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Research article

Classes of analytic functions involving the *q*-Ruschweyh operator and *q*-Bernardi operator

Khadeejah Rasheed Alhindi[1,*], **Khalid M. K. Alshammari**[2] **and Huda Ali Aldweby**[3]

[1] Department of Computer Science, College of Computer Science and Information Technology, Imam Abdulrahman Bin Faisal University, P. O. Box 1982, Dammam 31441, Saudi Arabia
[2] Department of Mathematics, College of Science, University of Ha'il, Ha'il 55425, Saudi Arabia
[3] Department of Mathematics, Faculty of Science, Alasmarya Islamic University, Libya

* **Correspondence:** Email: kralhindi@iau.edu.sa.

Abstract: In this paper, we introduced and studied two new classes of analytic functions using the concepts of subordination and q-calculus. We established inclusion relations for these q-classes and integral-preserving properties associated with the q-integral operator. We also determined certain convolution properties.

Keywords: q-derivative; q-integral operator; analytic functions; univalent functions; coefficient estimates; Ruschweyh operator; Bernardi operator
Mathematics Subject Classification: 30C45

1. Introduction

Quantum calculus, or q-calculus, is of great interest to academics for its diverse applications in scientific domains, especially in geometric function theory. In 1996, Ismail [1] was the first to define and study the class of q-starlike functions and established many properties associated with the class. The Russell operator, which is a generalization of the differential operator, plays a key role in characterizing subclasses of analytic functions by examining their geometric properties, such as starlikeness and convexity [2]. The Bernardi operator is similarly used to define classes of functions with geometric constraints that are important in the study of conformal mappings and their applications in complex analysis [3]. Later, in 2013, Mohammed and Darus [4] introduced the q-derivative operator, which uses the convolution structure of normalized analytic functions and q-hypergeometric functions. In 2014, Aldweby and Darus [5] introduced the q-analogue of the Ruscheweyh differential operator. Over the years, there has been a growing exploration of the connection between q-calculus and geometric function theory. The q-Salagean differential operator was introduced in [6]. Many authors successfully used these operators to investigate the properties of both known and new classes of analytic functions [7, 8].

In [9], Selvakumaran et al. developed q-integral operators for analytic functions using fractional q-calculus and examined the convex characteristics of these operators on specific classes of analytic functions. In [10], the authors introduced the q-Bernardi integral operator and studied its integral-preserving features. The q-analogue of the Noor integral operator was presented in [11]. In [12], the q-Srivastava Attiya operator and q-multiplier transformation were presented in relation to a specific q-Hurwitz-Lerch zeta function. By linking these q-operators with the idea of subordinations, several subclasses of analytic functions have been identified and examined.

Many researchers contributed to the theory by obtaining coefficient estimates that contain the initial coefficients of q-classes of biunivalent functions. The Fekete-Szego functional and Henkel determinants were studied for these classes. Some recent studies have also focused on new families of meromorphic functions [13].

The versatility and potential of q-calculus, as demonstrated by its ability to enhance our theoretical understanding of analytic functions and its applicability in various scientific domains, have made it an active area of research. The continued exploration and advancement of q-calculus in the field of geometric function theory are expected to yield valuable contributions to both the theoretical and practical aspects of this field.

This work explores new classes of analytic functions using the q-difference operator in the open unit disk, inspired by recent developments in q-calculus and its applications to analytic functions. Previous research has examined classical operator properties, but there is a gap in understanding q-operators, particularly in terms of inclusion relations, integral preservation, and convolution identities. To fill this gap, we examine the key characteristics of the new q-classes. In particular, we define inclusion relations across q-classes, investigate integral-preserving characteristics of the q-integral operator, and deduce convolution identities. These findings enhance the framework of q-calculus in geometric function theory, offering new insights and tools for future research.

2. Materials and methods

This section provides some mathematical preliminaries that are utilized in this paper.

Let $\mathcal{A}$ denote the class of the functions expressed as follows:

$$f(z) = z + \sum_{k=2}^{\infty} \mathfrak{a}_k z^k,$$

which are analytic in the open unit disk

$$U = \{z \in \mathbb{C} : |z| < 1\}.$$

Definition 2.1. *([14]) The convolution (Hadamard product) for two analytic functions $f(z), g(z) \in \mathcal{A}$ is defined as*

$$f(z) * g(z) = \left(z + \sum_{k=2}^{\infty} a_k z^k\right) * \left(z + \sum_{k=2}^{\infty} b_k z^k\right), \quad (z \in U).$$

In the following definition, we will refer to a well-known function with two key conditions, called a Schwarz function. These requirements are essential for applying various mathematical results and theorems related to analytic functions, ensuring predictable and consistent behavior within the unit disk.

Definition 2.2. *([14]) We say that two functions $f(z)$ and $g(z)$ are subordinate to one another and we write $f(z) \prec g(z)$ if there is a Schwarz function $w(z)$ with the conditions that*

$$|w(z)| \leq 1$$

and

$$f(z) = g(w(z)).$$

Additionally, in the case where the function $g(z)$ is univalent in U, the subsequent equivalent relationship is valid:

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \quad and \quad f(U) \subset g(U).$$

Definition 2.3. *([15]) For each non-negative integer k, the q-number, denoted by $[k]_q$, is defined by*

$$[k]_q = \frac{1 - q^k}{1 - q} = 1 + q + q^2 + ... + q^{k-1},$$

where

$$[0]_q = 0, \quad [1]_q = 1, \quad [k]_q! = [1]_q[2]_q...[k]_q$$

and

$$lim_{q \to 1^-}[k]_q = k.$$

Example 2.4.

$$[1]_{0.3} = 1, \quad [2]_{0.3} = 1.3, \quad [3]_{0.5} = 1.75, \quad [4]_{0.7} = 2.533$$

and

$$[3]_{0.8} = 2.44, \quad [5]_{0.9} = 4.0951, \quad [3]_{0.9}! = [3]_{0.9}[2]_{0.9}[1]_{0.9} = 4.61.$$

Example 2.5. *For non-negative integers r and s:*

$$[r + s]_q = [r]_q + q^r[s]_q = q^s[r]_q + [s]_q, \quad [r - s]_q = q^{-s}[r]_q - q^{-s}[s]_q.$$

In [16], Jackson defined the q-derivative and q-integral D_q: $\mathcal{A} \to \mathcal{A}$ as follows:

Definition 2.6. *([15]) The q-derivative operator of $f(z)$ is defined by the formula*

$$D_q f(z) = \frac{f(z) - f(qz)}{(1 - q)z}, \quad q \in (0, 1), \quad z \in U$$

and the q-integral is defined by the formula

$$\int_0^z f(t) d_q t = z(1 - q) \sum_{n=0}^{\infty} q^n f(zq^n),$$

provided that the series converges.

Therefore, for $f \in \mathcal{A}$, we conclude that:

$$D_q f(z) = 1 + \sum_{k=2}^{\infty} [k]_q a_k z^{[k-1]}$$

and

$$\int_0^z f(t) d_q t = \int_0^z \sum_{k=1}^{\infty} a_k t^k d_q t = \frac{z^2}{[2]_q} + \sum_{k=2}^{\infty} \frac{a_k}{[k+1]_q} z^{k+1}.$$

Let $\mathcal{P}$ be the class of functions $\phi(z)$, which map the unit disk U analytically onto the right-half plane. These functions play a major role in the field of geometric function theory. Many fundamental results have been established in regard to this class of functions. Any function ϕ belonging to the class $\mathcal{P}$ has the following representation form

$$\phi(z) = 1 + p_1 z + p_2 z^2 + ... + p_k z^k + ... = 1 + \sum_{k=1}^{\infty} p_k z^k,$$

such that

$$\phi(0) = 1$$

and

$$Re(\phi(z)) > 0, \quad z \in U.$$

These functions are usually called the Caratheodory functions or functions with a positive real part. We shall utilize these kind of functions with the q-derivative using the subordination concept in the following two definitions:

Definition 2.7. *Let $\phi \in \mathcal{P}$, $0 \leq \gamma < 1$. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}_q(\gamma, \phi)$, if and only if*

$$\frac{1}{1-\gamma}\left(\frac{zD_q(f(z))}{f(z)} - \gamma\right) \prec \phi(z), \tag{2.1}$$

where D_q is the q-derivative operator.

Definition 2.8. *Let $\phi \in \mathcal{P}$, $0 \leq \gamma < 1$. A function $f \in \mathcal{A}$ is said to be in the class $C_q(\gamma, \phi)$, if and only if*

$$\frac{1}{1-\gamma}\left(1 + \frac{qz\, D_q^2(f(z))}{D_q(f(z))} - \gamma\right) \prec \phi(z), \tag{2.2}$$

where D_q is the q-derivative operator.

We note that for special values of the parameter γ and the function ϕ, with $(q \to -1)$, we obtain the famous classes as follows:

(i)

$$\lim_{q \to 1^-} \mathcal{S}_q(0, \phi) = \mathcal{S}(\phi)$$

and

$$\lim_{q \to 1^-} C_q(0, \phi) = C(\phi),$$

then,

$$\left(\frac{z(f^`(z))}{f(z)}\right) \prec \phi(z), \quad \left(\frac{z(f^{``}(z))}{f^`(z)}\right) \prec \phi(z).$$

(ii)

$$\lim_{q\to 1^-} \mathcal{S}_q(0, \frac{1+z}{1-z}) = \mathcal{S}$$

and

$$\lim_{q\to 1^-} C_q(0, \frac{1+z}{1-z}) = C,$$

then,

$$\left(\frac{z(f^{`}(z))}{f(z)}\right) \prec \frac{1+z}{1-z}, \quad 1+\left(\frac{z(f^{``}(z))}{f^{`}(z)}\right) \prec \frac{1+z}{1-z}.$$

(iii)

$$\lim_{q\to 1^-} \mathcal{S}_q(\gamma, \frac{1+z}{1-z}) = \mathcal{S}(\gamma)$$

and

$$\lim_{q\to 1^-} C_q(\gamma, \frac{1+z}{1-z}) = C(\gamma),$$

then,

$$\frac{1}{1-\gamma}\left(\frac{zf^{`}(z)}{f(z)} - \gamma\right) \prec \frac{1+z}{1-z}, \quad \frac{1}{1-\gamma}\left(1+\frac{zf^{``}(z)}{f^{`}(z)} - \gamma\right) \prec \frac{1+z}{1-z}.$$

We recall the q-differential operator $\mathcal{R}_q^\lambda$, which was introduced in [5] and is also referred to as the q-analogue of the Rusheweyh operator, defined as follows:

$$\mathcal{R}_q^\lambda f(z) = z + \sum_{k=2}^{\infty} \frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!} a_k z^k, \tag{2.3}$$

where $f \in A$, $\lambda > -1$, and $q \in (0, 1)$.

As $q \to 1^{-1}$, we observe

$$\begin{aligned}\lim_{q\to 1} \mathcal{R}_q^\lambda f(z) &= z + \lim_{q\to 1}\left[\sum_{k=2}^{\infty} \frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!} a_k z^k\right] \\ &= z + \sum_{k=2}^{\infty} \frac{(k+\lambda-1)!}{(\lambda)!(k-1)!} a_k z^k \\ &= \mathcal{R}^\lambda f(z),\end{aligned}$$

where $\mathcal{R}^\lambda$ is the most familiar Ruscheweyh differential operator.

It can also be shown that this q-operator is q-hypergeometric in nature as

$$\mathcal{R}_q^\lambda f(z) = z\, {}_2\Phi_1(q^{\lambda+1}, q, q, q; z) * f(z),$$

where ${}_2\Phi_1$ is the Gauss q-hypergeometric function (see [17]).

The identity relation can be derived from Eq (2.3)

$$q^\lambda z\left(D_q(\mathcal{R}_q^\lambda f(z))\right) = [\lambda+1]_q \mathcal{R}_q^{\lambda+1} f(z) - [\lambda]_q \mathcal{R}_q^\lambda f(z). \tag{2.4}$$

To discuss our main results, we state the following lemma:

Lemma 2.9. *[18] Let $h(z)$ be convex in U with $h(0) = 1$, and let P: $U \to \mathbb{C}$ with $Re(P(z)) > 0$ in U. If*

$$p(z) = 1 + p_1(z) + p_2(z) + ...$$

is analytic in U, then

$$p(z) + P(z) * zD_q p(z) \prec h(z)$$

implies that $p(z) \prec h(z)$.

3. Results

This section presents the main results, which include introducing two new classes of analytic functions $\mathcal{S}_q^\lambda(\gamma;\phi)$ and $C_q^\lambda(\gamma;\phi)$ along with their inclusion relation, integral preserving properties, and convolution properties.

First, utilizing the q-operator $\mathcal{R}_q^\lambda$, we define distinct classes of analytic functions for $\phi \in P$ and $0 \leq \gamma < 1$,

$$\mathcal{S}_q^\lambda(\gamma;\phi) = \left\{f \in A : \mathcal{R}_q^\lambda f \in \mathcal{S}_q(\gamma;\phi)\right\} \tag{3.1}$$

and

$$C_q^\lambda(\gamma;\phi) = \left\{f \in A : \mathcal{R}_q^\lambda f \in C_q(\gamma;\phi)\right\}. \tag{3.2}$$

Moreover, we observe that

$$f \in C_q^\lambda(\gamma;\phi) \Leftrightarrow zD_q(f(z)) \in \mathcal{S}_q^\lambda(\gamma;\phi). \tag{3.3}$$

Next, we prove the following lemma with the help of Lemma 2.9.

Lemma 3.1. *Let β and η be complex numbers with $\beta \neq 0$ and let $\psi(z)$ be convex in U with*

$$\psi(0) = 1$$

and

$$\Re(\beta\psi(z) + \eta) > 0.$$

If

$$\mathfrak{p}(z) = 1 + \mathfrak{p}_1(z) + \mathfrak{p}_2(z) + \ldots$$

is analytic in U, then

$$\mathfrak{p}(z) + \frac{zD_q(\mathfrak{p}(z))}{\beta\mathfrak{p}(z) + \eta} \prec \psi(z)$$

implies that

$$\mathfrak{p}(z) \prec \psi(z).$$

Proof. By setting

$$P(z) = \frac{1}{\beta p(z) + \eta},$$

and we have

$$Re(\Re(\beta\psi(z) + \eta) > 0),$$

then

$$\Re(P(z)) > 0.$$

By Lemma 2.9, we get

$$\mathfrak{p}(z) \prec \psi(z).$$

This completes the proof. □

3.1. Inclusion relations

Theorem 3.2. *Let* $0 \leq \gamma < 1$ *and* $\phi \in \mathfrak{P}$ *with*

$$\Re(\phi(z)) > \max\left\{0, -\frac{[\lambda]_q/q^\lambda + \gamma}{1-\gamma}\right\}. \tag{3.4}$$

Then

$$\mathcal{S}_q^{\lambda+1}(\gamma;\phi) \subset \mathcal{S}_q^{\lambda}(\gamma;\phi). \tag{3.5}$$

Proof. Let $f \in \mathcal{S}_q^{\lambda+1}(\gamma;\phi)$ and suppose that

$$\psi(z) = \frac{1}{1-\gamma}\left(\frac{zD_q(\mathcal{R}_q^\lambda f(z))}{\mathcal{R}_q^\lambda f(z)} - \gamma\right) \quad (z \in U). \tag{3.6}$$

Then ψ is analytic in U with

$$\psi(0) = 1$$

and

$$\phi(z) \neq 0$$

for all $z \in U$. Combining Eqs (2.4) and (3.6), we get

$$([\lambda+1]_q/q^\lambda)\frac{\mathcal{R}_q^{\lambda+1} f(z)}{\mathcal{R}_q^\lambda f(z)} = (1-\gamma)\psi(z) + ([\lambda]_q/q^\lambda) + \gamma. \tag{3.7}$$

By employing logarithmic q-differentiation on both sides of Eqs (3.7) and (3.6), we obtain

$$\frac{\log\ q}{q-1}\left[\frac{1}{1-\gamma}\left(\frac{zD_q(\mathcal{R}_q^{\lambda+1} f(z))}{\mathcal{R}_q^{\lambda+1} f(z)} - \gamma\right)\right] = \frac{\log\ q}{q-1}\left[\psi(z) + \frac{zD_q(\psi(z))}{(1-\gamma)\psi(z) + ([\lambda]_q/q^\lambda) + \gamma}\right] \prec \phi(z). \tag{3.8}$$

Given the validity of Eq (3.4), applying Lemma 3.1 to Eq (3.8) results in

$$\psi(z) = \frac{1}{1-\gamma}\left(\frac{zD_q(\mathcal{R}_q^\lambda f(z))}{\mathcal{R}_q^\lambda f(z)} - \gamma\right) \prec \phi(z),$$

that is $f \in \mathcal{S}_q^\lambda(\gamma;\phi)$, which implies that the assertion (3.5) of Theorem 3.2 holds. □

Theorem 3.3. *Let* $0 \leq \gamma < 1$ *and* $\phi \in \mathcal{P}$. *Then*

$$C_q^{\lambda+1}(\gamma;\phi) \subset C_q^{\lambda}(\gamma;\phi). \tag{3.9}$$

Proof. Applying the relation (3.3) and Theorem 3.2, we have

$$\begin{aligned} f \in C_q^{\lambda+1}(\gamma;\phi) &\Leftrightarrow \mathcal{R}_q^{\lambda+1} f \in C_q(\gamma;\phi) \Leftrightarrow z(D_q(\mathcal{R}_q^{\lambda+1} f)) \in \mathcal{S}_q(\gamma;\phi) \\ &\Leftrightarrow \mathcal{R}_q^{\lambda+1}(zD_q(f)) \in \mathcal{S}_q(\gamma;\phi) \Leftrightarrow zD_q(f) \in \mathcal{S}_q^{\lambda+1}(\gamma;\phi) \\ &\Rightarrow zD_q(f) \in \mathcal{S}_q^{\lambda}(\gamma;\phi) \Leftrightarrow \mathcal{R}_q^{\lambda}(zD_q(f)) \in \mathcal{S}_q(\gamma;\phi) \\ &\Leftrightarrow z(D_q(\mathcal{R}_q^{\lambda} f)) \in \mathcal{S}_q(\gamma;\phi) \Leftrightarrow \mathcal{R}_q^{\lambda} f \in C_q(\gamma;\phi) \\ &\Leftrightarrow f \in C_q^{\lambda}(\gamma;\phi), \end{aligned}$$

which evidently proves Theorem 3.3. □

In this place, if we set

$$\phi(z) = \frac{1+z}{1-z}$$

in Theorems 3.2 and 3.3, we have the following consequence:

Corollary 3.4. *Let $\lambda > -1, 0 \le \gamma < 1$. Then*

$$\mathcal{S}_q^{\lambda+1}(\gamma; \frac{1+z}{1-z}) \subset \mathcal{S}_q^{\lambda}(\gamma; \frac{1+z}{1-z})$$

and

$$C_q^{\lambda+1}(\gamma; \frac{1+z}{1-z}) \subset C_q^{\lambda}(\gamma; \frac{1+z}{1-z}).$$

3.2. Integral preserving properties

In this section, we discuss some integral preserving properties for the q-integral operator defined in [10].

Theorem 3.5. *Let $f \in \mathcal{S}_q^{\lambda}(\gamma;\phi)$ with*

$$\Re((1-\gamma)\phi(z) + [\mu]_q/q^{\mu} + \gamma) > 0 \quad (z \in U).$$

Then $F(f) \in \mathcal{S}_q^{\lambda}(\gamma;\phi)$, where F_z is the q-Bernardi integral operator defined by

$$F(f)(z) = \frac{[\mu+1]_q}{z^{\mu}} \int_0^z t^{\mu-1} f(t) d_q t \quad (z \in U; \mu > -1). \tag{3.10}$$

Proof. Let $f \in \mathcal{S}_q^{\lambda}(\gamma;\phi)$. Then from Eq (3.10), we find that

$$z\, D_q(\mathcal{R}_q^{\lambda} F)(z) + [\mu]_q \mathcal{R}_q^{\lambda} F(z) = [\mu+1]_q \mathcal{R}_q^{\lambda} f(z). \tag{3.11}$$

By setting

$$\mathfrak{p}(z) = \frac{1}{1-\gamma}\left(\frac{z\, D_q(\mathcal{R}_q^{\lambda} F)(z)}{\mathcal{R}_q^{\lambda} F(z)} - \gamma\right), \tag{3.12}$$

we observe that $\mathfrak{p}$ is analytic in $\mathbb{U}$ with

$$\mathfrak{p}(0) = 1$$

and

$$\mathfrak{p}(z) \neq 0$$

for all $z \in U$. It follows from Eqs (3.11) and (3.12) that

$$\gamma + [\mu]_q/q^\mu + (1-\gamma)\mathfrak{p}(z) = [\mu+1]_q \frac{\mathcal{R}_q^\lambda f(z)}{\mathcal{R}_q^\lambda F(z)}. \tag{3.13}$$

Applying logarithmic q-differentiation on both sides of Eq (3.13), and using Eq (3.12), we obtain

$$\frac{\log}{q-1}\left[\mathfrak{p}(z) + \frac{zD_q(\mathfrak{p}(z))}{\gamma + [\mu]_q/q^\mu + (1-\gamma)\mathfrak{p}(z)}\right] = \frac{\log}{q-1}\left[\frac{1}{1-\gamma}\left(\frac{z\, D_q(\mathcal{R}_q^\lambda f)(z)}{\mathcal{R}_q^\lambda f(z)} - \gamma\right)\right] \prec \phi(z). \tag{3.14}$$

Since

$$\Re((1-\gamma)\phi(z) + [\mu]_q/q^\mu + \gamma) > 0 \ \ (z \in U),$$

an application of Lemma 3.1 to Eq (3.14) yields

$$\frac{1}{1-\gamma}\left(\frac{z\, D_q(\mathcal{R}_q^\lambda F)(z)}{\mathcal{R}_q^\lambda F(z)} - \gamma\right) \prec \phi(z)$$

and we readily deduce that the assertion of Theorem 3.5 holds true, which means that $F_z \in \mathcal{S}_q^\lambda(\gamma;\phi)$. This completes the proof. □

In the same manner of Theorem 3.3, one can get the next result:

Corollary 3.6. *Let $f \in C_q^\lambda(\gamma;\phi)$. Then $F_z(f) \in C_q^\lambda(\gamma;\phi)$.*

Theorem 3.7. *Let F be defined by (3.10). If $f \in \mathcal{S}_q^\lambda(\gamma;\phi)$, $\alpha > 0$, and*

$$\Re[D_q^{\mathfrak{n}}(\mathcal{R}_q^\lambda f(z))] \geq \alpha |zD_q^{\mathfrak{n}+1}(\mathcal{R}_q^\lambda f(z))|, \ \ \textit{for all} \ \ z \in U,$$

then

$$\begin{aligned}&\left|([\mu]_q + q^\mu[\mathfrak{n}]_q)D_q^{\mathfrak{n}}(\mathcal{R}_q^\lambda F(z)) + q^{\mu+\mathfrak{n}} zD_q^{\mathfrak{n}+1}(\mathcal{R}_q^\lambda F(z))\right| \\ &\geq \alpha\left|([\mu]_q + q^\mu[\mathfrak{n}+1]_q)D_q^{\mathfrak{n}+1}(\mathcal{R}_{\mathfrak{q}}^\lambda F(z)) + q^{\mu+\mathfrak{n}+2} zD_q^{\mathfrak{n}+2}(\mathcal{R}_q^\lambda F(z))\right|.\end{aligned}$$

Proof. By employing the operator $\mathcal{R}_q^\lambda$, we get

$$\frac{z^\mu}{[\mu+1]_q}(\mathcal{R}_q^\lambda F(z)) = \int_0^z t^{\mu-1}(\mathcal{R}_q^\lambda f(t))d_q t.$$

By taking the q-derivative, we have

$$\frac{1}{[1+\mu]_q}\left[q^\mu z^\mu D_q(\mathcal{R}_q^\lambda(F(z)) + [\mu]_q z^{\mu-1}(\mathcal{R}_q^\lambda F(z))\right] = z^{\mu-1}(\mathcal{R}_q^\lambda f(z)).$$

This relation is equivalent to

$$\frac{[\mu]_q}{[1+\mu]_q}(\mathcal{R}_q^\lambda F(z)) + \frac{q^\mu}{[1+\mu]_q} zD_q(\mathcal{R}_q^\lambda F(z)) = (\mathcal{R}_q^\lambda f(z)),$$

which implies that

$$\frac{[\mu]_q}{[1+\mu]_q}D_q(\mathcal{R}_q^\lambda F(z)) + \frac{q^{\mu+1}}{[1+\mu]_q}zD_q^2(\mathcal{R}_q^\lambda F(z)) + \frac{q^\mu}{[1+\mu]_q}D_q(\mathcal{R}_q^\lambda F(z)) = D_q(\mathcal{R}_q^\lambda f(z)),$$

and this is equivalent to

$$D_q(F(z)) + \frac{q^{\mu+1}}{[1+\mu]_q}zD_q^2(F(z)) = D_q(f(z)).$$

We obtain that

$$\frac{[\mu]_q + q^\mu[2]_q}{[\mu+1]_q}D_q^2(\mathcal{R}_q^\lambda F(z)) + \frac{q^{\mu+2}}{[1+\mu]_q}zD_q^3(\mathcal{R}_q^\lambda F(z)) = D_q^2(\mathcal{R}_q^\lambda f(z))$$
$$\frac{[\mu]_q + q^\mu[n]_q}{[\mu+1]_q}D_q^{\mathfrak{n}}(\mathcal{R}_q^\lambda F(z)) + \frac{q^{\mu+\mathfrak{n}}}{[\mu+1]_q}zD_q^{\mathfrak{n}+1}(\mathcal{R}_q^\lambda F(z)) = D_q^{\mathfrak{n}}(\mathcal{R}_q^\lambda f(z))$$

and

$$\frac{[\mu]_q + q^\mu[\mathfrak{n}+1]_q}{[\mu+1]_q}D_q^{\mathfrak{n}+1}(\mathcal{R}_q^\lambda F(z)) + \frac{q^{\mu+\mathfrak{n}+1}}{[\mu+1]_q}zD_q^{\mathfrak{n}+2}(\mathcal{R}_q^\lambda F(z)) = D_q^{\mathfrak{n}+1}(\mathcal{R}_q^\lambda f(z)).$$

If

$$\Re(D_q^{\mathfrak{n}}(\mathcal{R}_q^\lambda f(z)) \geq \alpha|zD_q^{\mathfrak{n}+1}(\mathcal{R}_q^\lambda f(z))|$$

for all $z \in U$, then

$$|D_q^{\mathfrak{n}}(\mathcal{R}_q^\lambda f(z))| \geq \alpha|zD_q^{\mathfrak{n}+1}(\mathcal{R}_q^\lambda f(z))|.$$

We employ $D_q^{\mathfrak{n}}$ and $D_q^{\mathfrak{n}+1}$ in the last inequality, we obtain

$$\Leftrightarrow \frac{1}{|[\mu+1]_q|}\left|([\mu]_q + q^\mu[\mathfrak{n}]_q)D_q^{\mathfrak{n}}(\mathcal{R}_q^\lambda F(z)) + q^{\mu+\mathfrak{n}}z\mathfrak{G}_{\mathfrak{q}}^{\mathfrak{n}+1}(\mathcal{R}_q^\lambda F(z))\right|$$
$$\geq \frac{\alpha}{|[\mu+1]_q|}\left|([\mu]_q + q^\mu[\mathfrak{n}+1]_q)zD_q^{\mathfrak{n}+1}(\mathcal{R}_q^\lambda F(z)) + q^{\mu+\mathfrak{n}+2}zD_q^{\mathfrak{n}+2}(\mathcal{R}_q^\lambda F(z))\right|$$

Hence, the proof is complete. □

3.3. Convolution properties

In this section, we derive certain convolution properties for the class $\mathcal{S}_q^\lambda(\gamma;\phi)$.

Theorem 3.8. *Let* $f \in \mathcal{S}_q^\lambda(\gamma;\phi)$. *Then*

$$f(z) = e^{\left(-\frac{q-1}{\log\ q}\right)}ze^{\left((1-\gamma)\int_0^z \frac{\phi(w(\zeta))-1}{\zeta}d_q\zeta\right)} * \left(z + \sum_{k=2}^{\infty}\frac{[\lambda]_q![k-1]_q!}{[k+\lambda-1]_q!}z^k\right), \tag{3.15}$$

where w *is analytic in* U *with*

$$w(z) = 0$$

and

$$|w(z)| < 1.$$

Proof. Suppose that $f \in \mathcal{S}_q^\lambda(\gamma;\phi)$ and Eq (3.1) can be written as follows:

$$\frac{zD_q(\mathcal{R}_q^\lambda f(z))}{\mathcal{R}_q^\lambda f(z)} = (1-\gamma)\phi(w(z)) + \gamma, \tag{3.16}$$

where w is analytic in U with

$$w(z) = 0$$

and

$$|w(z)| < 1 \quad (z \in U).$$

We now find from Eq (3.16) that

$$\frac{D_q(\mathcal{R}_q^\lambda f(z))}{\mathcal{R}_q^\lambda f(z)} = \frac{(1-\gamma)\phi(w(z))}{z} + \frac{\gamma}{z} - \frac{1}{z} + \frac{1}{z}, \tag{3.17}$$

$$\frac{D_q(\mathcal{R}_q^\lambda f(z))}{\mathcal{R}_q^\lambda f(z)} - \frac{1}{z} = (1-\gamma)\frac{\phi(w(z))-1}{z}, \tag{3.18}$$

which, upon q-integration, yields

$$\int_0^z \left[\frac{D_q(\mathcal{R}_q^\lambda f(\zeta))}{\mathcal{R}_q^\lambda f(\zeta)} - \frac{1}{\zeta}\right] d_q\zeta = (1-\gamma)\int_0^z \frac{\phi(w(\zeta))-1}{\zeta} d_q\zeta, \tag{3.19}$$

$$\frac{q-1}{\log\ q}\log\left(\frac{\mathcal{R}_q^\lambda f(z)}{z}\right) = (1-\gamma)\int_0^z \frac{\phi(w(\zeta))-1}{\zeta} d_q\zeta. \tag{3.20}$$

It follows from Eq (3.20) that

$$\mathcal{R}_q^\lambda f(z) = e^{\left(-\frac{q-1}{\log\ q}\right)} z e^{\left((1-\gamma)\int_0^z \frac{\phi(w(\zeta))-1}{\zeta} d_q\zeta.\right)}. \tag{3.21}$$

Now if we convolute both sides of Eq (3.21) by the expression $\left(z+\sum_{k=2}^{\infty}\frac{[\lambda]_q![k-1]_q!}{[k+\lambda-1]_q!}z^k\right)$, the assertion (3.15) of Theorem 3.8 is obtained. □

In the following, we derive a result related to functions in the class $\mathcal{S}_q^\lambda(\gamma;\phi)$ by examining their convolution with a special analytic function.

Theorem 3.9. *Let $f \in \mathcal{A}$ and $\phi \in \mathcal{P}$. Then $f \in \mathcal{S}_q^\lambda(\gamma;\phi)$ if and only if*

$$\left\{f * \left(z + \sum_{k=2}^{\infty}[k]_q\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}z^k\right)\right\} - \left[(1-\gamma)\phi(e^{if}) + \gamma\right]\left\{f * \left(z + \sum_{k=2}^{\infty}\frac{[k+\lambda-1]_q!}{[\lambda]_q![k-1]_q!}z^k\right)\right\} \neq 0. \tag{3.22}$$

Proof. Suppose that $f \in \mathcal{S}_q^\lambda(\gamma;\phi)$. Then Eq (3.1) is equivalent to

$$\frac{1}{1-\gamma}\left(\frac{z\,D_q(\mathcal{R}_q^\lambda f)(z)}{\mathcal{R}_q^\lambda f(z)} - \gamma\right) \neq \phi(e^{if}) \quad (z \in U; 0 \leq f < 2\pi). \tag{3.23}$$

The condition (3.23) can be written as follows:

$$\left\{zD_q(\mathcal{R}_q^\lambda f)(z)) - [(1-\gamma)\phi(e^{if}) + \gamma]\mathcal{R}_q^\lambda f(z)\right\} \neq 0 \quad (z \in \mathcal{U}; 0 \leq f < 2\pi), \tag{3.24}$$

$$\left\{f * zD_q(\mathcal{R}_q^\lambda)\right\} - [(1-\gamma)\phi(e^{if}) + \gamma]\left\{f * \mathcal{R}_q^\lambda\right\} \neq 0. \tag{3.25}$$

On the other hand, we have

$$zD_q(\mathcal{R}_q^\lambda f)(z) = z + \sum_{k=2}^{\infty} [k]_q \frac{[k+\lambda-1]_q!}{[\lambda]_q! \quad [k-1]_q!} a_k z^k, \tag{3.26}$$

$$\mathcal{R}_q^\lambda f(z) = z + \sum_{k=2}^{\infty} \frac{[k+\lambda-1]_q!}{[\lambda]_q!\,[k-1]_q!} a_k z^k. \tag{3.27}$$

Substituting Eqs (3.26) and (3.27) in Eq (3.25), we readily get the convolution property (3.22) asserted by Theorem 3.9.

The proof is complete. □

4. Conclusions

In this study, we discuss subclasses of starlike and convex functions associated with the q-Ruschewehy differential operator and the q-Bernardi integral operator. To define these q-classes of analytic functions, we use the concept of q-derivatives. For the newly defined classes, we investigate inclusion relations and integral preservation properties. We highlight several intriguing properties of convolution. In the future, this work will inspire other authors to make contributions in this area for numerous generalized subclasses of the q-classes of starlike and convex functions. Furthermore, extensions of the current work can be to consider higher-order q-derivatives or generalizations of the operators discussed. Lastly, they can apply these findings to multivalent and meromorphic functions.

Author contributions

K. R. Alhindi: conceptualization, validation, formal analysis, investigation, resources, writing–review and editing, visualization; K. M. K. Alshammari: resources, writing–review and editing; H. A. Aldweby: conceptualization, validation, formal analysis, investigation, resources, writing–original draft preparation, writing–review and editing. All authors have read and agreed to the published version of the manuscript.

Conflict of interest

The authors declare no conflicts of interest.

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