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Research article

Global exponential synchronization of discrete-time high-order BAM neural networks with multiple time-varying delays

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Abstract: The global exponential synchronization (GES) problem of a class of discrete-time high-order bidirectional associative memory neural networks (BAMNNs) with multiple time-varying delays (T-VDs) is studied. We investigate novel delay-dependent global exponential stability criteria for the error system by proposing a mathematical induction method. The global exponential stability criteria that have been obtained are described through linear scalar inequalities. These exponential synchronization conditions are very simple and convenient for verification based on standard software tools (such as YALMIP). Lastly, an instance is presented to demonstrate the validity of the theoretical findings.

Keywords: discrete-time high-order BAM neural network; multiple time-varying delays; global exponential synchronization; linear scalar inequalities; controller gains
Mathematics Subject Classification: 93D20

1. Introduction

In recent years, neural networks (NNs) have played an important role in fields such as addressable memory, pattern recognition, and optimization control due to their nonlinear computing capabilities and powerful parallel processing [1–4]. In 1986, Lee et al. [5] described a form of higher-order correlation formalism tensor for NNs, proposed high-order NNs based on ordinary low-order NNs. The model can simulate auto associative, heteroassociative, as well as multiassociative memory. In [6, 7], several higher-order connection-weight models with different structures are proposed. Recently, an increasing number of scholars working on NN research have turned their attention from low-order systems to high-order systems, trying to overcome the limitations of traditional NNs. These higher-order models have higher fault tolerance, larger storage capacity, faster convergence speed, and stronger approximation properties than first-order NNs.

As one of the interconnected NNs, bidirectional associative memory NNs (BAMNNs) [8, 9] consist of two-layers of heterogeneous associative circuits, which extend the functionality of single-layer neural networks and have the functions of memory and information association. For quite some time, most researchers have been focusing on low-order BAMNNs rather than high-order BAMNNs [10–15]. However, due to some shortcomings of low-order BAMNNs, it becomes very important to incorporate high-order interactions into such BAMNNs. Therefore, Simpson [7] put forward a class of higher-order delayed BAMNNs. In particular, second-order BAMNNs can enhance the storage capacity but require more connections. Since then, different architectures with high-order connections have been utilized to construct desirable BAMNNs [16–18].

It is widely known that synchronization is of utmost importance for BAMNNs. Identifying certain conditions to ensure the achievement of synchronization in drive-response systems is a significant research topic for BAMNNs. The term "synchronization" has a long history and denotes the state where two or more systems display common dynamic behaviors. When we study the dynamic behaviors of NNs, synchronization can play an extremely crucial role. In the past few years, the synchronization problem of various types of BAMNNs has drawn extensive attention due to their broad applications in multiple fields, including pattern recognition, associative memories, automatic control engineering, combinatorial optimization, signal processing, and parallel computation.

Up to now, it has been revealed that the research on the synchronization problem for delayed drive-response BAMNNs holds great significance for fundamental science. A large number of studies have emerged regarding the study of different types of synchronization problems for delayed BAMNNs. for example, long-time synchronization [19–22], finite-time synchronization [23–25] and fixed-time synchronization [26–29]. References [19] and [21] respectively studied the synchronization problems of single inertia BAMNN and delayed BAMNN by using matrix measure theory. They gave several criteria for GES by using Halanay inequality methods and Lyapunov stability theory, respectively. The proposed criteria are independent of delay parameters. Furthermore, valuable new insights into the stability and synchronization of BAMNNs are put forward. In reference [20], the authors concentrate impulsive synchronization of delayed memristive BAMNNs. By employing the so-called linear matrix inequality (LMI) approach, which is based on the time-varying Lyapunov function, the time-dependent impulsive results for the exponential stability of the error system are derived. In references [19–21], the Lyapunov functional method, LMI method, and matrix measure method are mainly applied to study synchronization problems. While reference [30] presents a new global synchronization research method and gives a criterion for global asymptotic synchronization of BAMNNs by using the integral inequality technique. This method and result extend the research on global synchronization of NNs. In reference [31], the general decay synchronization problem of BAMNNs with T-VD and distributed delay is studied by using Lyapunov method and useful inequality techniques. Sufficient criteria for the GDS of BAMNNs are given. In [32], sufficient conditions for global robust exponential synchronization of interval BAMNNs are given through a direct method based on the system solution. This method avoids the difficulty of establishing the Lyapunov–Krasovskii functional. Moreover, the derived global robust exponential synchronization criterion is simpler and easier to implement.

Regarding the achievement of exponential synchronization between dynamical systems, previous studies have employed various methods such as the matrix measure strategy and the method based on the finite-time stability theorem [24], the Lyapunov function method [33–36], the analytical method [37–39], the figure analysis method [40], and the integral inequality method [41–44]. However, there are few other methods apart from those mentioned above for studying the synchronization problem of delayed BAMNNs. This has motivated us to seek another method to investigate the GES problem of BAMNNs.

Furthermore, in computer simulation practices, discrete-time networks have certain advantages over continuous-time networks in transmitting digital information. Thus, researching the dynamic characteristics of discrete-time high-order NNs holds significant importance. However, aside from the aforementioned methods, there are very few other approaches for studying the synchronization problem of discrete-time delayed high-order BAMNNs. This spurs us to seek another method to investigate the GES problem of delayed discrete-time high-order BAMNNs. Based on the above discussion, we propose a novel method for synchronizing the drive-response system.

The primary objective of this paper is to design a control law that can globally exponentially synchronize the delayed discrete-time high-order BAMNNs. Additionally, it aims to obtain a time convergence that is more precise and has a high level of accuracy. The contributions of this article are:

(1) The synchronization criteria deduced by this approach are merely equivalent to solving a few straightforward linear scalar inequalities and are comparatively intuitive.
(2) The controller gains are in fact computed using the parameters of the discrete-time high-order BAMNN itself, and this can significantly decrease the computational complexity.
(3) The synchronization criterion derived is relatively straightforward and can be readily solved by utilizing YALMIP software.

The remaining portion is structured as follows: The elaboration and preparatory work of the problem will be conducted in the next section. The major achievements of this research, a new standard for GES, are presented in Section 3. Illustrative examples are provided in Section 4 to verify the validity of the obtained results. Finally, in Section 5, we present our conclusions.

Notations: The real number and integer set are denoted by $\mathbb{R}$ and $\mathbb{Z}$, respectively. For positive integers l and s where $l \leq s$, let $[l, s]_{\mathbb{Z}}$ be the set that contains all the positive integers ranging from l to s. The symbol $\mathbb{R}^{l\times s}$ denotes the set consisting of $l \times s$ matrices. $\mathbb{R}^{l\times s}_{\geq}$ and $\mathbb{R}^{l\times s}_{>}$ are subsets of $\mathbb{R}^{l\times s}$, with the former containing all nonnegative matrices and the latter containing all positive matrices. In a similar vein, we also utilize $\mathbb{R}_{\geq}$, $\mathbb{R}_{>}$, among others. When $s \to \infty$, the limit case of $[l, s]_{\mathbb{Z}}$ is denoted as $[l, \infty)_{\mathbb{Z}}$. For $S = [s_{ij}] \in \mathbb{R}^{l\times s}$ and $K = [k_{ij}] \in \mathbb{R}^{l\times s}$, the matrix $[s_{ij}k_{ij}]$, denoted by $S \circ K$, refers to the Hadamard product of S and K, and the notation $S \geq K$ (or $S \leq K$) denotes $s_{ij} \geq k_{ij}$ (or $s_{ij} \leq k_{ij}$). If $s_{ij} > k_{ij}$ (or $s_{ij} < k_{ij}$), we say $S \succ K$ (or $S \prec K$). Let $|S| = [|s_{ij}|]$. Then $|SK| \leq |S||K|$ for all $S \in \mathbb{R}^{p\times q}$ and $K \in \mathbb{R}^{q\times r}$. The notation $\|\cdot\|_2$ represents the Euclidean norm, and the notation $\otimes$ denotes the Kronecker product of matrices.

2. Preliminaries

We consider the following discrete-time high-order BAMNNs with multiple T-VDs:

$$\begin{aligned}\zeta_i(t+1) &= a_i\zeta_i(t) + \sum_{j=1}^{m}\left[b_{ij}f_j(\vartheta_j(t)) + c_{ij}f_j(\vartheta_j(t-\delta_{ij}(t)))\right]\\ &\quad+\sum_{j=1}^{m}\sum_{l=1}^{m} d_{ijl}g_j(\vartheta_j(t-\delta_{ijl}(t)))g_l(\vartheta_l(t-\delta_{ijl}(t)))\\ &\quad+I_i(t), i\in[1,n]_{\mathbb{Z}}, t\in[0,\infty)_{\mathbb{Z}},\end{aligned} \tag{2.1a}$$

$$\begin{aligned}\vartheta_j(t+1) &= \hat{a}_j\vartheta_j(t) + \sum_{i=1}^{n}\left[\hat{b}_{ji}\tilde{f}_i(\zeta_i(t)) + \hat{c}_{ji}\tilde{f}_i(\zeta_i(t-\sigma_{ji}(t)))\right]\\ &\quad+\sum_{i=1}^{n}\sum_{r=1}^{n} \hat{d}_{jir}\tilde{g}_i(\zeta_i(t-\sigma_{jir}(t)))\tilde{g}_r(\zeta_r(t-\sigma_{jir}(t)))\\ &\quad+\tilde{I}_j(t), j\in[1,m]_{\mathbb{Z}}, t\in[0,\infty)_{\mathbb{Z}},\end{aligned} \tag{2.1b}$$

where $a_i, \hat{a}_j \in (-1,1)$; Constants b_{ij}, $\hat{b}_{ji}$, c_{ij}, $\hat{c}_{ji}$, d_{ijl} and $\hat{d}_{jir}$ represent the connection weights; $\zeta_i(t)$ and $\vartheta_j(t)$ denote the ith and jth neuronal states of layer-X and layer-Y, respectively; $f_j : \mathbb{R} \to [-s_j^{(1)}, s_j^{(1)}]$, $g_j : \mathbb{R} \to [-s_j^{(2)}, s_j^{(2)}]$, $\tilde{f}_i : \mathbb{R} \to [-\tilde{s}_i^{(1)}, \tilde{s}_i^{(1)}]$ and $\tilde{g}_i : \mathbb{R} \to [-\tilde{s}_i^{(2)}, \tilde{s}_i^{(2)}]$ denote the neuronal activation functions; $s_j^{(1)}, s_j^{(2)}, \tilde{s}_i^{(1)}$ and $\tilde{s}_i^{(2)}$ are known positive constants; For the known positive integers $\bar{\delta}_{ij}$, $\bar{\delta}_{ijl}$, $\bar{\sigma}_{ji}$ and $\bar{\sigma}_{jir}$, the multiple T-VDs is represented as $\delta_{ij} : [0,\infty)_{\mathbb{Z}} \to [0,\bar{\delta}_{ij}]$, $\delta_{ijl} : [0,\infty)_{\mathbb{Z}} \to [0,\bar{\delta}_{ijl}]$, $\sigma_{ji} : [0,\infty)_{\mathbb{Z}} \to [0,\bar{\sigma}_{ji}]$ and $\sigma_{jir} : [0,\infty)_{\mathbb{Z}} \to [0,\bar{\sigma}_{jir}]$; and $I_i(t)$ and $\tilde{I}_j(t)$ are variable external input.

Remark 2.1. *The BAMNN extends the single-layer auto-associative Hebbian correlator to a two-layer pattern matched hetero-associative circuits, and extracts the complete and clear patterns stored in memory from incomplete or fuzzy patterns. The network, as in* (2.1)*, can store pairs of patterns or memories and search for them in both forward and backward directions. Therefore, the BAMNN exhibits relatively excellent information–memory and information–association capabilities.*

We require these assumptions.

Assumption 2.1. *For any $\alpha_1, \alpha_2 \in \mathbb{R}$ subject to $\alpha_1 \neq \alpha_2$, there are $\tilde{\beta}_i^{(1)}, \beta_j^{(1)} \in \mathbb{R}_>$ such that $\tilde{f}_i(0) = f_j(0)$, $0 \leq \frac{\tilde{f}_i(\alpha_1)-\tilde{f}_i(\alpha_2)}{\alpha_1-\alpha_2} \leq \tilde{\beta}_i^{(1)}$ and $0 \leq \frac{f_j(\alpha_1)-f_j(\alpha_2)}{\alpha_1-\alpha_2} \leq \beta_j^{(1)}$ where $\forall i \in [1,n]_{\mathbb{Z}}$, $\forall j \in [1,m]_{\mathbb{Z}}$.*

Assumption 2.2. *For any $\alpha_1, \alpha_2 \in \mathbb{R}$ subject to $\alpha_1 \neq \alpha_2$, there are $\tilde{\beta}_i^{(2)}, \beta_j^{(2)} \in \mathbb{R}_>$ such that $\tilde{g}_i(0) = g_j(0) = 0$, $0 \leq \frac{\tilde{g}_i(\alpha_1)-\tilde{g}_i(\alpha_2)}{\alpha_1-\alpha_2} \leq \tilde{\beta}_i^{(2)}$, and $0 \leq \frac{g_j(\alpha_1)-g_j(\alpha_2)}{\alpha_1-\alpha_2} \leq \beta_j^{(2)}$, $\forall i \in [1,n]_{\mathbb{Z}}$, $\forall j \in [1,m]_{\mathbb{Z}}$.*

We regard discrete-time high-order BAMNN (2.1) as a master drive system and consider the following form of slave matching response system:

$$\hat{\zeta}_i(t+1) = a_i\hat{\zeta}_i(t) + \sum_{j=1}^{m}\left[b_{ij}f_j(\hat{\vartheta}_j(t)) + c_{ij}f_j(\hat{\vartheta}_j(t-\delta_{ij}(t)))\right]$$

$$+\sum_{j=1}^{m}\sum_{l=1}^{m} d_{ijl} g_j(\hat{\vartheta}_j(t-\delta_{ijl}(t))) g_l(\hat{\vartheta}_l(t-\delta_{ijl}(t)))$$
$$+I_i(t)+U_i(t), i\in[1,n]_{\mathbb{Z}}, t\in[0,\infty)_{\mathbb{Z}}, \tag{2.2a}$$

$$\begin{aligned}\hat{\vartheta}_j(t+1) = {} & \hat{a}_j\hat{\vartheta}_j(t)+\sum_{i=1}^{n}\left[\hat{b}_{ji}\tilde{f}_i(\hat{\zeta}_i(t))+\hat{c}_{ji}\tilde{f}_i(\hat{\zeta}_i(t-\sigma_{ji}(t)))\right]\\ & +\sum_{i=1}^{n}\sum_{r=1}^{n}\hat{d}_{jir}\tilde{g}_i(\hat{\zeta}_i(t-\sigma_{jir}(t)))\tilde{g}_r(\hat{\zeta}_r(t-\sigma_{jir}(t)))\\ & +\tilde{I}_j(t)+V_j(t), j\in[1,m]_{\mathbb{Z}}, t\in[0,\infty)_{\mathbb{Z}},\end{aligned} \tag{2.2b}$$

where $U_i(t)$ and $V_j(t)$ are the controllers to realize the GES.

Let $\eta_i(t)=\zeta_i(t)-\hat{\zeta}_i(t)$, $\tilde{\eta}_j(t)=\vartheta_j(t)-\hat{\vartheta}_j(t)$. Take $\eta_i(t)$ and $\tilde{\eta}_j(t)$ as the synchronization error variables. Then, from Eqs (2.1) and (2.2), we can obtain that the error dynamical system:

$$\begin{aligned}\eta_i(t+1) = {} & a_i\eta_i(t)+\sum_{j=1}^{m}\left[b_{ij}f_j^*(\tilde{\eta}_j(t))+c_{ij}f_j^*(\tilde{\eta}_j(t-\delta_{ij}(t)))\right]\\ & +\sum_{j=1}^{m}\sum_{l=1}^{m} d_{ijl}\Big[g_j(\vartheta_j(t-\delta_{ijl}(t)))g_l^*(\tilde{\eta}_l(t-\delta_{ijl}(t)))\\ & +g_j^*(\tilde{\eta}_j(t-\delta_{ijl}(t)))g_l(\hat{\vartheta}_l(t-\delta_{ijl}(t)))\Big]\\ & -U_i(t), i\in[1,n]_{\mathbb{Z}}, t\in[0,\infty)_{\mathbb{Z}},\end{aligned} \tag{2.3a}$$

$$\begin{aligned}\tilde{\eta}_j(t+1) = {} & \hat{a}_j\tilde{\eta}_j(t)+\sum_{i=1}^{n}\left[\hat{b}_{ji}\tilde{f}_i^*(\eta_i(t))+\hat{c}_{ji}\tilde{f}_i^*(\eta_i(t-\sigma_{ji}(t)))\right]\\ & +\sum_{i=1}^{n}\sum_{r=1}^{n}\hat{d}_{jir}\Big[\tilde{g}_i(\zeta_i(t-\sigma_{jir}(t)))\tilde{g}_r^*(\eta_r(t-\sigma_{jir}(t)))\\ & +\tilde{g}_i^*(\eta_i(t-\sigma_{jir}(t)))\tilde{g}_r(\hat{\zeta}_r(t-\sigma_{jir}(t)))\Big]\\ & -V_j(t), j\in[1,m]_{\mathbb{Z}}, t\in[0,\infty)_{\mathbb{Z}},\end{aligned} \tag{2.3b}$$

where

$$\begin{aligned} f_j^*(\tilde{\eta}_j(\cdot)) &= f_j(\tilde{\eta}_j(\cdot)+\hat{\vartheta}_j(\cdot))-f_j(\hat{\vartheta}_j(\cdot)),\\ g_j^*(\tilde{\eta}_j(\cdot)) &= g_j(\tilde{\eta}_j(\cdot)+\hat{\vartheta}_j(\cdot))-g_j(\hat{\vartheta}_j(\cdot)),\\ \tilde{f}_i^*(\eta_i(\cdot)) &= \tilde{f}_i(\eta_i(\cdot)+\hat{\zeta}_i(\cdot))-\tilde{f}_i(\hat{\zeta}_i(\cdot)),\\ \tilde{g}_i^*(\eta_i(\cdot)) &= \tilde{g}_i(\eta_i(\cdot)+\hat{\zeta}_i(\cdot))-\tilde{g}_i(\hat{\zeta}_i(\cdot)).\end{aligned}$$

Due to Assumptions 2.1 and 2.2, we deduce that

$$\begin{aligned} & |f_j^*(\hbar)|\le\beta_j^{(1)}|\hbar|,\ |g_j^*(\hbar)|\le\beta_j^{(2)}|\hbar|,\ |\tilde{f}_i^*(\hbar)|\le\tilde{\beta}_i^{(1)}|\hbar|,\\ & |\tilde{g}_i^*(\hbar)|\le\tilde{\beta}_i^{(2)}|\hbar|,\ \hbar\in\mathbb{R},\ i\in[1,n]_{\mathbb{Z}}, j\in[1,m]_{\mathbb{Z}}.\end{aligned} \tag{2.4}$$

Let $\nu = \max_{1\le i,r\le n, 1\le j,l\le m} \max\left\{\bar{\delta}_{ij}, \bar{\sigma}_{ji}, \bar{\delta}_{ijl}, \bar{\sigma}_{jir}\right\}$. The symbol $C([-\nu, 0]_{\mathbb{Z}}, \mathbb{R}^n)$ denotes the set that consists of all functions $\varphi : [-\nu, 0]_{\mathbb{Z}} \to \mathbb{R}^n$. Let there be a norm $\|\cdot\|$ on $\mathbb{R}^n \times \mathbb{R}^m$. It is defined such that for $\|(\alpha, \beta)\| = (\|\alpha\|_2^2 + \|\beta\|_2^2)^{1/2}$, $\alpha \in \mathbb{R}^n$, $\beta \in \mathbb{R}^m$. Similarly, a norm $\|(\cdot,\cdot)\|_\nu$ can be defined on $C([-\nu, 0]_{\mathbb{Z}}, \mathbb{R}^n) \times C([-\nu, 0]_{\mathbb{Z}}, \mathbb{R}^m)$ by

$$\|(\phi, \tilde{\phi})\|_\nu = \sup_{s\in[-\nu,0]_{\mathbb{Z}}} \max\left\{\|\phi(s)\|_2, \|\tilde{\phi}(s)\|_2\right\},$$
$$\forall \phi \in C([-\nu, 0]_{\mathbb{Z}}, \mathbb{R}^n),\ \forall\, \tilde{\phi} \in C([-\nu, 0]_{\mathbb{Z}}, \mathbb{R}^m).$$

Definition 2.1. *[4] The response discrete-time high-order BAMNN* (2.2) *and the discrete-time high-order BAMNN* (2.1) *are said to be GES with a decay rate λ if there exist $\beta, \lambda \in \mathbb{R}_>$ and controllers $U_i(t)$ and $V_j(t)$ such that arbitrary solution $(\eta(t), \tilde{\eta}(t))$ of error dynamical system* (2.3) *satisfying*

$$\|(\eta(t), \tilde{\eta}(t))\| \le \beta e^{-\lambda t} \left\|(\phi,\ \tilde{\phi})\right\|_\nu, \forall t \in [\nu, \infty)_{\mathbb{Z}},$$

where $(\phi,\ \tilde{\phi}) \in C([-\nu, 0]_{\mathbb{Z}}, \mathbb{R}^n) \times C([-\nu, 0]_{\mathbb{Z}}, \mathbb{R}^m)$ is the initial functions corresponding to the solution $(\eta(t), \tilde{\eta}(t))$; $\eta(t) = [\eta_1(t)\ \ \dots\ \ \eta_n(t)]^T$ and $\tilde{\eta}(t) = [\tilde{\eta}_1(t)\ \ \dots\ \ \tilde{\eta}_m(t)]^T$.

The objective of this paper is to design a state feedback controller of the form:

$$\begin{aligned} &U_i(t) = (\rho_i - a_i)\eta_i(t),\ i \in [1, n]_{\mathbb{Z}}, t \in [0, \infty)_{\mathbb{Z}}, \\ &V_j(t) = (\hat{\rho}_j - \hat{a}_j)\tilde{\eta}_j(t), j \in [1, m]_{\mathbb{Z}}, t \in [0, \infty)_{\mathbb{Z}}, \end{aligned} \tag{2.5}$$

which makes the discrete-time high-order BAMNNs (2.1) and (2.2) achieve GES, where ρ_i and $\hat{\rho}_j$ are the controller gains to be determined.

Remark 2.2. *Discrete-time high-order BAMNN is a typical recurrent neural network, which plays an important role in pattern recognition and combinatorial optimization. At present, although many scholars have studied the dynamic characteristics of high-order BAMNN with time delays, most of them have focused on complex continuous-time network models, while the research on discrete-time network models is relatively scarce. Therefore, this paper conducts a study on the synchronization problem of a discrete-time high-order BAMNN model with multiple T-VDs. In addition, an important role is also played by BAMNN in the fields of signal processing and artificial intelligence.*

3. global exponential synchronization analysis

Set

$$\begin{aligned} &\Xi_\beta = e^{\beta\bar{\delta}} \circ |C|\Gamma_1 + |B|\Gamma_1 + \aleph,\ e^{\beta\bar{\delta}} = [e^{\beta\bar{\delta}_{ij}}],\ B = [b_{ij}],\ C = [c_{ij}], \\ &\aleph = G^T(E_\beta \circ |D| + \hat{E}_\beta \circ |\hat{D}|)\Gamma_2, G = I_m \otimes P, P = [s_1^{(2)} s_2^{(2)} \dots s_m^{(2)}]^T, \\ &D = [D_1^T D_2^T \dots D_m^T], \hat{D} = [D_1 D_2 \dots D_m], D_i = [d_{ijl}], \\ &E_\beta = [E_{\beta,1}^T E_{\beta,2}^T \dots E_{\beta,m}^T], \hat{E}_\beta = [E_{\beta,1} E_{\beta,2} \dots E_{\beta,m}], E_{\beta,i} = [e^{\beta\bar{\delta}_{ijl}}], \\ &\tilde{\Xi}_\beta = e^{\beta\bar{\sigma}} \circ |\tilde{C}|\tilde{\Gamma}_1 + |\tilde{B}|\tilde{\Gamma}_1 + \tilde{\aleph},\ e^{\beta\bar{\sigma}} = [e^{\beta\bar{\sigma}_{ji}}],\ \tilde{B} = [\hat{b}_{ji}],\ \tilde{C} = [\hat{c}_{ji}], \\ &\tilde{\aleph} = H^T(F_\beta \circ |\tilde{D}| + \hat{F}_\beta \circ |\hat{\tilde{D}}|)\tilde{\Gamma}_2, H = I_n \otimes Q, Q = [s_1^{(2)} s_2^{(2)} \dots s_n^{(2)}]^T, \end{aligned}$$

$$\tilde{D} = [\tilde{D}_1^T \tilde{D}_2^T \dots \tilde{D}_n^T], \hat{\tilde{D}} = [\tilde{D}_1 \tilde{D}_2 \dots \tilde{D}_n], \tilde{D}_j = [\hat{d}_{jil}],$$
$$F_\beta = [F_{\beta,1}^T F_{\beta,2}^T \dots F_{\beta,n}^T], \hat{F}_\beta = [F_{\beta,1} F_{\beta,2} \dots F_{\beta,n}], F_{\beta,j} = [e^{\beta\bar{\sigma}_{jil}}],$$
$$\Theta_{\beta,k} = -e^{-\beta} I_k,$$
$$\Gamma_1 = \mathrm{diag}(\beta_1^{(1)}, \dots, \beta_m^{(1)}),\ \tilde{\Gamma}_1 = \mathrm{diag}(\tilde{\beta}_1^{(1)}, \dots, \tilde{\beta}_n^{(1)}),$$
$$\Gamma_2 = \mathrm{diag}(\beta_1^{(2)}, \dots, \beta_m^{(2)}),\ \tilde{\Gamma}_2 = \mathrm{diag}(\tilde{\beta}_1^{(2)}, \dots, \tilde{\beta}_n^{(2)}).$$

Theorem 3.1. *If there exists $\varphi \in \mathbb{R}_>^n$, $\psi \in \mathbb{R}_>^m$, $\tilde{u} \in \mathbb{R}_>^n$, $\tilde{v} \in \mathbb{R}_>^m$ and $\beta \in \mathbb{R}_>$ such that*

$$\Theta_{\beta,n}\tilde{u} + \Xi_\beta \tilde{v} + \varphi \preceq 0, \tag{3.1}$$
$$\tilde{\Xi}_\beta \tilde{u} + \Theta_{\beta,m}\tilde{v} + \psi \preceq 0, \tag{3.2}$$

then the error system (2.3) *is globally exponentially stable, that is, the drive system discrete-time high-order BAMNN* (2.1) *and response system discrete-time high-order BAMNN* (2.2) *achieve GES via the controllers in* (2.5)*, where $\rho_i = \pm\varphi_i \tilde{u}_i^{-1}$ and $\hat{\rho}_j = \pm\psi_j \tilde{v}_j^{-1}$, where $\tilde{u}_i$, $\tilde{v}_j$, φ_i and ψ_j are the ith and jth components of $\tilde{u}$, $\tilde{v}$, φ and ψ, respectively.*

Proof. Choose $\Upsilon > 0$ such that

$$\Upsilon\tilde{u} \succ [1 \ \cdots \ 1]^T,\ \Upsilon\tilde{v} \succ [1 \ \cdots \ 1]^T.$$

For any fixed $\phi \in C([-\nu, 0]_{\mathbb{Z}}, \mathbb{R}^n)$ and $\tilde{\phi} \in C([-\nu, 0]_{\mathbb{Z}}, \mathbb{R}^m)$, define

$$\hat{u}(t) = \Upsilon\|(\phi, \tilde{\phi})\|_\nu e^{-\beta t}\tilde{u},\ t \in [-\nu, \infty)_{\mathbb{Z}}, \tag{3.3}$$
$$\hat{v}(t) = \Upsilon\|(\phi, \tilde{\phi})\|_\nu e^{-\beta t}\tilde{v},\ t \in [-\nu, \infty)_{\mathbb{Z}}, \tag{3.4}$$

where $(\phi, \tilde{\phi})$ serve as the initial functions.

Suppose $(\eta(t), \tilde{\eta}(t))$ is the solution of (2.3). Next, the following expression will be proved by using the mathematical induction method

$$|\eta(t)| \preceq \hat{u}(t),\ |\tilde{\eta}(t)| \preceq \hat{v}(t),\ t \in [-\nu, \infty)_{\mathbb{Z}}. \tag{3.5}$$

Obviously, in combination with the definition of $\|\cdot\|_\nu$ and the selection of Υ, we have

$$|\eta(k)| \preceq \hat{u}(k),\ |\tilde{\eta}(k)| \preceq \hat{v}(k),\ \forall k \in [-\nu, 0]_{\mathbb{Z}}.$$

Suppose that for any fixed $k \geq 0$, when $t \leq k$, the inequality (3.5) holds. For any $i \in [1, n]_{\mathbb{Z}}$, when $t = k + 1$, using (2.3a), (2.5), (2.4), and Assumptions 2.1 and 2.2, we obtain

$$\begin{aligned}|\eta_i(k+1)| \leq{}& |\rho_i||\eta_i(k)| + \sum_{j=1}^{m}\Big[|b_{ij}||f_j^*(\tilde{\eta}_j(k))| + |c_{ij}||f_j^*(\tilde{\eta}_j(k-\delta_{ij}(k)))|\Big] \\ &+ \sum_{j=1}^{m}\sum_{l=1}^{m}|d_{ijl}|\Big[|g_j(\vartheta_j(k-\delta_{ijl}(k)))||g_l^*(\tilde{\eta}_l(k-\delta_{ijl}(k)))| \\ &+ |g_j^*(\tilde{\eta}_j(k-\delta_{ijl}(k)))||g_l(\hat{\vartheta}_l(k-\delta_{ijl}(k)))|\Big]\end{aligned}$$

$$
\begin{aligned}
&\leq |\rho_i||\eta_i(k)| + \sum_{j=1}^{m}\Big[|b_{ij}|\beta_j^{(1)}|\tilde{\eta}_j(k)| + |c_{ij}|\beta_j^{(1)}|\tilde{\eta}_j(k-\delta_{ij}(k))|\Big] \\
&\quad + \sum_{j=1}^{m}\sum_{l=1}^{m}|d_{ijl}|\Big[|g_j(\vartheta_j(k-\delta_{ijl}(k)))|\beta_l^{(2)}|\tilde{\eta}_l(k-\delta_{ijl}(k))| \\
&\quad + \beta_j^{(2)}|\tilde{\eta}_j(k-\delta_{ijl}(k))||g_l(\hat{\vartheta}_l(k-\delta_{ijl}(k)))|\Big] \\
&\leq |\rho_i||\eta_i(k)| + \sum_{j=1}^{m}\Big[|b_{ij}|\beta_j^{(1)}|\tilde{\eta}_j(k)| + |c_{ij}|\beta_j^{(1)}|\tilde{\eta}_j(k-\delta_{ij}(k))|\Big] \\
&\quad + \sum_{j=1}^{m}\sum_{l=1}^{m}|d_{ijl}|\Big[s_j^{(2)}\beta_l^{(2)}|\tilde{\eta}_l(k-\delta_{ijl}(k))| + s_l^{(2)}\beta_j^{(2)}|\tilde{\eta}_j(k-\delta_{ijl}(k))|\Big].
\end{aligned}
$$

Utilizing the inductive hypothesis, we are able to obtain

$$
\begin{aligned}
|\eta_i(k+1)| &\leq |\rho_i|\hat{u}_i(k) + \sum_{j=1}^{m}\Big[|b_{ij}|\beta_j^{(1)}\hat{v}_j(k) + |c_{ij}|\beta_j^{(1)}\hat{v}_j(k-\delta_{ij}(k))\Big] \\
&\quad + \sum_{j=1}^{m}\sum_{l=1}^{m}|d_{ijl}|\Big[s_j^{(2)}\beta_l^{(2)}\hat{v}_l(k-\delta_{ijl}(k)) + s_l^{(2)}\beta_j^{(2)}\hat{v}_j(k-\delta_{ijl}(k))\Big]. \qquad (3.6)
\end{aligned}
$$

By substituting Eqs (3.3) and (3.4) into (3.6), we obtain

$$
\begin{aligned}
|\eta_i(k+1)| &\leq |\rho_i|\Upsilon\|(\phi,\tilde{\phi})\|_\nu e^{-\beta k}\tilde{u}_i + \sum_{j=1}^{m}\Big[|b_{ij}|\beta_j^{(1)}\Upsilon\|(\phi,\tilde{\phi})\|_\nu e^{-\beta k}\tilde{v}_j \\
&\quad + |c_{ij}|\beta_j^{(1)}\Upsilon\|(\phi,\tilde{\phi})\|_\nu e^{-\beta(k-\delta_{ij}(k))}\tilde{v}_j\Big] + \sum_{j=1}^{m}\sum_{l=1}^{m}|d_{ijl}|\Big[s_j^{(2)}\beta_l^{(2)}\Upsilon\|(\phi,\tilde{\phi})\|_\nu e^{-\beta(k-\delta_{ijl}(k))}\tilde{v}_l \\
&\quad + s_l^{(2)}\beta_j^{(2)}\Upsilon\|(\phi,\tilde{\phi})\|_\nu e^{-\beta(k-\delta_{ijl}(k))}\tilde{v}_j\Big] \\
&\leq \Upsilon\|(\phi,\tilde{\phi})\|_\nu e^{-\beta k}\times\Bigg\{|\rho_i|\tilde{u}_i + \sum_{j=1}^{m}(|b_{ij}|\beta_j^{(1)} + |c_{ij}|\beta_j^{(1)}e^{\beta\bar{\delta}_{ij}})\tilde{v}_j \\
&\quad + \sum_{j=1}^{m}\sum_{l=1}^{m}|d_{ijl}|\Big[s_j^{(2)}\beta_l^{(2)}e^{\beta\bar{\delta}_{ijl}}\tilde{v}_l + s_l^{(2)}\beta_j^{(2)}e^{\beta\bar{\delta}_{ijl}}\tilde{v}_j\Big]\Bigg\} \\
&= \Upsilon\|(\phi,\tilde{\phi})\|_\nu e^{-\beta k}\times\Bigg\{|\rho_i|\tilde{u}_i + \sum_{j=1}^{m}\Bigg(|b_{ij}|\beta_j^{(1)} + |c_{ij}|\beta_j^{(1)}e^{\beta\bar{\delta}_{ij}} + \sum_{l=1}^{m}|d_{ijl}|s_l^{(2)}\beta_j^{(2)}e^{\beta\bar{\delta}_{ijl}}\Bigg)\tilde{v}_j \\
&\quad + \sum_{j=1}^{m}\sum_{l=1}^{m}|d_{ijl}|s_j^{(2)}\beta_l^{(2)}e^{\beta\bar{\delta}_{ijl}}\tilde{v}_l\Bigg\} \\
&= \Upsilon\|(\phi,\tilde{\phi})\|_\nu e^{-\beta k}\times\Bigg\{|\rho_i|\tilde{u}_i + \sum_{j=1}^{m}\Bigg(|b_{ij}|\beta_j^{(1)} + |c_{ij}|\beta_j^{(1)}e^{\beta\bar{\delta}_{ij}} + \sum_{l=1}^{m}|d_{ijl}|s_l^{(2)}\beta_j^{(2)}e^{\beta\bar{\delta}_{ijl}}\Bigg)\tilde{v}_j \\
&\quad + \sum_{j=1}^{m}\sum_{l=1}^{m}|d_{ilj}|s_l^{(2)}\beta_j^{(2)}e^{\beta\bar{\delta}_{ilj}}\tilde{v}_j\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&= \Upsilon\|(\phi,\tilde{\phi})\|_{\nu}\mathrm{e}^{-\beta k}\times\Bigg\{|\rho_i|\tilde{u}_i+\sum_{j=1}^{m}\Bigg[|b_{ij}|\beta_j^{(1)}+|c_{ij}|\beta_j^{(1)}\mathrm{e}^{\beta\bar{\delta}_{ij}}+\sum_{l=1}^{m}|d_{ijl}|s_l^{(2)}\beta_j^{(2)}\mathrm{e}^{\beta\bar{\delta}_{ijl}}\\
&\quad+\sum_{l=1}^{m}|d_{ilj}|s_l^{(2)}\beta_j^{(2)}\mathrm{e}^{\beta\bar{\delta}_{ilj}}\Bigg]\tilde{v}_j\Bigg\}\\
&\leq \Upsilon\|(\phi,\tilde{\phi})\|_{\nu}\mathrm{e}^{-\beta k}\times\Bigg\{|\rho_i|\tilde{u}_i+\sum_{j=1}^{m}\Bigg[|b_{ij}|\beta_j^{(1)}+|c_{ij}|\beta_j^{(1)}\mathrm{e}^{\beta\bar{\delta}_{ij}}\\
&\quad+\sum_{l=1}^{m}s_l^{(2)}\left(|d_{ijl}|\mathrm{e}^{\beta\bar{\delta}_{ijl}}+|d_{ilj}|\mathrm{e}^{\beta\bar{\delta}_{ilj}}\right)\beta_j^{(2)}\Bigg]\tilde{v}_j\Bigg\}. \qquad (3.7)
\end{aligned}
$$

Note that $\varphi_i = |\rho_i|\tilde{u}_i$. Based on the arbitrariness of $i \in [1, n]_{\mathbb{Z}}$, we can conclude that (3.7) is equivalent to

$$|\eta(k+1)| \leq \Upsilon\|(\phi,\tilde{\phi})\|_{\nu}\mathrm{e}^{-\beta k}(\varphi+\Xi_{\beta}\tilde{v}).$$

By making use of (3.1) and (3.3), we obtain

$$|\eta(k+1)| \leq \Upsilon\|(\phi,\tilde{\phi})\|_{\nu}\mathrm{e}^{-\beta(k+1)}\tilde{u} = \hat{u}(k+1). \qquad (3.8)$$

Similarly, through a procedure similar to the one used in deriving (3.8), it is straightforward to that

$$|\tilde{\eta}(k+1)| \leq \Upsilon\|(\phi,\tilde{\phi})\|_{\nu}\mathrm{e}^{-\beta(k+1)}\tilde{v} = \hat{v}(k+1).$$

Therefore, (3.5) is true.

Then, in combination with Eqs (3.3)–(3.5), we obtain

$$
\begin{aligned}
\|(\eta(t),\tilde{\eta}(t))\| &= (\|\eta(t)\|_2^2+\|\tilde{\eta}(t)\|_2^2)^{\frac{1}{2}}\\
&\leq (\|\hat{u}(t)\|_2^2+\|\hat{v}(t)\|_2^2)^{\frac{1}{2}}\\
&= \Upsilon\mathrm{e}^{-\beta t}\|(\phi,\tilde{\phi})\|_{\nu}(\|\tilde{u}\|_2^2+\|\tilde{v}\|_2^2)^{\frac{1}{2}},\ \forall t\in[0,\infty)_{\mathbb{Z}}.
\end{aligned}
$$

Let $\mu = \Upsilon(\|\tilde{u}\|_2^2+\|\tilde{v}\|_2^2)^{\frac{1}{2}}$. Then

$$\|(\eta(t),\tilde{\eta}(t))\| \leq \mu\mathrm{e}^{-\beta t}\left\|(\phi,\tilde{\phi})\right\|_{\nu},\ \forall t\in[0,\infty)_{\mathbb{Z}}.$$

The arbitrariness of $\phi \in C([-\nu,0]_{\mathbb{Z}},\mathbb{R}^n)$ and $\tilde{\phi} \in C([-\nu,0]_{\mathbb{Z}},\mathbb{R}^m)$ guarantees GES of the error system (2.3), that is, the discrete-time high-order BAMNN (2.1) and (2.2) achieve GES via the controllers in (2.5). □

4. Numerical examples

Next, the effectiveness of the results given in this paper will be illustrated through a specific numerical example.

Example 4.1. *For $n = 2, m = 2$, consider the drive discrete-time high-order BAMNNs* (2.1) *and the response discrete-time high-order BAMNNs* (2.2) *with the controllers in* (2.5)*, and the parameters are as below:* $a_1 = 0.3, a_2 = 0.4$, $\hat{a}_1 = 0.5$, $\hat{a}_2 = 0.6$, $b_{11} = -1.6$, $b_{12} = 0.1$, $b_{21} = -4.1$, $b_{22} = 3.2$, $\hat{b}_{11} = 1.1$, $\hat{b}_{12} = 0.5$, $\hat{b}_{21} = -0.1$, $\hat{b}_{22} = -2.1$, $c_{11} = -1.4$, $c_{12} = 0.1$, $c_{21} = 0.2$, $c_{22} = -2.3$, $\hat{c}_{11} = -3$, $\hat{c}_{12} = -2.5$, $\hat{c}_{21} = 0.3$, $\hat{c}_{22} = -1.2$, $d_{111} = 5$, $d_{112} = 1$, $d_{121} = 0$, $d_{122} = 5$, $d_{211} = 2$, $d_{212} = 0$, $d_{221} = 0$, $d_{222} = 2$, $\hat{d}_{111} = 4$, $\hat{d}_{112} = 0$, $\hat{d}_{121} = 0$, $\hat{d}_{122} = 5$, $\hat{d}_{211} = 1$, $\hat{d}_{212} = 1$, $\hat{d}_{221} = 0$, $\hat{d}_{222} = 0$, *and*

$$
\begin{aligned}
&f_1(s) = f_2(s) = \tilde{f}_1(s) = \tilde{f}_2(s) = 0.04\text{tanh(s)},\\
&g_1(s) = g_2(s) = \tilde{g}_1(s) = \tilde{g}_2(s) = 0.4\text{tanh(s)}, s \in \mathbb{R},\\
&\sigma_{ji}(t) = p_{ji} + q_{ji}\cos(t\pi),\ \delta_{ij}(t) = r_{ij} + s_{ij}\sin(t\pi/2),\ i, j \in [1, 2]_{\mathbb{Z}}, t \in [0, \infty)_{\mathbb{Z}},
\end{aligned}
$$

where $r_{11} = r_{12} = r_{21} = r_{22} = 10$, $s_{11} = s_{12} = s_{21} = s_{22} = 10$, $p_{11} = p_{12} = p_{21} = p_{22} = 10$, $q_{11} = q_{21} = q_{12} = q_{22} = 10$, $\delta_{111} = \delta_{112} = \delta_{121} = \delta_{122} = \delta_{211} = \delta_{212} = \delta_{221} = \delta_{222} = 3$, $\sigma_{111} = \sigma_{112} = \sigma_{121} = \sigma_{122} = \sigma_{211} = \sigma_{212} = \sigma_{221} = \sigma_{222} = 3$.

Clearly, $\bar{\sigma}_{11} = \bar{\sigma}_{12} = \bar{\sigma}_{21} = \bar{\sigma}_{22} = \bar{\delta}_{11} = \bar{\delta}_{12} = \bar{\delta}_{21} = \bar{\delta}_{22} = 20$. *Furthermore, when* $\beta_1^{(1)} = \beta_2^{(1)} = 0.04$, $\tilde{\beta}_1^{(1)} = \tilde{\beta}_2^{(1)} = 0.04$, $\beta_1^{(2)} = \beta_2^{(2)} = 0.4$, *and* $\tilde{\beta}_1^{(2)} = \tilde{\beta}_2^{(2)} = 0.4$, *Assumptions 2.1 and 2.2 are satisfied.*

By solving the inequalities (3.1) and (3.2) in Theorem 3.1, the following feasible solutions are obtained:

$$\tilde{u} = [7.6343, 5.6956]^T, \tilde{v} = [11.0709, 3.1479]^T,$$

$$\varphi = [5.9 \times 10^3, 4.6 \times 10^3]^T, \psi = [5.3 \times 10^3, 3.7 \times 10^3]^T.$$

Consequently, the controller gains of the desired state feedback controllers are as follows:

$$
\begin{aligned}
&\rho_1 = 0.7709 \times 10^{-3},\ \rho_2 = 0.8025 \times 10^{-3},\\
&\hat{\rho}_1 = 0.5 \times 10^{-3},\ \hat{\rho}_2 = 1.2 \times 10^{-3}.
\end{aligned}
$$

It can be readily verified that the conditions of Theorem 3.1 in our paper are met. Consequently, based on Theorem 3.1 in our paper, the drive system given by (2.1) *and the response system given by* (2.2) *are GES under the controllers* (2.5)*. We choose the initial values of the state variables as* $\zeta(s) = [0.6428\ \ -0.1106]^T$, $\vartheta(s) = [0.2309\ \ 0.5839]^T$, $\hat{\zeta}(s) = [1.6873\ \ 0.9528]^T$ *and* $\hat{\vartheta}(s) = [-1.2949\ -0.3772]^T$, $s \in [-20, 0]_{\mathbb{Z}}$. *We also define the external input* $I_1 = I_2 = 0, J_1 = J_2 = 0$. *The error curves of drive–responses system* $\eta_1(t), \eta_2(t), \tilde{\eta}_1(t)$ *and* $\tilde{\eta}_2(t)$ *are shown in Figures 5 and 6, the curves of variables* $\zeta_1(t), \zeta_2(t), \vartheta_1(t), \vartheta_2(t)$, $\hat{\zeta}_1(t), \hat{\zeta}_2(t), \hat{\vartheta}_1(t), \hat{\vartheta}_2(t)$, *are shown in Figures 1–4. It is readily observable that the error states are rapidly converging to the equilibrium point of zero. Therefore, it can be concluded that, in accordance with Theorem 3.1, the considered discrete-time high-order BAMNNs model* (2.1) *can achieve GES with system* (2.2).

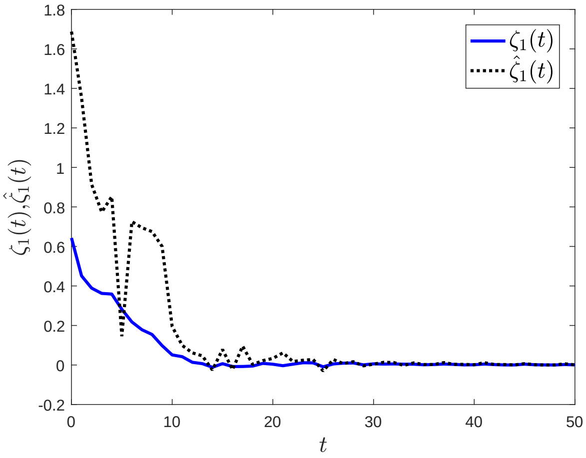

Figure 1. Synchronization curves of $\zeta_1(t)$ and $\hat{\zeta}_1(t)$.

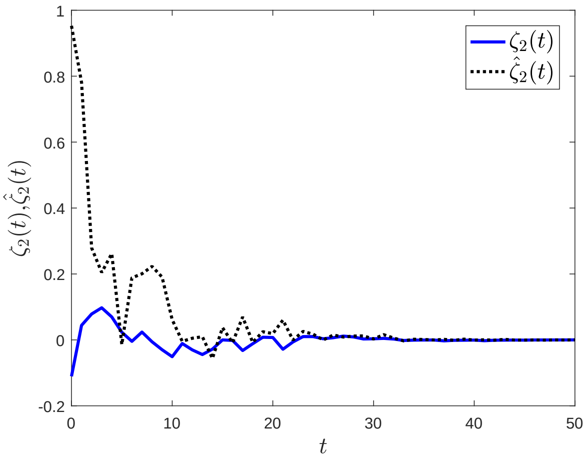

Figure 2. Synchronization curves of $\zeta_2(t)$ and $\hat{\zeta}_2(t)$.

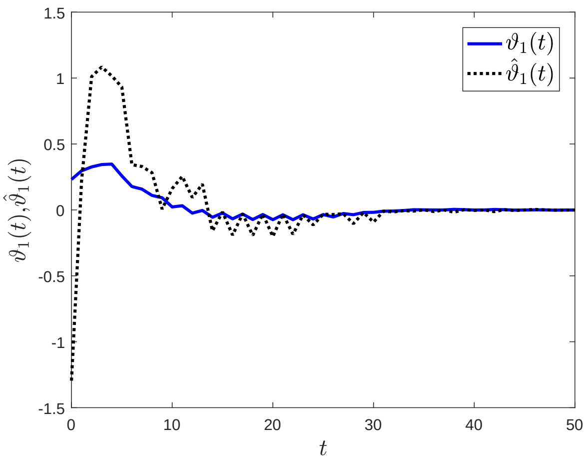

Figure 3. Synchronization curves of $\vartheta_1(t)$ and $\hat{\vartheta}_1(t)$.

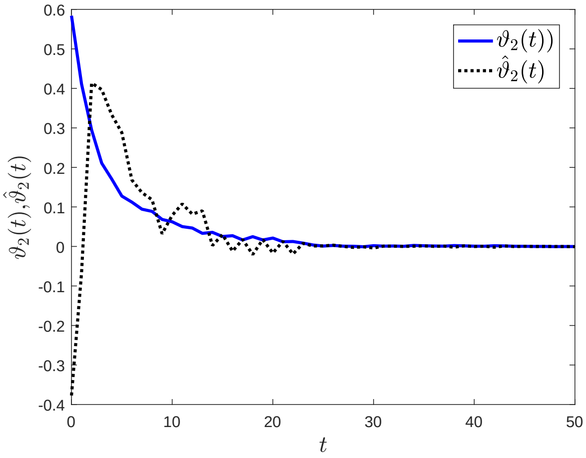

Figure 4. Synchronization curves of $\vartheta_2(t)$ and $\hat{\vartheta}_2(t)$.

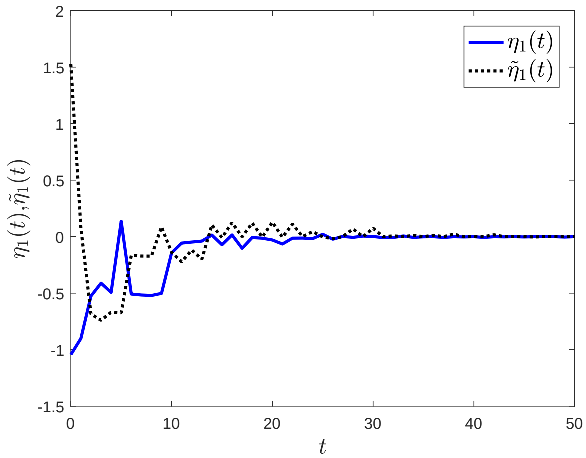

Figure 5. The error curves of $\eta_1(t)$ and $\tilde{\eta}_1(t)$.

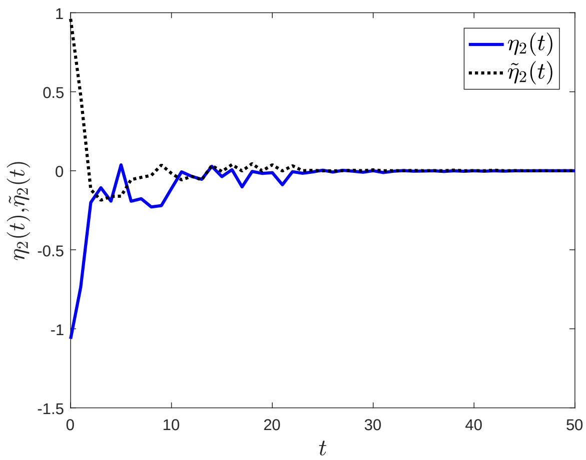

Figure 6. The error curves of $\eta_2(t)$ and $\tilde{\eta}_2(t)$.

5. Conclusions

This paper addresses the issue of GES for discrete-time high-order BAMNNs of multiple T-VD. Utilizing the definition of GES, we initially derive delay-dependent GES criteria for the error dynamical system. Subsequently, a controller gain is provided. Finally, we present illustrative examples to demonstrate the applicability of the conclusions. Compared with the previous research results, the proposed method has the following advantages:

(1) The method directly employs the definition of GES and avoids the construction of any Lyapunov-Krasovskii function;
(2) The obtained sufficient conditions are in the form of linear scalar inequalities, which are easy to solve;
(3) With a small modification, the proposed method is applicable to more general NNs models.

The stability of the error system considered in this paper is the stability in the Lyapunov sense. What is mainly considered is the long-term behavior of the neural network, and it cannot be directly applied to the research on the synchronization in finite time. Therefore, it is necessary to propose more effective analysis and design methods.

Author contributions

Er-yong Cong: Writing-review & editing, writing-original draft, validation, investigation, conceptualization; Li Zhu: Writing-review & editing, visualization, software, investigation, funding acquisition; Xian Zhang: Writing-review & editing, resources, methodology, conceptualization. All authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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