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Research article

On centrally extended mappings that are centrally extended additive

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Abstract: This paper aims to establish the following: Let Ω be a ring that satisfies some conditions and has an idempotent element $f \neq 0, 1$. We intend to show that if G is any suitable multiplicative generalized CE-derivation of Ω, then G is a centrally extended additive.

Keywords: ring; idempotent element; peirce decomposition; derivations; generalized derivation; centrally extended derivations
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1. Introduction

The investigation of centrally extended mappings in rings under certain conditions plays an increasingly important role in ring theory. The work of Bell and Daif [1] introduced the notion of centrally extended derivation as follows: Let Ω be a ring with center $\mathcal{Z}(\Omega)$. A map $\mathcal{D}$ of Ω is said to be a centrally extended derivation (CE-derivation) if, for all $v, u \in \Omega$, $\mathcal{D}(v+u) - \mathcal{D}(v) - \mathcal{D}(u) \in \mathcal{Z}(\Omega)$, and $\mathcal{D}(vu) - \mathcal{D}(v)u - v\mathcal{D}(u) \in \mathcal{Z}(\Omega)$. Moreover, they discussed the existence of such a map, which is not a derivation, as well as providing some findings regarding commutativity. Thenceforth, considerable findings about various types of maps have been discovered; for example, see [2–5].

The study of when multiplicative maps will be additive goes back to 1948, when Rickart [6] proved that any bijective and multiplicative mapping over a Boolean ring onto any arbitrary ring is additive. In 1969, the work of Martindale [7] was significant to generalize Rickart's main theorem when he demonstrated that every multiplicative isomorphism on a ring with a non-trivial idempotent is additive.

Inspired by Martindale's pioneering work, Daif [8] proved that a multiplicative derivation is additive under the existence of certain conditions on a ring.

Later on, the additivity of $n-$multiplicative maps on associative rings satisfying Martindale's conditions was proved by Wang [9]. In 2016 [10], Ferreira and Ferreira undertook a detailed study of a similar problem but within the framework of alternative rings. Inspired by these previous findings, [11] brilliantly proved the additivity of $n-$multiplicative isomorphisms and $n-$multiplicative derivations over Jordan rings. A great deal of work has been done in [12] and [13] concerning multiplicative left centralizer and multiplicative generalized derivations. Motivated by the role that centrally extended derivations play in the field of ring theory, we herein raised a question: When are multiplicative generalized CE-derivations additive?

The idea of a multiplicative generalized CE-derivation (MGCE-derivation) of a ring Ω is introduced in this note. This concept is defined as a mapping G of Ω into Ω so that $G(vu)-G(v)u-v\mathcal{D}(u) \in \mathcal{Z}(\Omega)$, $\forall\ v, u \in \Omega$, where $\mathcal{D} : \Omega \to \Omega$ is a CE-derivation. In other words, the maps G and $\mathcal{D}$ can be expressed as $G(vu) = G(v)u + v\mathcal{D}(u) + \delta(v, u)$ and $\mathcal{D}(vu) = \mathcal{D}(v)u + v\mathcal{D}(u) + \sigma(v, u)$, where $\delta(v, u)$ and $\sigma(v, u)$ are elements in $\mathcal{Z}(\Omega)$ and related with the mappings G and $\mathcal{D}$, respectively. For any ring Ω, a map $G : \Omega \to \Omega$ is called centrally extended additive (CE-additive) so that $G(v + u) - G(v) - G(u) \in \mathcal{Z}(\Omega)$, $\forall\ v, u \in \Omega$.

In this paper, we aim to find the answer to the following question: "Under what conditions does a multiplicative generalized CE-derivation become a centrally extended additive?" We will give a response to this query under appropriate circumstances.

2. Preliminaries

Throughout this paper, let Ω be a ring that does not necessarily have a unity, and let $f \in \Omega$ be an idempotent element such that $f \neq 1, f \neq 0$. Formally, we will set $f_1 = f$ and $f_2 = 1 - f$. The Peirce decomposition of Ω concerning the idempotent f can be expressed as $\Omega = f_1\Omega f_1 \oplus f_1\Omega f_2 \oplus f_2\Omega f_1 \oplus f_2\Omega f_2$. By letting $\Omega_{ij} = f_i\Omega f_j$: $i, j = 1, 2$, we could write $\Omega = \Omega_{11} \oplus \Omega_{12} \oplus \Omega_{21} \oplus \Omega_{22}$ (For further information, see Jacobson 1964 [14], Page 49). An element within the subring Ω_{ij} will be indicated by r_{ij}. If $\lambda = \lambda_{11} + \lambda_{12} + \lambda_{21} + \lambda_{22} \in \mathcal{Z}(\Omega)$, where $f\lambda = \lambda f$, then $\lambda_{12} = \lambda_{21} = 0$. Hence, we can conclude that $\mathcal{Z}(\Omega) \subseteq \Omega_{11} \oplus \Omega_{22}$. Additionally, we denote by $\mathcal{Z}_{ii}$ the subring $\Omega_{ii} \cap \mathcal{Z}(\Omega)$.

Applying the definition of $\mathcal{D}$, we observe that $\mathcal{D}(0) = \sigma(0, 0) \in \mathcal{Z}(\Omega)$. But $\mathcal{D}(0)\Omega$ is an ideal contained in the center of Ω. Since $f\mathcal{D}(0) \in \mathcal{Z}(\Omega)$, and $\sigma(0, 0) = \sigma_{11}(0, 0) + \sigma_{22}(0, 0)$, we have $f\mathcal{D}(0) = \sigma_{11}(0, 0) \in \mathcal{Z}_{11}$, and this provides $\sigma_{22}(0, 0) \in \mathcal{Z}_{22}$. Similarly, $G(0)\Omega$ is an ideal contained in the center of Ω, and $\delta_{11}(0, 0) \in \mathcal{Z}_{11}$, and $\delta_{22}(0, 0) \in \mathcal{Z}_{22}$.

Moreover, $\mathcal{D}(f) = \mathcal{D}(f^2) = \mathcal{D}(f)f + f\mathcal{D}(f) + \varphi$; $\varphi = \sigma(f, f) \in \mathcal{Z}(\Omega)$. If we express $\mathcal{D}(f) = d_{11} + d_{12} + d_{21} + d_{22}$ and apply the two ways of $\mathcal{D}(f)$, then we obtain $d_{22} = \varphi_{22}$ and $d_{11} = -\varphi_{11}$. Consequently, we have

$$\mathcal{D}(f) = d_{12} + d_{21} - \varphi_{11} + \varphi_{22}. \tag{2.1}$$

In a similar way, if $G : \Omega \to \Omega$ is a multiplicative generalized CE-derivation related with a CE-derivation $\mathcal{D}$, then $G(f) = G(f^2) = G(f)f + f\mathcal{D}(f) + \psi$, where $\psi = \delta(f, f) \in \mathcal{Z}(\Omega)$ and it is possible to write $G(f) = g_{11} + g_{12} + g_{21} + g_{22}$. By making use of the values of $G(f)$ and $\mathcal{D}(f)$, we conclude that $\psi_{11} = \varphi_{11}$, $g_{22} = \psi_{22}$ and $g_{12} = d_{12}$, so

$$G(f) = g_{11} + d_{12} + g_{21} + \psi_{22}. \tag{2.2}$$

To finish our task, we will need the following two facts:

Lemma 2.1. *$\psi_{ii} \in \mathcal{Z}_{ii}$ and $\varphi_{ii} \in \mathcal{Z}_{ii}$, for $i \in \{1, 2\}$.*

Proof. For all $r \in \Omega$, by figuring out the two sides of $G(fr) = G(f(fr))$, we obtain

$$G(f)r + \delta(f, r) = G(f)fr + f\mathcal{D}(f)r + f\sigma(f, r) + \delta(f, fr). \tag{2.3}$$

Now using (2.1) and (2.2) in (2.3), we obtain $\psi r = f\sigma(f, r) + \delta(f, fr) - \delta(f, r)$, where $\varphi_{11} = \psi_{11}$ and this means

$$\psi r \in \Omega_{11} \oplus \Omega_{22}. \tag{2.4}$$

Now, if we rewrite r as $r = r_{11} + r_{12} + r_{21} + r_{22}$ and using that $\psi \in \mathcal{Z}(\Omega)$, we obtain $\psi_{11}r_{11} = r_{11}\psi_{11}$ and $\psi_{22}r_{22} = r_{22}\psi_{22}$, which implies $\psi_{11} \in \mathcal{Z}(\Omega_{11})$ and $\psi_{22} \in \mathcal{Z}(\Omega_{22})$. And again (2.4) gives $\psi_{11}r_{12} + \psi_{22}r_{21} = 0$ and $r_{12}\psi_{22} + r_{21}\psi_{11} = 0$, which gives $\psi_{11}r_{12} = \psi_{22}r_{21} = 0$ and $r_{12}\psi_{22} = r_{21}\psi_{11} = 0$, this means that ψ is a left and right annihilator of the two subrings Ω_{12} and Ω_{21}. Now for any $r \in \Omega$, $\psi_{11}r = \psi_{11}r_{11} = r_{11}\psi_{11} = r\psi_{11}$, which gives $\psi_{11} \in \mathcal{Z}(\Omega)$. Since $\psi_{22} = \psi - \psi_{11}$, $\psi_{22} \in \mathcal{Z}(\Omega)$. Also, we obtain $\varphi_{11} = \psi_{11} \in \mathcal{Z}(\Omega)$, and $\varphi_{22} = (\varphi - \varphi_{11}) \in \mathcal{Z}(\Omega)$. □

To obtain our primary outcome, we presuppose that the ring Ω has an idempotent f and that Ω satisfies the following requirements:

(L_1) $\alpha\Omega f \subset \mathcal{Z}(\Omega)$ implies that $\alpha \in \mathcal{Z}(\Omega)$.
(L_2) $\alpha f\Omega(1 - f) \subset \mathcal{Z}(\Omega)$ implies that $\alpha \in \mathcal{Z}(\Omega)$.

And G is any multiplicative generalized CE-derivation of Ω related with a CE-derivation $\mathcal{D}$ of Ω.

Let us now present some examples of rings that meet the conditions (L_1) and (L_2), as well as those that do not meet these requirements.

Example 2.1. *Let $\Omega = M_2(\mathbb{C})$, the ring of 2×2 matrices over the field $\mathbb{C}$ of complex numbers. Taking $f = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \Omega$, which is a nontrivial idempotent element. Let $\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \Omega$. It is clear that $\alpha\beta f \in \mathcal{Z}(\Omega)$, and $\alpha f\beta(1 - f) \in \mathcal{Z}(\Omega)$ for all $\beta \in \Omega$ whenever $\alpha \notin \mathcal{Z}(\Omega)$. That is, this ring neither satisfy (L_1) nor (L_2).*

Example 2.2. *Let's take $M_2(\mathbb{H})$, the ring of 2×2 matrices over the quaternions $\mathbb{H}$. Let $f = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{H})$. If $\alpha\Omega f \subseteq \mathcal{Z}(\Omega)$, then α must be in the form $\alpha = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ which means $\alpha \in \mathcal{Z}(\Omega)$ and if $\alpha f\Omega(1 - f) \subseteq \mathcal{Z}(\Omega)$, then α must be in the form $\alpha = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}$ which means $\alpha \in \mathcal{Z}(\Omega)$. Thus, if $\alpha\Omega f \subseteq \mathcal{Z}(\Omega)$ or $\alpha f\Omega(1 - f) \subseteq \mathcal{Z}(\Omega)$ then $\alpha \in \mathcal{Z}(\Omega)$. That is, this ring satisfies both of (L_1) and (L_2).*

In the subsequent proofs, the following lemma is helpful.

Lemma 2.2. *The ideals $\Omega\psi$, $\Omega\psi_{ii}$, $\Omega\varphi$, $\Omega\varphi_{ii}$, and $\Omega\bar{\varphi}$ are contained in the center of Ω, in which $\psi = \delta(f, f) \in \mathcal{Z}(\Omega)$, $\varphi = \sigma(f, f) \in \mathcal{Z}(\Omega)$, and $\bar{\varphi} = \varphi_{22} - \varphi_{11} \in \mathcal{Z}(\Omega)$, where $i \in \{1, 2\}$.*

Proof. Starting with Lemma 2.1, for each $r_{11} \in \Omega_{11}$, we obtain $\psi r_{11} r_{12} = r_{11}\psi r_{12} = 0 \in \mathcal{Z}(\Omega)$ and using condition (L_2), we obtain $r_{11}\psi = \psi r_{11} \in \mathcal{Z}(\Omega)$.

Second, assume that $\mathcal{D}(r_{22}) = c_{11} + c_{12} + c_{21} + c_{22}$, and since $G(fr_{22}) = G(0) \in \mathcal{Z}(\Omega)$, using (2.2), we have $G(0) = G(f)r_{22} + f\mathcal{D}(r_{22}) + \delta(f, r_{22}) = d_{12}r_{22} + \psi_{22}r_{22} + c_{11} + c_{12} + \delta(f, r_{22})$, and this gives $d_{12}r_{22} + c_{12} = 0$ and $\psi_{22}r_{22} = \beta - c_{11}$, where $\beta = (G(0) - \delta(f, r_{22})) \in \mathcal{Z}(\Omega)$. Now, using Lemma 2.1, for any $s \in \Omega$, we get $\psi_{22}r_{22}s = r_{22}\psi_{22}s_{22} = \psi_{22}r_{22}s_{22} = (\beta - c_{11})s_{22} = \beta s_{22} = s_{22}\beta = s_{22}(\beta - c_{11}) = s_{22}\psi_{22}r_{22} = s\psi_{22}r_{22} = sr_{22}\psi_{22}$, and this gives $r_{22}\psi_{22} \in \mathcal{Z}(\Omega)$. And additionally, if $s, r \in \Omega$, then $rs\psi = r(s_{11}\psi + s_{22}\psi) = rs_{11}\psi + rs_{22}\psi = s_{11}\psi r + s_{22}\psi r = (s_{11} + s_{22})\psi r = s\psi r$. The other situations can be proven similarly. □

Using any fixed element d in Ω, we may construct an example of a CE-derivation, the map $\mathcal{D}_d : \Omega \to \Omega$ that fulfills $\mathcal{D}_d(r) - [r, d] \in \mathcal{K}$, where $\mathcal{K}$ is an ideal contained in the center of Ω, we may refer to it as an inner CE-derivation. At this point, with the use of Lemma 2.2 it is apparent that the map $\mathcal{D}_1$ given by $\mathcal{D}_1(s) = [s, d_{12} - d_{21}] + \bar{\varphi}$ is a CE-derivation, and applying (2.1), we obtain

$$\mathcal{D}_1(f) = d_{12} + d_{21} + \bar{\varphi} = \mathcal{D}(f). \tag{2.5}$$

Additionally, given any two fixed elements c and d in Ω, the map $G_{(c,d)} : \Omega \to \Omega$ that satisfies $G_{(c,d)}(r) - cr - rd \in \mathcal{N}$, where $\mathcal{N}$ is an ideal contained in the center of Ω, we may refer to it as an inner generalized CE-derivation related to the inner CE-derivation $\mathcal{D}_d$, which is given by $\mathcal{D}_d - [s, d] \in \mathcal{N}$.

Once more, applying Lemma 2.2, we can show that the map G_1 presented by $G_1(x) = (g_{11} + g_{21} - \psi_{11})x + x(d_{12} - d_{21}) + \psi$ is a generalized CE-derivation related to the inner CE-derivation $\mathcal{D}_1$, and with (2.2), we get,

$$G_1(f) = g_{11} + g_{21} + d_{12} + \psi_{22} = G(f). \tag{2.6}$$

For the sake of simplicity and without loss of generality, we will now substitute the CE-derivation $\mathcal{D}$ with the CE-derivation $\Phi = \mathcal{D} - \mathcal{D}_1$, which, by using (2.5), arrived us to $\Phi(f) = 0$ and the multiplicative generalized CE-derivation G by the multiplicative generalized CE-derivation $\Psi = G - G_1$ with $\Psi(f) = 0$, by (2.6). Also, $\Phi(0) = \mathcal{D}(0) - \mathcal{D}_1(0) = \mathcal{D}(0) - \bar{\varphi} = \theta \in \mathcal{Z}(\Omega)$ and $\Psi(0) = G(0) - G_1(0) = G(0) - \psi = \alpha \in \mathcal{Z}(\Omega)$. It is easy to show that both θ and α generate a central ideal in Ω.

The following lemmas are necessary for proving our primary theorem:

Lemma 2.3. *For any element $a_{ij} \in \Omega_{ij}$, there exists $b_{ij} \in \Omega_{ij}$ and $\rho_{ii}, \sigma_{ii} \in \mathcal{Z}_{ii},\ i, j \in \{1, 2\},\ i \neq j$ such that*
(1) $\Phi(a_{ii}) = b_{ii} + \rho_{jj}$, (2) $\Phi(a_{ij}) = b_{ij} + \rho_{ii} + \sigma_{jj}$.

Proof. In order to prove (1), We must prove two distinct cases:

(I) Suppose that $a_{11} \in \Omega_{11}$. Assume that $\Phi(a_{11}) = b_{11} + b_{12} + b_{21} + b_{22}$. Then $\Phi(a_{11}) = \Phi(fa_{11}) = f\Phi(a_{11}) + \rho,\ \rho \in \mathcal{Z}(\Omega)$, which gives $b_{21} = 0$, $\rho_{11} = 0$, and $b_{22} = \rho_{22} \in \mathcal{Z}_{22}$, so we get $\Phi(a_{11}) = b_{11} + b_{12} + \rho_{22}$. Similarly, $\Phi(a_{11}) = \Phi(a_{11}f) = \Phi(a_{11})f + \gamma,\ \gamma \in \mathcal{Z}(\Omega)$, which means $b_{12} = 0$, and we get $\Phi(a_{11}) = b_{11} + \delta_{22}$.

(II) Assume that $a_{22} \in \Omega_{22}$. Write $\Phi(a_{22}) = b_{11} + b_{12} + b_{21} + b_{22}$, so $\theta = \Phi(fa_{22}) = b_{11} + b_{12} + \gamma_1,\ \gamma_1 \in \mathcal{Z}(\Omega)$, so $b_{11} + b_{12} = \theta - \gamma_1 \in \mathcal{Z}(\Omega)$, which means $b_{12} = 0$ and $b_{11} \in \mathcal{Z}_{11}$. Likewise, $\theta = \Phi(a_{22}f) = b_{11} + b_{21} + \gamma_2,\ \gamma_2 \in \mathcal{Z}(\Omega)$, so $b_{11} + b_{21} = \theta - \gamma_2 \in \mathcal{Z}(\Omega)$, so that $b_{21} = 0$, and thus $\Phi(a_{22}) = b_{11} + b_{22}$, where $b_{11} \in \mathcal{Z}_{11}$.

Also, the proof of (2) has two separable cases:

(I) Assume that $\Phi(a_{12}) = b_{11} + b_{12} + b_{21} + b_{22}$, so that $f\Phi(a_{12}) = b_{11} + b_{12}$. Also, we have $\Phi(a_{12}) = \Phi(fa_{12}) = b_{11} + b_{12} + \sigma,\ \sigma \in \mathcal{Z}(\Omega)$, which gives $f\Phi(a_{12}) = b_{11} + b_{12} + \sigma_{11}$. Comparing the two values of $f\Phi(a_{12})$, we obtain $\sigma_{11} = 0$ and $\sigma = \sigma_{22} \in \mathcal{Z}_{22}$, and we obtain $\Phi(a_{12}) = b_{11} + b_{12} + \sigma_{22}$. Now $\theta = \Phi(a_{12}f) = \Phi(a_{12})f + \mu,\ \mu \in \mathcal{Z}(\Omega)$, hence, $\Phi(a_{12})f = (\theta - \mu) = \eta \in \mathcal{Z}(\Omega)$. This provides $\Phi(a_{12})f = b_{11} + b_{21} = \eta \in \mathcal{Z}(\Omega)$, which means $b_{21} = 0$ and $b_{11} = \eta_{11} \in \mathcal{Z}_{11}$. So we arrive at $\Phi(a_{12}) = b_{12} + \eta_{11} + \sigma_{22}$.

(II) Assume that $\Phi(a_{21}) = b_{11} + b_{12} + b_{21} + b_{22}$, so that $\Phi(a_{21})f = b_{11} + b_{21}$. Also, we have $\Phi(a_{21}) = \Phi(a_{21}f) = b_{11} + b_{21} + \kappa,\ \kappa \in \mathcal{Z}(\Omega)$, which gives $\Phi(a_{21})f = b_{11} + b_{21} + \kappa_{11}$. Comparing the two expressions of $\Phi(a_{21})f$, we get $\kappa_{11} = 0$, $\kappa = \kappa_{22} \in \mathcal{Z}_{22}$, and we obtain $\Phi(a_{21}) = b_{11} + b_{21} + \kappa_{22}$. Now $\theta = \Phi(fa_{21}) = f\Phi(a_{21}) + \nu,\ \nu \in \mathcal{Z}(\Omega)$, hence $f\Phi(a_{21}) = (\theta - \nu) = \zeta \in \mathcal{Z}(\Omega)$, and this gives $f\Phi(a_{21}) = \zeta \in \mathcal{Z}(\Omega)$, which means $b_{11} = \zeta_{11} \in \mathcal{Z}_{11}$, and we have $\Phi(a_{21}) = b_{21} + \zeta_{11} + \kappa_{22}$. □

Lemma 2.4. *For any element $a_{11} \in \Omega_{11}$, we have $\Psi(a_{11}) = b_{11} + \varphi_{22}$ for some $b_{11} \in \Omega_{11}$ and $\varphi_{22} \in \mathcal{Z}_{22}$.*

Proof. Since $\Psi(rs) = \Psi(r)s + r\Phi(s) + \gamma$, for each $r, s \in \Omega$ and $\gamma \in \mathcal{Z}(\Omega)$, it consequently concludes that, for each $a_{11} \in \Omega_{11}$ we have $\Psi(a_{11}) = \Psi(fa_{11}) = f\Phi(a_{11}) + \gamma_1,\ \gamma_1 \in \mathcal{Z}(\Omega)$ because $\Psi(f) = 0$, and by Lemma 2.3 $\Phi(\Omega_{11}) \subset \Omega_{11} + \mathcal{Z}(\Omega)$ and $\mathcal{Z}(\Omega) \subset \Omega_{11} + \Omega_{22}$, so we have that $\Psi\ |_{\Omega_{11}} \subset \Omega_{11} + \mathcal{Z}(\Omega)$. Now assume that $\Psi(a_{11}) = b_{11} + \varphi,\ \varphi \in \mathcal{Z}(\Omega)$. Then $\Psi(a_{11}) = \Psi(a_{11}f) = \Psi(a_{11})f + \gamma_2,\ \gamma_2 \in \mathcal{Z}(\Omega)$, which gives $\Psi(a_{11}) - \Psi(a_{11})f = b_{11} + \varphi - b_{11} - \varphi_{11} \in \mathcal{Z}(\Omega)$. We conclude that $\varphi_{22} \in \mathcal{Z}_{22}$ and $\Psi(a_{11}) = b_{11} + \varphi = b_{11} + \varphi_{11} + \varphi_{22} = c_{11} + \varphi_{22}$ with $c_{11} = b_{11} + \varphi_{11} \in \Omega_{11}$ and $\varphi_{22} \in \mathcal{Z}_{22}$, as required. □

Lemma 2.5. *For any $a_{12} \in \Omega_{12}$, $\Psi(a_{12}) = b_{12} + \vartheta_{11} + \vartheta_{22}$ for some $b_{12} \in \Omega_{12}$, $\vartheta_{11} \in \mathcal{Z}_{11}$ and $\vartheta_{22} \in \mathcal{Z}_{22}$.*

Proof. If $a_{12} \in \Omega_{12}$, then $\Psi(a_{12}) = \Psi(fa_{12}) = f\Phi(a_{12}) + \gamma,\ \gamma \in \mathcal{Z}(\Omega)$ so by Lemma 2.3, $\Psi(a_{12}) = b_{12} + \delta_{11} + \gamma = b_{12} + \vartheta$, for some $b_{12} \in \Omega_{12}$ and $\delta_{11},\ \vartheta \in \mathcal{Z}(\Omega)$. Also, $\Psi(0) = \Psi(a_{12}f) = \Psi(a_{12})f + a_{12}\Phi(f) + \gamma_1,\ \gamma_1 \in \mathcal{Z}(\Omega)$ so $\Psi(a_{12})f \in \mathcal{Z}(\Omega)$ and this gives $\vartheta_{11} \in \mathcal{Z}_{11}$ and since $\vartheta \in \mathcal{Z}(\Omega)$ we obtain $\vartheta_{22} \in \mathcal{Z}_{22}$. So finally, we arrived at $\Psi(a_{12}) = b_{12} + \vartheta_{11} + \vartheta_{22}$. □

Lemma 2.6. *For any $a_{21} \in \Omega_{21}$, we have $\Psi(a_{21}) = b_{11} + b_{21} + \theta_{22}$, for some $b_{11} \in \Omega_{11}$, $b_{21} \in \Omega_{21}$ and $\theta_{22} \in \mathcal{Z}_{22}$.*

Proof. Assume that $\Psi(a_{21}) = b_{11} + b_{12} + b_{21} + b_{22}$, for $a_{21} \in \Omega_{21}$. Then $\Psi(a_{21}) = \Psi(a_{21}f) = \Psi(a_{21})f + \theta,\ \theta \in \mathcal{Z}(\Omega)$, which gives $b_{12} = 0$, $\theta_{11} = 0$, and $b_{22} = \theta_{22} = \theta \in \mathcal{Z}(\Omega)$. So we have $\Psi(a_{21}) = \Psi(a_{21})f + \theta_{22} = b_{11} + b_{21} + \theta_{22},\ \theta_{22} \in \mathcal{Z}(\Omega)$. □

Lemma 2.7. *For any element $t \in (\Omega_{11} + \Omega_{21})$, $\Psi(t) = b_{11} + b_{21} + \delta_{22}$, for some $b_{11} \in \Omega_{11}$, $b_{21} \in \Omega_{21}$ and $\delta_{22} \in \mathcal{Z}_{22}$.*

Proof. Assuming that $t \in (\Omega_{11} + \Omega_{21})$ and $\Psi(t) = b_{11} + b_{12} + b_{21} + b_{22}$. Then $\Psi(t) = \Psi(a_{11} + a_{21}) = \Psi[(a_{11} + a_{21})f] = \Psi(a_{11} + a_{21})f + \delta,\ \delta \in \mathcal{Z}(\Omega)$. This gives $b_{12} = 0$, and $b_{22} = \delta = \delta_{22} \in \mathcal{Z}_{22}$ and we arrive at $\Psi(t) = b_{11} + b_{21} + \delta_{22}$. □

Lemma 2.8. *Ψ is CE-additive on Ω_{11}.*

Proof. If $a_{11}, b_{11} \in \Omega_{11}$, then $\Psi(a_{11} + b_{11}) = \Psi(f(a_{11} + b_{11})) = f\Phi(a_{11} + b_{11}) + \sigma_1 = \Phi[f(a_{11} + b_{11})] - \Phi(f)(a_{11} + b_{11}) + \sigma_2 = \Phi(a_{11} + b_{11}) + \sigma_2 = \Phi(a_{11}) + \Phi(b_{11}) + \sigma_3 = f\Phi(a_{11}) + f\Phi(b_{11}) + \sigma_4 = \Psi(fa_{11}) + \Psi(fb_{11}) + \sigma_5 = \Psi(a_{11}) + \Psi(b_{11}) + \sigma_5$, where $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ and $\sigma_5 \in \mathcal{Z}(\Omega)$. □

Lemma 2.9. *If $a_{11} \in \Omega_{11}$ and $a_{21} \in \Omega_{21}$, then we obtain $\Psi(a_{11} + a_{21}) - \Psi(a_{11}) - \Psi(a_{21}) \in \mathcal{Z}(\Omega)$.*

Proof. For any $w_{1n} \in \Omega_{1n}$ and $h_{12} \in \Omega_{12}$, $n = 1, 2$ we own $\{\Psi(a_{11} + a_{21}) - \Psi(a_{11}) - \Psi(a_{21})\}h_{12}w_{1n} = 0$, which means

$$\{\Psi(a_{11} + a_{21}) - \Psi(a_{11}) - \Psi(a_{21})\}h_{12}\Omega_{1n} = \{0\}. \tag{2.7}$$

Now, for any $w_{2n} \in \Omega_{2n}$ and $h_{12} \in \Omega_{12}$, $n = 1, 2$, we have got

$$\begin{aligned}
\Psi(a_{11} + a_{21})h_{12}w_{2n} &= \Psi((a_{11} + a_{21})h_{12}w_{2n}) - (a_{11} + a_{21})\Phi(h_{12}w_{2n}) + \eta_1 \\
&= \Psi[(a_{11}h_{12} + a_{21})(w_{2n} + h_{12}w_{2n})] - (a_{11} + a_{21})\Phi(h_{12}w_{2n}) + \eta_1 \\
&= \Psi(a_{11}h_{12} + a_{21})(w_{2n} + h_{12}w_{2n}) \\
&\quad +(a_{11}h_{12} + a_{21})\Phi(w_{2n} + h_{12}w_{2n}) - (a_{11} + a_{21})\Phi(h_{12}w_{2n}) + \eta_2 \\
&= \Gamma_2 + \Gamma_1, \quad \eta_1, \ \eta_2 \in \mathcal{Z}(\Omega),
\end{aligned} \tag{2.8}$$

where we assume that $\Gamma_1 = (a_{11}h_{12} + a_{21})\Phi(w_{2n} + h_{12}w_{2n}) - (a_{11} + a_{21})\Phi(h_{12}w_{2n}) + \eta_2$ and $\Gamma_2 = \Psi(a_{11}h_{12} + a_{21})(w_{2n} + h_{12}w_{2n})$.

Now, let us calculate the terms Γ_1 and Γ_2:

$$\begin{aligned}
\Gamma_1 &= (a_{11}h_{12} + a_{21})\Phi(w_{2n} + h_{12}w_{2n}) - (a_{11} + a_{21})\Phi(h_{12}w_{2n}) + \eta_2 \\
&= a_{11}h_{12}\Phi(w_{2n} + h_{12}w_{2n}) + a_{21}\Phi(w_{2n} + h_{12}w_{2n}) - (a_{11} + a_{21})\Phi(h_{12}w_{2n}) + \eta_2 \\
&= \Psi(a_{11}h_{12}(w_{2n} + h_{12}w_{2n})) - \Psi(a_{11}h_{12})(w_{2n} + h_{12}w_{2n}) + \Psi(a_{21}(w_{2n} + h_{12}w_{2n})) \\
&\quad -\Psi(a_{21})(w_{2n} + h_{12}w_{2n}) - a_{11}\Phi(h_{12}w_{2n}) - a_{21}\Phi(h_{12}w_{2n}) + \eta_3 \\
&= \{\Psi(a_{11}h_{12}w_{2n}) - \Psi(a_{11}h_{12})w_{2n} - a_{11}h_{12}\Phi(w_{2n})\} + a_{11}h_{12}\Phi(w_{2n}) \\
&\quad +\{\Psi(a_{21}h_{12}w_{2n}) - \Psi(a_{21})h_{12}w_{2n} - a_{21}\Phi(h_{12}w_{2n})\} - a_{11}\Phi(h_{12}w_{2n}) \\
&\quad -\Psi(a_{11}h_{12})h_{12}w_{2n} - \Psi(a_{21})w_{2n} + \eta_3 \\
&= a_{11}h_{12}\Phi(w_{2n}) - a_{11}\Phi(h_{12}w_{2n}) - \Psi(a_{11}h_{12})h_{12}w_{2n} - \Psi(a_{21})w_{2n} + \eta_4 \\
&= -a_{11}\Phi(h_{12})w_{2n} - \Psi(a_{11}h_{12})h_{12}w_{2n} - \Psi(a_{21}w_{2n}) + a_{21}\Phi(w_{2n}) + \eta_5 \\
&= -a_{11}\Phi(h_{12})w_{2n} + a_{21}\Phi(w_{2n}) + \eta_6, \quad \text{by Lemma 2.5},
\end{aligned} \tag{2.9}$$

where η_3, η_4, η_5 and $\eta_6 \in \mathcal{Z}(\Omega)$, so that we obtain

$$\Gamma_1 = -a_{11}\Phi(h_{12})w_{2n} + a_{21}\Phi(w_{2n}) + \eta_6. \tag{2.10}$$

Also, for Γ_2 we have:

$$\begin{aligned}
\Gamma_2 &= \Psi(a_{11}h_{12} + a_{21})(w_{2n} + h_{12}w_{2n}) = \Psi(a_{11}h_{12} + a_{21})w_{2n} + \Psi(a_{11}h_{12} + a_{21})h_{12}w_{2n} \\
&= \Psi((a_{11}h_{12} + a_{21})w_{2n}) - (a_{11}h_{12} + a_{21})\Phi(w_{2n}) + \Psi((a_{11}h_{12} + a_{21})h_{12}w_{2n}) \\
&\quad -(a_{11}h_{12} + a_{21})\Phi(h_{12}w_{2n}) + \eta_7 \\
&= \Psi(a_{11}h_{12}w_{2n}) + \Psi(a_{21}h_{12}w_{2n}) - a_{11}h_{12}\Phi(w_{2n}) - a_{21}\Phi(w_{2n}) - a_{11}h_{12}\Phi(h_{12}w_{2n}) \\
&\quad -a_{21}\Phi(h_{12}w_{2n}) + \eta_7 \\
&= \Psi(a_{11}h_{12})w_{2n} + \Psi(a_{21})h_{12}w_{2n} - a_{11}h_{12}\Phi(h_{12}w_{2n}) - a_{21}\Phi(w_{2n}) + \eta_8 \\
&= \Psi(a_{11})h_{12}w_{2n} + a_{11}\Phi(h_{12})w_{2n} + \Psi(a_{21})h_{12}w_{2n} - a_{21}\Phi(w_{2n}) + \eta_9, \\
&\quad \text{by Lemma 2.3, where } \eta_7, \ \eta_8, \text{ and } \eta_9 \in \mathcal{Z}(\Omega).
\end{aligned} \tag{2.11}$$

So, we obtain

$$\Gamma_2 = \Psi(a_{11})h_{12}w_{2n} + a_{11}\Phi(h_{12})w_{2n} + \Psi(a_{21})h_{12}w_{2n} - a_{21}\Phi(w_{2n}) + \eta_9. \tag{2.12}$$

Now, coming back to (2.10) and using (2.12) to collect the values of Γ_1 and Γ_2 and substituting in (2.8), we get $\Psi(a_{11}+a_{21})h_{12}w_{2n} = \Psi(a_{11})h_{12}w_{2n}+\Psi(a_{21})h_{12}w_{2n}+\eta_{10},\ \eta_{10} \in \mathcal{Z}(\Omega)$ which gives $\{\Psi(a_{11}+a_{21})-\Psi(a_{11}) - \Psi(a_{21})\}h_{12}w_{2n} \in \mathcal{Z}(\Omega)$ and so we obtain

$$\{\Psi(a_{11} + a_{21}) - \Psi(a_{11}) - \Psi(a_{21})\}h_{12}\Omega_{2n} \subset \mathcal{Z}(\Omega). \tag{2.13}$$

From (2.7) and (2.13) we obtain $\{\Psi(a_{11} + a_{21}) - \Psi(a_{11}) - \Psi(a_{21})\}h_{12}\Omega \subset \mathcal{Z}(\Omega)$. Using condition (L_1) we have $\{\Psi(a_{11} + a_{21}) - \Psi(a_{11}) - \Psi(a_{21})\}\Omega_{12} \subset \mathcal{Z}(\Omega)$. Using condition (L_2), we obtain $\Psi(a_{11} + a_{21}) - \Psi(a_{11}) - \Psi(a_{21}) \in \mathcal{Z}(\Omega)$. □

Lemma 2.10. *Ψ is CE-additive on Ω_{21}.*

Proof. For any $a_{21}, b_{21} \in \Omega_{21}, y_{12} \in \Omega_{12}$ and $y_{2n} \in \Omega_{2n}$ we have

$$\begin{aligned}
& \Psi(a_{21} + b_{21})y_{12}y_{2n} = \Psi((a_{21} + b_{21})y_{12}y_{2n}) - (a_{21} + b_{21})\Phi(y_{12}y_{2n}) + \pi_1 \\
= \ & \Psi(a_{21}y_{12}y_{2n} + b_{21}y_{12}y_{2n}) - (a_{21} + b_{21})\Phi(y_{12}y_{2n}) + \pi_1 \\
= \ & \Psi((a_{21}y_{12} + b_{21})(y_{2n} + y_{12}y_{2n})) - (a_{21} + b_{21})\Phi(y_{12}y_{2n}) + \pi_1 \\
= \ & \Psi(a_{21}y_{12} + b_{21})(y_{2n} + y_{12}y_{2n}) + (a_{21}y_{12} + b_{21})\Phi(y_{2n} + y_{12}y_{2n}) \\
& -(a_{21} + b_{21})\Phi(y_{12}y_{2n}) + \pi_2 \\
= \ & \Psi(a_{21}y_{12} + b_{21})y_{2n} + \Psi(a_{21}y_{12} + b_{21})y_{12}y_{2n} + a_{21}y_{12}\Phi(y_{2n} + y_{12}y_{2n}) \\
& +b_{21}\Phi(y_{2n} + y_{12}y_{2n}) - a_{21}\Phi(y_{12}y_{2n}) - b_{21}\Phi(y_{12}y_{2n}) + \pi_2 \\
= \ & \Psi(a_{21}y_{12}y_{2n}) - (a_{21}y_{12} + b_{21})\Phi(y_{2n}) + \Psi(b_{21}y_{12}y_{2n}) - (a_{21}y_{12} + b_{21})\Phi(y_{12}y_{2n}) \\
& +\Phi(a_{21}y_{12}y_{2n}) - \Phi(a_{21}y_{12})(y_{2n} + y_{12}y_{2n}) \\
& +\Phi(b_{21}y_{12}y_{2n}) - \Phi(b_{21})(y_{2n} + y_{12}y_{2n}) - a_{21}\Phi(y_{12}y_{2n}) - b_{21}\Phi(y_{12}y_{2n}) + \pi_3 \\
= \ & \{\Psi(a_{21}y_{12}y_{2n}) - a_{21}\Phi(y_{12}y_{2n})\} + \{\Psi(b_{21}y_{12}y_{2n}) - b_{21}\Phi(y_{12}y_{2n})\} \\
& -\{a_{21}y_{12}\Phi(y_{12}y_{2n}) + \Phi(a_{21}y_{12})y_{12}y_{2n}\} + \{\Phi(a_{21}y_{12}y_{2n}) - \Phi(a_{21}y_{12})y_{2n} \\
& -a_{21}y_{12}\Phi(y_{2n})\} + \{\Phi(b_{21}y_{12}y_{2n}) - \Phi(b_{21})y_{12}y_{2n} - b_{21}\Phi(y_{12}y_{2n})\} \\
& -\{b_{21}\Phi(y_{2n}) + \Phi(b_{21})y_{2n}\} + \pi_3 \\
= \ & \Psi(a_{21})y_{12}y_{2n} + \Psi(b_{21})y_{12}y_{2n} + \pi_4,
\end{aligned} \tag{2.14}$$

where $\pi_i \in \mathcal{Z}(\Omega),\ i \in \{1, 2, 3, 4\}$. So we have

$$[\Psi(a_{21} + b_{21}) - \Psi(a_{21}) - \Psi(b_{21})]\Omega_{12}\Omega_{2n} \subset \mathcal{Z}(\Omega). \tag{2.15}$$

Also, it is clear that

$$[\Psi(a_{21} + b_{21}) - \Psi(a_{21}) - \Psi(b_{21})]\Omega_{12}\Omega_{1n} \subset \mathcal{Z}(\Omega), \tag{2.16}$$

where $n = 1, 2$. From (2.15) and (2.16) we obtain $[\Psi(a_{21} + b_{21}) - \Psi(a_{21}) - \Psi(b_{21})]\Omega_{12}\Omega \subset \mathcal{Z}(\Omega)$. By condition (L_1) we have $[\Psi(a_{21} + b_{21}) - \Psi(a_{21}) - \Psi(b_{21})]\Omega_{12} \subset \mathcal{Z}(\Omega)$. Using condition (L_2), we obtain $\Psi(a_{21} + b_{21}) - \Psi(a_{21}) - \Psi(b_{21}) \in \mathcal{Z}(\Omega)$. □

Lemma 2.11. *Ψ is CE-additive on $\Omega_{11} + \Omega_{21} = \Omega f$.*

Proof. If $a_{11}, b_{11} \in \Omega_{11}$ and $a_{21}, b_{21} \in \Omega_{21}$, then Lemmas 2.8, 2.9 and 2.10 give

$$\begin{aligned}
\Psi((a_{11} + a_{21}) + (b_{11} + b_{21})) &= \Psi((a_{11} + b_{11}) + (a_{21} + b_{21})) \\
&= \Psi(a_{11} + b_{11}) + \Psi(a_{21} + b_{21}) + \phi_1 \\
&= \Psi(a_{11}) + \Psi(b_{11}) + \Psi(a_{21}) + \Psi(b_{21}) + \phi_2 \\
&= (\Psi(a_{11}) + \Psi(a_{21})) + (\Psi(b_{11}) + \Psi(b_{21})) + \phi_2 \\
&= \Psi(a_{11} + a_{21}) + \Psi(b_{11} + b_{21}) + \phi_3,
\end{aligned}$$

where $\phi_i \in \mathcal{Z}(\Omega),\ i \in \{1, 2, 3\}$. Thus, Ψ is CE-additive on $\Omega_{11} + \Omega_{21}$, as required. □

3. Results

We can now prove our primary result.

Theorem 3.1. *Suppose that Ω is a ring with a nontrivial idempotent f that satisfies requirements (L_1) and (L_2). If Ψ is any multiplicative generalized CE-derivation of Ω, then Ψ is CE-additive.*

Proof. Suppose that Ψ is any multiplicative generalized CE-derivation of Ω, i.e., $\Psi(ab) = \Psi(a)b + a\Phi(b) + \nu$, for every $a, b \in \Omega$ and $\nu \in \mathcal{Z}(\Omega)$ and some CE-derivation Φ of Ω. Consider $\Psi(a) + \Psi(b)$, where a and $b \in \Omega$. Take an element h in $\Omega f = \Omega_{11} + \Omega_{21}$. Thus, ah and $bh \in \Omega f$. Using Lemma 2.11, we obtain $(\Psi(a) + \Psi(b))h = \Psi(ah) + \Psi(bh) - (a + b)\Phi(h) + \nu_1 = \Psi(ah + bh) - (a + b)\Phi(h) + \nu_2 = \Psi((a + b)h) - (a + b)\Phi(h) + \nu_2 = \Psi(a + b)h + (a + b)\Phi(h) - (a + b)\Phi(h) + \nu_3 = \Psi(a + b)h + \nu_3$, where $\nu_i \in \mathcal{Z}(\Omega),\ i \in \{1, 2, 3\}$. Thus, $(\Psi(a) + \Psi(b))h - \Psi(a + b)h \in \mathcal{Z}(\Omega)$. Since h is an arbitrary element in Ωf, we obtain $(\Psi(a) + \Psi(b) - \Psi(a + b))\Omega f \in \mathcal{Z}(\Omega)$. Under condition (L_1), we obtain $\Psi(a + b) - \Psi(a) - \Psi(b) \in \mathcal{Z}(\Omega)$. It demonstrates that the multiplicative generalized CE-derivations Ψ and G are a CE-additive. □

Now, we are in a position to raise the following open problem. "Under what conditions does an MCE-derivation (or MGCE-derivation) become a centrally extended additive over an alternative ring?"

4. Conclusions

We showed that if G is an appropriate multiplicative generalized CE-derivation of a ring Ω, then G is CE-additive.

Author contributions

M. S. Tammam: conceptualization, methodology, validation, formal analysis, investigation, data curation, writing–original draft preparation, writing–review and editing, supervision; M. Almulhem: validation, formal analysis, writing–review and editing, supervision. All authors have read and agreed to the published version of the manuscript.

Conflict of interest

The authors declare no conflicts of interest.

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