcal{F}$-complete $\mathcal{F}$-MS (Ω, d) such that the set $\mathcal{U}_0$ is non-empty. If the combination $(\mathcal{U}, \mathcal{V})$ satisfies weak P-property (P-property) and $S : \mathcal{U} \to \mathcal{CB}(\mathcal{V})$ is a set-valued almost contractions, that is*

$$H(S\zeta, S\mathfrak{z}) \leq \lambda d(\zeta, \mathfrak{z}) + L(D(\mathfrak{z}, S\zeta) - dist(\mathcal{U}, \mathcal{V})) \tag{3.34}$$

for some $\lambda \in [0, 1), L \geq 0$ and for all $\zeta, \mathfrak{z} \in \mathcal{U}$ provided $S\zeta \subseteq \mathcal{V}_0$ for all $\zeta \in \mathcal{U}_0$, then S admits a BPP in $\mathcal{U}$.

Proof. By setting $\Upsilon(t) = \ln t$, for positive t, it is evident that any contraction fulfilling the conditions of (3.34) constitutes a set-valued almost Υ-contraction. Consequently, Theorem 13 (Corollary 6) guarantees the existence of a BPP for S in $\mathcal{U}$. □

Corollary 8. *Given two non-empty, closed subsets $\mathcal{U}$ and $\mathcal{V}$ embedded in the $\mathcal{F}$-complete $\mathcal{F}$-MS (Ω, d) such that the set $\mathcal{U}_0$ is non-empty. If the combination $(\mathcal{U}, \mathcal{V})$ satisfies weak P-property (P-property) and $S : \mathcal{U} \to \mathcal{CB}(\mathcal{V})$ is a set-valued contractions, that is*

$$H(S\zeta, S\mathfrak{z}) \leq \lambda d(\zeta, \mathfrak{z})$$

for some $\lambda \in [0, 1)$ and for all $\zeta, \mathfrak{z} \in \mathcal{U}$ provided $S\zeta \subseteq \mathcal{V}_0$, $\forall\, \zeta \in \mathcal{U}_0$, then S has a BPP in $\mathcal{U}$.

Proof. Take $L = 0$ in the Corollary 7. □

4. Connections to existing literature

4.1. Results in $\mathcal{F}$-metric spaces

(1) By assigning a unique element from set $\mathcal{V}$ to each element in set $\mathcal{U}$, the set-valued mapping $S : \mathcal{U} \to \mathcal{CB}(\mathcal{V})$ is reduced to a single-valued mapping $S : \mathcal{U} \to \mathcal{V}$. Moreover, setting $L = 0$ in Theorem 13 yields a result equivalent to that of Albargi et al. [21].

(2) Reducing $S : \mathcal{U} \to \mathcal{CB}(\mathcal{V})$ to $S : \mathcal{U} \to \mathcal{V}$, and subsequently setting $L = 0$ and choosing $\Upsilon(t) = \ln(t)$ for positive t in Theorem 13, leads to a result corresponding to the work of Lateef [20].

(3) Equating sets $\mathcal{U}$ and $\mathcal{V}$ as Ω reduces the concept of BPPs to that of FPs. Furthermore, reducing $S : \mathcal{U} \to \mathcal{CB}(\mathcal{V})$ to $S : \mathcal{U} \to \mathcal{U}$ and setting $L = 0$ recovers a primary result of Asif et al. [22] from Theorem 13.

(4) Considering $\mathcal{U} = \mathcal{V} = \Omega$, $L = 0$, and defining $\Upsilon(t) = \ln(t)$, for positive t yields a result equivalent to that of Işık et al. [26].

(5) By setting $\mathcal{U} = \mathcal{V} = \Omega$, $L = 0$, and defining $\Upsilon(t) = \ln(t)$, for positive t, while also reducing $S : \mathcal{U} \to \mathcal{CB}(\mathcal{V})$ to $S : \mathcal{U} \to \mathcal{V}$, we derive the principal theorem of Jleli et al. [3] as a special case of our main Theorem 13.

4.2. Results in metric spaces

In this subsection, considering $f(t) = \ln t$, for positive t and $\beta = \ln 1$ in Definition 1, the concept of $\mathcal{F}$-MS reduces to the classical MS. Consequently, we derive the following results:

(1) Our primary Theorem 13 directly implies the leading results of Patel et al. [18].

(2) Setting $L = 0$ in Theorem 13 recovers the principal result of Debnath [17].

(3) By taking $L = 0$ and $\Upsilon(t) = \ln t$, for positive t; the concept of set-valued Υ-contraction simplifies to set-valued contraction, yielding the leading result of Abkar et al. [16] as a corollary of Theorem 13.

(4) Equating sets $\mathcal{U}$ and $\mathcal{V}$ as Ω in Theorem 13, transforms the BPP problem into a FP problem, recovering the main result of Altun et al. [9].

(5) Setting $\mathcal{U} = \mathcal{V} = \Omega$ and $L = 0$ in Theorem 13 reproduces the theorem of Altun et al. [8].

(6) Choosing $\mathcal{U} = \mathcal{V} = \Omega$ and defining $\Upsilon(t) = \ln t$, for positive t, allows us to derive the main theorem of Berinde et al. [7] as a special case of our Theorem 13.

(7) By setting $\mathcal{U} = \mathcal{V} = \Omega$, $L = 0$ and defining $\Upsilon(t) = \ln t$, for positive t, we obtain the main theorem of Nadler [6] as a consequence of our Theorem 13.

(8) Finally, by setting $\mathcal{U} = \mathcal{V} = \Omega$, $L = 0$, and reducing $S : \mathcal{U} \to \mathcal{CB}(\mathcal{V})$ to $S : \mathcal{U} \to \mathcal{V}$, we recover the main theorem of Wardowski [5] from our Theorem 13, which is itself a generalization of BCP [4].

5. Conclusions

In the context of $\mathcal{F}$-MSs, a novel concept of set-valued almost Υ-contractions is presented in this paper and established the BPPs for the aforementioned contractions. By generalizing existing contraction conditions, our findings integrate and extend numerous distinguished results from the literature, including those of Banach [4], Wardowski [5], Nadler [6], Berinde et al. [7], Altun et al. [8, 9], Abkar et al. [16], Debnath [17], and Patel et al. [18] in the setting of classical MS. We also obtain the prime results of Jleli et al. [3], Asif et al. [22], Lateef [20], Albargi et al. [21] and Işık et al. [26] in the framework of $\mathcal{F}$-MS as a consequence. A concrete example demonstrates the practical application of our theorems.

The outcomes of this study pave the way for future investigations into the BPP theory for fuzzy mappings in the context of $\mathcal{F}$-MSs. Moreover, exploring the potential of our results in addressing real-world problems such as fractional and ordinary differential equations is a promising avenue for further research. Additionally, examining the interplay between our findings and the emerging field of orthogonal $\mathcal{F}$-MSs offers exciting opportunities for future advancements.

Acknowledgments

The author would like to thank the editor and the referees for their careful reading and helpful comments.

Conflict of interest

The author declares that he has no conflicts of interest.

References

1. M. Frechet, Sur quelques points du calcul fonctionnel, *Rendiconti del Circolo Matematico di Palermo*, **22** (1906), 1–72.
2. S. Czerwik, Contraction mappings in *b*-metric spaces, *Acta Math. Inform. Univ. Ostra.*, **1** (1993), 5–11.
3. M. Jleli, B. Samet, On a new generalization of metric spaces, *J. Fixed Point Theory Appl.*, **20** (2018), 128. https://doi.org/10.1007/s11784-018-0606-6
4. S. Banach, Sur les operations dans les ensembles abstraits et leur applications aux equations integrales, *Fundam. Math.*, **3** (1922), 133–181.
5. D. Wardowski, Fixed point theory of a new type of contractive mappings in complete metric spaces, *Fixed Point Theory Appl.*, **2012** (2012), 94. https://doi.org/10.1186/1687-1812-2012-94
6. Jr. S. B. Nadler, Multi-valued contraction mappings, *Pacific J. Math.*, **30** (1969), 475–478.
7. M. Berinde, V. Berinde, On a general class of multivalued weakly Picard mappings, *J. Math. Anal. Appl.*, **326** (2007), 772–782. https://doi.org/10.1016/j.jmaa.2006.03.016

8. I. Altun, G. Minak, H. Dag, Multivalued $\mathcal{F}$-contractions on complete metric spaces, *J. Nonlinear Convex Anal.*, **16** (2015), 659–666.

9. I. Altun, G. Durmaz, G. Mınak, S. Romaguera, Multivalued almost F-contractions on complete metric spaces, *Filomat*, **30** (2016), 441–448. https://doi.org/10.2298/FIL1602441A

10. B. Ali, H. A. Butt, M. De la Sen, Existence of fixed points of generalized set-valued F-contractions of b-metric spaces, *AIMS Math.*, **7** (2022), 17967–17988. https://doi.org/10.3934/math.2022990

11. L. B. Ćirić, Generalized contractions and fixed point theorems, *Publ. Inst. Math.*, **12** (1971), 19–26.

12. K. Fan, Extensions of two fixed point theorems of F. E. Brower, In: *Fleischman, W.M. (eds) Set-Valued Mappings, Selections and Topological Properties of 2x, Lecture Notes in Mathematics, Springer, Berlin, Heidelberg*. https://doi.org/10.1007/BFb0069713

13. S. S. Basha, Extensions of Banach's contraction principle, *Numer. Func. Anal. Opt.*, **31** (2010), 569–576. https://doi.org/10.1080/01630563.2010.485713

14. M. Omidvari, S. M. Vaezpour, R. Saadati, Best proximity point theorems for F-contractive non-self mappings, *Miskolc Math. Notes*, **15** (2014), 615–623. https://doi.org/10.18514/MMN.2014.1011

15. H. Şahin, A new kind of F-contraction and some best proximity point results for such mappings with an application, *Turk. J. Math.*, **46** (2022), 2151–2166. https://doi.org/10.55730/1300-0098.3260

16. A. Abkar, M. Gabeleh, The existence of best proximity points formultivalued non self mappings, *RACSAM*, **107** (2013), 319–325. https://doi.org/10.1007/s13398-012-0074-6

17. P. Debnath, Optimization through best proximity points for multivalued $\mathcal{F}$-contractions, *Miskolc Math. Notes*, **22** (2021), 143–151. https://doi.org/10.18514/MMN.2021.3355

18. D. K. Patel, Bhupeshwar, Finding the best proximity point of generalized multivalued contractions with applications, *Numer. Func. Anal. Opt.*, **44** (2023), 1602–1627. https://doi.org/10.1080/01630563.2023.2267294

19. M. De La Sen, M. Abbas, N. Saleem, On optimal fuzzy best proximity coincidence points of proximal contractions involving cyclic mappings in non-archimedean fuzzy metric spaces, *Mathematics,* **5** (2017), 22. https://doi.org/10.3390/math5020022

20. D. Lateef, Best proximity points in $\mathcal{F}$-metric spaces with applications, *Demonstratio Math.*, **56** (2023), 1–14. https://doi.org/10.1515/dema-2022-0191

21. A. H. Albargi, J. Ahmad, Integral equations: New solutions via generalized best proximity methods, *Axioms*, **13** (2024), 467. https://doi.org/10.3390/axioms13070467

22. A. Asif, M. Nazam, M. Arshad, S. O. Kim, $\mathcal{F}$-metric, F-contraction and common fixed-point theorems with applications, *Mathematics*, **7** (2019), 586. https://doi.org/10.3390/math7070586

23. A. Bera, H. Garai, B. Damjanović, A. Chanda, Some interesting results on $\mathcal{F}$-metric spaces, *Filomat*, **33** (2019), 3257–3268. https://doi.org/10.2298/FIL1910257B

24. D. Lateef, J. Ahmad, Dass and Gupta's fixed point theorem in $\mathcal{F}$-metric spaces, *J. Nonlinear Sci. Appl.*, **12** (2019), 405–411. https://doi.org/10.22436/jnsa.012.06.06

25. A. Hussain, H. Al-Sulami, N. Hussain, H. Farooq, Newly fixed disc results using advanced contractions on $\mathcal{F}$-metric space, *J. Appl. Anal. Comput.*, **10** (2020), 2313–2322. https://doi.org/10.11948/20190197

26. H. Işık, N. Hussain, A. R. Khan, Endpoint results for weakly contractive mappings in $\mathcal{F}$-metric spaces with application, *Int. J. Nonlinear Anal. Appl.*, **11** (2020), 351–361. http://dx.doi.org/10.22075/ijnaa.2020.20368.2148

27. M. Gabeleh, H. P. A. Künzi, Equivalence of the existence of best proximity points and best proximity pairs for cyclic and noncyclic nonexpansive mappings, *Demonstr. Math.*, **53** (2020), 38–43. https://doi.org/10.1515/dema-2020-0005

AIMS Press

© 2024 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (https://creativecommons.org/licenses/by/4.0)