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Research article

Stability of stationary solutions to outflow problem for compressible viscoelastic system in one dimensional half space

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Abstract: The system of equations describing motion of compressible viscoelastic fluids is considered in a one dimensional half space under the outflow boundary condition. We investigate the existence and stability of stationary solutions. It is shown that the stationary solution exists for large Mach number and small number of propagation speed of elastic wave. We next show that the stationary solution is asymptotically stable, provided that the initial perturbation is sufficiently small.

Keywords: compressible viscoelastic system; energy method; outflow problem; stationary solution; asymptotic stability
Mathematics Subject Classification: 35Q35, 35B40, 76N10

1. Introduction

This paper studies the compressible viscoelastic system in the one-dimensional half space $\mathbb{R}_+ = (0, \infty)$:

$$\rho_t + (\rho v)_x = 0, \tag{1.1}$$

$$(\rho v)_t + (\rho v^2)_x - \nu v_{xx} + P(\rho)_x = \beta^2(\rho F^2)_x, \tag{1.2}$$

$$F_t + vF_x = v_x F. \tag{1.3}$$

Here $\rho = \rho(t, x)$, $v = v(t, x)$, and $F = F(t, x)$ are the unknown density, velocity field, and deformation tensor, respectively, at time $t \geq 0$ and position $x \in \mathbb{R}_+$; $P(\rho)$ stands for the pressure assumed to be a smooth function of ρ satisfying $P'(\rho) > 0$ and $P''(\rho) > 0$ for $\rho > 0$; $\nu > 0$ is the viscosity coefficient; $\beta > 0$ is the strength of the elasticity. In particular, if we set $\beta = 0$, the systems (1.1) and (1.2) become the usual compressible Navier-Stokes equation.

We impose the initial condition and boundary conditions at $x = \infty$ and $x = 0$:

$$(\rho, v, F)|_{t=0} = (\rho_0, v_0, F_0), \quad \inf_{x\in\mathbb{R}_+} \rho_0(x) > 0, \tag{1.4}$$

$$\lim_{x\to\infty}(\rho, v, F) = (\rho_+, v_+, F_+), \tag{1.5}$$

$$v(t, 0) = v_b. \tag{1.6}$$

Here, the end states ρ_+, v_+ and F_+ are given constants with $\rho_+ > 0$, and v_b is a given constant assumed to be $v_b < 0$ for considering the situation that the fluid flows out from the boundary $x = 0$. Throughout this paper, we consider the initial boundary problems (1.1)–(1.6), called the outflow problem. The aim of this paper is to show the existence and asymptotic stability of stationary solutions for the outflow problems (1.1)–(1.6) and clarify the interaction between the effect of the elastic force $\beta^2(\rho F^2)_x$ and outflow boundary condition (1.6).

We first discuss the existence and properties for the stationary solution $(\tilde{\rho}, \tilde{v}, \tilde{F})(x)$, called the boundary layer solution, solving the system:

$$(\tilde{\rho}\tilde{v})_x = 0, \tag{1.7}$$

$$(\tilde{\rho}\tilde{v}^2)_x - \nu\tilde{v}_{xx} + P(\tilde{\rho})_x = \beta^2(\tilde{\rho}\tilde{F}^2)_x, \tag{1.8}$$

$$\tilde{v}\tilde{F}_x = \tilde{v}_x\tilde{F}, \tag{1.9}$$

with the conditions:

$$\lim_{x\to\infty}(\tilde{\rho}, \tilde{v}, \tilde{F}) = (\rho_+, v_+, F_+), \quad \inf_{x\in\mathbb{R}_+} \tilde{\rho}(x) > 0, \tag{1.10}$$

$$\tilde{v}(0) = v_b. \tag{1.11}$$

It is shown that the solution of the problems (1.7)–(1.11) exists uniquely if and only if $M_\beta \geq 1$ and $v_b < v_*$ hold. Here, M_β is the modified Mach number given by $M_\beta := |v_+|/\sqrt{c_+^2 + \beta^2 F_+^2}$, where $c_+ := \sqrt{P'(\rho_+)}$ stands for sound speed; v_* is a certain negative constant determined in Section 2. In addition, the solution satisfies the following estimate:

$$|(\tilde{\rho} - \rho_+, \tilde{v} - v_+, \tilde{F} - F_+)(x)| \leq \begin{cases} C\delta e^{-cx}, & M_\beta > 1, \\ C\delta(1 + \delta x)^{-1}, & M_\beta = 1, \end{cases}$$

where δ denotes $\delta := |v_b - v_+|$. We call $(\tilde{\rho}, \tilde{v}, \tilde{F})(x)$ the non-degenerate stationary solution tending to the end state exponentially when $M_\beta > 1$, while we say $(\tilde{\rho}, \tilde{v}, \tilde{F})(x)$ by the degenerate stationary solution converging to the end state algebraically when $M_\beta = 1$. We also note that if we take β large so that $M_\beta < 1$ under the fixed end state, then the stationary solution does not exist. This means that the stationary outflow does not occur due to the recoiling effect of strong elastic force.

We next establish the asymptotic stability of the stationary solution under the small initial perturbation $(\rho_0, v_0, F_0) - (\tilde{\rho}, \tilde{v}, \tilde{F})$, provided that δ is sufficiently small. This follows from the local-in-time solvability of (1.1)–(1.6) and the a priori estimates for the perturbation in $H^1(\mathbb{R}_+)$. Since the systems (1.1)–(1.3) is classified by a quasilinear parabolic-hyperbolic system, the local-in-time solvability is shown by the iteration method and theory of weak solutions to linear transport equations and parabolic equations, inspired by Kagei and Kawashima's paper [11].

To derive the *a priori* estimate, we need to deal with the term $\beta^2(\rho F^2 - \tilde{\rho}\tilde{F}^2)_x$ when we consider the problem for a usual perturbation $(\rho, v, F) - (\tilde{\rho}, \tilde{v}, \tilde{F})$. However, it seems to be difficult to control $\rho F - \rho_+ F_+$ appearing from this term. To overcome this difficulty, we assume the following condition for (ρ_0, F_0):

$$\rho_0 F_0 = \rho_+ F_+ \tag{1.12}$$

for $x \in \mathbb{R}_+$. We then see from (1.1), (1.3), (1.6) and (1.12) that $\rho F = \rho_+ F_+$ holds for $t \geq 0$ and $x \geq 0$. This constraint is a one-dimensional version of the following equality

$$\rho \det \mathbf{F} = \rho_+ \det \mathbf{F}_+, \ t \geq 0, \ x = {}^{\top}(x_1, x_2, \ldots, x_n) \in \mathbb{R}_+ \times \mathbb{R}^{n-1},$$

which is equivalent to the conservation law of mass in the Lagrange coordinate. Here, $n \geq 2$; $\mathbf{F}_+$ stands for given $n \times n$ matrix-valued constant; $\mathbf{F}(t, x) = (F^{j,k}(t, x))_{1 \leq j,k \leq n}$ denotes an $n \times n$ matrix-valued deformation tensor, respectively.

Therefore, rewriting $F = \rho_+ F_+ \rho^{-1}$, the problems (1.1)–(1.6) for (ρ, v, F) is reduced to the system for (ρ, v):

$$\rho_t + (\rho v)_x = 0, \tag{1.13}$$

$$(\rho v)_t + (\rho v^2)_x - \nu v_{xx} + P(\rho)_x = \beta^2 (\rho_+ F_+)^2 \left(\frac{1}{\rho} \right)_x \tag{1.14}$$

with the initial condition and boundary conditions at $x = \infty$ and $x = 0$:

$$(\rho, v)|_{t=0} = (\rho_0, v_0), \quad \inf_{x \in \mathbb{R}_+} \rho_0(x) > 0 \tag{1.15}$$

$$\lim_{x \to \infty} (\rho, v) = (\rho_+, v_+), \tag{1.16}$$

$$v(t, 0) = v_b. \tag{1.17}$$

We then prepare the reduced perturbation $(\phi, \psi) := (\rho - \tilde{\rho}, v - \tilde{v})$ and carry out its estimate as two steps. For (ϕ, ψ) itself, based on the idea of Kawashima, Nishibata and Zhu's paper [13] and the fact that $P(\rho) - \beta^2(\rho_+ F_+)^2/\rho$ monotonically increases for $\rho > 0$, we construct a suitable energy form equivalent to $|(\phi, \psi)|^2$, and use its equation and the properties of the stationary solution. The effect of the term $\beta^2(\rho_+ F_+)^2(1/\rho)_x$ is mainly involved to the proof of this estimate. Indeed, the convexity of $P(\rho) - \beta^2(\rho_+ F_+)^2/\rho$ around $\rho = \rho_+$ requires for the degenerate case $M_\beta = 1$. In order to get this convexity, we need to add the condition for the Mach number $1 < M_+ < \sqrt{\rho_+(P''(\rho_+)/2P'(\rho_+)) + 1}$, where $M_+ := |v_+|/c_+$. For the 1-st order spatial derivative of (ϕ, ψ), we utilize the structure of (1.13) and (1.14) and monotonicity of $P(\rho) - \beta^2(\rho_+ F_+)^2/\rho$ for $\rho > 0$.

We remark the reason why we only assume the regularity condition $(\rho_0 - \tilde{\rho}, v_0 - \tilde{v}, F_0 - \tilde{F}) \in H^1(\mathbb{R}_+)$ in contrast of the case $\beta = 0$ in [13]. In the argument in [13], we need the equation

$$\int_0^\infty v(t, x)\phi_{xx}(t, x)\phi_x(t, x)dx = -\frac{1}{2} v_b |\phi_x(t, 0)|^2 - \frac{1}{2} \int_0^\infty v_x(t, x)|\phi_x(t, x)|^2 dx \tag{1.18}$$

to construct the a priori estimate for ϕ_x. This means that we need to take care of $\phi_{xx}(t, x)$ and $|\phi_x(t, 0)|$. The equation (1.18) is obtained by the integration by parts with (1.6) and makes sense under $\psi \in C([0, T]; H^1(\mathbb{R}_+))$ and $\phi \in C([0, T]; H^k(\mathbb{R}_+))$ for $T > 0$ and $k \geq 2$ because $H^k(\mathbb{R}_+) \subset C^1([0, \infty))$

holds for $k \geq 2$. In our setting, the equation (1.18) is not valid since we restrict the situation that $\phi \in C([0,T]; H^1(\mathbb{R}_+))$ which is not sufficient to define $\phi_{xx}(t,x)$ and $\phi_x(t,0)$. To deal with this difficulty, Kawashima, Nishibata and Zhu in [13] additionally assumed that the initial perturbation belongs to the Hölder space for guaranteeing $\phi_x(t,0)$, and then applied the method of difference quotient to avoid appearing higher derivatives such as $\phi_{xx}(t,x)$. Later, Kagei and Kawashima introduced the weak forms of the parabolic equation and first-order transport equation to show the local-in-time existence of the quasilinear parabolic-hyperbolic system in [11], and then used their theory to obtain the estimates for the higher order derivatives without the Hölder regularity of initial perturbation in [12]. Therefore, inspired the idea of [12], we can conclude that it is enough to suppose that initial perturbation is small only in $H^1(\mathbb{R}_+)$ to show the asymptotic stability result.

Known results. The systems (1.1)–(1.3) describing the motion of compressible viscoelastic fluid is governed in the macroscopic scale by the variational modeling. Indeed the second equation (1.2) is treated as the conservation law of momentum following from the energy dissipation law with the free energy induced by elastic solids. Here, the free energy is taken as the derivative of $\beta^2\rho W'(F)F = \beta^2\rho F^2$, where $W(F) = \beta^2 F^2/2$ denotes linear isentropic elasticity. The other equations (1.1) and (1.3) are the kinematic assumptions for ρ and F. For more physical detail, we refer to [3–5, 17, 25]. Starting with Sideris and Thomases [25], the mathematical analysis of the systems (1.1)–(1.3) has been progressed mainly on the stability of the trivial motionless state. In fact, its stability is investigated by [1, 7, 8, 16, 24] in the three dimensional whole space and is studied by [2, 9, 23] in the three-dimensional bounded domain with smooth boundary case under the Dirichlet boundary condition. For the stability of non-trivial flows with non-zero velocity, the dynamics of solutions around them becomes more complicated than the trivial motionless case since the advection terms in (1.1)–(1.3) produce the additional hyperbolic aspect. Therefore, comparing to the the trivial motionless case, there are few results on their stability as follows. Ishigaki [10] and Haruki and Ishigaki [6] investigated the stability of parallel flows in the three dimensional layer, and Morando, Trakhinin and Trebeschi [19] and Trakhinin [26] studied the stability of shock waves in the two-dimensional whole space without the viscous effect.

We next review the mathematical analysis of outflow problem. For the case $\beta = 0$, it is natural to expect that the behavior of the solution in the half space is closely related to the boundary and the end states v_b, v_+ and ρ_+. Then, Matsumura [18] suggested that the long time asymptotic states will be composed of the rarefaction wave, the viscous shock wave and the boundary layer solution. The stability of stationary solution is related to the case that the asymptotic state is given by the boundary layer only. Kawashima, Nishibata and Zhu [13] characterized the existence of stationary solutions by determining suitable conditions for v_b, v_+ and ρ_+, and then showed its asymptotic stability by the energy method in the Eulerian coordinate. Nakamura, Nishibata and Yuge [21] established the convergence rate toward the stationary solutions as $t \to \infty$ under the small initial perturbation belonging to the weighted L^2 Sobolev space. Later that, Nakamura, Ueda and Kawashima [22] refined the convergence rate toward the degenerate stationary solution discussed in [13, 21]. Furthermore, Kawashima and Kagei [11], and Nakamura and Nishibata [20] extended these stability results to the multidimensional case. On the other hand, if v_b, v_+ and ρ_+ do not satisfy the conditions for the existence of stationary solutions, the asymptotic state of the solution becomes different from the boundary layer solution. For the details, we refer to [14, 15, 27] when the rarefaction wave involves its time asymptotic state. Concerning the case $\beta > 0$, as far as the authors know, it remains open.

Outline of this paper. This paper is organized as four sections and one appendix. In Section 2, notations of several function spaces and lemmata are explained. In Section 3, we give the detailed necessary conditions for the existence of a stationary solution and provide its properties. In Section 4, the detail of the main result in this paper is stated. In Section 5, we show the asymptotic stability of the stationary solutions. In Appendix A, we prove the local-in-time existence around the stationary solutions.

2. Notations

In this section, we introduce several function spaces and important lemmata.

For $1 \le p \le \infty$, the symbol L^p stands for the usual Lebesgue space on $\mathbb{R}_+$, and its norm is denoted by $\|\cdot\|_{L^p}$. For a non-negative integer $m \ge 0$, we define H^m as the m-th order L^2 Sobolev space on $\mathbb{R}_+$, and its norm is denoted by $\|\cdot\|_{H^m}$. For simplicity, we write $\mathbf{L}^p = L^p \times L^p \times L^p$ (resp., $\mathbf{H}^m = H^m \times H^m \times H^m$). The symbol $C_0^1(\mathbb{R}_+)$ denotes the set of all $C^1(\mathbb{R}_+)$ functions whose supports are compact in $\mathbb{R}_+$. We call $H_0^1(\mathbb{R}_+)$ the completion of $C_0^1(\mathbb{R}_+)$ in $H^1(\mathbb{R}_+)$. For $T > 0$, we define $C_0^1((0,T) \times \mathbb{R}_+)$ (resp., $C_0^1([0,T)\times\mathbb{R}_+)$) as the set of all $C^1((0,T)\times\mathbb{R}_+)$ (resp., $C^1([0,T)\times\mathbb{R}_+)$) functions whose supports are compact in $(0,T)\times\mathbb{R}_+$ (resp., $[0,T)\times\mathbb{R}_+$).

For $0 \le a < b \le \infty$, a Banach space $\mathcal{X}$ endowed with norm $\|\cdot\|_{\mathcal{X}}$ and a non-negative integer k, we define $C^k([a,b];\mathcal{X})$ that

$$C^k([a,b];\mathcal{X}) := \{f : [a,b] \to \mathcal{X};\ f \text{ is a } C^k \text{ function in } [a,b] \text{ satisfying } \|f\|_{C^k([a,b];\mathcal{X})} < \infty\},$$

where

$$\|f\|_{C^k([a,b];\mathcal{X})} := \sum_{l=0}^{k} \sup_{t\in[a,b]} \|\partial_t^l f(t)\|_{\mathcal{X}}.$$

Here, we identify $[a,\infty] := [a,\infty)$ in the case $b = \infty$. For simplicity, we write $C([a,b];\mathcal{X}) := C^0([a,b];\mathcal{X})$.

For $0 \le a < b \le \infty$, a Banach space $\mathcal{X}$ endowed with norm $\|\cdot\|_{\mathcal{X}}$ and a non-negative integer k, $L^2(a,b;\mathcal{X})$ and $H^k(a,b;\mathcal{X})$ denote

$$L^2(a,b;\mathcal{X}) := \{f : [a,b] \to \mathcal{X};\ f \text{ is a measurable function in } [a,b] \text{ satisfying } \|f\|_{L^2(a,b;\mathcal{X})} < \infty\},$$

where

$$\|f\|_{L^2(a,b;\mathcal{X})} := \left(\int_a^b \|f(t)\|_{\mathcal{X}}^2 dt\right)^{1/2},$$

and

$$H^k(a,b;\mathcal{X}) := \{f \in L^2(a,b;\mathcal{X});\ f \text{ is a } k\text{-th times weakly differentiable function in } (a,b) \text{ satisfying } \|f\|_{H^k(a,b;\mathcal{X})} < \infty\},$$

where

$$\|f\|_{H^k(a,b;\mathcal{X})} := \left(\sum_{l=0}^{k} \int_a^b \|\partial_t^l f(t)\|_{\mathcal{X}}^2 d\tau\right)^{1/2}.$$

For a real number α, we set $[\alpha]$ as its integer part. Throughout this paper, we simply regard the letters c, C, $\tilde{c}$ and $\tilde{C}$ as positive various constants.

We next state several lemmata to establish our asymptotic stability result. We first introduce the well-known Gagliardo–Nirenberg inequality to obtain a priori estimate in $H^1(\mathbb{R}_+)$.

Lemma 2.1. *Let $f \in H^1(\mathbb{R}_+)$. Then, $f \in C([0,\infty))$ and it satisfies*

$$\|f\|_{L^\infty} \leq \sqrt{2}\|f\|_{L^2}^{1/2}\|f_x\|_{L^2}^{1/2}.$$

To show the asymptotic stability of the stationary solution, the following lemma plays a role.

Lemma 2.2. [12, Lemma 4.5.] *Let $T > 0$ be an arbitrary number, and $f = f(t,x)$ be $f \in C([0,T];H_0^1(\mathbb{R}_+)) \cap L^2(0,T;H^2(\mathbb{R}_+)) \cap H^1(0,T;L^2(\mathbb{R}_+))$. Then f satisfies*

$$\|f_x(t_2)\|_{L^2}^2 \leq C\left(\|f_x(t_1)\|_{L^2}^2 + \int_{t_1}^{t_2} \|f_t(\tau)\|_{L^2}\|f_x(\tau)\|_{H^1}d\tau\right)$$

for $0 \leq t_1 \leq t_2 \leq T$.

We give definitions of weak solutions to the linear transport equation and the parabolic equation for studying the local-in-time existence of the unique solution to (1.1)–(1.6) around the stationary solution.

For $0 \leq a < b \leq \infty$ and $k = 0, 1$, we define the function spaces $X^k(a,b)$ and $Y^k(a,b)$ as

$$X^k(a,b) := C([a,b];H^{k-l}(\mathbb{R}_+)),$$
$$Y^k(a,b) := C([a,b];\widetilde{H}^k(\mathbb{R}_+)) \cap \bigcap_{l=0}^{k} H^l(a,b;\widetilde{H}^{k+1-2l}(\mathbb{R}_+)),$$

respectively. Here, $\widetilde{H}^k(\mathbb{R}_+)$ is given by $\widetilde{H}^k(\mathbb{R}_+) := H^k(\mathbb{R}_+) \cap H_0^1(\mathbb{R}_+)$ when $k = 1, 2$, and $\widetilde{H}^0(\mathbb{R}_+) := L^2(\mathbb{R}_+)$. We set the corresponding norms such that

$$\|f\|_{X^k(a,b)} := \sum_{l=0}^{k} \|f\|_{C^l([a,b];H^{k-l}(\mathbb{R}_+))},$$
$$\|g\|_{Y^k(a,b)} := \left(\|g\|_{C([a,b];H^k(\mathbb{R}_+))}^2 + \sum_{l=0}^{k} \|g\|_{H^l(a,b;H^{k+1-2l}(\mathbb{R}_+))}^2\right)^{1/2}.$$

We introduce the function space $Z^k(a,b)$ as

$$Z^k(a,b) := X^k(a,b) \times Y^k(a,b) \times X^k(a,b),$$

and the norms of $u = (\phi,\psi,\zeta)$ in $Z^k(a,b)$ are given by

$$\|u\|_{Z^k(a,b)} := \left(\|\phi\|_{X^k(a,b)}^2 + \|\psi\|_{Y^k(a,b)}^2 + \|\zeta\|_{X^k(a,b)}^2\right)^{1/2}.$$

For simplicity, we write $X^k(T) := X^k(0,T)$, $Y^k(T) := Y^k(0,T)$ and $Z^k(T) := Z^k(0,T)$ for $T > 0$. We also set

$$X_M^k(T) := \{f \in X^k(T);\ \|f\|_{X^k(T)} \leq M\},$$

$$Y_M^k(T) := \{g \in Y^k(T);\ \|g\|_{Y^k(T)} \le M\},$$
$$Z_M^k(T) := \{u \in Z^k(T);\ \|u\|_{Z^k(T)} \le M\}.$$

for $M > 0$.

We define weak solutions to the linear transport equation and the parabolic equation associated with the stationary solution of (1.1)–(1.6). The stationary solution $(\tilde{\rho}, \tilde{v}, \tilde{F})$ is the smooth solution to (1.7)–(1.11), and the existence of solutions will be discussed in Section 3. Using this stationary solution, we introduce functions $a(\psi) := \tilde{v} + \psi$ and $b(\phi) := \tilde{\rho} + \phi$, as well as a linear operator B : $H_0^1(\mathbb{R}_+) \to H^{-1}(\mathbb{R}_+)$ defined by $\langle B\psi, \varphi\rangle := \nu(\psi_x, \varphi_x)_{L^2}$ for $\psi, \varphi \in H_0^1(\mathbb{R}_+)$. Here, $H^{-1}(\mathbb{R}_+)$ is the dual space of $H_0^1(\mathbb{R}_+)$, $\langle\cdot,\cdot\rangle$ that stands for the pairing between $H^{-1}(\mathbb{R}_+)$ and $H_0^1(\mathbb{R}_+)$, and $(\cdot,\cdot)_{L^2}$ denotes the usual $L^2(\mathbb{R}_+)$-inner product. We also note that B is identified as a usual expression $B\psi = -\nu\psi_{xx}$ if $\psi \in \widetilde{H}^2(\mathbb{R}_+)$ holds. Then, we define weak solutions as follows.

Definition 2.3. *Let $T > 0$ and let $\tilde{\psi} = \tilde{\psi}(t,x)$ be a given function. For given $\phi_0 \in L^2(\mathbb{R}_+)$ and $f \in L^2(0,T;L^2(\mathbb{R}_+))$, we call ϕ a weak solution of the initial value problem*

$$\phi_t + a(\tilde{\psi})\phi_x = f, \quad \phi|_{t=0} = \phi_0, \quad \phi|_{x=\infty} = 0 \tag{2.1}$$

if ϕ belongs to $X^0(T)$ and satisfies the weak form

$$-\int_0^T (\phi, \varphi_t + (a(\tilde{\psi})\varphi)_x)_{L^2} dt = (\phi_0, \varphi(0))_{L^2} + \int_0^T (f,\varphi)_{L^2} dt \tag{2.2}$$

for all $\varphi \in C_0^1([0,T) \times \mathbb{R}_+)$.

Definition 2.4. *Let $T > 0$ and let $\tilde{\phi} = \tilde{\phi}(t,x)$ be a given function. For given $\psi_0 \in L^2(\mathbb{R}_+)$ and $g \in L^2(0,T;H^{-1}(\mathbb{R}_+))$, we call ψ a weak solution of the initial-boundary value problem*

$$b(\tilde{\phi})\psi_t + B\psi = g, \quad \psi|_{t=0} = \psi_0, \quad \psi|_{x=0} = \psi|_{x=\infty} = 0 \tag{2.3}$$

if ψ belongs to $Y^0(T)$ and satisfies the weak form

$$\begin{aligned} &-\int_0^T (b(\tilde{\phi})\psi, \varphi)_{L^2} h' dt - \int_0^T \left\langle \tilde{\phi}_t \psi, \varphi \right\rangle h dt + \int_0^T \langle B\psi, \varphi\rangle h dt \\ &= (b(\tilde{\phi}(0,\cdot))\psi_0, \varphi)_{L^2} h(0) + \int_0^T \langle g, \varphi\rangle h dt \end{aligned} \tag{2.4}$$

for all $\varphi \in H_0^1(\mathbb{R}_+)$ and $h \in C_0^1([0,T))$.

The existence, regularity and estimates of weak solutions to (2.1) and (2.3) are stated as the following lemma.

Lemma 2.5. *Let $T > 0$ be a positive constant. Assume that $\tilde{\psi} = \tilde{\psi}(t,x)$ satisfies $\tilde{\psi} \in Y^1(T)$. Then, for any $\phi_0 \in H^k(\mathbb{R}_+)$ and $f \in L^2(0,T;H^k(\mathbb{R}_+))$ with $k = 0, 1$, there exists a unique weak solution $\phi \in X^k(T)$ to satisfying*

$$\begin{aligned} \|\phi(t_2)\|_{H^k}^2 \le& \|\phi(t_1)\|_{H^k}^2 + \sum_{l=0}^{k} \int_{t_1}^{t_2} \{(a(\widetilde{\psi})_x(\tau), |\partial_x^l \phi(\tau)|^2)_{L^2} + 2l((a(\widetilde{\psi})_x \phi_x)(\tau), \phi_x(\tau))_{L^2}\} d\tau \\ &+ 2\sum_{l=0}^{k} \int_{t_1}^{t_2} (\partial_x^l f(\tau), \partial_x^l \phi^f(\tau))_{L^2} d\tau \end{aligned} \tag{2.5}$$

for all $0 \le t_1 \le t_2 \le T$.

In addition, in the case $k = 1$, ϕ *belongs to* $C^1([0,T]; L^2(\mathbb{R}_+))$ *and is controlled as*

$$\max_{t\in[0,T]} \|\phi(t) - \phi_0(\tilde{y}(0;t,\cdot))\|_{L^\infty} \le T^{\frac{1}{2}} \left(\int_0^T \|f(\tau)\|_{H^1}^2 d\tau \right)^{\frac{1}{2}}. \tag{2.6}$$

Here $\tilde{y} = \tilde{y}(\tau; t, x) \in \mathbb{R}_+$ *is a unique solution of the problem*

$$\frac{d\tilde{y}}{d\tau}(\tau; t, x) = a(\tilde{\psi}(\tau, \tilde{y}(\tau; t, x))),\ 0 \le \tau \le t \le T, \quad \tilde{y}(t; t, x) = x.$$

Lemma 2.6. *Let* T, M *and* m *be positive constants. Assume that* $\tilde{\phi} = \tilde{\phi}(t, x)$ *satisfies*

$$\tilde{\phi} \in X_M^1(T), \quad \tilde{\phi} \in C^1([0,T]; L^2(\mathbb{R}_+)), \qquad \inf_{(t,x)\in[0,T]\times\mathbb{R}_+} \tilde{\phi}(t,x) \ge (m-1) \inf_{x\in\mathbb{R}_+} \tilde{\rho}(x).$$

Then, for any $\psi_0 \in \widetilde{H}^k(\mathbb{R}_+)$ *and* $g \in L^2(0,T; H^{k-1}(\mathbb{R}_+))$ *with* $k = 0, 1$, *there exists a unique weak solution* $\psi \in Y^k(T)$ *to* (2.3) *satisfying*

$$\begin{aligned} &\|\psi(t)\|_{H^k}^2 + C_1(\delta, M, m) \int_0^t (\|\psi(\tau)\|_{H^{k+1}}^2 + k\|\psi_t(\tau)\|_{L^2}^2) d\tau \\ &\le \|\psi_0\|_{H^k}^2 + C_2(\delta, M, m) \int_0^t (\|g(\tau)\|_{H^{k-1}}^2 + \|\psi(\tau)\|_{L^2}^2) d\tau \end{aligned} \tag{2.7}$$

for all $0 \le t \le T$. *Here* $C_1(\delta, M, m)$ *is a positive constant decreasing in* δ, M *and increasing in* m, *and* $C_2(\delta, M, m)$ *is a positive constant increasing in* δ, M *and decreasing in* m.

Lemmas 2.5 and 2.6 are proved by the same method as [11], so we omit the details.

Remark 2.7. (i) *If the same assumptions as in Lemma 2.5 with* $k = 1$ *are satisfied, then* (2.2) *for any* $\varphi \in C_0^1([0,T) \times \mathbb{R}_+)$ *becomes equivalent to the first equation of* (2.1) *in* $C([0,T]; L^2(\mathbb{R}_+))$.

(ii) *If the same assumptions as in Lemma 2.6 with* $k = 1$ *hold, then* (2.4) *for any* $\varphi \in H_0^1(\mathbb{R}_+)$ *and* $h \in C_0^1([0,T))$ *becomes equivalent to the first equation of* (2.3) *in* $L^2(0,T; L^2(\mathbb{R}_+))$.

3. Stationary problem

In this section, we discuss the existence and the convergence rate of the stationary solution $(\tilde{\rho}, \tilde{v}, \tilde{F})$ satisfying the following stationary problems (1.7)–(1.11). To solve this problem, we analyze the properties of the solutions and derive the reduced problem. Integrating (1.7) over $[x, \infty)$ for $x > 0$, we have

$$\tilde{\rho} = \rho_+ \frac{v_+}{\tilde{v}}. \tag{3.1}$$

Letting $x \to 0$ in (3.1), we obtain

$$v_+ = \frac{\tilde{\rho}(0)}{\rho_+} v_b.$$

Therefore, $v_b < 0$ and (1.10) give the fact that $v_+ < 0$ is necessary for the existence of the stationary solution to the problems (1.7)–(1.11). Furthermore, because of (1.10) and (3.1), we regard as $\tilde{v} < 0$.

On the other hand, (1.9) gives $(\tilde{F}/\tilde{v})_x = 0$. Thus, integrating the resultant inequality over $[x, \infty)$ for $x > 0$ and employing (1.10), we have

$$\tilde{F} = \frac{F_+}{v_+}\tilde{v} = \frac{\rho_+ F_+}{\tilde{\rho}}. \tag{3.2}$$

This equality means that

$$\begin{aligned} &\tilde{F}(x) > 0 \quad &\text{if} \quad &F_+ > 0, \\ &\tilde{F}(x) < 0 \quad &\text{if} \quad &F_+ < 0, \\ &\tilde{F}(x) = 0 \quad &\text{if} \quad &F_+ = 0 \end{aligned}$$

for any $x \in \mathbb{R}$. Furthermore, we also get

$$\tilde{F}(0) = \frac{F_+}{v_+} v_b.$$

Namely, in the case $F_+ = 0$, our problem is reduced to the stationary problem of the compressible Navier-Stokes equation, which problem was studied in [13, 21]. Thus, we mainly consider the case $F_+ \neq 0$ in this paper. We will discuss the case $F_+ = 0$ at the end of this section.

By integrating (1.8) over $[x, \infty)$ for $x > 0$ and substituing (3.1) and (3.2) into (1.10), we arrive at the following problem:

$$\nu \tilde{v}_x = I_\beta(\tilde{v}), \tag{3.3}$$

$$\lim_{x\to\infty} \tilde{v}(x) = v_+, \tag{3.4}$$

$$\tilde{v}(0) = v_b, \tag{3.5}$$

where

$$I_\beta(z) := \rho_+ v_+ \left(1 - \frac{\beta^2 F_+^2}{|v_+|^2}\right)(z - v_+) + P\left(\frac{\rho_+ v_+}{z}\right) - P(\rho_+). \tag{3.6}$$

Here, we remark that $I_\beta(v_+) = 0$. Our main purpose of this section is to construct the solution to (3.3)–(3.5). To this end, we analyze the profile of $I_\beta(z)$. It is easy to get $\lim_{z\to -0} I_\beta(z) = \infty$ and

$$\lim_{z\to-\infty} I_\beta(z) = \begin{cases} \infty & \text{if} \quad 0 < \beta < |v_+/F_+|, \\ P(0) - P(\rho_+) & \text{if} \quad \beta = |v_+/F_+|, \\ -\infty & \text{if} \quad \beta > |v_+/F_+|. \end{cases} \tag{3.7}$$

For the derivative, we calculate

$$I'_\beta(z) = \rho_+ v_+ \left(1 - \frac{\beta^2 F_+^2}{v_+^2}\right) - P'\left(\frac{\rho_+ v_+}{z}\right)\frac{\rho_+ v_+}{z^2}, \tag{3.8}$$

and this function leads to

$$I'_\beta(v_+) = \frac{\rho_+ c_+^2}{v_+}(M_+^2 - 1) - \beta^2 \frac{\rho_+ F_+^2}{v_+}. \tag{3.9}$$

Furthermore, we also obtain

$$I''_\beta(z) = \left\{2P'\left(\frac{\rho_+ v_+}{z}\right) + P''\left(\frac{\rho_+ v_+}{z}\right)\frac{\rho_+ v_+}{z}\right\}\frac{\rho_+ v_+}{z^3} > 0$$

for $z < 0$, which means that $I_\beta(z)$ is a convex function. In the case $M_+ > 1$, (3.9) gives

$$I'_\beta(v_+) < 0 \quad \text{if} \quad 0 < \beta < \beta_c, \tag{3.10}$$

$$I'_\beta(v_+) = 0 \quad \text{if} \quad \beta = \beta_c, \tag{3.11}$$

$$I'_\beta(v_+) > 0 \quad \text{if} \quad \beta > \beta_c, \tag{3.12}$$

where

$$\beta_c := \frac{c_+}{|F_+|}\sqrt{M_+^2 - 1}. \tag{3.13}$$

In view of the classification of the sign of $I'_\beta(v_+)$, we use the modified Mach number M_β to see that the conditions (3.10)–(3.12) are rewritten as

$$\begin{aligned} I'_\beta(v_+) < 0 \quad &\text{if} \quad M_\beta > 1, \\ I'_\beta(v_+) = 0 \quad &\text{if} \quad M_\beta = 1, \\ I'_\beta(v_+) > 0 \quad &\text{if} \quad M_\beta < 1. \end{aligned}$$

Remark that $\beta_c \leq |v_+/F_+|$. Especially, in the case $0 < \beta < \beta_c$, there exists v_* such that $v_+ < v_* < 0$ and $I_\beta(v_*) = 0$. Furthermore, we also have $I_\beta(z) > 0$ for $z < v_+$ and $I_\beta(z) < 0$ for $v_+ < z < v_*$. These properties are important to solve the problems (3.3)–(3.5).

Then the existence and property for the solution to (3.3)–(3.5) are described as the following key lemma.

Lemma 3.1. *Assume $v_+ < 0$. Then, the following facts hold true.*

(i) (subsonic case) *Assume that $M_\beta < 1$ holds. Then, the problems* (3.3)–(3.5) *with $v_+ \neq v_b$ has no solution.*

(ii) *Assume that $M_\beta \geq 1$ holds. Then, the following assertions hold.*

(ii-i) (supersonic case) *Suppose $M_\beta > 1$. Then, there exists a unique solution to* (3.3)–(3.5) *satisfying the following decay estimates if, and only if, $v_b < v_*$:*

$$C\delta e^{-cx} \leq |(\tilde{v} - v_+)(x)| \leq \tilde{C}\delta e^{-\tilde{c}x}, \tag{3.14}$$

$$|\partial_x^k(\tilde{v} - v_+)(x)| \leq C\delta e^{-cx}, \quad k = 1, 2. \tag{3.15}$$

Furthermore, the solution $\tilde{v}$ monotonically increases if, and only if, $v_b < v_+$, and monotonically decreases if, and only if, $v_+ < v_b < v_*$.

(ii-ii) (transonic case) *Suppose $M_\beta = 1$. Then, there exists a unique solution to* (3.3)–(3.5) *if, and only if, $v_b < v_+$. Furthermore, the solution $\tilde{v}$ monotonically increases and satisfies the following estimates*

$$\frac{c\delta}{1 + \delta x} \leq (v_+ - \tilde{v})(x) \leq \frac{C\delta}{1 + \delta x}, \tag{3.16}$$

$$|\partial_x^k(\tilde{v} - v_+)(x)| \leq \frac{C\delta^{k+1}}{(1 + \delta x)^{k+1}}, \quad k = 1, 2. \tag{3.17}$$

Remark 3.2. *The generalized Mach number M_β gives the following characterization. Let Ω_{super}, Γ_{trans} and Ω_{sub} be the sets that*

$$\Omega_{super} := \{(M_+,\beta) \,|\, M_\beta > 1\} = \{(M_+,\beta) \,|\, M_+ > 1,\ 0 < \beta < \beta_c\},$$

$$\Gamma_{trans} := \{(M_+,\beta) \,|\, M_\beta = 1\} = \{(M_+,\beta) \,|\, M_+ > 1,\ \beta = \beta_c\},$$

$$\Omega_{sub} := \{(M_+,\beta) \,|\, M_\beta < 1\} = \{(M_+,\beta) \,|\, M_+ > 1,\ \beta > \beta_c\} \cup \{(M_+,\beta) \,|\, M_+ \leq 1\}$$

for $(M_+,\beta) \in \mathbb{R}_+ \times \mathbb{R}_+$. Then, we find that the set $\{(M_+,\beta) \,|\, M_+ > 0,\ \beta > 0\}$ is separated to the three sets Ω_{super}, Ω_{sub} and Γ_{trans}. More precisely, these sets are drawn in the (M_+,β)-plain as follows (Figure 1).

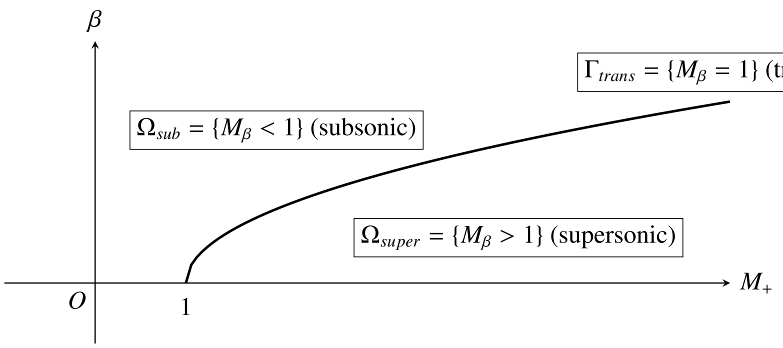


Figure 1. Classification of stationary solutions.

Proof. Proof of Lemma 3.1 (i) Since (3.9), we have $I'_\beta(v_+) > 0$ if $M_\beta < 1$. Therefore, using (3.3), we immediately conclude that $\tilde{v}(x)$ can not approach to v_+ as $x \to \infty$, and there does not exist a solution to (3.3)–(3.5) (see also the two graphs of $I_\beta(z)$ displayed in the below figures).

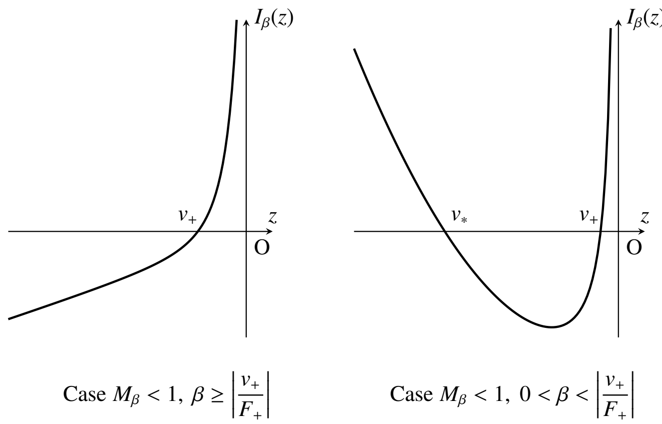
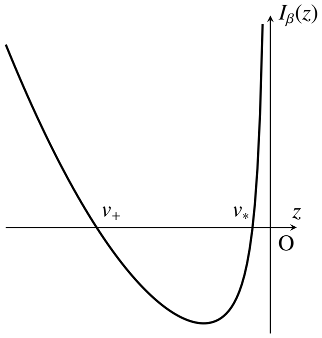


Figure 2. Graphs of $I_\beta(z)$.

(ii-i) In the case $M_\beta > 1$, employing (3.3) and the properties of $I_\beta(z)$ mentioned before, we see that there exists a monotonically increasing solution $\tilde{v}$ to (3.3)–(3.5) if $v_b < v_+$. Similarly, we also find that there exists a monotonically decreasing solution $\tilde{v}$ to (3.3)–(3.5) if $v_+ < v_b < v_*$ (see also the graph of $I_\beta(z)$ displayed in the below figure).



Case $M_\beta > 1$

Figure 3. Graphs of $I_\beta(z)$.

The uniqueness and smoothness of the solutions are derived from the standard argument for ordinary differential equations.

We shall derive the convergence estimate. Taylor's theorem gives

$$I_\beta(z) = I_\beta(v_+) + I'_\beta(v_+)(z - v_+) + \frac{1}{2} I''_\beta(v_+ + \theta(z - v_+))(z - v_+)^2 \tag{3.18}$$

for some $\theta \in (0, 1)$. Since $I_\beta(v_+) = 0$, and $I'_\beta(v_+) < 0$ for $M_\beta > 1$, there exist positive constants δ_*, c_* and C_* such that

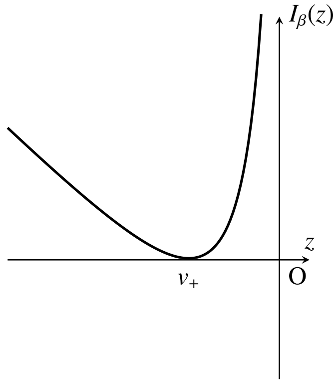
$$c_*(\tilde{v} - v_+) \leq -I_\beta(\tilde{v}) \leq C_*(\tilde{v} - v_+) \tag{3.19}$$

for $|\tilde{v} - v_+| \leq \delta_*$. Thus, combining (3.3) and (3.19), and solving the resultant problem, we obtain

$$\tilde{v}(x) - v_+ \leq (v_b - v_+)e^{-c_* x/\nu}, \qquad \tilde{v}(x) - v_+ \geq (v_b - v_+)e^{-C_* x/\nu} \tag{3.20}$$

for $|\tilde{v} - v_+| \leq \delta_*$. Now, we had already obtained the existence of the global solution $\tilde{v}$ which approached to v_+. This means that there exists $x_* > 0$ such that $|\tilde{v}(x) - v_+| \leq \delta_*$ for $x \geq x_*$. This fact and (3.20) yields to (3.16). For the higher derivatives of $\tilde{v}$, we can apply the same argument and omit it in detail.

(ii-ii) In the case $M_\beta = 1$, using the same argument as before, we find that there exists a monotonically increasing solution $\tilde{v}$ to (3.3)–(3.5) if, and only if, $v_b < v_+$ (see also the graph of $I_\beta(z)$ displayed in the below figure).



Case $M_\beta = 1$

Figure 4. Graphs of $I_\beta(z)$.

We consider the convergence estimate. In this case, it is not possible to derive the estimate (3.19), and it should be analyzed more carefully. Using (3.18) with $I_\beta(v_+) = I'_\beta(v_+) = 0$ and the convexity of $I_\beta(z)$, we obtain

$$\tilde{c}_*(z - v_+)^2 \le I_\beta(z) \le \tilde{C}_*(z - v_+)^2, \tag{3.21}$$

where $\tilde{c}_*$ and $\tilde{C}_*$ are positive constants defined by

$$\tilde{c}_* = \frac{1}{2} \inf_{z\in[v_b,v_+],\, \theta\in[0,1]} I''_\beta(v_+ + \theta(z - v_+)), \quad \tilde{C}_* = \frac{1}{2} \sup_{z\in[v_b,v_+],\, \theta\in[0,1]} I''_\beta(v_+ + \theta(z - v_+)).$$

Then, combining (3.3) and (3.21), and solving the resultant problem, we also get

$$v_+ - \tilde{v}(x) \le \frac{v_+ - v_b}{1 + \tilde{c}_*(v_+ - v_b)x/\nu}, \qquad v_+ - \tilde{v}(x) \ge \frac{v_+ - v_b}{1 + \tilde{C}_*(v_+ - v_b)x/\nu},$$

which means (3.16). For the higher derivatives of $\tilde{v}$, we employ (3.3) and (3.21) again and conclude (3.17). □

Lemma 3.1 immediately gives the following proposition for $(\tilde{\rho}, \tilde{v}, \tilde{F})$.

Proposition 3.3. *Assume $v_+ < 0$. Then, the following facts hold true.*

(i) (subsonic case) *Assume that $M_\beta < 1$ holds. Then, the problems* (1.7)–(1.11) *with $v_+ \ne v_b$ has no solution.*

(ii) *Assume that $M_\beta \ge 1$ holds. Then, the following assertions hold.*

(ii-i) (supersonic case) *Suppose $M_\beta > 1$. Then, there exists a unique solution to* (1.7)–(1.11) *if, and only if, $v_b < v_*$, and the following decay estimates are satisfied:*

$$|\partial_x^k(\tilde{\rho} - \rho_+, \tilde{v} - v_+, \tilde{F} - F_+)(x)| \le C\delta e^{-cx}, \quad k = 0, 1, 2. \tag{3.22}$$

Furthermore, the solution $\tilde{v}$ monotonically increases if, and only if, $v_b < v_+$ and monotonically decreases if, and only if, $v_+ < v_b < v_*$.

(ii-ii) (transonic case) *Suppose $M_\beta = 1$. Then, there exists a unique solution to* (1.7)–(1.11) *if, and only if, $v_b < v_+$. Furthermore, the solution $\tilde{v}$ monotonically increases and satisfies the following estimates:*

$$|\partial_x^k(\tilde{\rho} - \rho_+, \tilde{v} - v_+, \tilde{F} - F_+)(x)| \le \frac{C\delta^{k+1}}{(1 + \delta x)^{k+1}}, \quad k = 0, 1, 2. \tag{3.23}$$

Remark 3.4. (i) *In the case $F_+ = 0$, the solution $\tilde{F}(x)$ to* (1.7)–(1.11) *becomes $\tilde{F}(x) \equiv 0$.*

(ii) *The proof of Lemma 3.1 shows that taking β large so that $\beta \ge |v_+/F_+|$ at least, the stationary solution does not exist regardless of the Mach number M_+ (see for the proof). This indicates that the strong recoiling effect of elastic force disturbs the stationary outflow of the fluid.*

(iii) *It follows from* (3.22) *and* (3.23) *that $(\tilde{\rho}, \tilde{v}, \tilde{F})$ satisfies*

$$\|(\tilde{\rho}, \tilde{v}, \tilde{F})\|_{L^\infty} \le |(\rho_+, v_+, F_+)| + C\delta,$$

$$\|\partial_x^k(\tilde{\rho}, \tilde{v}, \tilde{F})\|_{L^\infty} \le \begin{cases} C\delta, & M_\beta > 1, \\ C\delta^{k+1}, & M_\beta = 1, \end{cases}$$

$$\|\partial_x^k(\tilde{\rho}, \tilde{v}, \tilde{F})\|_{L^2} \le \begin{cases} C\delta, & M_\beta > 1, \\ C\delta^{k+\frac{1}{2}}, & M_\beta = 1 \end{cases}$$

for $k = 1, 2$. These estimates will be repeatedly used in Section 5 and Appendix A.

Proof of Proposition 3.3. The equalities (3.1) and (3.2) tell us that $(\tilde{\rho}, \tilde{F})$ is rewritten by $\tilde{v}$, which is the solution to (3.3)–(3.5). Then, it is easy to confirm that $(\tilde{\rho}, \tilde{v}, \tilde{F})$ is a solution to (1.7)–(1.11). Furthermore, employing (3.15), (3.17), and the fact that

$$\tilde{\rho} - \rho_+ = -\frac{\rho_+}{\tilde{v}}(\tilde{v} - v_+), \qquad \tilde{F} - F_+ = \frac{F_+}{v_+}(\tilde{v} - v_+),$$

the estimates (3.22) and (3.23) are also obtained. □

4. Main result

This section is devoted to introducing the main result of this paper.

We first introduce the compatibility condition. We once assume that a smooth solution $(\rho, v, F)(t, x)$ of the problems (1.1)–(1.6) exists and the perturbation $(\rho - \tilde{\rho}, v - \tilde{v}, F - \tilde{F})$ belongs to $Z^1(T)$ with some positive time $T > 0$. Then, since $v - \tilde{v} \in C([0, T]; H_0^1(\mathbb{R}_+))$ holds, $(v - \tilde{v})(0, \cdot)$ belongs to $H_0^1(\mathbb{R}_+)$. Therefore, it is necessary to impose the compatibility condition of 0-th order:

$$v_0 - \tilde{v} \in H_0^1(\mathbb{R}_+). \tag{4.1}$$

We state the following theorem related to the asymptotic stability of the stationary solution.

Theorem 4.1. *Assume that at least one of the following two cases holds:*

(ND) $M_\beta > 1$ *and* $v_b < v_*$.

(D) $M_\beta = 1$, $1 < M_+ < \sqrt{\dfrac{\rho_+ P''(\rho_+)}{2P'(\rho_+)} + 1}$ *and* $v_b < v_+$.

Suppose that the initial data (ρ_0, v_0, F_0) and the boundary data v_b satisfy (1.4)–(1.6), *compatibility condition* (4.1) *and $(\rho_0 - \tilde{\rho}, v_0 - \tilde{v}) \in H^1(\mathbb{R}_+)$. Furthermore, assume that the initial data also satisfies* (1.12) *in $H^1(\mathbb{R}_+)$. Then, there exists a positive small number $\varepsilon_1 > 0$ such that if*

$$\|(\rho_0 - \tilde{\rho}, v_0 - \tilde{v})\|_{H^1} + \delta \le \varepsilon_1, \quad \inf_{x \in \mathbb{R}_+} \phi_0(x) \ge -\frac{1}{2} \inf_{x \in \mathbb{R}_+} \tilde{\rho}(x),$$

then the problems (1.1)–(1.6) *has a unique solution $(\rho, v, F)(t, x)$ satisfying $(\rho - \tilde{\rho}, v - \tilde{v}, F - \tilde{F}) \in C([0, \infty); H^1)$. Furthermore, the solution converges to the stationary solution as a time goes to infinity, that is,*

$$\lim_{t \to \infty} \|(\rho - \tilde{\rho}, v - \tilde{v}, F - \tilde{F})(t)\|_{L^\infty} = 0. \tag{4.2}$$

Remark 4.2. (i) The restriction $1 < M_+ < (\rho_+(P''(\rho_+)/2P'(\rho_+)) + 1)^{1/2}$ in the condition (D) arises from the convexity of the function $P(\rho) - \beta_c^2 \rho_+^2 F_+^2/\rho$ around $\rho = \rho_+$ to show the stability of the degenerate stationary solution.
(ii) Under the second condition of (1.4), we can reformulate (1.12) as $F_0 - \tilde{F} = -(\rho_0 - \tilde{\rho})\tilde{F}/\rho_0$, which gives the smallness of $F_0 - \tilde{F}$ in $H^1(\mathbb{R}_+)$, provided that $\rho_0 - \tilde{\rho}$ is small in $H^1(\mathbb{R}_+)$. This fact and Theorem 4.1 mean the smallness of $(\rho_0 - \tilde{\rho}, v_0 - \tilde{v}, F_0 - \tilde{F})$ and the asymptotic stability of $(\tilde{\rho}, \tilde{v}, \tilde{F})$.

5. Asymptotic stability of non-degenerate stationary solution

Our purpose of this section is to prove Theorem 4.1. The argument is based on the combination of the local existence theory and the corresponding *a priori* estimate for the solution.

To consider the stability of the stationary solutions, we introduce a new function by

$$J(t,x) := \rho(t,x)F(t,x).$$

Then, the Eqs (1.1) and (1.3) lead to

$$J_t + vJ_x = 0. \tag{5.1}$$

For the Eq (5.1), we assign the initial data $J(0,x) = J_0(x)$ in $x \in \mathbb{R}_+$, where $J_0 := \rho_0 F_0$. The Eq (5.1) has useful properties, and we often utilize (5.1) instead of (1.3). Indeed, the stationary solution $\widetilde{J}(x)$ of (5.1) satisfies $\tilde{v}\widetilde{J}_x = 0$ and this gives $\widetilde{J}(x) = J_+$, where $J_+ := \rho_+ F_+$. This fact will be used later.

On the other hand, it is useful to employ the different expression for (1.2), that is,

$$\rho v_t + \rho v v_x - \nu v_{xx} + P_\beta(\rho)_x - \beta^2 Q(\rho, J)_x = 0,$$

where

$$P_\beta(\rho) := P(\rho) - \beta^2 \frac{J_+^2}{\rho}, \qquad Q(\rho, J) := \frac{1}{\rho}(J^2 - J_+^2).$$

Remark that $P'_\beta(\rho) = P'(\rho) + \beta^2 J_+^2/\rho^2 > 0$ holds true.

We set a perturbation from the stationary solution as

$$(\phi, \psi)(t,x) := (\rho, v)(t,x) - (\tilde{\rho}, \tilde{v})(x), \qquad \zeta(t,x) := J(t,x) - J_+.$$

Coupling (1.1), (1.2) and (5.1) with (1.4)–(1.11), the perturbation (ϕ, ψ, ζ) satisfies the system

$$\begin{aligned} \phi_t + v\phi_x + \rho\psi_x &= f_1, \\ \rho\psi_t + \rho v\psi_x + P'_\beta(\rho)\phi_x - \nu\psi_{xx} &= f_2, \\ \zeta_t + v\zeta_x &= 0 \end{aligned} \tag{5.2}$$

with the initial conditions and boundary conditions at $x = \infty$ and $x = 0$:

$$(\phi, \psi, \zeta)|_{t=0} = (\phi_0, \psi_0, \zeta_0), \quad \inf_{x \in \mathbb{R}_+} (\tilde{\rho}(x) + \phi_0(x)) > 0, \tag{5.3}$$

$$\lim_{x \to \infty} (\phi, \psi, \zeta) = (0, 0, 0), \tag{5.4}$$

$$\psi(t, 0) = 0. \tag{5.5}$$

Here, the initial perturbation is defined by

$$(\phi_0, \psi_0)(x) := (\rho_0 - \tilde{\rho}, v_0 - \tilde{v})(x), \qquad \zeta_0(x) := J_0(x) - J_+,$$

and the functions f_1 and f_2 are given by

$$\begin{aligned} f_1 &:= -(\phi\tilde{v}_x + \psi\tilde{\rho}_x), \\ f_2 &:= -(\rho v - \tilde{\rho}\tilde{v})\tilde{v}_x - (P'_\beta(\rho) - P'_\beta(\tilde{\rho}))\tilde{\rho}_x + \beta^2 Q(\rho, J)_x. \end{aligned}$$

Firstly, we mention the local-in-time existence of the solution of (5.2) with boundary conditions (5.4) and (5.5), and initial and compatibility conditions at $t = \tau$:

$$(\phi, \psi, \zeta)|_{t=\tau} = (\phi_\tau, \psi_\tau, \zeta_\tau), \quad \inf_{x\in\mathbb{R}_+} (\tilde{\rho}(x) + \phi_\tau(x)) > 0, \tag{5.6}$$

$$\psi_\tau \in H_0^1(\mathbb{R}_+). \tag{5.7}$$

Here, $\tau \geq 0$ is an arbitrary non-negative number.

Proposition 5.1. *There exists a positive small number $\varepsilon \in (0, 1)$ independent of τ such that the following assertion holds true:*

Let $\tau \geq 0$ and $M_0 \in (0, \varepsilon]$. If $(\phi_\tau, \psi_\tau, \zeta_\tau)$ satisfies the compatibility condition of 0-th order (5.7) *and*

$$\|(\phi_\tau, \psi_\tau, \zeta_\tau)\|_{H^1} \leq M_0, \quad \inf_{x\in\mathbb{R}_+} \phi_\tau(x) \geq -\frac{1}{2} \inf_{x\in\mathbb{R}_+} \tilde{\rho}(x),$$

then there exists $T = T(\delta, M_0) > 0$ independent of τ such that the problem (5.2) *with* (5.4)–(5.6) *has a unique solution $(\phi, \psi, \zeta) \in Z^1(\tau, \tau + T)$ satisfying*

$$\inf_{(t,x)\in[\tau,\tau+T]\times\mathbb{R}_+} \phi(t, x) \geq -\frac{3}{4} \inf_{x\in\mathbb{R}_+} \tilde{\rho}(x), \tag{5.8}$$

$$(\phi, \psi, \zeta) \in X^1_{3M_0}(\tau, \tau + T), \quad \phi, \zeta \in C^1([\tau, \tau + T]; L^2(\mathbb{R}_+)). \tag{5.9}$$

Remark 5.2. *Under* (5.9)*, the first and third equations of* (5.2) *hold in $C([\tau, \tau + T]; L^2(\mathbb{R}_+))$, and the second equation of* (5.2) *makes sense in $L^2(\tau, \tau + T; L^2(\mathbb{R}_+))$.*

Proposition 5.1 will be proved in Appendix A for the case $\tau = 0$ by using the iteration argument.

We next focus on the *a priori* estimate in the Sobolev space. To state this, we introduce the following notations:

$$N(t)^2 := \sup_{0\leq\tau\leq t} \left(\|\phi(\tau)\|^2_{H^1} + \|\psi(\tau)\|^2_{H^1}\right), \qquad N_0(t) := \sup_{0\leq\tau\leq t} \|J(\tau)\|_{L^\infty},$$

$$M(t)^2 := \int_0^t \left(\|\phi_x(\tau)\|^2_{L^2} + \|\psi_x(\tau)\|^2_{H^1} + |\phi(\tau, 0)|^2\right) d\tau.$$

Then, the *a priori* estimate is summarized by the following proposition.

Proposition 5.3. *Suppose that the same assumptions in Theorem* 4.1 *hold true. Let (ϕ, ψ, ζ) be a solution to* (5.2)–(5.5) *in a time interval $[0, T]$ with $(\phi, \psi, \zeta) \in Z^1(T)$. Then there exist positive constants ε_0 and C_0, such that if $N(T) + \delta \leq \varepsilon_0$, then the following estimates hold uniformly for $t \in [0, T]$:*

$$N(t)^2 + M(t)^2 \leq C_0 \left(\|\phi_0\|^2_{H^1} + \|\psi_0\|^2_{H^1}\right), \quad \|\zeta(t)\|_{H^1} = 0, \tag{5.10}$$

$$\inf_{(t,x)\in[0,t]\times\mathbb{R}_+} \phi(t, x) \geq -\frac{1}{2} \inf_{x\in\mathbb{R}_+} \tilde{\rho}(x). \tag{5.11}$$

To construct the *a priori* estimate, we have to employ the properties for the stationary solutions constructed in Lemma 3.1. Specifically, the following lemma is important to derive the *a priori* estimate.

Lemma 5.4 ([13]). *Suppose that the same assumptions as in Proposition 5.3 hold. Then, the following estimates are obtained.*

(i) *Let $M_\beta > 1$. Then, the following estimates hold true:*

$$\begin{aligned} &\int_0^t \int_0^\infty |\partial_x^k \tilde{v}|^j |\phi|^2 dx d\tau \le C\delta^j \int_0^t \left(\|\phi_x(\tau)\|_{L^2}^2 + |\phi(\tau,0)|^2 \right) d\tau, \\ &\int_0^t \int_0^\infty |\partial_x^k \tilde{v}|^j |\psi|^2 dx d\tau \le C\delta^j \int_0^t \|\psi_x(\tau)\|_{L^2}^2 d\tau \end{aligned} \tag{5.12}$$

for $t \in [0,T]$ and $k, j \in \mathbb{N}$.

(ii) *Let $M_\beta = 1$. Then the following estimates hold true:*

$$\begin{aligned} &\int_0^t \int_0^\infty |\partial_x^k \tilde{v}|^j |\phi|^2 dx d\tau \le C\delta^{(k+1)j-2} \int_0^t \left(\|\phi_x(\tau)\|_{L^2}^2 + |\phi(\tau,0)|^2 \right) d\tau, \\ &\int_0^t \int_0^\infty |\partial_x^k \tilde{v}|^j |\psi|^2 dx d\tau \le C\delta^{(k+1)j-2} \int_0^t \|\psi_x(\tau)\|_{L^2}^2 d\tau \end{aligned} \tag{5.13}$$

for $t \in [0,T]$ and $k, j \in \mathbb{N}$ with $k + j \ge 3$.

Proof. The proof is based on (3.22) and (3.23). The argument is the same as the one in [13], and we omit it in detail. □

Proposition 3.3 and the smallness assumptions on ε_0 in Proposition 5.3 ensure that if ε_0 is sufficiently small, then (5.11) is obtained, and there exist certain positive constants c_ρ, C_ρ, c_v and C_v such that

$$c_\rho \le \rho(t,x) \le C_\rho, \qquad c_v \le -v(t,x) \le C_v, \qquad \tilde{\rho}(x) \ge \frac{\rho_+}{2} \tag{5.14}$$

for $(t,x) \in [0,T] \times \mathbb{R}_+$.

Here, we have noticed the facts $\rho_+ - C(\delta + \|\phi\|_{L^\infty(\mathbb{R}_+)}) \le \rho \le \rho_+ + C(\delta + \|\phi\|_{L^\infty(\mathbb{R}_+)})$ and $|v_+| - C(\delta + \|\psi\|_{L^\infty(\mathbb{R}_+)}) \le -v \le |v_+| + C(\delta + \|\psi\|_{L^\infty(\mathbb{R}_+)})$ by using Proposition 3.3, and then applying Lemma 2.1. To derive (5.10) in Proposition 5.3, we often utilize this boundedness for ρ and $\tilde{\rho}$.

To construct the *a priori* estimate, we need an important property of ζ. More precisely, the following key lemma is shown.

Lemma 5.5. *Suppose that the same assumptions as in Proposition 5.3 hold true. Then, $\zeta(t) \equiv 0$ in $H^1(\mathbb{R}_+)$ for all $t \in [0,T]$.*

Proof. In view of Remark 5.2, we first show that ζ is the weak solution of

$$\zeta_t + a(\psi)\zeta_x = 0, \quad \zeta|_{t=0} = \zeta_0, \quad \zeta|_{x=\infty} = 0.$$

In fact, multiplying third equation of (5.2) by an arbitrary function $\varphi \in C_0^1([0,T) \times \mathbb{R}_+)$, integrating over $(0,T) \times \mathbb{R}_+$ and applying integration by parts, we have the following weak form

$$-\int_0^T (\zeta, \varphi_t + (a(\psi)\varphi)_x)_{L^2} d\tau = (\zeta_0, \varphi(0,\cdot))_{L^2}.$$

This gives the desired fact. We then apply Lemma 2.5 with $k = 1,\ \tilde{\psi} = \psi,\ \phi = \zeta,\ t_1 = 0,\ t_2 = t$ and $f = 0$ to obtain

$$\begin{aligned}\|\zeta(t)\|_{H^1}^2 &\le \|\zeta_0\|_{H^1}^2 + \sum_{l=0}^{1}\int_0^t\int_0^\infty (v_x|\partial_x^l\zeta|^2 + 2lv_x|\zeta_x|^2)dxd\tau\\ &\le \|\zeta_0\|_{H^1}^2 + C\int_0^t \|v_x(\tau)\|_{L^\infty}\|\zeta(\tau)\|_{H^1}^2 d\tau,\end{aligned}$$

and the Gronwall inequality gives

$$\|\zeta(t)\|_{H^1}^2 \le \|\zeta_0\|_{H^1}^2 \exp\left(Ct^{\frac{1}{2}}\left\{\int_0^t \|v_x(\tau)\|_{L^\infty}^2 d\tau\right\}^{\frac{1}{2}}\right)$$

for $t \in [0, T]$. Therefore, using the condition (1.12), Remark 3.4 (iii) and (5.9), we obtain $\|\zeta(t)\|_{H^1} = 0$ for all $t \in [0, T]$, and this completes the proof. □

Remark 5.6. *Lemma 5.5 means that the solution F is represented by $F = \rho_+F_+/\rho$, once the solution ρ is constructed.*

In view of Lemma 5.5, it is enough to concentrate the derivation of *a priori* estimate for $(\phi, \psi)(t)$ only. We first show the basic estimate for $(\phi, \psi)(t)$. The proof is based on its suitable energy form.

Lemma 5.7. *Suppose that the same assumptions as in Proposition 5.3 hold true. Then the following estimate holds:*

$$\begin{aligned}&\|\phi(t)\|_{L^2}^2 + \|\psi(t)\|_{L^2}^2 + \int_0^t \left(\|\psi_x(\tau)\|_{L^2}^2 + |\phi(\tau, 0)|^2\right)d\tau\\ &\le C(\|\phi_0\|_{L^2}^2 + \|\psi_0\|_{L^2}^2) + C\delta M(t)^2 + \beta^4 C\left(1 + N_0(t)^2\right)\int_0^t \|\zeta(\tau)\|_{L^2}^2 d\tau\end{aligned} \tag{5.15}$$

for $t \in [0, T]$.

Proof. To derive the desired estimate, we employ the energy form. We introduce the useful energy function as follows:

$$\mathcal{E} := \Phi(\rho, \tilde{\rho}) + \frac{1}{2}\psi^2, \qquad \Phi(\rho, \tilde{\rho}) := \int_{\tilde{\rho}}^{\rho} \frac{P_\beta(\eta) - P_\beta(\tilde{\rho})}{\eta^2} d\eta. \tag{5.16}$$

Then the function $\Phi(\rho, \tilde{\rho})$ has the following expansion.

$$\Phi(\rho, \tilde{\rho}) = \Phi(\rho) - \Phi(\tilde{\rho}) - \partial_{\tilde{w}}\Phi(\tilde{\rho})(w - \tilde{w}), \quad \Phi(\rho) := \int^{\rho} \frac{P_\beta(\eta)}{\eta^2} d\eta,$$

where $w = 1/\rho$ and $\tilde{w} = 1/\tilde{\rho}$. This expression with $\partial_w\Phi(\rho) = -P_\beta(\rho)$ and $\partial_w^2\Phi(\rho) = \rho^2 P'_\beta(\rho) > 0$ lead to the fact that $\rho\Phi(\rho, \tilde{\rho})$ is equivalent to $|\rho - \tilde{\rho}|^2$ for small $|\rho - \tilde{\rho}|$, and there exist positive constants c_0 and C_0 such that

$$c_0(\phi^2 + \psi^2) \le \rho\mathcal{E} \le C_0(\phi^2 + \psi^2). \tag{5.17}$$

This energy function satisfies the following energy form:

$$(\rho\mathcal{E})_t + \mathcal{F}_x + v\psi_x^2 = \mathcal{R}_1 + \mathcal{R}_2 - \beta^2\psi_x Q(\rho, J), \tag{5.18}$$

where

$$\mathcal{F} := \rho v \mathcal{E} + (P_\beta(\rho) - P_\beta(\tilde{\rho}))\psi - v\psi\psi_x - \beta^2\psi Q(\rho, J),$$
$$\mathcal{R}_1 := -v\frac{\tilde{v}_{xx}}{\tilde{\rho}}\phi\psi, \quad \mathcal{R}_2 := -\{\rho\psi^2 + P_\beta(\rho) - P_\beta(\tilde{\rho}) - P'_\beta(\tilde{\rho})\phi\}\tilde{v}_x.$$

Integrating (5.18) over $(0, t) \times \mathbb{R}_+$ and employing the boundary conditions (5.4) and (5.5), we get

$$\begin{aligned}&\int_0^\infty \rho(t, x)\mathcal{E}(t, x)dx - \int_0^t (\rho v\mathcal{E})(\tau, 0)d\tau + v\int_0^t \|\psi_x(\tau)\|_{L^2}^2 d\tau \\ &= \int_0^\infty \rho_0(x)\mathcal{E}(0, x)dx + \int_0^t\int_0^\infty (\mathcal{R}_1(\tau, x) + \mathcal{R}_2(\tau, x))dxd\tau \\ &\quad - \beta^2\int_0^t\int_0^\infty \psi_x(\tau, x)Q(\rho, J)(\tau, x)dxd\tau.\end{aligned} \tag{5.19}$$

Because of $v_b < 0$, the second term on the left-hand side of (5.19) is handled as

$$-(\rho v\mathcal{E})(\tau, 0) = |v_b|\rho(\tau, 0)\mathcal{E}(\tau, 0) \geq c_0|v_b||\phi(\tau, 0)|^2.$$

For the remainder terms $\mathcal{R}_1$ and $\mathcal{R}_2$, we estimate

$$|\mathcal{R}_1| \leq C|\tilde{v}_{xx}||\phi||\psi|, \qquad |\mathcal{R}_2| \leq C|\tilde{v}_x|(\psi^2 + \phi^2).$$

Therefore, for the non-degenerate case $M_\beta > 1$, we apply (5.12) and obtain

$$\int_0^t\int_0^\infty (|\mathcal{R}_1| + |\mathcal{R}_2|)dxd\tau \leq C\delta\int_0^t \left(\|\phi_x(\tau)\|_{L^2}^2 + \|\psi_x(\tau)\|_{L^2}^2 + |\phi(\tau, 0)|^2\right)d\tau \leq C\delta M(t)^2. \tag{5.20}$$

On the other hand, we can not employ the same argument for $\mathcal{R}_2$ in the degenerate case $M_\beta = M_{\beta_c} = 1$. To overcome difficulty, we reformulate $\mathcal{R}_2$ as

$$\begin{aligned}\mathcal{R}_2 = &-\tilde{v}_x\rho\psi^2 - \frac{1}{2}P''_\beta(\rho_+)\tilde{v}_x\phi^2 - \frac{1}{2}\left(P''_{\beta_c}(\tilde{\rho}) - P''_{\beta_c}(\rho_+)\right)\tilde{v}_x\phi^2 \\ &- \left(P_{\beta_c}(\rho) - P_{\beta_c}(\tilde{\rho}) - P'_{\beta_c}(\tilde{\rho})\phi - \frac{1}{2}P''_{\beta_c}(\tilde{\rho})\phi^2\right)\tilde{v}_x.\end{aligned} \tag{5.21}$$

Due to $\tilde{v}_x > 0$, the first term of the right hand side in (5.21) is negative. Furthermore, using the fact that $P''_{\beta_c}(\rho_+) > 0$ is satisfied if, and only if, $0 < \beta_c < \beta_*$, where

$$\beta_* := \sqrt{\frac{\rho_+ P''(\rho_+)}{2F_+^2}},$$

the second term of the right hand side in (5.21) is also negative if $0 < \beta_c < \beta_*$. Namely, $\mathcal{R}_2$ is estimated as

$$\tilde{v}_x\rho\psi^2 + c\tilde{v}_x\phi^2 + \mathcal{R}_2 \leq 0 \tag{5.22}$$

for suitably small δ and $|\phi|$. Therefore, using (5.22), (5.20) also holds for the case $M_\beta = M_{\beta_c} = 1$.

For the last term in (5.19), using the fact that $\|Q(\rho, J)\|_{L^2} \leq (|J_+| + \|J\|_{L^\infty})\|\zeta\|_{L^2}/c_\rho$, we obtain

$$\beta^2 \int_0^\infty \psi_x(\tau, x) Q(\rho, J)(\tau, x) dx \leq \varepsilon \|\psi_x\|_{L^2}^2 + \beta^4 C_\varepsilon \|Q(\rho, J)\|_{L^2}^2$$
$$\leq \varepsilon \|\psi_x\|_{L^2}^2 + \beta^4 C_\varepsilon (1 + \|J\|_{L^\infty}^2) \|\zeta\|_{L^2}^2.$$

Finally, substituting the above estimate into (5.19), we arrive at the desired estimate and complete the proof. □

The next goal is to derive the estimate for the first-order derivatives of the solution. To obtain the estimate for ϕ_x, we need to deal with ϕ_{xx} and $\phi(t, 0)$ after differentiating the first equation of (5.2) in x and applying integration by parts, formally. However, since we only treat $\phi(t) \in H^1(\mathbb{R}_+)$ for $0 \leq t \leq T$, ϕ_{xx} and $\phi(t, 0)$ do not always exist. Therefore, the formal argument cannot make sense in our setting. To overcome this difficulty, we recall Definition 2.3 and Lemma 2.5 to follow the theory of weak solutions to transport equations. This merit is that ϕ_{xx} and $\phi(t, 0)$ are not needed in their statements and the proof of the estimate for ϕ_x. This argument is inspired in [12].

Lemma 5.8. *Suppose that the same assumptions as in Proposition 5.3 hold true. Then the following estimate holds:*

$$\begin{aligned} \|\phi_x(t)\|_{L^2}^2 + \int_0^t \|\phi_x(\tau)\|_{L^2}^2 d\tau \leq C(\|\psi_0\|_{L^2}^2 + \|\phi_{0,x}\|_{L^2}^2) + C\left(\|\psi(t)\|_{L^2}^2 + \int_0^t \|\psi_x(\tau)\|_{L^2}^2 d\tau\right) \\ + C\,(N(t) + \delta)\, M(t)^2 + \beta^4 C\,(1 + N_0(t))^2\,(1 + N(t))^2 \int_0^t \|\zeta(\tau)\|_{H^1}^2 d\tau \end{aligned} \tag{5.23}$$

for $t \in [0, T]$.

Proof. We first claim that $\overline{\phi} := \phi_x \in C([0, T]; L^2(\mathbb{R}_+))$ is the weak solution of

$$\overline{\phi}_t + a(\psi)\overline{\phi}_x = \tilde{f}_1, \quad \overline{\phi}|_{t=0} = \phi_{0,x}, \quad \overline{\phi}|_{x=\infty} = 0, \tag{5.24}$$

where $\tilde{f}_1 := -v_x\phi_x + (f_1 - \rho\psi_x)_x \in L^2(0, T; L^2(\mathbb{R}_+))$. To show this fact, we notice that the first equation of (5.2) holds for almost every $(t, x) \in (0, T) \times \mathbb{R}_+$ because of Remark 5.2. Let φ be an arbitrary function belonging to $C_0^1([0, T) \times \mathbb{R}_+)$. Multiplying the first equation of (5.2) by $-\varphi_x$ and integrating over $(0, T) \times \mathbb{R}_+$ we have

$$-\int_0^T \int_0^\infty (\phi_t + a(\psi)\phi_x)\varphi_x dx dt = -\int_0^T \int_0^\infty (-\rho\psi_x + f_1)\varphi_x dx dt.$$

Applying integration by parts and using $a(\psi)\varphi_x = (a(\psi)\varphi)_x - v_x\varphi$ in we arrive at the weak form

$$-\int_0^T (\phi_x, \varphi_t + (a(\psi)\varphi)_x)_{L^2} d\tau = (\phi_{0,x}, \varphi(0, \cdot))_{L^2} + \int_0^T (\tilde{f}_1, \varphi)_{L^2} d\tau,$$

which yields the desired fact. Therefore, we are able to apply Lemma 2.5 with $k = 0$, $\tilde{\psi} = \psi$, $\phi = \phi_x$, $t_1 = 0$, $t_2 = tt$ and $f = \tilde{f}_1$ to give

$$\begin{aligned} \|\phi_x(t)\|_{L^2}^2 \leq \|\phi_{0,x}\|_{L^2}^2 - \int_0^t \int_0^\infty v_x \phi_x^2 dx d\tau - 2\int_0^t \int_0^\infty \rho\phi_x\psi_{xx} dx d\tau \\ - 2\int_0^t \int_0^\infty \rho_x\psi_x\phi_x dx d\tau + 2\int_0^t \int_0^\infty \partial_x f_1 \phi_x dx d\tau \end{aligned} \tag{5.25}$$

for $0 \le t \le T$. We next eliminate $\rho\phi_x\psi_{xx}$ in the right-hand side of (5.25). Multiplying the second equation in (5.2) by $\rho\phi_x$ yields

$$\rho^2\phi_x\psi_t + \rho P'_\beta(\rho)\phi_x^2 - \nu\rho\phi_x\psi_{xx} + \rho^2 v\phi_x\psi_x = \rho\phi_x f_2.$$

Then, integrating above equation over $(0,t)\times\mathbb{R}_+$ and using the formula

$$\int_0^t\int_0^\infty \rho^2\phi_x\psi_t dxd\tau = \int_0^\infty \rho^2\phi_x\psi dx - \int_0^\infty \rho_0^2\phi_{0,x}\psi_0 dx + \int_0^t\int_0^\infty (2\tilde{\rho}_x\psi + \rho\psi_x)\rho\phi_t dxd\tau,$$

we have

$$\begin{aligned}
&\int_0^\infty \rho^2\phi_x\psi dx + \int_0^t\int_0^\infty \rho P'_\beta(\rho)\phi_x^2 dxd\tau - \nu\int_0^t\int_0^\infty \rho\phi_x\psi_{xx}dxd\tau \\
&= \int_0^\infty \rho_0^2\phi_{0,x}\psi_0 dx - \int_0^t\int_0^\infty \rho^2 v\phi_x\psi_x dxd\tau - \int_0^t\int_0^\infty (2\tilde{\rho}_x\psi + \rho\psi_x)\rho\phi_t dxd\tau \\
&+ \int_0^t\int_0^\infty \rho\phi_x f_2 dxd\tau.
\end{aligned} \tag{5.26}$$

Here, the integral of $\rho^2\phi_x\psi_t$ is calculated via mollification with respect to t and integration by parts. This calculation is standard, so we omit the detail. Then, combining (5.25) and (5.26) to eliminate $\rho\phi_x\psi_{xx}$, we get

$$\begin{aligned}
&\frac{\nu}{2}\|\phi_x\|_{L^2}^2 + \int_0^\infty \rho^2\phi_x\psi dx + \int_0^t\int_0^\infty \rho P'_\beta(\rho)\phi_x^2 dxd\tau \\
&\le \frac{\nu}{2}\|\phi_{0,x}\|_{L^2}^2 + \int_0^\infty \rho_0^2\phi_{0,x}\psi_0 dx + \int_0^t\int_0^\infty R_1 dxd\tau,
\end{aligned} \tag{5.27}$$

where

$$R_1 := \nu\phi_x\partial_x f_1 + \rho\phi_x f_2 - \frac{\nu}{2}v_x\phi_x^2 - (\nu\rho_x + \rho^2 v)\phi_x\psi_x - (2\tilde{\rho}_x\psi + \rho\psi_x)\rho\phi_t.$$

We estimate the remainder term R_1. Since $Q(\rho,J)_x = 2JJ_x/\rho - (J^2 - J_+^2)\rho_x/\rho^2$, we have

$$\begin{aligned}
\|Q(\rho,J)_x\|_{L^2} &\le \frac{2}{c_\rho}\|J\|_{L^\infty}\|\zeta_x\|_{L^2} + \frac{1}{c_\rho^2}(|J_+| + \|J\|_{L^\infty})(\|\tilde{\rho}_x\zeta\|_{L^2} + \|\phi_x\zeta\|_{L^2}) \\
&\le C(1 + \|J\|_{L^\infty})(1 + \|\phi_x\|_{L^2})\|\zeta\|_{H^1}.
\end{aligned}$$

Thus, this estimate and

$$\begin{aligned}
&|f_1| \le |\tilde{v}_x||\phi| + |\tilde{\rho}_x||\psi|, \quad |\partial_x f_1| \le |\tilde{v}_{xx}||\phi| + |\tilde{\rho}_{xx}||\psi| + |\tilde{v}_x||\phi_x| + |\tilde{\rho}_x||\psi_x|, \\
&|f_2| \le |\tilde{v}_x|(|\tilde{v}||\phi| + |\rho||\psi|) + C|\tilde{\rho}_x||\phi| + \beta^2|Q(\rho,J)_x|
\end{aligned}$$

give

$$\begin{aligned}
\int_0^t \|f_1\|_{L^2}^2 d\tau &\le 2\int_0^t\int_0^\infty (\tilde{v}_x^2\phi^2 + \tilde{\rho}_x^2\psi^2)dxd\tau \\
&\le C\delta^2\int_0^t \left(\|\phi_x(\tau)\|_{L^2}^2 + \|\psi_x(\tau)\|_{L^2}^2 + |\phi(\tau,0)|^2\right)d\tau,
\end{aligned} \tag{5.28}$$

$$\int_0^t \|\partial_x f_1\|_{L^2}^2 d\tau \leq 4\int_0^t \int_0^\infty (\tilde{v}_{xx}^2\phi^2 + \tilde{\rho}_{xx}^2\psi^2 + \tilde{v}_x^2\phi_x^2 + \tilde{\rho}_x^2\psi_x^2)dxd\tau$$
$$\leq C\delta^2 \int_0^t \left(\|\phi_x(\tau)\|_{L^2}^2 + \|\psi_x(\tau)\|_{L^2}^2 + |\phi(\tau,0)|^2\right) d\tau, \tag{5.29}$$

and

$$\int_0^t \|f_2\|_{L^2}^2 d\tau \leq C\int_0^t \int_0^\infty \left(\tilde{v}_x^2(\phi^2+\psi^2) + \tilde{\rho}_x^2\phi^2\right) dxd\tau + \beta^4 C\int_0^t \|Q(\rho,J)_x\|_{L^2}^2 d\tau$$
$$\leq C\delta^2 \int_0^t \left(\|\phi_x(\tau)\|_{L^2}^2 + \|\psi_x(\tau)\|_{L^2}^2 + |\phi(\tau,0)|^2\right) d\tau \tag{5.30}$$
$$+ \beta^4 C(1+N_0(t))^2(1+N(t))^2 \int_0^t \|\zeta\|_{H^1}^2 d\tau.$$

Here, Lemmas 3.1 and 5.4, and (5.14) are also applied. Using the first equation in (5.2), we have

$$|R_1| \leq C|\phi_x|(|\partial_x f_1| + |f_2|) + C|v_x|\phi_x^2 + C(|\rho_x| + |v|)|\phi_x||\psi_x|$$
$$+ C(|\tilde{\rho}_x||\psi| + |\psi_x|)(|f_1| + |v||\phi_x| + |\psi_x|),$$

and this gives

$$\int_0^\infty |R_1|dx \leq \sigma\|\phi_x\|_{L^2}^2 + C_\sigma(\|\psi_x\|_{L^2}^2 + \|f_1\|_{H^1}^2 + \|f_2\|_{L^2}^2)$$
$$+ C\int_0^\infty |\tilde{v}_x|\phi_x^2 dx + C\int_0^\infty |\psi_x|\phi_x^2 dx + C_\sigma \int_0^\infty |\tilde{\rho}_x|^2|\psi|^2 dx$$

for all $\sigma > 0$. Thus, the estimates (5.28), –(5.30) and Lemma 5.4 yield

$$\int_0^t \int_0^\infty |R_1|dxd\tau \leq (\sigma + C\delta)\int_0^t \|\phi_x\|_{L^2}^2 d\tau + C_\sigma \int_0^t \|\psi_x\|_{L^2}^2 d\tau + C_\sigma \delta^2 M(t)^2$$
$$+ CN(t)M(t)^2 + \beta^4 C_\sigma (1+N_0(t))^2(1+N(t))^2 \int_0^t \|\zeta\|_{H^1}^2 d\tau. \tag{5.31}$$

Here, we used the fact that

$$\int_0^t \int_0^\infty |\psi_x|\phi_x^2 dxd\tau \leq \int_0^t \|\psi_x\|_{H^1}\|\phi_x\|_{L^2}^2 d\tau \leq \sup_{\tau\in[0,t]} \|\phi_x(\tau)\|_{L^2} \int_0^t (\|\psi_x(\tau)\|_{H^1}^2 + \|\phi_x(\tau)\|_{L^2}^2) d\tau.$$

Therefore, (5.27) with (5.31) leads to

$$c\|\phi_x\|_{L^2}^2 + \tilde{c}_\rho \int_0^t \|\phi_x\|_{L^2}^2 d\tau \leq C(\|\psi_0\|_{L^2}^2 + \|\phi_{0,x}\|_{L^2}^2) + C\|\psi\|_{L^2}^2 + (\sigma + C\delta)\int_0^t \|\phi_x\|_{L^2}^2 d\tau$$
$$+ C_\sigma \int_0^t \|\psi_x\|_{L^2}^2 d\tau + C_\sigma \delta^2 M(t)^2 + CN(t)M(t)^2 + \beta^4 C_\sigma (1+N_0(t))^2(1+N(t))^2 \int_0^t \|\zeta\|_{H^1}^2 d\tau,$$

where $\tilde{c}_\rho := \min_{\rho\in[c_\rho, C_\rho]} \rho P'_\beta(\rho)$. Thus, letting σ suitably small, we arrive at the desired estimate (5.23) and the proof is completed. □

We next focus on ψ_x whose estimate is given in the following lemma.

Lemma 5.9. *Suppose that the same assumptions as in Proposition 5.3 hold true. Then, the following inequality holds:*

$$\begin{aligned}\|\psi_x(t)\|_{L^2}^2 + \int_0^t \|\psi_{xx}(\tau)\|_{L^2}^2 d\tau \le C\|\psi_{0,x}\|_{L^2}^2 + C\int_0^t (\|\phi_x(\tau)\|_{L^2}^2 + \|\psi_x(\tau)\|_{L^2}^2)\, d\tau + C\delta^2 M(t)^2 \\ + \beta^4 C(1+N_0(t))^2(1+N(t))^2 \int_0^t \|\zeta(\tau)\|_{H^1}^2 d\tau\end{aligned} \tag{5.32}$$

for $t \in [0, T]$.

Proof. Multiplying the second equation in (5.2) by $-\psi_{xx}/\rho$ yields

$$-\psi_x\psi_{xx} + \frac{\nu}{\rho}\psi_{xx}^2 = R_2,$$

where

$$R_2 := -\frac{\psi_{xx}}{\rho} f_2 + v\psi_x\psi_{xx} + \frac{P_\beta'(\rho)}{\rho}\phi_x\psi_{xx}.$$

Then, integrating this equality over $(0,t)\times\mathbb{R}_+$ and using the formula

$$-\int_0^t \int_0^\infty \psi_x\psi_{xx} dx d\tau = \frac{1}{2}(\|\psi_x\|_{L^2}^2 - \|\psi_{0,x}\|_{L^2}^2),$$

we obtain

$$\frac{1}{2}\|\psi_x\|_{L^2}^2 + \frac{\nu}{C_\rho}\int_0^t \|\psi_{xx}\|_{L^2}^2 d\tau \le \frac{1}{2}\|\psi_{0,x}\|_{L^2}^2 + \int_0^t \int_0^\infty |R_2| dx d\tau. \tag{5.33}$$

Here, the integral of $\psi_x\psi_{xx}$ is calculated via mollification with respect to t and integration by parts. This calculation is standard, so we omit the detail. We then estimate the remainder term as $|R_2| \le C(|\phi_x| + |\psi_x| + |f_2|)|\psi_{xx}|$, and this gives

$$\begin{aligned}\int_0^t \int_0^\infty |R_2| dx d\tau &\le \sigma \int_0^t \|\psi_{xx}\|_{L^2}^2 d\tau + C_\sigma \int_0^t \left(\|\phi_x\|_{L^2}^2 + \|\psi_x\|_{L^2}^2 + \|f_2\|_{L^2}^2\right) d\tau \\ &\le \sigma \int_0^t \|\psi_{xx}\|_{L^2}^2 d\tau + C_\sigma \int_0^t (\|\phi_x\|_{L^2}^2 + \|\psi_x\|_{L^2}^2) d\tau \\ &\quad + C_\sigma\delta^2 \int_0^t \left(\|\phi_x\|_{L^2}^2 + \|\psi_x\|_{L^2}^2 + |\phi(\tau,0)|^2\right) d\tau \\ &\quad + \beta^4 C_\sigma (1+N_0(t))^2(1+N(t))^2 \int_0^t \|\zeta\|_{H^1}^2 d\tau\end{aligned} \tag{5.34}$$

for all $\sigma > 0$. The estimate (5.33) with (5.34) leads to the desired estimate (5.32). This completes the proof. □

Proof of Proposition 5.3. Combining (5.15) and (5.23), we have

$$\begin{aligned}&\|\phi(t)\|_{H^1}^2 + \|\psi(t)\|_{L^2}^2 + \int_0^t (\|\phi_x(\tau)\|_{L^2}^2 + \|\psi_x(\tau)\|_{L^2}^2 + |\phi(\tau,0)|^2) d\tau \\ &\le C(\|\phi_0\|_{H^1}^2 + \|\psi_0\|_{L^2}^2) + C\,(N(t)+\delta)\, M(t)^2 + \beta^4 C\,(1+N_0(t))^2\,(1+N(t))^2 \int_0^t \|\zeta(\tau)\|_{H^1}^2 d\tau.\end{aligned}$$

Furthermore, substituting the resultant estimate and (5.32), we obtain

$$\begin{aligned}
&\|\phi(t)\|_{H^1}^2 + \|\psi(t)\|_{H^1}^2 + \int_0^t (\|\phi_x(\tau)\|_{L^2}^2 + \|\psi_x(\tau)\|_{H^1}^2 + |\phi(\tau,0)|^2) d\tau \\
&\le C(\|\phi_0\|_{H^1}^2 + \|\psi_0\|_{H^1}^2) + C\,(N(t)+\delta)\,M(t)^2 + \beta^4 C\,(1+N_0(t))^2\,(1+N(t))^2 \int_0^t \|\zeta(\tau)\|_{H^1}^2 d\tau.
\end{aligned}$$

Namely this gives

$$\begin{aligned}
N(t)^2 + M(t)^2 \le\ & C(\|\phi_0\|_{H^1}^2 + \|\psi_0\|_{H^1}^2) + C\,(N(t)+\delta)\,M(t)^2 \\
& + \beta^4 C\,(1+N_0(t))^2\,(1+N(t))^2 \int_0^t \|\zeta(\tau)\|_{H^1}^2 d\tau
\end{aligned}$$

for $t \in [0,T]$. Consequently, by using Lemma 5.5 and taking ε_0 suitably small, we arrive at the desired estimate (5.10). This completes the proof. □

In order to complete the proof of Theorem 4.1, it remains to derive the estimate for $F - \tilde{F}$, which is summarized as the following lemma.

Lemma 5.10. *Suppose that the same assumptions as in Theorem 4.1 hold. Then, F satisfies*

$$\|(F-\tilde{F})(t)\|_{H^1} \le C\|(\phi_0,\psi_0)\|_{H^1} \tag{5.35}$$

for all $t \in [0,T]$.

Proof. Using (3.2) and Lemma 5.5, $F - \tilde{F}$ is rewritten as

$$F - \tilde{F} = -\frac{\tilde{F}}{\rho}\phi. \tag{5.36}$$

Therefore, we easily see from Proposition 3.3, Proposition 5.3, (5.14) and (5.36) that

$$\|(F-\tilde{F})(t)\|_{L^2} \le C\|\phi(t)\|_{L^2} \le C\|(\phi_0,\psi_0)\|_{H^1} \tag{5.37}$$

for all $t \in [0,T]$. We next study $(F-\tilde{F})_x$. Differentiating (5.36) in x gives

$$(F-\tilde{F})_x = -\frac{\tilde{F}_x}{\rho}\phi + \frac{\tilde{F}}{\rho^2}\rho_x\phi - \frac{\tilde{F}}{\rho}\phi_x.$$

It then follows from Proposition 3.3, Proposition 5.3, (5.14) and (5.36) that

$$\|(F-\tilde{F})_x(t)\|_{L^2} \le C\|\phi(t)\|_{H^1} \le C\|(\phi_0,\psi_0)\|_{H^1} \tag{5.38}$$

for all $t \in [0,T]$. Consequently, the desired estimate (5.35) immediately follows from (5.37) and (5.38). This completes the proof. □

Proof of Theorem 4.1. To show the global existence result, we use a bootstrap method based on the combination of Proposition 5.1, Proposition 5.3 and Lemma 5.10. Let ε_0 in Proposition 5.3 be $\varepsilon_0 \in (0, \varepsilon]$, where ε is introduced in Proposition 5.1. We then set

$$\varepsilon_1 := \frac{\varepsilon_0}{6} \min\left\{1, \frac{1}{C_0}\right\} \le \varepsilon.$$

We start from the assumptions

$$\|(\phi_0, \psi_0, \zeta_0)\|_{H^1} + \delta \le \varepsilon_1, \quad \inf_{x\in\mathbb{R}_+} \phi_0(x) \ge -\frac{1}{2} \inf_{x\in\mathbb{R}_+} \tilde{\rho}(x).$$

Since $\|(\phi_0, \psi_0, \zeta_0)\|_{H^1} \le \varepsilon_0/6 \le \varepsilon$ clearly holds, we are able to apply Proposition 5.1 with $\tau = 0$, $M_0 = \varepsilon_0/6$ $(\le \varepsilon)$ and $(\phi_\tau, \psi_\tau, \zeta_\tau) = (\phi_0, \psi_0, \zeta_0)$ to show that (5.2)–(5.5) has a unique solution $(\phi, \psi, \zeta) \in Z^1(0, T) = Z^1(T)$ satisfying (5.8) and (5.9) with $T = T(\delta, \varepsilon_0/6)$, $\tau = 0$, $M_0 = \varepsilon_0/6$ and $(\phi_\tau, \psi_\tau, \zeta_\tau) = (\phi_0, \psi_0, \zeta_0)$. Moreover, it follows from (5.9) and $\delta \le \varepsilon_1 \le \varepsilon_0/6$ that

$$\sup_{t\in[0,T]} \|(\phi, \psi, \zeta)(t)\|_{H^1} + \delta \le 3 \cdot \frac{\varepsilon_0}{6} + \frac{\varepsilon_0}{6} \le \varepsilon_0. \tag{5.39}$$

Therefore, we can apply Proposition 5.3 to obtain

$$\sup_{t\in[0,T]} \|(\phi, \psi, \zeta)(t)\|_{H^1} \le C_0\|(\phi_0, \psi_0, \zeta_0)\|_{H^1} \le C_0\varepsilon_1 \le \frac{\varepsilon_0}{6},$$
$$\inf_{(t,x)\in[0,T]\times\mathbb{R}_+} \phi(t, x) \ge -\frac{1}{2} \inf_{x\in\mathbb{R}_+} \tilde{\rho}(x),$$

which means

$$\|(\phi, \psi, \zeta)(T)\|_{H^1} \le \frac{\varepsilon_0}{6}(\le \varepsilon), \quad \inf_{x\in\mathbb{R}_+} \phi(T, x) \ge - -\frac{1}{2} \inf_{x\in\mathbb{R}_+} \tilde{\rho}(x).$$

Here, we use the fact $\varepsilon_1 \le \varepsilon_0/(6C_0)$. Therefore, we are able to construct a unique solution $(\phi^1, \psi^1, \zeta^1) \in Z^1(T, 2T)$ of (5.2)–(5.5) satisfying (5.8) and (5.9) with $\tau = T$, $M_0 = \varepsilon_0/6$ and $(\phi_\tau, \psi_\tau, \zeta_\tau) = (\phi, \psi, \zeta)(T)$ by applying Proposition 5.1 with $\tau = T = T(\delta, \varepsilon_0/6)$, $M_0 = \varepsilon_0/6$ and $(\phi_\tau, \psi_\tau, \zeta_\tau) = (\phi, \psi, \zeta)(T)$. Moreover, we see from (5.9) and $\delta \le \varepsilon_1 \le \varepsilon_0/6$ that

$$\sup_{t\in[T,2T]} \|(\phi^1, \psi^1, \zeta^1)(t)\|_{H^1} + \delta \le 3 \cdot \frac{\varepsilon_0}{6} + \frac{\varepsilon_0}{6} \le \varepsilon_0. \tag{5.40}$$

Therefore, we are able to extend the solution of (5.2)–(5.5) in $Z^1(T)$ to that in $Z^1(2T)$ by defining $(\phi, \psi, \zeta)(t, x) := (\phi^1, \psi^1, \zeta^1)(t, x)$ for $(t, x) \in [T, 2T] \times \mathbb{R}_+$. Combining (5.39) and (5.40) to confirm

$$\sup_{t\in[0,2T]} \|(\phi, \psi, \zeta)(t)\|_{H^1} + \delta \le \varepsilon_0,$$

we can apply Proposition 5.3 with $T = 2T$ and notice the fact $\varepsilon_1 \le \varepsilon_0/(6C_0)$ to obtain

$$\sup_{t\in[0,2T]} \|(\phi, \psi, \zeta)(t)\|_{H^1} \le C_0\|(\phi_0, \psi_0, \zeta_0)\|_{H^1} \le C_0\varepsilon_1 \le \frac{\varepsilon_0}{6},$$
$$\inf_{(t,x)\in[0,2T]\times\mathbb{R}_+} \phi(t, x) \ge -\frac{1}{2} \inf_{x\in\mathbb{R}_+} \tilde{\rho}(x)$$

which yields

$$\|(\phi,\psi,\zeta)(2T)\|_{H^1} \le \frac{\varepsilon_0}{6}\ (\le \varepsilon), \quad \inf_{x\in\mathbb{R}_+}\phi(2T,x) \ge -\frac{1}{2}\inf_{x\in\mathbb{R}_+}\tilde{\rho}(x).$$

Therefore, it follows from Proposition 5.1 with $\tau = 2T = 2T(\delta,\varepsilon_0/6)$ and $(\phi_\tau,\psi_\tau,\zeta_\tau) = (\phi,\psi,\zeta)(2T)$ that we are able to extend the solution (ϕ,ψ,ζ) of (5.2)–(5.5) in $Z^1(2T)$ to that in $Z^1(3T)$. Consequently, repeating the above argument we have a global-in-time solution (ϕ,ψ,ζ) of (5.2)–(5.5) in $[0,\infty)\times\mathbb{R}_+$ with the desired properties

$$\begin{aligned}&(\phi,\psi,\zeta)\in C([0,\infty);H^1(\mathbb{R}_+)), \quad \sup_{t\in[0,\infty)}\|(\phi,\psi,\zeta)(t)\|_{H^1} \le C_0\|(\phi_0,\psi_0,\zeta_0)\|_{H^1},\\ &\inf_{(t,x)\in[0,\infty)\times\mathbb{R}_+}\phi(t,x) \ge -\frac{1}{2}\inf_{x\in\mathbb{R}_+}\tilde{\rho}(x).\end{aligned} \tag{5.41}$$

We next prove the limit (4.2). We see from (5.41) and the definition of ε_1 that $\|(\phi,\psi,\zeta)(t)\|_{H^1}+\delta \le \varepsilon_0$ holds for all $t \ge 0$. Therefore, Proposition 5.3 leads to

$$M(\infty)^2 := \int_0^\infty (\|\phi_x(\tau)\|_{L^2}^2 + \|\psi_x(\tau)\|_{H^1}^2 + |\phi(\tau,0)|^2)d\tau \le C\|(\phi_0,\psi_0,\zeta_0)\|_{H^1}^2 < \infty, \tag{5.42}$$

which implies that there exists a monotone increasing sequence $\{t_m\}_{m=1}^\infty$ with $t_m \to \infty$ $(m\to\infty)$ such that the following limit holds:

$$\lim_{m\to\infty}(\|\phi_x(t_m)\|_{L^2}^2 + \|\psi_x(t_m)\|_{H^1}^2) = 0. \tag{5.43}$$

In view of Lemma 2.1, it suffices to prove $\|(\phi,\psi)_x(t)\|_{L^2} \to 0$ as $t\to\infty$. Since ψ belongs to same function space as Lemma 2.2 with an arbitrary T, we have the estimate

$$\|\psi_x(t)\|_{L^2}^2 \le C\left\{\|\psi_x(t_m)\|_{L^2}^2 + \left(\int_{t_m}^t \|\psi_t(\tau)\|_{L^2}^2 d\tau\right)^{\frac{1}{2}}\left(\int_{t_m}^t \|\psi_x(\tau)\|_{H^1}^2 d\tau\right)^{\frac{1}{2}}\right\} \tag{5.44}$$

for $t \ge t_m$. It is directly seen from the second equation of (5.2) and inequalities (5.14), (5.30) and (5.42) that ψ_t satisfies

$$\int_0^\infty \|\psi_t(\tau)\|_{L^2}^2 d\tau \le CM(\infty)^2 \le C\|(\phi_0,\psi_0,\zeta_0)\|_{H^1}^2 < \infty.$$

Therefore, for any $\eta > 0$, there exists $m_1 \in \mathbb{N}$ such that if $t \ge t_{m_1}$ holds, then (5.44) leads to $\|\psi_x(t)\|_{L^2}^2 < \eta$, which yields $\|\psi_x(t)\|_{L^2} \to 0$ as $t\to\infty$. To show $\|\phi_x(t)\|_{L^2} \to 0$ as $t\to\infty$, we make use the fact that ϕ_x is the weak solution of the problem (5.24) for an arbitrary number $T > 0$. Applying Lemma 2.5 with $k=0$, $\tilde{\psi}=\psi$, $\phi=\phi_x$, $t_1=t_m$, $t_2=t$ and $f=\tilde{f}_1$ we obtain

$$\|\phi_x(t)\|_{L^2}^2 \le \|\phi_x(t_m)\|_{L^2}^2 + \int_{t_m}^t (\|v_x(\tau)\|_{L^\infty}\|\phi_x(\tau)\|_{L^2} + 2\|\tilde{f}_1(\tau)\|_{L^2})\|\phi_x(\tau)\|_{L^2}d\tau \tag{5.45}$$

for $t \ge t_m$. It follows from (3.22), (5.14), (5.28), (5.41) and (5.42) that the integrand in the right-hand side of (5.45) satisfies

$$\int_0^\infty (\|v_x(\tau)\|_{L^\infty}\|\phi_x(\tau)\|_{L^2} + 2\|g_1(\tau)\|_{L^2})\|\phi_x(\tau)\|_{L^2}d\tau \le CM(\infty)^2 \le C\|(\phi_0,\psi_0,\zeta_0)\|_{H^1}^2 < \infty.$$

Therefore, for any $\eta > 0$, there exists $m_2 \in \mathbb{N}$ such that if $t \geq t_{m_2}$ holds, then (5.45) gives $\|\phi_x(t)\|_{L^2}^2 < \eta$, which leads to $\|\phi_x(t)\|_{L^2} \to 0$ as $t \to \infty$. Consequently we have $\|(\phi, \psi)(t)\|_{L^\infty} \to 0$ as $t \to \infty$. The limit $\|(F - \tilde{F})(t)\|_{L^\infty} \to 0$ as $t \to \infty$ and property $F - \tilde{F} \in C([0, \infty); H^1(\mathbb{R}_+))$ directly follow from (5.35) and (5.36), and hence we arrive at the desired results $(\rho - \tilde{\rho}, v - \tilde{v}, F - \tilde{F}) \in C([0, \infty); H^1(\mathbb{R}_+))$ and (4.2). This completes the proof of Theorem 4.1. □

Author contributions

Yusuke Ishigaki and Yoshihiro Ueda: Conceptualization, Formal analysis, Investigation, Methodology, Validation, Writing-original draft, Writing-review & editing. Both authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

The authors declare no conflict of interests.

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A. Proof of Proposition 5.1

In this appendix, we show Proposition 5.1 by using standard iteration method.

Without loss of geneality, we only consider the case $\tau = 0$. Throughout this appendix, we set $u(t,x) := (\phi, \psi, \zeta)(t,x)$ and $u_0(x) := (\phi_0, \psi_0, \zeta_0)(x)$.

In order to perform iteration method, we rewrite (5.2)–(5.5) in the following form

$$\phi_t + a(\psi)\phi_x = g_1(\phi, \psi, \psi_x), \quad b(\phi)\psi_t + B\psi = g_2(u, u_x), \quad \zeta_t + a(\psi)\zeta_x = 0, \tag{A.1}$$

$$u|_{t=0} = u_0, \quad u|_{x=\infty} = 0, \quad \psi|_{x=0} = 0, \tag{A.2}$$

$$\inf_{x\in\mathbb{R}_+} (\tilde{\rho}(x) + \phi_0(x)) > 0, \tag{A.3}$$

where the functions a, b and the operator $B\psi$ are introduced in Section 2, and g_1 and g_2 are defined by

$$g_1(\phi, \psi, \psi_x) := -(\phi + \tilde{\rho})\psi_x - \phi\tilde{v}_x - \psi\tilde{\rho}_x,$$

$$\begin{aligned} g_2(u, u_x) := &-(\phi + \tilde{\rho})(\psi + \tilde{v})\psi_x - (\phi\psi + \tilde{\rho}\psi + \tilde{v}\phi)\,\tilde{v}_x \\ &- (P_\beta(\phi + \tilde{\rho}) - P_\beta(\tilde{\rho}))_x + \beta^2(Q(\phi + \tilde{\rho}, \zeta + \tilde{J}) - Q(\tilde{\rho}, \tilde{J}))_x. \end{aligned}$$

Our purpose of this appendix is to show the local-in-time solvability of (A.1)–(A.3). We note that the first and third equations of (A.1) are transport equations of ϕ and ζ, and the second equation of (A.1) is a parabolic equation of ψ . In view of this consideration, we construct the sequence of functions $\{u^{(n)}\}_{n=0}^{\infty} \subset Z^1(T)$ with some positive T to approximate u by the iteration argument.

We first define $u^{(0)} = (\phi^{(0)}, \psi^{(0)}, \zeta^{(0)})$ by the following two steps. The first step is to take $\psi^{(0)}$ by solving the problem

$$b(\phi_0)\psi_t^{(0)} + B\psi^{(0)} = g_2(u_0, u_{0,x}), \quad \psi^{(0)}|_{t=0} = \psi_0, \quad \psi^{(0)}|_{x=\infty} = \psi^{(0)}|_{x=0} = 0. \tag{A.4}$$

The second step is to determine $\phi^{(0)}$ and $\zeta^{(0)}$ by solving the problem

$$\begin{gathered} \phi_t^{(0)} + a(\psi^{(0)})\phi_x^{(0)} = g_1(\phi_0, \psi^{(0)}, \psi_x^{(0)}), \quad \zeta_t^{(0)} + a(\psi^{(0)})\zeta_x^{(0)} = 0, \\ (\phi^{(0)}, \zeta^{(0)})|_{t=0} = (\phi_0, \zeta_0), \ (\phi^{(0)}, \zeta^{(0)})|_{x=\infty} = (0, 0) \end{gathered} \tag{A.5}$$

by using u_0 and $\psi^{(0)}$. We next set $u^{(n)} = (\phi^{(n)}, \psi^{(n)}, \zeta^{(n)})$, for $n \in \mathbb{N}$, inductively by the following process. Assuming that $u^{(n-1)}$ is obtained, we define $u^{(n)}$ as the solution of the problem

$$\begin{aligned} &\phi_t + a(\psi^{(n-1)})\phi_x = g_1(\phi^{(n-1)}, \psi^{(n-1)}, \psi_x^{(n-1)}), \\ &b(\phi^{(n-1)})\psi_t + B\psi = g_2(u^{(n-1)}, u_x^{(n-1)}), \\ &\zeta_t + a(\psi^{(n-1)})\zeta_x = 0 \end{aligned} \tag{A.6}$$

with (A.2) and (A.3). If the obtained sequence $\{u^{(n)}\}_{n=0}^{\infty}$ has a limit $u = (\phi, \psi, \zeta)$ in some function space, then it will be a solution to (A.1)–(A.3). To derive this fact, we prepare the following lemma.

Lemma A.1. *Let $u_0 = (\phi_0, \psi_0, \zeta_0)$ satisfy the assumptions as in Proposition 5.1 with $\tau = 0$ except the smallness condition $M_0 \leq \varepsilon$. Then the following assertions hold true.*

(i) *The pair of three functions $u^{(0)} = (\phi^{(0)}, \psi^{(0)}, \zeta^{(0)})$ solving* (A.4) *and* (A.5) *exists uniquely in $Z^1(T_0)$ with some $T_0 = T_0(\delta, M_0)$. Furthermore, $u^{(0)}$ satisfies $u^{(0)} \in Z^1_{C_0M_0}(T_0) \cap X^1_{3M_0}(T_0)^3$, $(\phi^{(0)}, \zeta^{(0)}) \in C^1([0, T_0]; L^2(\mathbb{R}_+))$, and*

$$\phi^{(0)}(t, x) \geq -\frac{3}{4} \inf_{x \in \mathbb{R}_+} \tilde{\rho}(x).$$

Here, $C_0 = C_0(\delta, M_0)$ is a certain positive constant increasing in $\delta, M_0 > 0$.

(ii) *Let $n \in \{0\} \cup \mathbb{N}$. Then, there exists $T_1 = T_1(\delta, M_0) \in (0, T_0]$ independent of n such that* (A.6) *has a unique solution $u^{(n)} = (\phi^{(n)}, \psi^{(n)}, \zeta^{(n)})$ in $Z^1(T_1)$ satisfying $u^{(n)} \in Z^1_{C_0M_0}(T_1) \cap X^1_{3M_0}(T_1)^3$, $(\phi^{(n)}, \zeta^{(n)}) \in C^1([0, T_1]; L^2(\mathbb{R}_+))$,*

$$\inf_{(t,x) \in [0,T_1] \times \mathbb{R}_+} \phi^{(n)}(t, x) \geq -\frac{3}{4} \inf_{x \in \mathbb{R}_+} \tilde{\rho}(x).$$

Here T_0 and C_0 are same constants as in (i).

(iii) *Let $n, m \in \mathbb{N}$. Then, there exists $T_2 = T_2(\delta, M_0) \in (0, T_1]$ independent of n and m such that the following estimates hold:*

$$\|u^{(n+1)} - u^{(n)}\|_{Z^0(T_2)} \leq C(M_0^2 + T_2)\|u^{(n)} - u^{(n-1)}\|_{Z^0(T_2)}, \tag{A.7}$$

$$\|\phi_t^{(n)} - \phi_t^{(m)}\|_{C([0,T_2];H^{-1}(\mathbb{R}_+))} \leq C(\|u^{(n)} - u^{(m)}\|_{Z^0(T_2)} + \|u^{(n-1)} - u^{(m-1)}\|_{Z^0(T_2)}). \tag{A.8}$$

Here, T_1 is the same constant as in (ii)*, and $C = C(T_2, \delta, M_0)$ is a certain positive constant increasing in T_2, δ and M_0.*

Before proving Lemma A.1, we prepare the following lemmata which will play a crutial role.

Lemma A.2. *Let T, M and m be positive constants, and let $\psi_0 \in H^1_0(\mathbb{R}_+)$. Then, the following facts are obtained.*

(i) *Let $\overline{u} = (\overline{\phi}, \overline{\psi}, \overline{\zeta})$ be a given pair of functions satisfying*

$$\begin{aligned} &\overline{u} \in X^1_M(T)^3, \quad \overline{\phi} \in C^1([0, T]; L^2(\mathbb{R}_+)), \quad \overline{\psi}(t) \in H^1_0(\mathbb{R}_+), \\ &\inf_{(t,x) \in [0,T] \times \mathbb{R}_+} \overline{\phi}(t, x) \geq (m - 1) \inf_{x \in \mathbb{R}_+} \tilde{\rho}(x) \end{aligned} \tag{A.9}$$

for $0 \leq t \leq T$. Then, there exists a unique solution $\psi \in Y^1(T)$ of the problem

$$b(\overline{\phi})\psi_t + B\psi = g_2(\overline{u}, \overline{u}_x), \quad \psi|_{t=0} = \psi_0, \quad \psi|_{x=0} = \psi|_{x=\infty} = 0. \tag{A.10}$$

Furthermore, ψ satisfies

$$\|\psi\|^2_{X^1(T)} + C_1(\delta, M, m) \int_0^t (\|\psi(\tau)\|^2_{H^2} + \|\psi_t(\tau)\|^2_{L^2}) d\tau \leq e^{C_1(\delta,M,m)T}(\|\psi_0\|^2_{H^1} + C_1(\delta, M, m)T). \tag{A.11}$$

Here, $C_1(\delta, M, m)$ is a positive constant taken from (2.7) *in Lemma 2.6, and $C_1(\delta, M, m)$ is a positive constant increasing in δ, M and decreasing in m.*

(ii) *For* $j = 1, 2$, *let* $\overline{u}^{(j)} = (\overline{\phi}^{(j)}, \overline{\psi}^{(j)}, \overline{\zeta}^{(j)})$ *be given pairs of functions satisfying* (A.9), *and let* $\psi^{(j)} \in Y^1(T)$ *be solutions of* (A.10) *with* $\overline{u} = \overline{u}^{(j)}$ *respectively. Then,* $\psi^{(1)} - \psi^{(2)}$ *satisfies*

$$\|\psi^{(1)} - \psi^{(2)}\|^2_{Y^0(T)} \leq C_2(T, \delta, M, m) \left(\|\psi^{(2)}\|^2_{Y^1(T)} + T\right) \|\overline{u}^{(1)} - \overline{u}^{(2)}\|^2_{Z^0(T)}. \tag{A.12}$$

Here, $C_2(T, \delta, M, m)$ *is a positive constant increasing in* T, δ, M *and decreasing in* m.

Lemma A.3. *Let* T, $\widetilde{M}$ *and* M *be positive constants, and let* $\phi_0, \zeta_0 \in H^1(\mathbb{R}_+)$. *Then, the following facts are obtained.*

(i) *Let* $\overline{u} = (\overline{\phi}, \overline{\psi}, \overline{\zeta})$ *be a given pair of functions satisfying* $\overline{u} \in Z^1_{\widetilde{M}}(T) \cap X^1_M(T)^3$. *Then, there exists a unique pair of solutions* $(\phi, \zeta) \in X^1(T)^2$ *of the two problems*

$$\phi_t + a(\overline{\psi})\phi_x = g_1(\overline{\phi}, \overline{\psi}, \overline{\psi}_x), \quad \phi|_{t=0} = \phi_0, \quad \phi|_{x=\infty} = 0, \tag{A.13}$$

and

$$\zeta_t + a(\overline{\psi})\zeta_x = 0, \quad \zeta|_{t=0} = \zeta_0, \quad \zeta|_{x=\infty} = 0. \tag{A.14}$$

Furthermore, ϕ *and* ζ *satisfy* $\phi, \zeta \in C^1([0, T]; L^2(\mathbb{R}_+))$ *and the following estimates*

$$\begin{aligned} &\|\phi\|^2_{X^1(T)} + \|\zeta\|^2_{X^1(T)} \\ &\leq e^{C_3(T,\delta,\widetilde{M},M)\sqrt{T}} (2\|\phi_0\|^2_{H^1} + 2\|\zeta_0\|^2_{H^1} + C_3(T, \delta, \widetilde{M}, M)\sqrt{T}), \end{aligned} \tag{A.15}$$

$$\inf_{(t,x)\in[0,T]\times\mathbb{R}_+} \phi(t, x) \geq \inf_{x\in\mathbb{R}_+} \phi_0(x) - C_3(T, \delta, \widetilde{M}, M)\sqrt{T}. \tag{A.16}$$

Here, $C_3(T, \delta, \widetilde{M}, M)$ *is a positive constant increasing in* $T, \delta, \widetilde{M}, M$.

(ii) *For* $j = 1, 2$, *let* $\overline{u}^{(j)} = (\overline{\phi}^{(j)}, \overline{\psi}^{(j)}, \overline{\zeta}^{(j)})$ *be given pairs of functions satisfying* $\overline{u}^{(j)} \in Z^1_{\widetilde{M}}(T) \cap X^1_M(T)^3$, *and let* $(\phi^{(j)}, \zeta^{(j)}) \in (X^1(T))^2$ *be pairs of solutions of* (A.13) *and* (A.14) *with* $\overline{u} = \overline{u}^{(j)}$ *respectively.*

Then, $\phi^{(1)} - \phi^{(2)}$ *and* $\zeta^{(1)} - \zeta^{(2)}$ *satisfy*

$$\|\phi^{(1)} - \phi^{(2)}\|^2_{X^0(T)} + \|\zeta^{(1)} - \zeta^{(2)}\|^2_{X^0(T)} \leq C_4(T, \delta, \widetilde{M}, M) T (\|\phi^{(2)}\|_{X^1(T)} + \|\zeta^{(2)}\|_{X^1(T)} + 1) \|\overline{u}^{(1)} - \overline{u}^{(2)}\|^2_{Z^0(T)}, \tag{A.17}$$

$$\|\phi^{(1)}_t - \phi^{(2)}_t\|^2_{C([0,T];H^{-1}(\mathbb{R}_+))} \leq C_5(\delta, M)\|\phi^{(1)} - \phi^{(2)}\|^2_{X^0(T)} + C_5(\delta, M)(\|\phi^{(2)}\|_{X^1(T)} + 1)\|\overline{u}^{(1)} - \overline{u}^{(2)}\|^2_{Z^0(T)}. \tag{A.18}$$

Here, $C_4(T, \delta, \widetilde{M}, M)$ *and* $C_5(\delta, M)$ *are positive constants increasing in* T, δ, $\widetilde{M}$, M.

Proof of Lemma A.2. (i) We first note that $g_2(\overline{u}, \overline{u}_x)$ belongs to $L^2(0, T; L^2(\mathbb{R}_+))$ by using (A.9). Then, applying Lemma 2.6 with $\tilde{\phi} = \overline{\phi}$, $g = g_2(\overline{u}, \overline{u}_x)$ and $k = 1$, the problem (A.10) has a unique solution $\psi \in Y^1(T)$ satisfying

$$\begin{aligned} &\|\psi(t)\|^2_{H^1} + C_1(\delta, M, m) \int_0^t \left(\|\psi(\tau)\|^2_{H^2} + \|\psi_t(\tau)\|^2_{L^2}\right) d\tau \\ &\leq \|\psi_0\|^2_{H^1} + C_2(\delta, M, m) \int_0^t \left(\|\psi(\tau)\|^2_{L^2} + \|g_2(\bar{u}, \bar{u}_x)(\tau)\|^2_{L^2}\right) d\tau \end{aligned}$$

for $0 \le t \le T$. Here $C_1(\delta, M, m)$ and $C_2(\delta, M, m)$ are taken in (2.7). We then use the Gronwall inequality to earn

$$\|\psi\|_{X^1(T)}^2 + C_1(\delta, M, m) \int_0^T \left(\|\psi(\tau)\|_{H^2}^2 + \|\psi_t(\tau)\|_{L^2}^2\right) d\tau \le e^{C_2(\delta,M,m)T} \left(\|\psi_0\|_{H^1}^2 + \int_0^T \|g_2(\bar{u}, \bar{u}_x)(\tau)\|_{L^2}^2 d\tau\right).$$

Therefore, since $g_2(\overline{u}, \overline{u}_x)$ satisfies

$$\int_0^T \|g_2(\overline{u}, \overline{u}_x)(\tau)\|_{L^2}^2 d\tau \le C_{1,1}(\delta, M, m)T$$

with some constant $C_{1,1}(\delta, M, m)$ increasing in δ, M and decreasing in m, we arrive at the desired inequality (A.11). This completes the proof of (i).

(ii) We first see from (A.10) that $\psi^{(1)} - \psi^{(2)}$ is a weak solution of

$$b(\overline{\phi}^{(1)})(\psi^{(1)} - \psi^{(2)})_t + B(\psi^{(1)} - \psi^{(2)}) = g_2(\overline{u}^{(1)}, \overline{u}_x^{(1)}) - g_2(\overline{u}^{(2)}, \overline{u}_x^{(2)}) - (\overline{\phi}^{(1)} - \overline{\phi}^{(2)})\psi_t^{(2)},$$

$$(\psi^{(1)} - \psi^{(2)})|_{t=0} = 0, \quad (\psi^{(1)} - \psi^{(2)})|_{x=0} = (\psi^{(1)} - \psi^{(2)})|_{x=\infty} = 0.$$

Applying (2.7) in Lemma 2.6 with $k = 0$, $\tilde{\phi} = \overline{\phi}^{(1)}$, $\psi = \psi^{(1)} - \psi^{(2)}$, $g = g_2(\overline{u}^{(1)}, \overline{u}_x^{(1)}) - g_2(\overline{u}^{(2)}, \overline{u}_x^{(2)}) - (\overline{\phi}^{(1)} - \overline{\phi}^{(2)})\psi_t^{(2)}$, we obtain

$$\begin{aligned}
&\|(\psi^{(1)} - \psi^{(2)})(t)\|_{L^2}^2 + \overline{c}(M, m) \int_0^t \|(\psi^{(1)} - \psi^{(2)})(\tau)\|_{H^1}^2 d\tau \\
&\le C_{2,1}(M, m) \int_0^t \left(\|((\overline{\phi}^{(1)} - \overline{\phi}^{(2)})\psi_t^{(2)})(\tau)\|_{H^{-1}}^2 + \|(g_2(\overline{u}^{(1)}, \overline{u}_x^{(1)}) - g_2(\overline{u}^{(2)}, \overline{u}_x^{(2)}))(\tau)\|_{H^{-1}}^2\right) d\tau \\
&\quad + C_{2,1}(\delta, M, m) \int_0^t \|(\psi^{(1)} - \psi^{(2)})(\tau)\|_{L^2}^2 d\tau,
\end{aligned}$$

for $0 \le t \le T$, where $C_{2,1}(\delta, M, m)$ is a positive constant increasing in δ, M and decreasing in m. We then use the Gronwall inequality to rewrite this inequality as

$$\begin{aligned}
&\|(\psi^{(1)} - \psi^{(2)})(t)\|_{X^0(T)}^2 + \int_0^T \|(\psi^{(1)} - \psi^{(2)})(\tau)\|_{H^1}^2 d\tau \\
&\le C_{2,1}(\delta, M, m)e^{C_{2,1}(\delta,M,m)T} \int_0^t \left(\|((\overline{\phi}^{(1)} - \overline{\phi}^{(2)})\psi_t^{(2)})(\tau)\|_{L^2}^2 + \|(g_2(\overline{u}^{(1)}, \overline{u}_x^{(1)}) - g_2(\overline{u}^{(2)}, \overline{u}_x^{(2)}))(\tau)\|_{H^{-1}}^2\right) d\tau.
\end{aligned} \tag{A.19}$$

We next estimate the right-hand side of (A.19). The function $(\bar{\phi}^{(1)} - \bar{\phi}^{(2)})\psi_t^{(2)}$ is controlled as

$$\int_0^T \|((\overline{\phi}^{(1)} - \overline{\phi}^{(2)})\psi_t^{(2)})(\tau)\|_{L^2}^2 d\tau \le \|\overline{\phi}^{(1)} - \overline{\phi}^{(2)}\|_{X^0(T)}^2 \int_0^t \|\psi_t^{(2)}(\tau)\|_{L^2}^2 d\tau \le \|\overline{u}^{(1)} - \overline{u}^{(2)}\|_{Z^0(T)}^2 \|\psi_t^{(2)}\|_{Y^1(T)}^2.$$

Using (A.9) with $\overline{u} = \overline{u}^{(j)}$ with $j = 1, 2$ and integration by parts, $g_2(\overline{u}^{(1)}, \overline{u}_x^{(1)}) - g_2(\overline{u}^{(2)}, \overline{u}_x^{(2)})$ is estimated as

$$\int_0^T \|(g_2(\overline{u}^{(1)}, \overline{u}_x^{(1)}) - g_2(\overline{u}^{(2)}, \overline{u}_x^{(2)}))(\tau)\|_{H^{-1}}^2 d\tau \le C_{2,2}(\delta, M, m)T\|\overline{u}^{(1)} - \overline{u}^{(2)}\|_{Z^0(T)}^2,$$

where $C_{2,2}(\delta, M, m)$ is a positive constant increasing in δ, M and decreasing in m. Therefore, together with (A.19) and the estimates for $(\overline{\phi}^{(1)} - \overline{\phi}^{(2)})\psi_t^{(2)}$ and $g_2(\overline{u}^{(1)}, \overline{u}_x^{(1)}) - g_2(\overline{u}^{(2)}, \overline{u}_x^{(2)})$, we earn (A.12). This completes the proof of (ii). □

Proof of Lemma A.3. (i) We first note that $g_1(\overline{\phi}, \overline{\psi}, \overline{\psi}_x)$ belongs to $L^2(0, T; H^1(\mathbb{R}_+))$ and $a(\overline{\psi})$ satisfies the same assumptions for $\tilde{\psi}$ as in Lemma 2.5 by using the assumptions of $\bar{u}$. We then apply Lemma 2.5 with $k = 1$, $\tilde{\psi} = \overline{\psi}$ and $f = g_1(\overline{\phi}, \overline{\psi}, \overline{\psi}_x)$ to show that (A.13) has a unique solution $\phi \in X^1(T)$ with $\phi \in C^1([0, T]; L^2(\mathbb{R}_+))$. As in this argument, (A.14) has a unique solution $\zeta \in X^1(T)$ with $\zeta \in C^1([0, T]; L^2(\mathbb{R}_+))$ by replacing ϕ and $g_1(\overline{\phi}, \overline{\psi}, \overline{\psi}_x)$ as ζ and 0, respectively.

We next derive (A.15). It follows from Lemma 2.1 and (2.5) in Lemma 2.5 with $k = 1$, $\tilde{\psi} = \overline{\psi}$, $f = g_1(\overline{\phi}, \overline{\psi}, \overline{\psi}_x)$, $t_1 = 0$ and $t_2 = t$ that ϕ satisfies

$$\|\phi(t)\|_{H^1}^2 \leq \|\phi_0\|_{H^1}^2 + C_{3,1}(\delta) \int_0^T \|g_1(\overline{\phi}, \overline{\psi}, \overline{\psi}_x)\|_{H^1} \|\phi(\tau)\|_{H^1} d\tau + C_{3,1}(\delta) \int_0^t (1 + \|\overline{\psi}_x(\tau)\|_{H^1}) \|\phi(\tau)\|_{H^1}^2 d\tau$$

for $0 \leq t \leq T$, where $C_{3,1}(\delta)$ is a positive constant increasing in δ. Applying the Gronwall inequality to the resultant inequality and using the fact $\int_0^t \|\bar{\psi}_x(\tau)\|_{H^1} d\tau \leq \sqrt{T}\|\overline{u}\|_{Z^1(T)} \leq \sqrt{T}\widetilde{M}$ yields

$$\|\phi(t)\|_{H^1}^2 \leq e^{C_{3,2}(T,\delta,\widetilde{M})\sqrt{T}} \left\{ \|\phi_0\|_{H^1}^2 + C_{3,1}(\delta) \int_0^T \|g_1(\overline{\phi}, \overline{\psi}, \overline{\psi}_x)\|_{H^1} \|\phi(\tau)\|_{H^1} d\tau \right\}$$

for $0 \leq t \leq T$, where $C_{3,2}(T, \delta, \widetilde{M}) := C_{3,1}(\delta)\sqrt{T} + \widetilde{M}$. Similarly, ζ is estimated as

$$\|\zeta(t)\|_{H^1}^2 \leq e^{C_{3,2}(T,\delta,\widetilde{M})\sqrt{T}} \|\zeta_0\|_{H^1}^2$$

for $0 \leq t \leq T$. We then add these inequalities to give

$$\begin{aligned} &\|\phi(t)\|_{H^1}^2 + \|\zeta(t)\|_{H^1}^2 \\ &\leq e^{C_{3,2}(T,\delta,\widetilde{M})\sqrt{T}} \left\{ \|\phi_0\|_{H^1}^2 + \|\zeta_0\|_{H^1}^2 + C_{3,1}(\delta) \int_0^T \|g_1(\overline{\phi}, \overline{\psi}, \overline{\psi}_x)\|_{H^1} \|\phi(\tau)\|_{H^1} d\tau \right\} \end{aligned}$$

for $0 \leq t \leq T$. To complete the derivation of (A.15), it remains to estimate the right-hand side of this inequality. From the assumption of $\overline{u}$, $g_1(\overline{\phi}, \overline{\psi}, \overline{\psi}_x)$ satisfies

$$\int_0^T \|g_1(\overline{\phi}, \overline{\psi}, \overline{\psi}_x)\|_{H^1} d\tau \leq \sqrt{T} \left(\int_0^T \|g_1(\overline{\phi}, \overline{\psi}, \overline{\psi}_x)\|_{H^1}^2 d\tau \right)^{\frac{1}{2}} \leq C_{3,3}(T, \delta, \widetilde{M}, M)\sqrt{T}, \tag{A.20}$$

where $C_{3,3}(T, \delta, \widetilde{M}, M)$ is a positive constant increasing in $T, \delta, \widetilde{M}, M > 0$. Hence (A.20) leads to

$$\begin{aligned} &C_{3,1}(\delta) e^{C_{3,2}(T,\delta,\widetilde{M})\sqrt{T}} \int_0^T \|g_1(\overline{\phi}, \overline{\psi}, \overline{\psi}_x)(\tau)\|_{H^1} \|\phi(\tau)\|_{H^1} d\tau \\ &\leq C_{3,1}(\delta) e^{C_{3,2}(T,\delta,\widetilde{M})\sqrt{T}} (\|\phi\|_{X^1(T)}^2 + \|\zeta\|_{X^1(T)}^2)^{\frac{1}{2}} \int_0^T \|g_1(\overline{\phi}, \overline{\psi}, \overline{\psi}_x)(\tau)\|_{H^1} d\tau \\ &\leq \frac{1}{2} (\|\phi\|_{X^1(T)}^2 + \|\zeta\|_{X^1(T)}^2) + C_{3,4}(T, \delta, \widetilde{M}, M) e^{C_{3,2}(T,\delta,\widetilde{M})\sqrt{T}} T, \end{aligned}$$

which yields (A.15). Here $C_{3,4}(T, \delta, \widetilde{M}, M)$ is a positive constant increasing in $T, \delta, \widetilde{M}, M$. We finally show (A.16) via a characteristic method. Let $\overline{y} = \overline{y}(\tau; t, x) \in \mathbb{R}_+$ be a unique solution of

$$\frac{d\overline{y}}{d\tau}(\tau; t, x) = a(\overline{\psi}(\tau, \overline{y}(\tau; t, x))), \ 0 \leq \tau \leq t \leq T, \qquad \overline{y}(t; t, x) = x.$$

Decomposing $\phi(t,x) = \phi_0(\overline{y}(0;t,x)) + (\phi(t,x) - \phi_0(\overline{y}(0;t,x)))$ and using (2.6) in Lemma 2.5 and (A.20) immediately yields

$$\begin{aligned}\inf_{(t,x)\in[0,T]\times\mathbb{R}_+} \phi(x) &\geq \inf_{x\in\mathbb{R}_+} \phi_0(x) - \|\phi(t,\cdot) - \phi_0(\overline{y}(0;t,\cdot))\|_{L^\infty} \\ &\geq \inf_{x\in\mathbb{R}_+} \phi_0(x) - C_{3,3}(T,\delta,\widetilde{M},M)\sqrt{T},\end{aligned}$$

which leads to (A.16). This completes the proof of (i).

(ii) We first see from (A.13) that $\phi^{(1)} - \phi^{(2)}$ is a weak solution of

$$\begin{gathered}(\phi^{(1)} - \phi^{(2)})_t + a(\overline{\psi}^{(1)})(\phi^{(1)} - \phi^{(2)})_x = -(\overline{\psi}^{(1)} - \overline{\psi}^{(2)})\phi_x^{(2)} + g_1(\overline{\phi}^{(1)}, \overline{\psi}^{(1)}, \overline{\psi}_x^{(1)}) - g_1(\overline{\phi}^{(2)}, \overline{\psi}^{(2)}, \overline{\psi}_x^{(2)}), \\ (\phi^{(1)} - \phi^{(2)})|_{t=0} = 0, \quad (\phi^{(1)} - \phi^{(2)})|_{x=\infty} = 0.\end{gathered}$$

Applying (2.5) in Lemma 2.5 with $k = 0$, $\tilde{\psi} = \overline{\psi}$, $\phi = \phi^{(1)} - \phi^{(2)}$, $f = -(\overline{\psi}^{(1)} - \overline{\psi}^{(2)})\phi_x^{(2)} + g_1(\overline{\phi}^{(1)}, \overline{\psi}^{(1)}, \overline{\psi}_x^{(1)}) - g_1(\overline{\phi}^{(2)}, \overline{\psi}^{(2)}, \overline{\psi}_x^{(2)})$, $t_1 = 0$ and $t_2 = T$, we obtain

$$\begin{aligned}\|(\phi^{(1)} - \phi^{(2)})(t)\|_{L^2}^2 \leq C_{4,1}(\delta)\Bigg\{&\int_0^T \|((\overline{\psi}^{(1)} - \overline{\psi}^{(2)})\phi_x^{(2)})(\tau)\|_{L^2}\|(\phi^{(1)} - \phi^{(2)})(\tau)\|_{L^2}d\tau \\ &+ \int_0^T \|(g_1(\overline{\phi}^{(1)}, \overline{\psi}^{(1)}, \overline{\psi}_x^{(1)}) - g_1(\overline{\phi}^{(2)}, \overline{\psi}^{(2)}, \overline{\psi}_x^{(2)}))(\tau)\|_{L^2}\|(\phi^{(1)} - \phi^{(2)})(\tau)\|_{L^2}d\tau \\ &+ \int_0^t (1 + \|\overline{\psi}_x^{(1)}(\tau)\|_{H^1})\|(\phi^{(1)} - \phi^{(2)})(\tau)\|_{L^2}^2 d\tau\Bigg\}\end{aligned}$$

for $0 \leq t \leq T$, where $C_{4,1}(\delta)$ is a positive constant depending only on δ. We then use the Gronwall inequality to read the resultant inequality as

$$\begin{aligned}\|(\phi^{(1)} - \phi^{(2)})(t)\|_{L^2}^2 \leq C_{4,1}(\delta)e^{C_{4,2}(T,\delta,\widetilde{M})\sqrt{T}}\Bigg\{&\int_0^T \|((\overline{\psi}^{(1)} - \overline{\psi}^{(2)})\phi_x^{(2)})(\tau)\|_{L^2}\|(\phi^{(1)} - \phi^{(2)})(\tau)\|_{L^2}d\tau \\ &+ \int_0^T \|(g_1(\overline{\phi}^{(1)}, \overline{\psi}^{(1)}, \overline{\psi}_x^{(1)}) - g_1(\overline{\phi}^{(2)}, \overline{\psi}^{(2)}, \overline{\psi}_x^{(2)}))(\tau)\|_{L^2}\|(\phi^{(1)} - \phi^{(2)})(\tau)\|_{L^2}d\tau\Bigg\}\end{aligned} \tag{A.21}$$

for $0 \leq t \leq T$. Here $C_{4,2}(T,\delta,\widetilde{M})$ is a positive constant increasing in $T, \delta, \widetilde{M}$. We next focus on the right-hand side of (A.21). We directly compute the term involving $(\overline{\psi}^{(1)} - \overline{\psi}^{(2)})\phi_x^{(2)}$ as

$$\begin{aligned}&C_{4,1}(\delta)e^{C_{4,2}(T,\widetilde{M})\sqrt{T}} \int_0^T \|((\overline{\psi}^{(1)} - \overline{\psi}^{(2)})\phi_x^{(2)})(\tau)\|_{L^2}\|(\phi^{(1)} - \phi^{(2)})(\tau)\|_{L^2}d\tau \\ &\leq C_{4,1}(\delta)e^{C_{4,2}(T,\widetilde{M})\sqrt{T}} \int_0^T \|(\overline{\psi}^{(1)} - \overline{\psi}^{(2)})(\tau)\|_{L^\infty}d\tau\|\phi_x^{(2)}\|_{X^1(T)}\|\phi^{(1)} - \phi^{(2)}\|_{X^0(T)} \\ &\leq C_{4,1}(\delta)e^{C_{4,2}(T,\widetilde{M})\sqrt{T}}\sqrt{T}\|\overline{\psi}^{(1)} - \overline{\psi}^{(2)}\|_{Y^0(T)}\|\phi_x^{(2)}\|_{X^1(T)}\|\phi^{(1)} - \phi^{(2)}\|_{X^0(T)} \\ &\leq \frac{1}{4}\|\phi^{(1)} - \phi^{(2)}\|_{X^0(T)}^2 + C_{4,3}(T,\widetilde{M},M)T\|\phi_x^{(2)}\|_{X^1(T)}\|\overline{u}^{(1)} - \overline{u}^{(2)}\|_{Z^0(T)},\end{aligned}$$

where $C_{4,3}(T,\delta,\widetilde{M},M)$ denotes $C_{4,3}(T,\delta,\widetilde{M},M) := 2C_{4,1}(\delta)e^{2C_{4,2}(T,\delta,\widetilde{M})\sqrt{T}}$. For the term containing $g_1(\overline{\phi}^{(1)}, \overline{\psi}^{(1)}, \overline{\psi}_x^{(1)}) - g_1(\overline{\phi}^{(2)}, \overline{\psi}^{(2)}, \overline{\psi}_x^{(2)})$, we use the assumptions of $\bar{u}^{(j)}$ $(j = 1, 2)$ to deduce

$$\int_0^T \|(g_1(\overline{\phi}^{(1)}, \overline{\psi}^{(1)}, \overline{\psi}_x^{(1)}) - g_1(\overline{\phi}^{(2)}, \overline{\psi}^{(2)}, \overline{\psi}_x^{(2)}))(\tau)\|_{L^2}d\tau \leq C_{4,4}(T,\delta,\widetilde{M},M)\sqrt{T}\|\overline{u}^{(1)} - \overline{u}^{(2)}\|_{Z^0(T)},$$

where $C_{4,4}(T,\delta,\widetilde{M},M)$ is a positive constant increasing in $T,\delta,\widetilde{M},M$. This estimate leads to

$$\begin{aligned}
&C_{4,1}(\delta)e^{C_{4,2}(T,\delta,\widetilde{M})\sqrt{T}}\int_0^T \|(g_1(\overline{\phi}^{(1)},\overline{\psi}^{(1)},\overline{\psi}_x^{(1)})-g_1(\overline{\phi}^{(2)},\overline{\psi}^{(2)},\overline{\psi}_x^{(2)}))(\tau)\|_{L^2}\|(\phi^{(1)}-\phi^{(2)})(\tau)\|_{L^2}d\tau\\
&\le C_{4,5}(T,\delta,\widetilde{M},M)\int_0^T \|(g_1(\overline{\phi}^{(1)},\overline{\psi}^{(1)},\overline{\psi}_x^{(1)})-g_1(\overline{\phi}^{(2)},\overline{\psi}^{(2)},\overline{\psi}_x^{(2)}))(\tau)\|_{L^2}d\tau\|\phi^{(1)}-\phi^{(2)}\|_{X^0(T)}\\
&\le C_{4,5}(T,\delta,\widetilde{M},M)\sqrt{T}\|\overline{u}^{(1)}-\overline{u}^{(2)}\|_{Z^0(T)}\|\phi^{(1)}-\phi^{(2)}\|_{X^0(T)}\\
&\le \frac{1}{4}\|\phi^{(1)}-\phi^{(2)}\|_{X^0(T)}^2+C_{4,5}(T,\delta,\widetilde{M},M)T\|\overline{u}^{(1)}-\overline{u}^{(2)}\|_{Z^0(T)},
\end{aligned}$$

where $C_{4,5}(T,\delta,\widetilde{M},M)$ is a positive constant increasing in $T,\delta,\widetilde{M},M$. Therefore, combining (A.21) and these estimates, we arrive at

$$\|\phi^{(1)}-\phi^{(2)}\|_{X^0(T)}^2\le C_{4,6}(T,\delta,\widetilde{M},M)T(\|\phi_x^{(2)}\|_{X^1(T)}+1)\|\overline{u}^{(1)}-\overline{u}^{(2)}\|_{Z^0(T)}^2.$$

Here, $C_{4,6}(T,\delta,\widetilde{M},M)$ is a positive constant increasing in $T,\delta,\widetilde{M},M$. Similarly, the following estimate for $\zeta^{(1)}-\zeta^{(2)}$ is obtained by replacing $\phi^{(j)}$ and $g_1(\overline{\phi}^{(j)},\overline{\psi}^{(j)},\overline{\psi}_x^{(j)})$ as $\zeta^{(j)}$ and 0, respectively for $j=1,2$ in the above argument:

$$\|\zeta^{(1)}-\zeta^{(2)}\|_{X^0(T)}^2\le C_{4,6}(T,\delta,\widetilde{M},M)T\|\zeta_x^{(2)}\|_{X^1(T)}\|\overline{u}^{(1)}-\overline{u}^{(2)}\|_{Z^0(T)}^2.$$

Hence, combining the above two estimates leads to (A.17). We finally prove (A.18) by using the equality

$$\begin{aligned}
\left\langle \phi_t^{(1)}-\phi_t^{(2)},\varphi\right\rangle &= (\phi^{(1)}-\phi^{(2)},(a(\overline{\psi}^{(1)})\varphi)_x)_{L^2}-((\overline{\psi}^{(1)}-\overline{\psi}^{(2)})\phi_x^{(2)},\varphi)_{L^2}\\
&\quad+(g_1(\overline{\phi}^{(1)},\overline{\psi}^{(1)},\overline{\psi}_x^{(1)})-g_1(\overline{\phi}^{(2)},\overline{\psi}^{(2)},\overline{\psi}_x^{(2)}),\varphi)_{L^2}
\end{aligned}$$

for any $\varphi\in H_0^1(\mathbb{R}_+)$ with $\|\varphi\|_{H^1}\le 1$. By virtue of $\overline{u}^{(j)}\in X_M^1(T)^3,\ j=1,2$ and integration by parts, each terms in the right-hand side of this equality are controlled as

$$\begin{aligned}
|(g_1(\overline{\phi}^{(1)},\overline{\psi}^{(1)},\overline{\psi}_x^{(1)})-g_1(\overline{\phi}^{(2)},\overline{\psi}^{(2)},\overline{\psi}_x^{(2)}),\varphi)_{L^2}|&\le C_5(\delta,M)\|\overline{u}^{(1)}-\overline{u}^{(2)}\|_{Z^0(T)},\\
|(\phi^{(1)}-\phi^{(2)},(a(\overline{\psi}^{(1)})\varphi)_x)_{L^2}|&\le C_5(\delta,M)\|\phi^{(1)}-\phi^{(2)}\|_{X^0(T)},\\
|((\overline{\psi}^{(1)}-\overline{\psi}^{(2)})\phi_x^{(2)},\varphi)_{L^2}|&\le C_5(\delta,M)\|\phi_x^{(2)}\|_{X^1(T_2)}\|\overline{u}^{(1)}-\overline{u}^{(2)}\|_{Z^0(T)}
\end{aligned}$$

for $0\le t\le T$, where $C_5(\delta,M)$ is a positive constant increasing in δ,M. Therefore, using these estimates, we have (A.18) This completes the proof of (ii). □

Proof of Lemma A.1. (i) We first focus on the unique existence and properties of $\psi^{(0)}$. Let $T_{0,1}>0$ be a constant taken suitably small later. Since u_0 satisfies (A.9) in Lemma A.2 with $\bar{u}=u_0$, $T=T_{0,1}$, $M=M_0$ and $m=1/2$, (A.4) has a unique solution $\psi^{(0)}\in Y^1(T_{0,1})$. Furthermore, using (A.11) with $T=T_{0,1}$, the following estimate for $\psi=\psi^{(0)}$ is obtained:

$$\begin{aligned}
&\|\psi^{(0)}\|_{X^1(T_{0,1})}^2+C_1\left(\delta,M_0,\frac{1}{2}\right)\int_0^{T_{0,1}}(\|\psi^{(0)}(\tau)\|_{H^2}^2+\|\psi_t^{(0)}(\tau)\|_{L^2}^2)d\tau\\
&\le e^{C_1\left(\delta,M_0,\frac{1}{2}\right)T_{0,1}}\left(M_0^2+C_1\left(\delta,M_0,\frac{1}{2}\right)T_{0,1}\right).
\end{aligned}$$

Therefore using the properties $C_1(\delta, M_0, 1/2) \geq C_1(\delta, 3M_0, 1/4)$ and $C_1(\delta, M_0, 1/2) \leq C_1(\delta, 3M_0, 1/4)$, and then letting $T_{0,1} = T_{0,1}(\delta, M_0)$ small such that $e^{C_1(\delta,3M_0,1/4)T_{0,1}}(M_0^2 + C_1(\delta, 3M_0, 1/4)T_{0,1}) \leq 9M_0^2$, we have

$$\|\psi^{(0)}\|^2_{X^1(T_{0,1})} + C_1\left(\delta, 3M_0, \frac{1}{4}\right)\int_0^{T_{0,1}} (\|\psi^{(0)}(\tau)\|^2_{H^2} + \|\psi^{(0)}_t(\tau)\|^2_{L^2})d\tau \leq 9M_0^2, \tag{A.22}$$

which yields $\psi^{(0)} \in Y^1_{\widetilde{M}_0}(T_{0,1}) \cap X^1_{3M_0}(T_{0,1})$. Here $\widetilde{M}_0$ denotes $\widetilde{M}_0 := 3\{1 + C_1(\delta, 3M_0, 1/4)^{-1}\}^{1/2}M_0$. We next investigate the existence and properties of $(\phi^{(0)}, \zeta^{(0)})$. Let $\widetilde{u}^{(0)} = \widetilde{u}^{(0)}(t,x)$ be $\widetilde{u}^{(0)} := (\phi_0, \psi^{(0)}, \zeta_0)$ and let T_0 be a constant $T_0 \in (0, T_{0,1}]$ determined later. Then in view of (A.22), $\widetilde{u}^{(0)}$ satisfies $\widetilde{u}^{(0)} \in Z^1_{\widetilde{M}_0}(T_0) \cap X^1_{3M_0}(T_0)^3$. We then apply Lemma A.3 (i) with $\overline{u} = \widetilde{u}^{(0)}$, $T = T_0$, $\widetilde{M} = \widetilde{M}_0$ and $M = 3M_0$ to show that (A.5) has a unique solution $(\phi^{(0)}, \zeta^{(0)}) \in X^1(T_0)^2$ satisfying $\phi^{(0)}, \zeta^{(0)} \in C^1([0, T_0]; L^2(\mathbb{R}_+))$ and

$$\|\phi^{(0)}\|^2_{X^1(T_0)} + \|\zeta^{(0)}\|^2_{X^1(T_0)} \leq e^{\widetilde{C}_{3,0}(T_0,\delta,M_0)\sqrt{T_0}}(4M_0^2 + \widetilde{C}_{3,0}(T_0, \delta, M_0)\sqrt{T_0}),$$
$$\inf_{(t,x)\in[0,T_0]\times\mathbb{R}_+} \phi^{(0)}(t,x) \geq -\frac{1}{2}\inf_{x\in\mathbb{R}_+}\tilde{\rho}(x) - \widetilde{C}_{3,0}(T_0, \delta, M_0)\sqrt{T_0}.$$

Here, $\widetilde{C}_{3,0}(T_0, \delta, M_0)$ is defined as $\widetilde{C}_{3,0}(T_0, \delta, M_0) := C_3(T_0, \delta, \widetilde{M}_0, 3M_0)$ by using $C_3(T, \delta, \widetilde{M}, M)$ defined in Lemma A.3 (ii). We also note that $\widetilde{C}_{3,0}(T_0, \delta, M_0)$ increases inT_0, δ, M_0. Therefore taking $T_0 = T_0(\delta, M_0)$ small such that $e^{\widetilde{C}_{3,0}(T_0,\delta,M_0)\sqrt{T_0}}(4M_0^2 + \widetilde{C}_{3,0}(T_0, \delta, M_0)\sqrt{T_0}) \leq 9M_0^2$ and $\widetilde{C}_{3,0}(T_0, \delta, M_0)\sqrt{T_0} \leq (1/4)\inf_{x\in\mathbb{R}_+}\tilde{\rho}(x)$ in the above inequalities, we have

$$\begin{aligned} &\|\phi^{(0)}\|^2_{X^1(T_0)} + \|\zeta^{(0)}\|^2_{X^1(T_0)} \leq 9M_0^2, \\ &\inf_{(t,x)\in[0,T_0]\times\mathbb{R}_+} \phi^{(0)}(t,x) \geq -\frac{3}{4}\inf_{x\in\mathbb{R}_+}\tilde{\rho}(x). \end{aligned} \tag{A.23}$$

As a result, adding (A.22) to (A.23) imply $u^{(0)} \in Z^1_{C_0M_0}(T_0) \cap X^1_{3M_0}(T_0)^3$. Here $C_0 = C_0(\delta, M_0)$ is given by $C_0 := 3\{3 + C_1(\delta, 3M_0, 1/4)^{-1}\}^{1/2}$ increasing in δ, M_0. This completes the proof of Lemma A.1 (i).

(ii) We employ the induction argument with respect to n. The case $n = 0$ is already true. We assume that the properties $u^{(n-1)} \in Z^1_{C_0M_0}(T_1) \cap X^1_{3M_0}(T_1)^3$, $\phi^{(n-1)}, \zeta^{(n-1)} \in C^1([0, T_1]; L^2(\mathbb{R}_+))$ and

$$\inf_{(t,x)\in[0,T_1]\times\mathbb{R}_+} \phi^{(n-1)}(t,x) \geq -\frac{3}{4}\inf_{x\in\mathbb{R}_+}\tilde{\rho}(x)$$

hold with some $n \in \mathbb{N}$ and $T_1 = T_1(\delta, M_0) \in (0, T_0]$ independent of n. Then using Lemma A.2 (i) and Lemma A.3 (i), $u^{(n)} = (\phi^{(n)}, \psi^{(n)}, \zeta^{(n)})$ is uniquely determined by solving (A.6) and satisfies $u^{(n)} \in Z^1(T_1)$ and $\phi^{(n)}, \zeta^{(n)} \in C^1([0, T_1]; L^2(\mathbb{R}_+))$. In addition, it follows (A.11) and (A.15) with $u = u^{(n)}$, $\overline{u} = u^{(n-1)}$, $T = T_1$, $\widetilde{M} = C_0M_0$, $M = 3M_0$ and $m = 1/4$ that

$$\|\psi^{(n)}\|^2_{X^1(T_1)} + C_1\left(\delta, 3M_0, \frac{1}{4}\right)\int_0^{T_1} (\|\psi^{(n)}(\tau)\|^2_{H^2} + \|\psi^{(n)}_t(\tau)\|^2_{L^2})d\tau$$
$$\leq e^{C_1\left(\delta,3M_0,\frac{1}{4}\right)T_1}\left(M_0^2 + C_1\left(\delta, 3M_0, \frac{1}{4}\right)T_1\right),$$
$$\|\phi^{(n)}\|^2_{X^1(T_1)} + \|\zeta^{(n)}\|^2_{X^1(T_1)} \leq e^{\widetilde{C}_3(T_1,\delta,M_0)\sqrt{T_1}}(4M_0^2 + \widetilde{C}_3(\delta, T_1, M_0)\sqrt{T_1}),$$

$$\inf_{(t,x)\in[0,T_1]\times\mathbb{R}_+}\phi^{(n)}(t,x)\geq-\frac{1}{2}\inf_{x\in\mathbb{R}_+}\tilde{\rho}(x)-\widetilde{C}_3(T_1,\delta,M_0)\sqrt{T_1}.$$

Here, $\widetilde{C}_3(T_1,\delta,M_0)$ is defined as $\widetilde{C}_3(T_1,\delta,M_0):=C_3(T_1,\delta,C_0M_0,3M_0)$ by using $C_3(T,\delta,\widetilde{M},M)$ defined in Lemma A.3 (i). Therefore, taking $T_1=T_1(\delta,M_0)$ suitably small without the dependence of n, $u^{(n)}$ also belongs to $Z^1_{C_0M_0}(T_1)\cap X^1_{3M_0}(T_1)^3$, and satisfies

$$\inf_{(t,x)\in[0,T_1]\times\mathbb{R}_+}\phi^{(n)}(t,x)\geq-\frac{3}{4}\inf_{x\in\mathbb{R}_+}\tilde{\rho}(x).$$

As a result, the statement for n becomes true. This completes the proof of Lemma A.1 (ii).

(iii) We first estimate $\psi^{(n+1)}-\psi^{(n)}$. Replacing $(\psi^{(1)},\psi^{(2)})$ and $(\overline{u}^{(1)},\overline{u}^{(2)})$ as $(\psi^{(n+1)},\psi^{(n)})$ and $(u^{(n)},u^{(n-1)})$ in Lemma A.2 (ii) respectively, (A.12) is read as

$$\|\psi^{(n+1)}-\psi^{(n)}\|_{Y^0(T_2)}\leq\widetilde{C}_2(T_2,\delta,M_0)(\|\psi^{(n)}\|^2_{Y^1(T)}+T_2)\|u^{(n)}-u^{(n-1)}\|_{Z^0(T_2)}.$$

Here, we set $\widetilde{C}_2(T_2,\delta,M_0):=C_2(T_2,\delta,3M_0,1/4)$. Then, using $u^{(n)}\in Z^1_{C_0M_0}(T_2)$ for all $n\in\mathbb{N}\cup\{0\}$ to earn $\|\psi^{(n)}\|_{Y^1(T)}\leq C_0M_0$, we have

$$\|\psi^{(n+1)}-\psi^{(n)}\|_{Y^0(T_2)}\leq C_6(T_2,\delta,M_0)(M_0^2+T_2)\|u^{(n)}-u^{(n-1)}\|_{Z^0(T_2)},$$

where $C_6(T_2,\delta,M_0)$ is a positive constant increasing in T_2,δ,M_0. We next focus on $\phi^{(n+1)}-\phi^{(n)}$ and $\zeta^{(n+1)}-\zeta^{(n)}$. Changing $(\phi^{(1)},\phi^{(2)})$, $(\zeta^{(1)},\zeta^{(2)})$ and $(\bar{u}^{(1)},\bar{u}^{(2)})$ as $(\phi^{(n+1)},\phi^{(n)})$, $(\zeta^{(n+1)},\zeta^{(n)})$ and $(u^{(n)},u^{(n-1)})$ in (A.17) in Lemma A.3 (ii), we have

$$\begin{aligned}&\|\phi^{(n+1)}-\phi^{(n)}\|^2_{X^0(T_2)}+\|\zeta^{(n+1)}-\zeta^{(n)}\|^2_{X^0(T_2)}\\&\leq\widetilde{C}_4(T_2,\delta,M_0)T_2(\|\phi^{(n)}\|_{X^1(T)}+\|\zeta^{(n)}\|_{X^1(T)}+1)\|u^{(n)}-u^{(n-1)}\|^2_{Z^0(T_2)}.\end{aligned}$$

Here, we put $\widetilde{C}_4(T_2,\delta,M_0):=C_4(T_2,\delta,C_0M_0,3M_0)$. Then noticing from (ii) of this lemma that $u^{(n)}\in X^1_{3M_0}(T_2)^3$ holds for all $n\in\mathbb{N}$, this inequality is rewritten as

$$\|\phi^{(n+1)}-\phi^{(n)}\|^2_{X^0(T_2)}+\|\zeta^{(n+1)}-\zeta^{(n)}\|^2_{X^0(T_2)}\leq C_7(T_2,\delta,M_0)T_2\|u^{(n)}-u^{(n-1)}\|_{Z^0(T_2)},$$

where $C_7(T_2,\delta,M_0)$ is a positive constant increasing in T_2,δ,M_0. Consequently, combining the above estimates we arrive at (A.7). Similarly (A.8) is obtained by replacing $(\phi^{(1)},\phi^{(2)})$, $(\bar{u}^{(1)},\bar{u}^{(2)})$ and M as $(\phi^{(n)},\phi^{(m)})$, $(u^{(n-1)},u^{(m-1)})$ and $3M_0$, respectively in (A.17) in Lemma A.3 (ii) to derive

$$\|\phi_t^{(n)}-\phi_t^{(m)}\|^2_{C([0,T_2];H^{-1}(\mathbb{R}_+))}\leq C_5(\delta,3M_0)\|\phi^{(n)}-\phi^{(m)}\|^2_{X^0(T_2)}+C_5(\delta,3M_0)(\|\phi^{(m)}\|_{X^1(T)}+1)\|u^{(n)}-u^{(m)}\|^2_{Z^0(T_2)}$$

for any $n,m\in\mathbb{N}$, and then applying the fact $u^{(m)}\in X^1_{3M_0}(T_2)^3$ for any $m\in\mathbb{N}$. This completes the proof of Lemma A.1 (iii). □

Proof of Proposition 5.1. Let ε and T_2 be the constants stated in Proposition 5.1 and Lemma A.1 (iii). For simplicity, we rewrite T_2 as T, and let ε and T be $\varepsilon,T\in(0,1)$.

We first see from (A.7) in Lemma A.1 (iii) that if u_0 satisfies $\|u_0\|_{H^1}\leq M_0\leq\varepsilon$, then $\{u^{(n)}\}_{n=0}^{\infty}$ is a Cauchy sequence in $Z^0(T)$, provided that ε and $T=T(\delta,M_0)$ are chosen as $C(1,\delta,1)(\varepsilon^2+T)<$

1, respectively. Furthermore, owing to this fact and (A.8), $\{\phi_t^{(n)}\}_{n=0}^{\infty}$ becomes a Cauchy sequence in $C([0,T];H^{-1}(\mathbb{R}_+))$. Therefore $\{u^{(n)}\}_{n=0}^{\infty}$ has a limit $u=(\phi,\psi,\zeta)$ such that

$$u^{(n)} \to u \text{ in } Z^0(T), \quad \phi_t^{(n)} \to \phi_t \text{ in } C([0,T];H^{-1}(\mathbb{R}_+)). \tag{A.24}$$

Moreover, Lemma A.1 (ii) and (A.24) lead to

$$g_1(\phi^{(n-1)},\psi^{(n-1)},\psi_x^{(n-1)}) \to g_1(\phi,\psi,\psi_x)\ n\to\infty \text{ in } L^2(0,T;L^2(\mathbb{R}_+)),$$
$$g_2(u^{(n-1)},u_x^{(n-1)}) \to g_2(u,u_x) \text{ in } L^2(0,T;H^{-1}(\mathbb{R}_+)),$$

and it follows from Lemma A.1 (ii) that there exists a subsequence $\{u^{(n_k)}\}_{k=0}^{\infty}\subset\{u^{(n)}\}_{n=0}^{\infty}$ satisfying

$$u^{(n_k)} \overset{*}{\rightharpoonup} u \text{ weakly-}* \text{ in } L^\infty(0,T;H^1(\mathbb{R}_+)), \tag{A.25}$$
$$\psi^{(n_k)} \rightharpoonup \psi \text{ weakly in } L^2(0,T;H^2(\mathbb{R}_+))\cap H^1(0,T;L^2(\mathbb{R}_+)). \tag{A.26}$$

We are now going to claim that u is a solution of (A.1) in $(0,T)\times\mathbb{R}_+$ with $u\in Z^1(T)$. We first investigate ϕ and ζ. Since the first equation of (A.6) is satisfied in $C[0,T];L^2(\mathbb{R}_+))$ for $n\in\mathbb{N}$, the following weak form holds for any $\varphi\in C_0^1([0,T)\times\mathbb{R}_+)$:

$$-\int_0^T(\phi^{(n)},\varphi_t+(a(\psi^{(n-1)})\varphi)_x)_{L^2}dt=(\phi_0,\varphi(0))_{L^2}+\int_0^T(g_1(\phi^{(n-1)},\psi^{(n-1)},\psi_x^{(n-1)}),\varphi)_{L^2}dt.$$

Therefore, it is not difficult to see from taking the limit $n\to\infty$ in this equality and using Lemma A.1 (ii) and (A.24) that ϕ satisfies

$$-\int_0^T(\phi,\varphi_t+(a(\psi)\varphi)_x)_{L^2}dt=(\phi_0,\varphi(0))_{L^2}+\int_0^T(g_1(\phi,\psi,\psi_x),\varphi)_{L^2}dt. \tag{A.27}$$

This means that ϕ is a weak solution of (2.1) with $\tilde{\psi}=\psi$ and $f=g_1(\phi,\psi,\psi_x)$ in Definition 2.3. Similarly, since $\zeta^{(n)}$ solves the third equation of (A.6) for $n\in\mathbb{N}$ in $C[0,T];L^2(\mathbb{R}_+))$, ζ satisfies

$$-\int_0^T(\zeta,\varphi_t+(a(\psi)\varphi)_x)_{L^2}dt=(\zeta_0,\varphi(0))_{L^2}. \tag{A.28}$$

This implies that ζ is a weak solution of (2.1) with $\tilde{\psi}=\psi$ and $f=0$ in Definition 2.3.

We next check $g_1(\phi,\psi,\psi_x)\in L^2(0,T;H^1(\mathbb{R}_+))$ and $\psi\in Y^1(T)$, which imply $\phi,\zeta\in X^1(T)$ and $\phi,\zeta\in C^1([0,T];L^2(\mathbb{R}_+))$ by applying the regularity properties of weak solutions in Lemma 2.5. From (A.24)–(A.26), $g_1(\phi,\psi,\psi_x)\in L^2(0,T;H^1(\mathbb{R}_+))$ holds true. Moreover $\psi\in Y^1(T)$ is also satisfied by (A.24), (A.26) and $C([0,T];L^2(\mathbb{R}_+))\cap L^2(0,T;\widetilde{H}^2(\mathbb{R}_+))\subset C([0,T];H_0^1(\mathbb{R}_+))$. Therefore $\phi,\zeta\in X^1(T)$ and $\phi,\zeta\in C^1([0,T];L^2(\mathbb{R}_+))$ hold ture, and the properties $\phi,\zeta\in X^1_{3M_0}(T)$ and $\inf_{(t,x)\in[0,T]\times\mathbb{R}_+}\phi(t,x)\geq-(3/4)\inf_{x\in\mathbb{R}_+}\tilde{\rho}(x)$ follow from Lemma A.1 (ii), (A.24) and (A.25). As a result, restricting $\varphi\in C_0^1((0,T)\times\mathbb{R}_+)$ in (A.27) and (A.28), and then applying integration by parts, we see that $u=(\phi,\psi,\zeta)$ satisfies the first and third equations of (A.1) in $C([0,T];L^2(\mathbb{R}_+))$.

We next study ψ. Since $\psi^{(n)}$ is the solution of the second equation of (A.6) for $n\in\mathbb{N}$, the following weak form holds for all $\varphi\in H_0^1(\mathbb{R}_+)$ and $h\in C_0^1([0,T))$:

$$-\int_0^T(b(\phi^{(n-1)})\psi^{(n)},\varphi)_{L^2}h'dt-\int_0^T\left\langle\phi_t^{(n-1)}\psi^{(n)},\varphi\right\rangle hdt+\int_0^T\left\langle B\psi^{(n)},\varphi\right\rangle hdt$$
$$=(b(\phi(0,\cdot))\psi_0,\varphi)_{L^2}h(0)+\int_0^T\left\langle g_2(u^{(n-1)},u_x^{(n-1)}),\varphi\right\rangle hdt.$$

Therefore taking the limit $n \to \infty$ in this equality and using Lemma A.1 (ii) and (A.24), we obtain

$$\begin{aligned}&-\int_0^T (b(\phi)\psi, \varphi)_{L^2} h' dt - \int_0^T \langle \phi_t \psi, \varphi \rangle\, hdt + \int_0^T \langle B\psi, \varphi\rangle\, hdt \\ &= (b(\phi(0,\cdot))\psi_0, \varphi)_{L^2} h(0) + \int_0^T \langle g_2(u, u_x), \varphi \rangle\, hdt.\end{aligned} \tag{A.29}$$

This shows that ψ is a weak solution of (2.3) with $\tilde{\phi} = \phi$ and $g = g_2(u, u_x)$ in Definition 2.4. The properties $\psi \in Y^1_{C_0 M_0}(T) \cap X^1_{3M_0}(T)$ and $g_2(u, u_x) \in L^2(0, T; L^2(\mathbb{R}_+))$ are confirmed by Lemma A.1 (ii),(A.25) and (A.26). From the above discussion, the assumptions for $\tilde{\phi} = \phi$ in Lemma 2.6 with $M = 3M_0$ and $m = 1/4$ are also justified. Therefore, we gurantee these properties and then obtain the second equation of (A.1) in $L^2(0, T; L^2(\mathbb{R}_+))$ by restricting $h \in C^1_0(0, T)$ in (A.29) and applying integration by parts.

Consequently, we prove the local-in-time existence of the solution to (5.2) with (5.4)–(5.6) satisfying $u \in X^1_{3M_0}(T)$, (5.8) and (5.9) with $\tau = 0$. The uniqueness of the solution is confirmed in a similar manner to the proof of Lemma A.1 (iii). This completes the proof of Proposition 5.1. □