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Research article

Exploring the flexibility of m-point quaternary approximating subdivision schemes with free parameter

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Abstract: In this study, we proposed a family of m-point quaternary approximating subdivision schemes, characterized by an explicit formula involving three parameters. One of these parameters served as a shape control parameter, allowing for flexible curve design, while the other two parameters identify different members of the family and determined the smoothness of the resulting limit curves. We conducted a thorough analysis of the proposed schemes, covering their smoothness properties, polynomial generation, and reproduction capabilities. Additionally, we examined the behavior of the Gibbs phenomenon within the family both theoretically and graphically, highlighting the advantages of the proposed schemes in eliminating undesirable oscillations. A comparative study with existing subdivision schemes demonstrated the effectiveness and versatility of our approach. The results indicated that the proposed family offered enhanced smoothness and control, making it suitable for a wide range of applications in computer graphics and geometric modeling.

Keywords: quaternary; subdivision scheme; shape control parameter; Laurent polynomial; Gibbs phenomenon
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1. Introduction

Subdivision schemes (SS) can precisely characterize smooth curves and surfaces from the given set of control points through iterative refinement. The most significant, influential, and extensively used technique of Computer Aided Geometric Design is the SS. The popularity of SSs is due to their effectiveness and simplicity. They play an important role in computer graphics due to their wide range of applications in several fields, including computer animation and the design of curves or surfaces. They are also essential for preserving the shape of data, geometric objects, and images in image processing. Recently, Liu et al. [1] reviewed the theory and applications of refinement schemes, which shows that refinement curves and surfaces are widely used in geometric modeling.

SSs can be broadly classified into two primary categories: approximating and interpolating. Approximating schemes generate new control points during the refinement process, collectively shaping the limit curve without necessarily passing through the original control points. This approach often yields smoother curves with higher continuity orders. Interpolating schemes, on the other hand, ensure that the limit curve passes through the original control points. This provides more precise shape control and is commonly employed in engineering applications. One can achieve higher smoothness by using approximating schemes with smaller support, but interpolating schemes do not meet the requirement for smoothness [2]. The arity of an SS is the number of points inserted at a refinement level, say $k + 1$, between two consecutive points from level k. The arity of a scheme directly affects the smoothness of the limit curves or surfaces. Higher arity schemes generally offer higher smoothness compared to lower arity schemes, making them essential for applications requiring fine geometric detailing. In [3], it has been proved that the large support and higher arity schemes may outperform the small support and lower arity schemes. Consequent to this, the research communities are interested in introducing higher arity schemes (i.e., quaternary) which give better results and less computational cost.

Initial work on the subdivision was started by De Rham [4] when he presented a corner-cutting algorithm for curve modeling. Chaikin [5] was the second one who presented another corner cutting SS. They [6] defined a symmetric iterative interpolation process. Dyn [7] provides a foundational perspective on how Laurent polynomials can be applied to analyze subdivision schemes. The work by Dyn and Levin [8] on subdivision schemes has been highly influential in both theoretical research and practical applications in computer graphics and geometric design. The interpolating 4-point C^2 ternary stationary subdivision scheme developed by Hassan et al. [9] and represents a significant advancement in subdivision methods, particularly for applications that demand high smoothness and accuracy in curve interpolation.

The paper by Hongchan et al. [10] contributes a powerful tool to the field of geometric modeling, particularly in Computer Aided Design/Computer Aided Manufacturing applications. By combining the flexibility of ternary subdivision with adjustable control parameters, the authors provide a subdivision scheme that balances high smoothness with user-defined shape control. Mustafa et al. [11] presented a significant contribution to subdivision scheme theory by introducing an n-ary interpolating subdivision scheme with odd-point masks.

Ghaffar et al. [12] makes a valuable contribution to subdivision scheme theory by presenting a flexible 4-point a-ary approximating scheme. This generalization allows designers to adjust the level of smoothness and computational demand, making the scheme adaptable to various applications in graphics and geometric modeling. Ashraf et al. [13] presented a significant advancement in subdivision schemes by introducing a nonstationary four-point ternary interpolating scheme that emphasizes shape preservation. The shape-preserving variant of the Lane-Riesenfeld algorithm introduced by Ashraf et al. [14] enhances the traditional algorithm by ensuring that specific geometric features are maintained throughout the refinement process. Finally, Zouaoui et al. [15] contributed to the field of subdivision schemes by introducing novel n-point ternary schemes that mitigate the Gibbs phenomenon, leading to smoother and more accurate curve approximations near sharp features.

Mustafa and Khan [16] pioneered the exploration of a 4-point quaternary SS characterized by a single shape parameter, resulting in C^3 limit curves. Ko [17] examined the convergence and regularity properties of a quaternary approximating SS by using the Laurent polynomials technique. An algorithm to introduce $3n$-point quaternary approximation SS was developed by Bari et al. [18]. Using shape parameters, Pervez [19] proposed a 3-point approximation approach that shows continuity from C^0 to C^3. A generalized formula for 5-point approximating SS of any arity was created by Hussain et al. [20]. An algorithm for developing a novel 7-point quaternary approximation SS using shape parameters was developed by Nawaz et al. [21]. Yao et al. [22] showed interest in fractal and convexity analysis of a 4-point quaternary SS.

1.1. Role and motivation

Based on the literature review, we observe that a vast body of research has been dedicated to binary and ternary SSs. However, quaternary SSs have remained less explored. There is a room to define a family of quaternary approximating SSs showing diverse characteristics to meet the different requirements of end users. The introduction of a new family of quaternary SSs is fundamental because it provides an opportunity to improve the smoothness of the resulting curves without significantly increasing the computational complexity. High-arity schemes, such as quaternary schemes, strike a balance between smoothness and computational efficiency, making them particularly advantageous in applications where both are critical. The motivation behind introducing a new family of quaternary SSs stems from the need for increased smoothness and flexibility in designing curves. The development of such schemes with free shape parameters allows for adjustable smoothness and enhanced control over the generated geometry. Additionally, these schemes are crucial for eliminating undesirable features such as the Gibbs phenomenon, which causes oscillations near discontinuities in the limit function. Addressing this phenomenon is essential for improving the visual quality and accuracy of the generated curves in practical applications.

In this study, we develop an explicit formula for building a family of quaternary approximating SSs. The formulae involve three parameters, in which one parameter plays the role of the shape control parameter and the rest of the two parameters identifies the different members of the proposed family and smoothness level of limit curves. The remainder of the paper is organized as follows: Section 2 provided basic concepts and preliminaries. In Section 3, we present the explicit formulae to create a family of m-point quaternary approximating SSs. The smoothness analysis of the proposed family of schemes is given in Section 4. In Section 5, polynomial generation and reproduction property of the proposed schemes is discussed. Section 6 presents theoretical and graphical analysis of the Gibbs phenomenon of the proposed family. A comparative analysis of the proposed schemes is presented in Section 7, followed by the conclusion in Section 8.

2. Preliminaries

A quaternary subdivision scheme δ can be defined in the term of mask consisting of a finite set of nonzero coefficients $c = \{c_i : i \in \mathbb{Z}\}$, as follows

$$q_i^{k+1} = \sum_{j\in\mathbb{Z}} c_{i-4j} q_j^k, \quad i \in \mathbb{Z}, \ j \in \mathbb{Z}. \tag{2.1}$$

The number of points added between two successive control points from level k to $k + 1$ indicates the arity of the SS. When 4 points are put in this instance, it is referred to as quaternary. An SS is said to be uniformly convergent, if for any initial data $q^0 = \{q_i^0 : i \in \mathbb{Z}\}$, there exists a continuous function q, such that, for any closed interval $\mathbb{I} \subset \mathbb{R}$, it satisfies convergent.

$$\lim_{k\to\infty} \sup_{i\in 4^k I} |q_i^k - q(4^{-k}i)| = 0.$$

The limit function q is denoted by $q = \delta^\infty q^0$. A symbol also called the Laurent polynomial of the mask $c = \{c_i : i \in \mathbb{Z}\}$, of the scheme (2.1) is defined as $C(z) = \sum_{i\in\mathbb{Z}} c_i z^i$. The Laurent polynomial of the convergent quaternary subdivision scheme satisfies the following conditions

$$C(1) = 4, \qquad C(e^{\frac{2ip\pi}{4}}) = 0, \qquad P = 1, 2, 3.$$

Definition 2.1. [23] *Let g be a punctually discontinuous function and its sampling g_h be defined by $g_{ih} = g(ih)$. The Gibbs phenomenon in the refinement scheme deals with the convergence of $(\delta^\infty g_h)$ toward g when h goes to* 0*. It can be delimited by two properties:*
P1. *Away from the discontinuity, the convergence is rather slow, and for any point x,*

$$|g(x) - (\delta^\infty g_h)(x)| = O(h).$$

P2. *There is an overstepped, close to discontinuity, that does not reduce with the reduction of h. Thus,* $\max_{x\in\mathbb{R}} |g(x) - (\delta^\infty g_h)(x)|$ *does not tend to zero with h.*

Theorem 2.1. [24] Let g be any function defined by

$$\begin{aligned} g(x) &= g_-(x), g_- \in C^n(]-\infty, \varphi]), \ \ \forall x \le \varphi, \\ g(x) &= g_+(x), g_+ \in C^n(]-\infty, \varphi]), \ \ \forall x \ge \varphi, \end{aligned}$$

with $n \ge 2$, $0 \le \varphi \le h$, and $g_-(\varphi) > g_+(\varphi)$. Let δ_c be a univariate stationary refinement scheme with;

$$\xi_t^{[k]}(i) = \begin{cases} \sum_{\tau\le i} c_{4\tau+t}{}^{[k]}, & \text{if } i < 0, \\ 0, & \text{if } i = 0, \\ \sum_{\tau\ge i} c_{4\tau+t}{}^{[k]}, & \text{if } i > 0, \end{cases}$$

where $c^{[k]}$ defined as $c(z)^{[k]} = \sum_{j\in\mathbb{Z}} c_{j-4i}$ and $0 \le t < 4^k$. Then, if $\xi_t^{[k]}(i) \ge 0 \ \forall i, k$, and if h is sufficiently small, we have the following two conditions:
C1. If $|x| \ge \max\left\{\left|\frac{M-1}{2}\right|, \left|\frac{M+N}{2} + 1\right|\right\} h$, then

$$|g(x) - (\delta_c^\infty g_h)| = O(h^n),$$

$n \ge 2$.
C2. If $|x| \ge \max\left\{\left|\frac{M-1}{2}\right|, \left|\frac{M+N}{2} + 1\right|\right\} h$, there exists $\beta_h = O(h)$ such that:

$$g_{1,h} - \beta_h \le g_+(h) - \beta_h \le (H_c^\infty)(x) \le g_-(0) + \beta_h = g_{0,h} + \beta_h.$$

3. A new family of m-point *QRS*

In this section, we present a new family of m-point *QRS*, known as ($\delta_{\alpha,\theta,m}$) with a shape control parameter α. The family of the Laurent polynomials of the scheme $\delta_{\alpha,\theta,m}$ is defined as

$$C_{\alpha,\theta,m}(z) = \frac{1}{z^m}\left(\frac{1+z+z^2+z^3}{4}\right)^{\theta+1}(4\alpha(1+z^4)+(z+z^3)+2(1-4\alpha)z^2), \tag{3.1}$$

where $\theta = 0, 1, 2, \ldots$, and $m = 2, 3, 4, \ldots$, identify the different subfamilies of $\delta_{\alpha,\theta,m}$. Specifically, the parameter θ governs both the smoothness of the scheme and the number of points m in the subdivision rules. The value of m depends upon θ by the relation

$$m = \left\lceil \theta + \frac{3}{2} \right\rceil. \tag{3.2}$$

The relation (3.2) suggests that as θ increases, the smoothness of the scheme improves and m grows stepwise. The ceiling function $\lceil a \rceil$ returns the largest integer greater than or equal to a. In this case, it ensures that m remains an integer, as the number of points in a refinement scheme cannot be fractional. By varying values of θ in (3.2), we get corresponding values of m to construct the subfamilies of $\delta_{\alpha,\theta,m}$. Table 1 shows the set of values for m depending on θ.

Table 1. Set of values for m depending on θ.

θ	0	1	2	3	4	…
m	2	3	4	5	6	…

3.1. Subfamily $\delta_{\alpha,0,2}$

A subfamily of 2-point schemes $\delta_{\alpha,0,2}$ can be derived by substituting $\theta = 0$ and $m = 2$ in (3.1), which results in the following Laurent polynomial:

$$\begin{aligned} C_{\alpha,0,2}(z) &= \alpha z^5 + \left(\frac{1}{4}+\alpha\right)z^4 + \left(\frac{3}{4}-\alpha\right)z^3 + (1-\alpha)\,z^2 + (1-\alpha)\,z + \left(\frac{3}{4}-\alpha\right) \\ &\quad + \left(\frac{1}{4}+\alpha\right)z^{-1} + \alpha z^{-2}. \end{aligned} \tag{3.3}$$

The Laurent polynomial (3.3) defines the refinement scheme $\delta_{\alpha,0,2}$, which is given by

$$\begin{cases} q_{4i}^{k+1} = (1-\alpha)\,q_i^k + \alpha q_{i+1}^k, \\ q_{4i+1}^{k+1} = \left(\frac{3}{4}-\alpha\right)q_i^k + \left(\frac{1}{4}+\alpha\right)q_{i+1}^k, \\ q_{4i+2}^{k+1} = \left(\frac{1}{4}+\alpha\right)q_i^k + \left(\frac{3}{4}-\alpha\right)q_{i+1}^k, \\ q_{4i+3}^{k+1} = \alpha q_i^k + (1-\alpha)q_{i+1}^k. \end{cases} \tag{3.4}$$

Remark 3.1. *$\delta_{\frac{7}{8},0,2}$ coincides with the scheme proposed in [25].*

3.2. *Subfamily* $\delta_{\alpha,1,3}$

By setting $\theta = 1$ and $m = 3$ in (3.1), we generate a sub-family of 3-point schemes $\delta_{\alpha,1,3}$. The Laurent polynomial of $\delta_{\alpha,1,3}$ is as follows:

$$\begin{aligned} C_{\alpha,1,3}(z) &= \frac{1}{4}\alpha z^7 + \left(\frac{1}{16} + \frac{1}{2}\alpha\right) z^6 + \left(\frac{1}{4} + \frac{1}{4}\alpha\right) z^5 + \frac{1}{2}z^4 + \left(\frac{3}{4} - \frac{1}{2}\alpha\right) z^3 + \left(\frac{7}{8} - \alpha\right) z^2 + \left(\frac{3}{4} - \frac{1}{2}\alpha\right) z \\ &\quad + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\alpha\right) z^{-1} + \left(\frac{1}{16} + \frac{1}{2}\alpha\right) z^{-2} + \frac{1}{4}\alpha z^{-3}. \end{aligned} \tag{3.5}$$

The refinement scheme $\delta_{\alpha,1,3}$ corresponding to the Laurent polynomial (3.5) is given by

$$\begin{cases} q_{4i}^{k+1} = \frac{1}{2}q_i^k + \frac{1}{2}q_{i+1}^k, \\ q_{4i+1}^{k+1} = \left(\frac{1}{4} + \frac{1}{4}\alpha\right) q_i^k + \left(\frac{3}{4} - \frac{1}{2}\alpha\right) q_{i+1}^k + \frac{1}{4}q_{i+2}^k, \\ q_{4i+2}^{k+1} = \left(\frac{1}{16} + \frac{1}{2}\alpha\right) q_i^k + \left(\frac{7}{8} - \alpha\right) q_{i+1}^k + \left(\frac{1}{16} + \frac{1}{2}\alpha\right) q_{i+2}^k, \\ q_{4i+3}^{k+1} = \frac{1}{4}q_i^k + \left(\frac{3}{4} - \frac{1}{2}\alpha\right) q_{i+1}^k + \left(\frac{1}{4} + \frac{1}{4}\alpha\right) q_{i+2}^k. \end{cases} \tag{3.6}$$

3.3. *Subfamily* $\delta_{\alpha,2,4}$

By substituting $\theta = 2$ and $m = 4$ in (3.1), we establish a subfamily of 4-point schemes $\delta_{\alpha,2,4}$, which leads to the Laurent polynomial

$$\begin{aligned} C_{\alpha,2,4}(z) &= \frac{1}{16}\alpha z^9 + \left(\frac{1}{64} + \frac{3}{16}\alpha\right) z^8 + \left(\frac{5}{64} + \frac{1}{4}\alpha\right) z^7 + \left(\frac{13}{64} + \frac{1}{4}\alpha\right) z^6 + \left(\frac{25}{64} + \frac{1}{16}\alpha\right) z^5 \\ &\quad + \left(\frac{19}{32} - \frac{5}{16}\alpha\right) z^4 + \left(\frac{23}{32} - \frac{1}{2}\alpha\right) z^3 + \left(\frac{23}{32} - \frac{1}{2}\alpha\right) z^2 + \left(\frac{19}{32} - \frac{5}{16}\alpha\right) z + \left(\frac{25}{64} + \frac{1}{16}\alpha\right) \\ &\quad + \left(\frac{13}{64} + \frac{1}{4}\alpha\right) z^{-1} + \left(\frac{5}{64} + \frac{1}{4}\alpha\right) z^{-2} + \left(\frac{1}{64} + \frac{3}{16}\alpha\right) z^{-3} + \frac{1}{16}\alpha z^{-4}. \end{aligned} \tag{3.7}$$

The refinement scheme $\delta_{\alpha,2,4}$ linked with the Laurent polynomial (3.7) is described below

$$\begin{cases} q_{4i}^{k+1} = \left(\frac{13}{64} + \frac{1}{4}\alpha\right) q_i^k + \left(\frac{23}{32} - \frac{1}{2}\alpha\right) q_{i+1}^k + \left(\frac{5}{64} + \frac{1}{4}\alpha\right) q_{i+2}^k, \\ q_{4i+1}^{k+1} = \left(\frac{5}{64} + \frac{1}{4}\alpha\right) q_i^k + \left(\frac{23}{32} - \frac{1}{2}\alpha\right) q_{i+1}^k + \left(\frac{13}{64} + \frac{1}{4}\alpha\right) q_{i+2}^k, \\ q_{4i+2}^{k+1} = \left(\frac{1}{64} + \frac{3}{16}\alpha\right) q_i^k + \left(\frac{19}{32} - \frac{5}{16}\alpha\right) q_{i+1}^k + \left(\frac{25}{64} + \frac{1}{16}\alpha\right) q_{i+2}^k + \frac{1}{16}\alpha q_{i+3}^k, \\ q_{4i+3}^{k+1} = \frac{1}{16}\alpha q_i^k + \left(\frac{25}{64} + \frac{1}{16}\alpha\right) q_{i+1}^k + \left(\frac{19}{32} - \frac{5}{16}\alpha\right) q_{i+2}^k + \left(\frac{1}{64} + \frac{3}{16}\alpha\right) q_{i+3}^k. \end{cases} \tag{3.8}$$

3.4. *Subfamily* $\delta_{\alpha,3,5}$

By setting $\theta = 3$ and $m = 5$ in (3.1), we drive a subfamily of 5-point schemes $\delta_{\alpha,3,5}$, which results in the following Laurent polynomial:

$$C_{\alpha,3,5}(z) = \frac{1}{64}\alpha z^{11} + \left(\frac{1}{256} + \frac{1}{16}\alpha\right) z^{10} + \left(\frac{3}{128} + \frac{1}{8}\alpha\right) z^9 + \left(\frac{19}{256} + \frac{3}{16}\alpha\right) z^8 + \left(\frac{11}{64} + \frac{3}{16}\alpha\right) z^7$$

$$+\left(\frac{81}{256}+\frac{1}{16}\alpha\right)z^{6}+\left(\frac{61}{128}-\frac{1}{8}\alpha\right)z^{5}+\left(\frac{155}{256}-\frac{5}{16}\alpha\right)z^{4}+\left(\frac{21}{32}-\frac{13}{16}\alpha\right)z^{3}$$
$$+\left(\frac{155}{256}-\frac{5}{16}\alpha\right)z^{2}+\left(\frac{61}{128}-\frac{1}{8}\alpha\right)z+\left(\frac{81}{256}+\frac{1}{16}\alpha\right)+\left(\frac{11}{64}+\frac{3}{16}\alpha\right)z^{-1}$$
$$+\left(\frac{19}{256}+\frac{3}{16}\alpha\right)z^{-2}+\left(\frac{3}{128}+\frac{1}{8}\alpha\right)z^{-3}+\left(\frac{1}{256}+\frac{1}{16}\alpha\right)z^{-4}+\frac{1}{64}\alpha z^{-5}. \tag{3.9}$$

The refinement scheme $\delta_{\alpha,3,5}$ corresponding to the Laurent polynomial (3.9) is given by

$$\begin{cases} q^{k+1}_{4i} = \left(\frac{19}{256}+\frac{3}{16}\alpha\right)q^k_i+\left(\frac{155}{256}-\frac{5}{16}\alpha\right)q^k_{i+1}+\left(\frac{81}{256}+\frac{1}{16}\alpha\right)q^k_{i+2}+\left(\frac{1}{256}+\frac{1}{16}\alpha\right)q^k_{i+3},\\ q^{k+1}_{4i+1} = \left(\frac{3}{128}+\frac{1}{8}\alpha\right)q^k_i+\left(\frac{61}{128}-\frac{1}{8}\alpha\right)q^k_{i+1}+\left(\frac{61}{128}-\frac{1}{8}\alpha\right)q^k_{i+2}+\left(\frac{3}{128}+\frac{1}{8}\alpha\right)q^k_{i+3},\\ q^{k+1}_{4i+2} = (\frac{1}{256}+\frac{1}{16}\alpha)q^k_i+(\frac{81}{256}+\frac{1}{16}\alpha)q^k_{i+1}+\left(\frac{155}{256}-\frac{5}{16}\alpha\right)q^k_{i+2}+\left(\frac{19}{256}+\frac{3}{16}\alpha\right)q^k_{i+3},\\ q^{k+1}_{4i+3} = \frac{1}{64}\alpha q^k_i+\left(\frac{11}{64}+\frac{3}{16}\alpha\right)q^k_{i+1}+\left(\frac{21}{32}-\frac{13}{16}\alpha\right)q^k_{i+2}+\left(\frac{11}{64}+\frac{3}{16}\alpha\right)q^k_{i+3}+\frac{1}{64}\alpha q^k_{i+4}. \end{cases} \tag{3.10}$$

Remark 3.2. *$\delta_{-\frac{11}{16},3,5}$ coincides with the scheme proposed in [26], when $\mu = \frac{11}{20}$.*

By proceeding with the same process for further values of θ and its corresponding exponent $m > 5$, we can obtain new subfamilies of $\delta_{\alpha,\theta,m}$.

4. Analysis of convergence and smoothness

The fundamental criteria for selecting an optimal refinement scheme are its convergence, smoothness, and support width of the limit function. Here, we focus on the investigation of the convergence and smoothness of $\delta_{\alpha,\theta,m}$, with a specific emphasis on its alignment with the given conditions.

For a given refinement scheme δ_c, let δ_{c_1} be the associated refinement scheme for the divided differences of the primary points, ensuring that it satisfies

$$\mathfrak{D}q^{k+1} = \delta_{c_1}\mathfrak{D}q^k,$$

where $q^k = \delta_c^k q^0$ and $(\mathfrak{D}q^k)_i = 4^k(q^k_{i+1} - q^k_i)$. The symbol associated with the difference scheme δ_{c_1} is given by

$$C_1(z) = \frac{4z^3}{1+z+z^2+z^3}C(z),$$

where $C(z)$ is the symbol associated with the scheme δ_c. Now, we recall some celebrated results for evaluating the convergence and smoothness of a refinement scheme.

Theorem 4.1. *[16] Let δ_c and δ_{c_1} be the refinement schemes with the symbols $C(z)$ and $C_1(z)$, respectively. Then, the scheme δ_c is uniformly convergent (i.e., C^0) if, and only if, there exist an integer $s > 0$ such that $\left\|\left(\frac{1}{4}\delta_{a_1}\right)^s\right\|_\infty < 1$.*

Furthermore, to evaluate smoothness of the scheme δ_c, we have the following result:

Theorem 4.2. *[16] Let δ_c and δ_{c_1} be the refinement schemes with the symbols $C(z)$ and $C_1(z)$ respectively, such that*

$$C(z) = \left(\frac{1 + z + z^2 + z^3}{4z^3}\right)^n C_1(z).$$

If the scheme δ_{c_1} is contractive, then the scheme δ_c is C^n for any initial data.

Now, we are ready to present the main results of this section for the convergence and smoothness of the subfamilies of the scheme $\delta_{\alpha,\theta,m}$.

4.1. Convergence

Theorem 4.3. *The subfamilies of $\delta_{\alpha,\theta,m}$ presented in (3.4), (3.6), (3.8), and (3.10), converge when $\alpha \in \left(-\frac{1}{4}, \frac{1}{2}\right)$, $\left(-\frac{7}{4}, \frac{9}{4}\right)$, $\left(-\frac{13}{8}, \frac{19}{8}\right)$, and $\left(-\frac{57}{16}, \frac{71}{16}\right)$, respectively.*

Proof. The family of Laurent polynomials of the scheme $\delta_{\alpha,\theta,m}$ given in (3.1) can be expressed as

$$C_{\alpha,\theta,m}(z) = \left(\frac{1 + z + z^2 + z^3}{4z^3}\right) d_{\alpha,\theta,m}(z),$$

where

$$d_{\alpha,\theta,m}(z) = z^{3-m}\left(\frac{1 + z + z^2 + z^3}{4}\right)^{\theta} (4\alpha(1 + z^4) + (z + z^3) + 2(1 - 4\alpha), \tag{4.1}$$

is the Laurent polynomial of the first-order divided difference of the scheme $\delta_{\alpha,\theta,m}$.
Case-I: For the convergence of the subfamily $\delta_{\alpha,0,2}$, we use $\theta = 0$ and $m = 2$ in (4.1) and have

$$d_{\alpha,0,2}(z) = 4\alpha z^5 + z^4 + (2 - 8\alpha)z^3 + z^2 + 4\alpha z.$$

Thus, the mask of the scheme $\delta_{d_{\alpha,0,2}}$, which is related to the Laurent polynomial $d_{\alpha,0,2}(z)$, is $\{4\alpha, 1, (2 - 8\alpha), 1, 4\alpha\}$, since it can be easily verified that for $\alpha \in \left(-\frac{1}{4}, \frac{1}{2}\right)$,

$$\left\|\frac{1}{4}\delta_{d_{\alpha,0,2}}\right\|_{\infty} = \frac{1}{4}\max\{2|4\alpha|, |1|, |2 - 8\alpha|\} < 1.$$

Thus, the subfamily $\delta_{\alpha,0,2}$ converges when $\alpha \in \left(-\frac{1}{4}, \frac{1}{2}\right)$.
Case-II: For the convergence of the subfamily $\delta_{\alpha,1,3}$, we use $\theta = 1$ and $m = 3$ in (4.1) and have

$$d_{\alpha,1,3}(z) = \alpha z^9 + \left(\frac{1}{4} + \alpha\right) z^8 + \left(\frac{3}{4} - \alpha\right) z^7 + (1 - \alpha)z^6 + (1 - \alpha)z^5 + \left(\frac{3}{4} - \alpha\right) z^4 + \left(\frac{1}{4} + \alpha\right) z^3 + \alpha z^2.$$

Thus, the mask of the scheme $\delta_{d_{\alpha,1,3}}$, which is related to the Laurent polynomial $d_{\alpha,1,3}(z)$, is

$$\left\{\alpha, \left(\frac{1}{4} + \alpha\right), \left(\frac{3}{4} - \alpha\right), (1 - \alpha), (1 - \alpha), \left(\frac{3}{4} - \alpha\right), \left(\frac{1}{4} + \alpha\right), \alpha\right\}.$$

It can be easily verified that for $\alpha \in \left(-\frac{7}{4}, \frac{9}{4}\right)$,

$$\left\|\frac{1}{4}\delta_{d_{\alpha,1,3}}\right\|_{\infty} = \frac{1}{4}\max\left\{|\alpha| + |(1 - \alpha)|, \left|\left(\frac{1}{4} + \alpha\right)\right| + \left|\left(\frac{3}{4} - \alpha\right)\right|\right\} < 1.$$

Thus, the subfamily $\delta_{\alpha,1,3}$ converges when $\alpha \in \left(-\frac{7}{4}, \frac{9}{4}\right)$.

Case-III: For the convergence of the subfamily $\delta_{\alpha,2,4}$, we use $\theta = 2$ and $m = 4$ in (4.1) and have

$$\begin{aligned} d_{\alpha,2,4}(z) &= \left\{\frac{1}{4}\alpha z^9 + \left(\frac{1}{16} + \frac{1}{2}\alpha\right) z^8 + \left(\frac{1}{4} + \frac{1}{4}\alpha\right) z^7 + \frac{1}{2} z^6 + \left(\frac{3}{4} - \frac{1}{2}\alpha\right) z^5 + \left(\frac{7}{8} - \alpha\right) z^4 \right. \\ &\quad \left. + \left(\frac{3}{4} - \frac{1}{2}\alpha\right) z^3 + \frac{1}{2} z^2 + \left(\frac{1}{4} + \frac{1}{4}\alpha\right) z + \left(\frac{1}{16} + \frac{1}{2}\alpha\right) + \frac{1}{4}\alpha z^{-1}\right\}. \end{aligned}$$

Thus, the mask of the scheme $\delta_{d_{\alpha,2,4}}$ corresponding to the Laurent polynomial $d_{\alpha,2,4}(z)$, is

$$\left\{\frac{1}{4}\alpha, \left(\frac{1}{16} + \frac{1}{2}\alpha\right), \left(\frac{1}{4} + \frac{1}{4}\alpha\right), \frac{1}{2}, \left(\frac{3}{4} - \frac{1}{2}\alpha\right), \left(\frac{7}{8} - \alpha\right), \left(\frac{3}{4} - \frac{1}{2}\alpha\right), \frac{1}{2}, \left(\frac{1}{4} + \frac{1}{4}\alpha\right), \left(\frac{1}{16} + \frac{1}{2}\alpha\right), \frac{1}{4}\alpha\right\}.$$

It is evident that for $\alpha \in \left(-\frac{7}{4}, \frac{9}{4}\right)$

$$\left\|\frac{1}{4}\delta_{d_{\alpha,2,4}}\right\|_\infty = \frac{1}{4} \max \left\{\left|\frac{1}{4}\alpha\right| + \left|\left(\frac{3}{4} - \frac{1}{2}\alpha\right)\right| + \left|\left(\frac{1}{4} + \frac{1}{4}\alpha\right)\right|, \left|\left(\frac{1}{16} + \frac{1}{2}\alpha\right)\right| + \left|\left(\frac{7}{8} - \alpha\right)\right| + \left|\left(\frac{1}{16} + \frac{1}{2}\alpha\right)\right|, 1\right\} < 1.$$

Thus, the subfamily $\delta_{\alpha,1,3}$ converges when $\alpha \in \left(-\frac{7}{4}, \frac{9}{4}\right)$.

Case-IV: For the convergence of the subfamily $\delta_{\alpha,3,5}$, we use $\theta = 3$ and $m = 5$ in (4.1) and have

$$\begin{aligned} d_{\alpha,3,5}(z) &= \frac{1}{16}\alpha z^{11} + \left(\frac{1}{64} + \frac{3}{16}\alpha\right) z^{10} + \left(\frac{5}{64} + \frac{1}{4}\alpha\right) z^9 + \left(\frac{13}{64} + \frac{1}{4}\alpha\right) z^8 + \left(\frac{25}{64} + \frac{1}{16}\alpha\right) z^7 \\ &\quad + \left(\frac{19}{32} - \frac{5}{16}\alpha\right) z^6 + \left(\frac{23}{32} - \frac{1}{2}\alpha\right) z^5 + \left(\frac{23}{32} - \frac{1}{2}\alpha\right) z^4 + \left(\frac{19}{32} - \frac{5}{16}\alpha\right) z^3 + \left(\frac{25}{64} + \frac{1}{16}\alpha\right) z^2 \\ &\quad + \left(\frac{13}{64} + \frac{1}{4}\alpha\right) z + \left(\frac{5}{64} + \frac{1}{4}\alpha\right) + \left(\frac{1}{64} + \frac{3}{16}\alpha\right) z^{-1} + \frac{1}{16}\alpha z^{-2}. \end{aligned}$$

Thus, the mask of the scheme $\delta_{\alpha,3,5}$, which is related to the Laurent polynomial $d_{\alpha,3,5}(z)$, is

$$\begin{aligned} &\left\{\frac{1}{16}\alpha, \left(\frac{1}{64} + \frac{3}{16}\alpha\right), \left(\frac{5}{64} + \frac{1}{4}\alpha\right), \left(\frac{13}{64} + \frac{1}{4}\alpha\right), \left(\frac{25}{64} + \frac{1}{16}\alpha\right), \left(\frac{19}{32} - \frac{5}{16}\alpha\right), \left(\frac{23}{32} - \frac{1}{2}\alpha\right), \right. \\ &\left. \left(\frac{23}{32} - \frac{1}{2}\alpha\right), \left(\frac{19}{32} - \frac{5}{16}\alpha\right), \left(\frac{25}{64} + \frac{1}{16}\alpha\right), \left(\frac{13}{64} + \frac{1}{4}\alpha\right), \left(\frac{5}{64} + \frac{1}{4}\alpha\right), \left(\frac{1}{64} + \frac{3}{16}\alpha\right), \frac{1}{16}\alpha\right\}. \end{aligned}$$

It is clear that for $\alpha \in \left(-\frac{57}{16}, \frac{71}{16}\right)$,

$$\begin{aligned} \left\|\frac{1}{4}\delta_{d_{\alpha,3,5}}\right\|_\infty &= \frac{1}{4} \max \left\{\left|\frac{1}{16}\alpha\right| + \left|\frac{25}{64} + \frac{1}{16}\alpha\right| + \left|\frac{19}{32} - \frac{5}{16}\alpha\right| + \left|\frac{1}{64} + \frac{3}{16}\alpha\right|, \left|\frac{13}{64} + \frac{1}{4}\alpha\right| + \right. \\ &\quad \left. \left|\frac{23}{32} - \frac{1}{2}\alpha\right| + \left|\frac{5}{64} + \frac{1}{4}\alpha\right|\right\} < 1. \end{aligned}$$

Thus, the subfamily $\delta_{\alpha,3,5}$ converges when $\alpha \in \left(-\frac{57}{16}, \frac{71}{16}\right)$. □

4.2. Smoothness

The smoothness of a refinement scheme depends upon its continuity. The following result shows that the family of schemes $\delta_{\alpha,\theta,m}$ maintains a level of C^θ continuity.

Theorem 4.4. *The family of schemes* $\delta_{\alpha,\theta,m}$ *has smoothness* $C^\theta\ \forall\ \ \alpha \in (-\frac{1}{4}, \frac{1}{2})$.

Proof. The family of the Laurent polynomials (3.1) can be expressed as

$$C_{\alpha,\theta,m}(z) = \left(\frac{1+z+z^2+z^3}{4z^3}\right)^\theta d_{\alpha,\theta,m}(z),$$

where

$$d_{\alpha,\theta,m}(z) = 4z^{3\theta+3-m}\left\{\alpha + \frac{1}{4}z + \left(\frac{1}{2} - 2\alpha\right)z^2 + \frac{1}{4}z^3 + \alpha z^4\right\}. \tag{4.2}$$

Thus, the mask of the scheme $\delta_{d_{\alpha,\theta,m}}$ corresponding to the Laurent polynomial $d_{\alpha,\theta,m}(z)$ is $\{4\alpha, 1, (2-8\alpha), 1, 4\alpha\}$. It can be easily verified that for $\alpha \in \left(-\frac{1}{4}, \frac{1}{2}\right)$,

$$\left\|\frac{1}{4}\delta_{d_{\alpha,\theta,m}}\right\|_\infty \frac{1}{4}\max\{2|4\alpha|, |1|, |2-8\alpha|\} < 1.$$

Thus, the $\delta_{\alpha,\theta,m}$ is C^θ when $\alpha \in \left(-\frac{1}{4}, \frac{1}{2}\right)$. □

4.3. Hölder regularity

Hölder regularity is an extension of the notion of continuity which gives more information for subdivision schemes like these, rather than just quoting the number of derivatives that converge. A function $\varphi : \mathbb{R} \to \mathbb{R}$ is defined to be regular of order $\tau + \alpha$ (for $\tau \in \mathbb{N}_0$ and $0 < \alpha \leq 1$) if it is τ times continuously differentiable and φ^τ is Lipschitz of order α, i.e.,

$$\left|\varphi^{(\tau)}(v+h) - \varphi^{(\tau)}(v)\right| \leq c\,|h|^\alpha\,,$$

for all v and h in $\mathbb{R}$ and some constant c.

Continuity of a subdivision curve is defined by just saying that if the nth derivative of a curve exists everywhere in an interval and it is continuous, then the curve is said to be C^n continuous in that interval. However, the Hölder regularity of a subdivision curve is a measure of how many derivatives are continuous, and of how continuous the highest continuous derivative is. Therefore, Hölder regularity is essential to determine the overall smoothness characteristics of the schemes. According to Riouls method [27], the Hölder regularity of the $\delta_{\alpha,\theta,m}$ can be computed as follows:

Theorem 4.5. *The Hölder regularity of the family of the schemes* $\delta_{\alpha,\theta,m}$ *is*

$$R_\theta = \begin{cases} \theta - \log_4\left(\frac{1}{2} - 2\alpha\right), & \text{if } -\frac{1}{4} < \alpha < \frac{1}{8}, \\ \theta - \log_4(2\alpha), & \text{if } \ \frac{1}{8} < \alpha < \frac{1}{2}, \\ \theta + 1, & \text{if } \ \alpha = \frac{1}{8}. \end{cases}$$

Proof. Hölder regularity is the extended continuity of a refinement scheme. Using the Rioul's approach [27], the $\delta_{\alpha,\theta,m}$ has Hölder regularity $R_\theta = \theta + \psi^G$, for all $G \geq 1$, where ψ^G is determined by

$$4^{-G\psi^G} = \left\| \left(\frac{1}{4} \delta_{d_{\alpha,\theta,m}} \right)^G \right\|_\infty .$$

The proof of the Theorem 4.4, indicates that

$$\left\| \frac{1}{4} \delta_{d_{\alpha,\theta,m}} \right\|_\infty = \begin{cases} \frac{1}{2} - 2\alpha, & \text{if } -\frac{1}{4} < \alpha < \frac{1}{8}, \\ 2\alpha, & \text{if } \frac{1}{8} < \alpha < \frac{1}{2}, \\ \frac{1}{4}, & \text{if } \alpha = \frac{1}{8}. \end{cases}$$

For $G = 1$, we have

$$\eta = \begin{cases} -\log_4 \left(\frac{1}{2} - 2\alpha \right), & \text{if } -\frac{1}{4} < \alpha < \frac{1}{8}, \\ -\log_4 (2\alpha), & \text{if } \frac{1}{8} < \alpha < \frac{1}{2}, \\ 1, & \text{if } \alpha = \frac{1}{8}. \end{cases}$$

Hence, the Hölder regularity of the scheme $\delta_{\alpha,\theta,m}$ is

$$R_\theta = \begin{cases} \theta - \log_4 \left(\frac{1}{2} - 2\alpha \right), & \text{if } -\frac{1}{4} < \alpha < \frac{1}{8}, \\ \theta - \log_4 (2\alpha), & \text{if } \frac{1}{8} < \alpha < \frac{1}{2}, \\ \theta + 1, & \text{if } \alpha = \frac{1}{8}. \end{cases}$$

□

4.4. Support of $\delta_{\alpha,\theta,m}$

The support of a subdivision scheme represents the area of the limit curve affected by the displacement of a single control point from its initial place. The support of a refinement scheme reflects the effect of local control on its limiting curves. For the sake of computations, we only discuss the support of the first family member $\delta_{\alpha,0,2}$ in detail. The support of the rest of the family members can be easily calculated on the same principles.

Lemma 4.1. When we apply the $\delta_{\alpha,0,2}$ scheme to the initial data,

$$q_i^0 = \begin{cases} 1, & i = 0, \\ 0, & i \neq 0. \end{cases} \tag{4.3}$$

Following the initial subdivision step, the nonzero points are $q^1_{-4}, q^1_{-3}, \dots, q^1_2, q^1_3$.

Lemma 4.2. Applying the $\delta_{\alpha,0,2}$ scheme to the initial data as in (4.3) and then following the second refinement step, the nonzero points are $q^2_{-20}, q^2_{-19}, \dots, q^2_{14}, q^2_{15}$.

Lemma 4.3. *When we apply the $\delta_{\alpha,0,2}$ scheme with the initial data as in (4.3), after the third subdivision step, the points with nonzero values are $q^3_{-84}, q^3_{-83}, \dots, q^3_{62}, q^3_{63}$.*

We choose not to include the support derivation due to its obviousness.

Theorem 4.6. *The support of $\delta_{\alpha,0,2}$ is $\frac{7}{3}$, which implies that it vanishes outside the interval $\left[-\frac{7}{6},\frac{7}{6}\right]$.*

Proof. By using the result of Lemmas 4.1, 4.2, and 4.3, we prove the above result. Consider a set

$$G_k = \left\{\frac{j}{4^k} : \forall j \in \mathbb{Z}\right\},$$

so that

$$\vartheta\left(\frac{j}{4^k}\right) = q_j^k, \forall j \in \mathbb{Z}.$$

Using Lemma 4.1, we can deduce that if we apply the $\delta_{\alpha,0,2}$ scheme on the initial data, the position of the first nonzero point on the left after the initial subdivision step is

$$q_{-4}^1 = \vartheta\left(-\frac{4}{4}\right),$$

and the last nonzero point on the right is

$$q_3^1 = \vartheta\left(\frac{3}{4}\right).$$

Based on Lemma 4.2, it follows that when we use the $\delta_{\alpha,0,2}$ scheme for the initial data, the position of the first nonzero point on the left after second subdivision step is as follows

$$q_{-20}^2 = q_{-4(1+4)}^2 = \vartheta\left(-\frac{4(1+4)}{4^2}\right),$$

and the last nonzero point on the right is

$$q_{15}^2 = q_{3(1+4)}^2 = \vartheta\left(\frac{3(1+4)}{4^2}\right).$$

Similarly, from Lemma 4.3, if we apply $\delta_{\alpha,0,2}$ scheme to the initial data, the position of the first nonzero point on the left after the third subdivision step is

$$q_{-84}^3 = q_{-4(1+4+4^2)}^3 = \vartheta\left(-\frac{4(1+4+4^2)}{4^3}\right),$$

and the last nonzero point on the right is

$$q_{63}^3 = q_{3(1+4+4^2)}^3 = \vartheta\left(\frac{3(1+4+4^2)}{4^3}\right).$$

Continuing this procedure, the position of the first nonzero point on the left after the k-th subdivision step is

$$q_{-4(1+4+\ldots+4^{k-1})}^k = \vartheta\left(-\frac{4(1+4+\ldots+4^{k-1})}{4^k}\right),$$

and the last nonzero point on the right is

$$q_{3(1+4+\ldots+4^{k-1})}^k = \vartheta\left(\frac{3(1+4+\ldots+4^{k-1})}{4^k}\right).$$

The difference between the nonzero points on the left and right after the k-th subdivision step is

$$\begin{aligned} l &= \left[\frac{3(1+4+\ldots+4^{k-1})}{4^k} - \frac{-4(1+4+\ldots+4^{k-1})}{4^k}\right] \\ &= \left[\frac{3(1+4+\ldots+4^{k-1})}{4^k} + \frac{4(1+4+\ldots+4^{k-1})}{4^k}\right] \\ &= \left[(3+4)\left(\frac{1+4+\ldots+4^{k-1}}{4^k}\right)\right] \\ &= \left[\frac{7}{4}\left(1+\frac{1}{4}+\frac{1}{4^2}+\ldots+\frac{1}{4^{k-1}}\right)\right]. \end{aligned}$$

Since $\frac{1}{4} < 1$, the sum of the geometric sequence allows us to determine the accumulated extent on each side. Consequently, we deduce that as we approach the limit, the complete impact of the initial non-zero vertex will progressively spread further along.

$$\frac{7}{4}\left(\sum_{j=0}^{k-1}\frac{1}{4^j}\right) = \frac{7}{4}\left(\frac{1}{1-\frac{1}{4}}\right) = \frac{7}{3}.$$

Hence, the support width is $\frac{7}{3}$, which implies that it vanishes outside the interval $\left[-\frac{7}{6}, \frac{7}{6}\right]$. □

Figure 1 shows basic limit curves generated by $\delta_{\frac{1}{8},\theta,m}$ for $\theta = 0,\ldots,4$ and $m = 2,\ldots,6$, respectively.

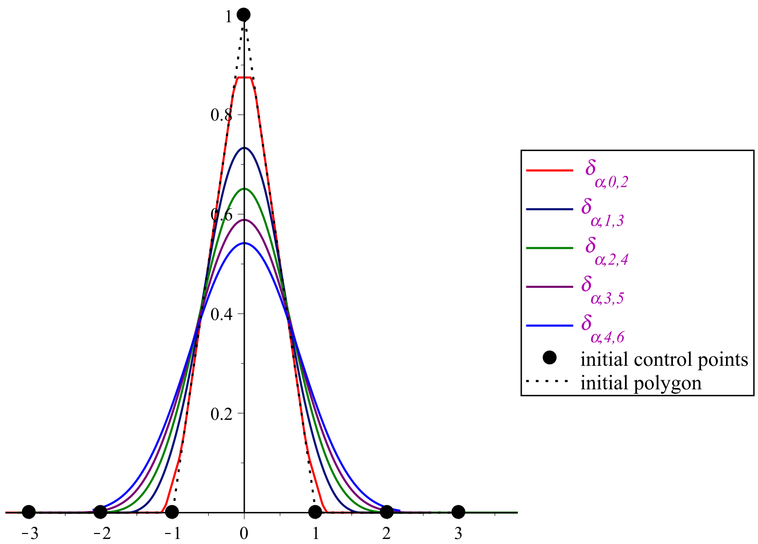


Figure 1. Basic limit curves generated by $\delta_{\alpha,\theta,m}$ at $\alpha = \frac{1}{8}$.

In Table 2, we summarize the continuity and support of the $\delta_{\alpha,\theta,m}$ schemes.

Table 2. Continuity and support of $\delta_{\alpha,\theta,m}$.

Scheme	Continuity	Support
$\delta_{\alpha,1,3}$	C^1 when $\alpha \in \left(-\frac{1}{4}, \frac{1}{2}\right)$	10/3
$\delta_{\alpha,2,4}$	C^2 when $\alpha \in \left(-\frac{1}{4}, \frac{1}{2}\right)$	13/3
$\delta_{\alpha,3,5}$	C^3 when $\alpha \in \left(-\frac{1}{4}, \frac{1}{2}\right)$	16/3
$\delta_{\alpha,4,6}$	C^4 when $\alpha \in \left(-\frac{1}{4}, \frac{1}{2}\right)$	19/3

5. Polynomial generation and reproduction

The generation and the reproduction of polynomials are two significant features of a refinement scheme. The generation degree is the highest degree of polynomials that a refinement scheme can produce during its iterative process and the reproduction degree is the highest degree of polynomials that a refinement scheme can precisely reproduce. Let us recall some basic results regarding polynomial generation and reproduction.

Definition 5.1. [28] Let $\mathbf{p}^0 = \{p(n) : n \in \mathbb{Z}\}$ where $p \in \Pi_d$ with $d \in \mathbb{N}_0$. We say that a stationary subdivision scheme δ reproduces polynomials in $\prod_d$, if δ is convergent and $p = \delta^\infty \mathbf{p}^0$. Also, the subdivision scheme δ is said to be $\prod_d$-generating if $\delta^\infty \mathbf{p}^0 \in \prod_d$.

Definition 5.2. [29] For a refinement scheme δ_c, we denote by $\tau = \frac{C'(1)}{4}$, the corresponding parametric shift and attach the data q_n^k for $n \in \mathbb{N}, k \in \mathbb{N}$ to the parameter values.

$$x_n^k = x_0^k + \frac{n}{4^k} \quad with \quad x_0^k = x_0^{k-1} - \frac{\tau}{4^k}. \tag{5.1}$$

Theorem 5.1. *[30] A convergent subdivision scheme δ with any arity $r \geq 2$ reproduces polynomials of degree $d \geq 1$, with respect to the parametrization in (5.1) if, and only if,*

$$\sum_{j \in \mathbb{Z}} j^k c_{mj+i} = \left(\frac{\tau - i}{m}\right)^k, \; i = 0, 1, \ldots, m-1, \; for\, k = 1, 2, \ldots, d, \tag{5.2}$$

where $\tau = \frac{C'(1)}{r}$.

Theorem 5.2. *[31] A convergent subdivision scheme that reproduces polynomial P_n has an approximation order of* $n + 1$.

In the case of a *QRS* (where $m = 4$), (5.2) takes the form as

$$\begin{cases} \sum_{j \in \mathbb{Z}} j^k c_{4j} = \left(\frac{\tau}{4}\right)^k, \\ \sum_{j \in \mathbb{Z}} j^k c_{4j+1} = \left(\frac{\tau-1}{4}\right)^k, \\ \sum_{j \in \mathbb{Z}} j^k c_{4j+2} = \left(\frac{\tau-2}{4}\right)^k, \\ \sum_{j \in \mathbb{Z}} j^k c_{4j+3} = \left(\frac{\tau-3}{4}\right)^k, \end{cases} \tag{5.3}$$

for $k = 1, 2, \ldots, d$, and $\tau = \frac{C'_{\alpha,\theta,m}(1)}{4}$. Now we are ready to present the main results of this section.

Theorem 5.3. *The family of $\delta_{\alpha,\theta,m}$ schemes generates polynomials up to degree θ.*

Proof. The family of Laurent polynomials (3.1), can be expressed as

$$C_{\alpha,\theta,m}(z) = (1 + z + z^2 + z^3)^{(\theta+1)} d_{\alpha,\theta,m}(z),$$

where

$$d_{\alpha,\theta,m}(z) = \frac{\alpha + \frac{1}{4}z + \left(\frac{1}{2} - 2\alpha\right) z^2 + \frac{1}{4}z^3 + \alpha z^4}{4^\theta z^m}.$$

Since $d_{\alpha,\theta,m}(1) = \frac{1}{4^\theta}$, the family of $\delta_{\alpha,\theta,m}$ schemes generates polynomials up to degree θ. □

Theorem 5.4. *The subfamily $\delta_{\alpha,0,2}$ reproduces a linear polynomial when $\alpha = \frac{1}{8}$.*

Proof. From (3.3), the Laurent polynomial of the scheme $\delta_{\alpha,0,2}$ is given by

$$\begin{aligned} C_{\alpha,0,2}(z) &= \sum_{j=-2}^{5} c_j z^j \\ &= \alpha z^5 + \left(\frac{1}{4} + \alpha\right) z^4 + \left(\frac{3}{4} - \alpha\right) z^3 + (1 - \alpha) z^2 + (1 - \alpha) z \\ &\quad + \left(\frac{3}{4} - \alpha\right) + \left(\frac{1}{4} + \alpha\right) z^{-1} + \alpha z^{-2}. \end{aligned} \tag{5.4}$$

Differentiating (5.4) and then evaluating at $z = 1$ leads us to $C'_{\alpha,0,2}(1) = 6$ and $\tau = \frac{3}{2}$. By simplifying (5.3) for $\tau = \frac{3}{2}$, we get the system as given below:

$$\begin{cases} (1)^k c_4 = \left(\frac{3}{8}\right)^k, \\ (1)^k c_5 = \left(\frac{1}{8}\right)^k, \\ (-1)^k c_{-2} = \left(-\frac{1}{8}\right)^k, \\ (-1)^k c_{-1} = \left(-\frac{3}{8}\right)^k. \end{cases} \tag{5.5}$$

Since (5.5) is verified for $k = 1$, the scheme $\delta_{\alpha,0,2}$ reproduces the linear polynomials. □

Corollary 5.1. The subfamily $\delta_{\alpha,0,2}$ has approximation order 2.

Theorem 5.5. *The subfamily $\delta_{\alpha,1,3}$ reproduces linear polynomial $\forall\ \alpha \in (-\frac{7}{4}, \frac{9}{4})$.*

Proof. From (3.3), the Laurent polynomial of the scheme $\delta_{\alpha,1,3}$ is given by

$$\begin{aligned} C_{\alpha,1,3}(z) &= \sum_{j=-3}^{7} c_j z^j \\ &= \frac{1}{4}\alpha z^7 + \left(\frac{1}{16} + \frac{1}{2}\alpha\right) z^6 + \left(\frac{1}{4} + \frac{1}{4}\alpha\right) z^5 + \frac{1}{2} z^4 + \left(\frac{3}{4} - \frac{1}{2}\alpha\right) z^3 + \left(\frac{7}{8} - \alpha\right) z^2 + \left(\frac{3}{4} - \frac{1}{2}\alpha\right) z \\ &\quad + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\alpha\right) z^{-1} + \left(\frac{1}{16} + \frac{1}{2}\alpha\right) z^{-2} + \frac{1}{4}\alpha z^{-3}. \end{aligned} \tag{5.6}$$

Differentiating (5.6) and then evaluating at $z = 1$ leads us to $c'_{\alpha,1,3}(1) = 8$ and, thus, $\tau = 4$. By simplifying (5.3) for $\tau = 4$, we get the system as given below:

$$\begin{cases} c_4 = \left(\frac{1}{2}\right)^k, \\ (-1)^k c_{-3} + c_5 = \left(\frac{1}{4}\right)^k, \\ (-1)^k c_{-2} + c_6 = 0, \\ (-1)^k c_{-1} + c_7 = \left(-\frac{1}{4}\right)^k. \end{cases} \tag{5.7}$$

Since (5.7) is verified only for $k = 1$, $\forall\ \alpha \in (-\frac{7}{4}, \frac{9}{4})$, thus the scheme $\delta_{\alpha,1,3}$ reproduces the linear polynomials. □

Corollary 5.2. The subfamily $\delta_{\alpha,1,3}$ has approximation order 2, $\forall\ \alpha \in (-\frac{7}{4}, \frac{9}{4})$.

Theorem 5.6. *The subfamily $\delta_{\alpha,2,4}$ reproduces linear polynomial* $\forall\ \alpha \in (-\frac{13}{8}, \frac{19}{8})$.

Proof. From (3.7), the Laurent polynomial of the scheme $\delta_{\alpha,2,4}$ is given by

$$\begin{aligned} C_{\alpha,2,4}(z) &= \sum_{j=-4}^{9} c_j z^j \\ &= \frac{1}{16}\alpha z^9 + \left(\frac{1}{64} + \frac{3}{16}\alpha\right) z^8 + \left(\frac{5}{64} + \frac{1}{4}\alpha\right) z^7 + \left(\frac{13}{64} + \frac{1}{4}\alpha\right) z^6 + \left(\frac{25}{64} + \frac{1}{16}\alpha\right) z^5 \\ &\quad + \left(\frac{19}{32} - \frac{5}{16}\alpha\right) z^4 + \left(\frac{23}{32} - \frac{1}{2}\alpha\right) z^3 + \left(\frac{23}{32} - \frac{1}{2}\alpha\right) z^2 + \left(\frac{19}{32} - \frac{5}{16}\alpha\right) z + \left(\frac{25}{64} + \frac{1}{16}\alpha\right) \\ &\quad + \left(\frac{13}{64} + \frac{1}{4}\alpha\right) z^{-1} + \left(\frac{5}{64} + \frac{1}{4}\alpha\right) z^{-2} + \left(\frac{1}{64} + \frac{3}{16}\alpha\right) z^{-3} + \frac{1}{16}\alpha z^{-4}. \end{aligned} \tag{5.8}$$

Differentiating (5.8) and then evaluating at $z = 1$ leads us to $C'_{\alpha,2,4}(1) = 10$ and $\tau = \frac{5}{2}$. By simplifying (5.3) for $\tau = \frac{5}{2}$, we get the system as given below

$$\begin{cases} c_4 + (2)^k c_8 = \left(\frac{5}{8}\right)^k, \\ (-1)^k c_{-3} + c_5 + (2)^k c_9 = \left(\frac{3}{8}\right)^k, \\ (-1)^k c_{-2} + c_6 + (2)^k c_{10} = \left(\frac{1}{8}\right)^k, \\ (-1)^k c_{-1} + c_7 = \left(-\frac{1}{8}\right)^k. \end{cases} \tag{5.9}$$

Since (5.9) is verified for $k = 1$, $\forall\ \alpha \in (-\frac{7}{4}, \frac{9}{4})$, thus the scheme $\delta_{\alpha,2,4}$ reproduces the linear polynomials. □

Corollary 5.3. The subfamily of the schemes $\delta_{\alpha,2,4}$ has approximation order 2, $\forall\ \alpha \in (-\frac{7}{4}, \frac{9}{4})$.

Theorem 5.7. *The subfamily of the schemes $\delta_{\alpha,3,5}$ reproduces linear polynomial $\forall\ \alpha \in (-\frac{13}{8}, \frac{19}{8})$.*

Proof. From (3.7), the Laurent polynomial of the scheme $\delta_{\alpha,3,5}$ is given by

$$\begin{aligned} C_{\alpha,3,5}(z) &= \sum_{j=-5}^{11} c_j z^j \\ &= \frac{1}{64}\alpha z^{11} + \left(\frac{1}{256} + \frac{1}{16}\alpha\right)z^{10} + \left(\frac{3}{128} + \frac{1}{8}\alpha\right)z^9 + \left(\frac{19}{256} + \frac{3}{16}\alpha\right)z^8 + \left(\frac{11}{64} + \frac{3}{16}\alpha\right)z^7 \\ &\quad + \left(\frac{81}{256} + \frac{1}{16}\alpha\right)z^6 + \left(\frac{61}{128} - \frac{1}{8}\alpha\right)z^5 + \left(\frac{155}{256} - \frac{5}{16}\alpha\right)z^4 + \left(\frac{21}{32} - \frac{13}{16}\alpha\right)z^3 \\ &\quad + \left(\frac{155}{256} - \frac{5}{16}\alpha\right)z^2 + \left(\frac{61}{128} - \frac{1}{8}\alpha\right)z + \left(\frac{81}{256} + \frac{1}{16}\alpha\right) + \left(\frac{11}{64} + \frac{3}{16}\alpha\right)z^{-1} \\ &\quad + \left(\frac{19}{256} + \frac{3}{16}\alpha\right)z^{-2} + \left(\frac{3}{128} + \frac{1}{8}\alpha\right)z^{-3} + \left(\frac{1}{256} + \frac{1}{16}\alpha\right)z^{-4} + \frac{1}{64}\alpha z^{-5}. \end{aligned} \tag{5.10}$$

Differentiating (5.10) and then evaluating at $z = 1$ leads us to $C'_{\alpha,2,4}(1) = 14$ and $\tau = \frac{7}{2}$. By simplifying (5.3) for $\tau = \frac{7}{2}$, we get the system as given below:

$$\begin{cases} (-1)^k c_{-4} + c_4 + (2)^k c_8 = \left(\frac{3}{4}\right)^k, \\ (-1)^k c_{-3} + c_5 + (2)^k c_9 = \left(\frac{1}{2}\right)^k, \\ (-1)^k c_{-2} + c_6 + (2)^k c_{10} = \left(\frac{1}{4}\right)^k, \\ (-2)^k c_{-5} + (-1)^k c_{-1} + c_7 + (2)^k c_{11} = 0. \end{cases}$$

Since (5.11) is verified for $k = 1$, $\forall\ \alpha \in (-\frac{7}{4}, \frac{9}{4})$, thus the scheme $\delta_{\alpha,3,5}$ reproduces the linear polynomials. □

Corollary 5.4. The subfamily $\delta_{\alpha,3,5}$ has approximation order 2, $\forall\ \alpha \in (-\frac{7}{4}, \frac{9}{4})$.

Similarly, by using the same process, we get the polynomial reproduction degree and approximation order of the rest of the family members as given in Table 3.

Table 3. Polynomial generation/reproduction and Approximation order of $\delta_{\alpha,\theta,m}$.

Scheme	Degree of PG	Degree of PR	AO
$\delta_{\alpha,1,3}$	1	1 when $\alpha \in (-\frac{7}{4}, \frac{9}{4})$	2
$\delta_{\alpha,2,4}$	2	1 when $\alpha = 0$	2
$\delta_{\alpha,3,5}$	3	3 when $\alpha = -\frac{11}{16}$	4
$\delta_{\alpha,4,6}$	4	3 when $\alpha = -\frac{27}{32}$	4

For better understanding, we display the graphical representation of the polynomial reproduction property.

Example 5.1. In this example, we use a linear polynomial $f(x)$ to obtain the initial control points

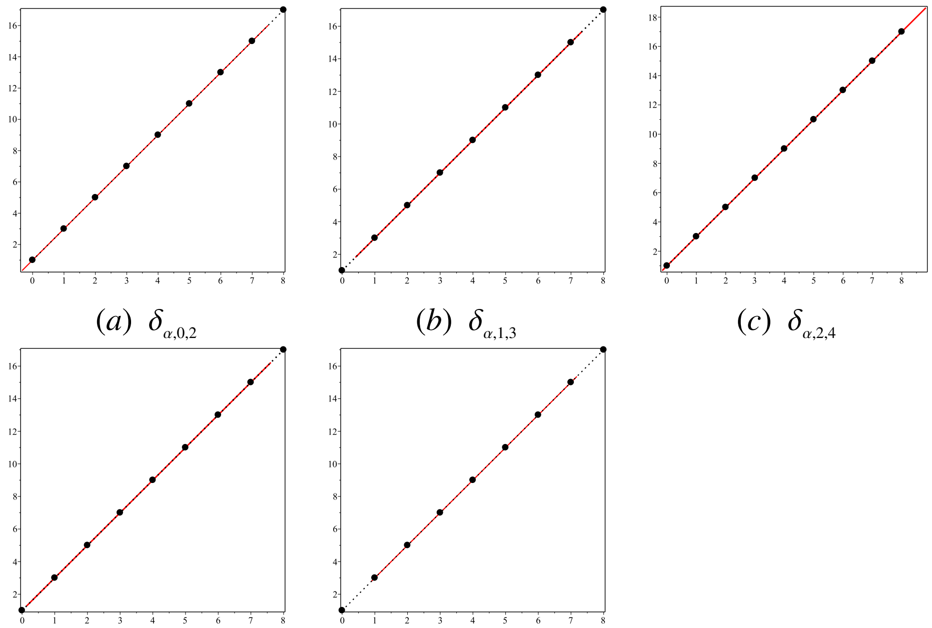
$$f(x) = 2x + 1, \quad x \in (0, 18). \tag{5.11}$$

We evaluate the property of polynomial reproduction. The graphical representation in Figure 2a–e illustrates the behavior of $\delta_{\alpha,\theta,m}$ with $\alpha = \frac{1}{8}$.

Example 5.2. In this example, we use a cubic polynomial $g(x)$ to obtain the initial control points

$$g(x) = x^3, \quad x \in (0, 6). \tag{5.12}$$

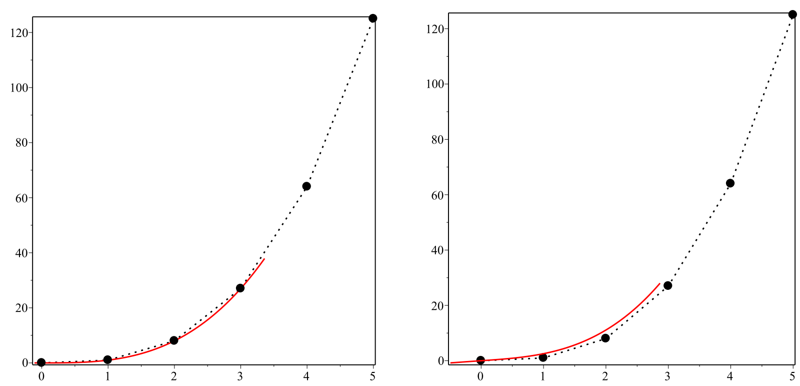
We evaluate the property of polynomial reproduction. The graphical representation in Figure 3a,c and Figure 3b,d illustrates the behavior of $\delta_{\alpha,\theta,m}$ for $\theta = 3$, $\alpha = -\frac{11}{16}$ and $\theta = 4$, $\alpha = -\frac{27}{32}$ respectively.



(a) $\delta_{\alpha,0,2}$ (b) $\delta_{\alpha,1,3}$ (c) $\delta_{\alpha,2,4}$

(d) $\delta_{\alpha,3,5}$ (e) $\delta_{\alpha,4,6}$

Figure 2. The limit functions obtained by using a linear polynomial. The results shown in (a)–(e) are obtained by $\delta_{\alpha,\theta,m}$, for $\alpha = 1/8$.



(a) *cubic polynomial* (b) *cubic polynomial*

Figure 3. The limit functions obtained by the scheme $\delta_{\alpha,\theta,m}$, employing the cubic polynomial specified in (5.12), for the values of $\alpha = -27/32$.

6. Theoretical and graphical analysis of the Gibbs phenomenon

This section focuses on the analysis of the Gibbs phenomenon in the curves generated by the scheme $\delta_{\alpha,\theta,m}$. The Gibbs phenomenon refers to a mathematical phenomenon that occurs when attempting to approximate a sharp jump or discontinuity in a function using a Fourier series, leading to persistent overshoot or oscillations near the discontinuity. Eliminating these oscillations is crucial for improving accuracy, visual quality, and overall performance of a subdivision scheme. To address this, we evaluate the characteristics of the masks in these subdivision schemes by applying appropriate criteria (as given in [24]), including partial sums of the nonnegative masks, to determine whether Gibbs oscillations occur near discontinuities. Recently, a family of 7-point binary subdivision schemes derived in [32] similarly addresses the Gibbs phenomenon. The transition from a binary to a quaternary framework allows our family of m-point schemes to encompass a wider variety of subdivision behaviors. This generalization demonstrates that the same principles for avoiding the Gibbs phenomenon apply effectively across different types of schemes, including both binary and quaternary settings.

Theorem 6.1. *The subfamily $\delta_{\alpha,0,2}$ does not produce Gibbs oscillations close to the discontinuity for $\alpha \in [0, \infty)$.*

Proof. By Theorem 2.1, a stationary refinement scheme avoids the Gibbs phenomenon occurring near a discontinuity if

$$\xi_t^{[k]}(i) \geq 0 \ \forall i, k, \tag{6.1}$$

where

$$\xi_t^{[k]}(i) = \begin{cases} \sum_{\tau \leq i} c_{4\tau+t}^{[k]}, & \text{if } i < 0, \\ 0, & \text{if } i = 0, \\ \sum_{\tau \geq i} c_{4\tau+t}^{[k]}, & \text{if } i > 0, \end{cases} \tag{6.2}$$

and, $0 \leq t < 4^k$.

The Laurent polynomial of $\delta_{\alpha,0,2}$ is

$$C_{\alpha,\theta,m}(z) = \sum_{j=-2}^{5} c_j z^j. \tag{6.3}$$

Without loss of generality, we focus on the case where $k = 1$, as the coefficients of the subdivision scheme do not depend on k, simplifying the analysis without affecting the generality of the results. Therefore, by setting $k = 1$ and substituting $c_j^{[1]} = c_j$ for $j = -2, -1, \ldots, 5$, in (6.2), we obtain

$$\xi_t^{[1]}(i) = \begin{cases} \sum_{\tau \leq i} c_{4\tau+t}^{[1]}, & \text{if } i < 0, \\ 0, & \text{if } i = 0, \\ \sum_{\tau \geq i} c_{4\tau+t}^{[1]}, & \text{if } i > 0, \end{cases} \tag{6.4}$$

where $0 \leq t \leq 4$. Thus, for $t = 0$, we get

$$\xi_0^{[1]}(i) = \begin{cases} \sum_{\tau \leq i} c_{4\tau}^{[1]}, & \text{if } i < 0, \\ 0, & \text{if } i = 0, \\ \sum_{\tau \geq i} c_{4\tau}^{[1]}, & \text{if } i > 0. \end{cases} \tag{6.5}$$

By substituting $i = -1, 0, 1$ in (6.5), we have

$$\xi_0^{[1]}(-1), \xi_0^{[1]}(0), \xi_0^{[1]}(1) \geq 0 \quad \forall \quad \alpha \in [-0.25, \infty). \tag{6.6}$$

Similarly for $t = 1$, (6.4) becomes

$$\xi_1^{[1]}(i) = \begin{cases} \sum_{\tau \le i} c_{4\tau+1}^{[1]}, & \text{if } i < 0, \\ 0, & \text{if } i = 0, \\ \sum_{\tau \ge i} c_{4\tau+1}^{[1]}, & \text{if } i > 0. \end{cases} \tag{6.7}$$

By taking $i = -1, 0, 1$ in (6.7), we get

$$\xi_1^{[1]}(-1), \xi_1^{[1]}(0), \xi_1^{[1]}(1) \ge 0 \quad \forall \quad \alpha \in [0, \infty). \tag{6.8}$$

Similarly for $t = 2$, (6.4) becomes

$$\xi_2^{[1]}(i) = \begin{cases} \sum_{\tau \le i} c_{4\tau+2}^{[1]}, & \text{if } i < 0, \\ 0, & \text{if } i = 0, \\ \sum_{\tau \ge i} c_{4\tau+2}^{[1]}, & \text{if } i > 0. \end{cases} \tag{6.9}$$

By taking $i = -1, 0, 1$ in (6.9), we get

$$\xi_2^{[1]}(-1), \xi_2^{[1]}(0), \xi_2^{[1]}(1) \ge 0 \quad \forall \quad \alpha \in [0, \infty). \tag{6.10}$$

Finally for $t = 3$, (6.4) takes the form

$$\xi_3^{[1]}(i) = \begin{cases} \sum_{\tau \le i} c_{4\tau+3}^{[1]}, & \text{if } i < 0, \\ 0, & \text{if } i = 0, \\ \sum_{\tau \ge i} c_{4\tau+3}^{[1]}, & \text{if } i > 0. \end{cases} \tag{6.11}$$

By taking $i = -1, 0, 1$ in (6.11), we get

$$\xi_3^{[1]}(-1), \xi_3^{[1]}(0), \xi_3^{[1]}(1) \ge 0 \quad \forall \quad \alpha \in [-0.25, \infty). \tag{6.12}$$

By combining (6.6), (6.8), (6.10), and (6.12), we conclude that the subfamily of the schemes $\delta_{\alpha,0,2}$ does not exhibit the Gibbs phenomenon $\forall \ \alpha \in [0, \infty)$. □

Similarly, by using same the process to find the interval where Gibbs phenomenon does not appear, we get the following results of other $\delta_{\alpha,\theta,m}$ schemes given in Table 4.

Table 4. Absence of Gibbs phenomenon for some value of α.

Scheme	Range for absence of Gibbs phenomenon
$\delta_{\alpha,1,3}$	$\alpha \in [0, \infty)$
$\delta_{\alpha,2,4}$	$\alpha \in \left[0, \frac{39}{8}\right]$
$\delta_{\alpha,3,5}$	$\alpha \in \left[0, \frac{39}{8}\right]$
$\delta_{\alpha,4,6}$	$\alpha \in \left[0, \frac{751}{132}\right]$

We present some figures which graphically demonstrate the smoothness of the proposed schemes. These figures are created by applying the proposed schemes to different open and closed polygons. We offer numerical illustrations of continuous functions. We also provide some numerical examples of discontinuous functions after eliminating the Gibbs phenomenon for some particular choice of parameter.

Example 6.1. In this example, we derive the initial control points by utilizing a continuous and discontinuous function $f(x)$ and $g(x)$, respectively,

$$f(x) = x^2 \sin(x), \quad x \in (-10, 10), \tag{6.13}$$

and

$$g(x) = \begin{cases} \tan(x), & \text{if } x < -\frac{\pi}{2}, \\ \tan(x), & \text{if } -\frac{\pi}{2} < x < \frac{\pi}{2}, \\ \tan(x), & \text{if } x > \frac{\pi}{2}. \end{cases} \tag{6.14}$$

In Figure 4, the black dotted lines depict the original control polygon, while the black solid circles represent the initial control points. The solid red lines illustrate the behavior of the proposed SS when $\alpha = \frac{1}{8}$. Figure 4a visualizes the C^0 limit curves produced by $\delta_{\frac{1}{8},0,2}$. Figure 4b exhibits the C^1 limit curves generated by $\delta_{\frac{1}{8},1,3}$. Figure 4c displays the C^2 limit curves resulting from $\delta_{\frac{1}{8},2,4}$. Figure 4d showcases the C^3 limit curves produced by $\delta_{\frac{1}{8},3,5}$. Lastly, Figure 4e demonstrates the C^4 limit curves created by $\delta_{\frac{1}{8},4,6}$. Figure 4f–j provides a comprehensive overview of the behavior of $\delta_{\frac{1}{8},\theta,m}$ for $\theta = 0, \ldots, 4$ and $m = 2, \ldots, 6$, respectively. The observed results indicate that the limit function does not exhibit Gibbs oscillations near the discontinuity. This is an important finding, as it suggests that the function maintains stability and smoothness even in the presence of discontinuities.

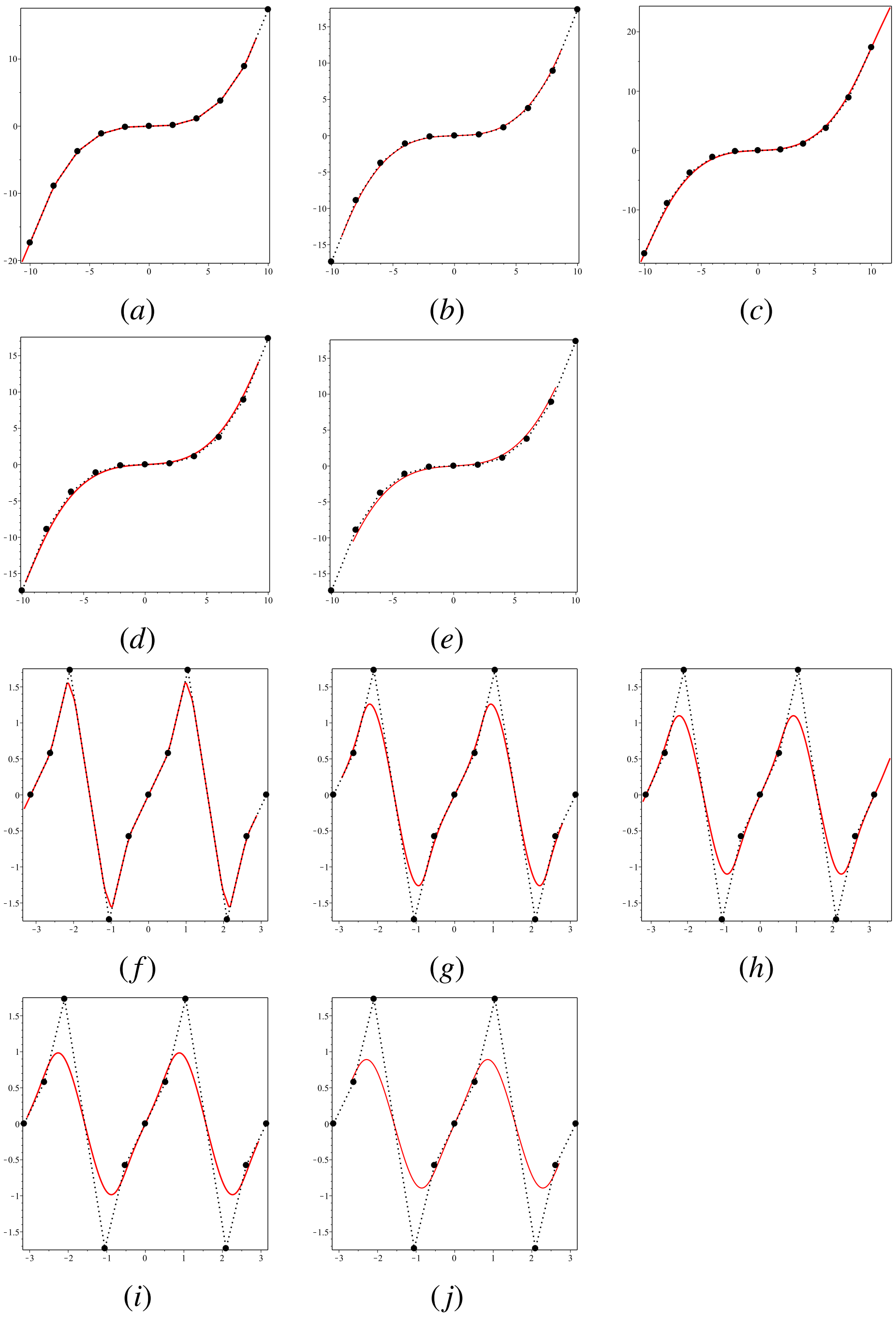
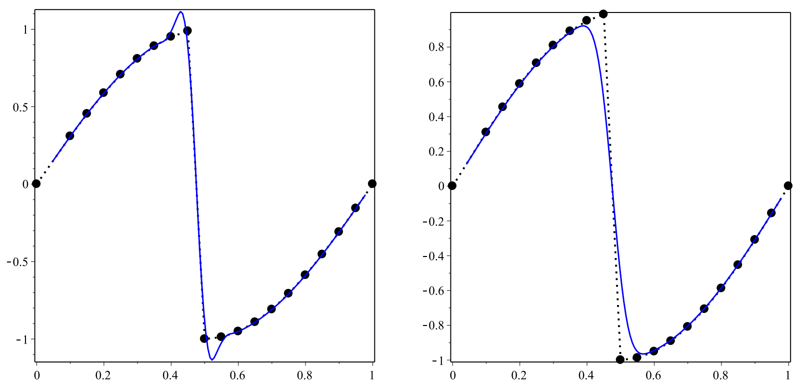


Figure 4. The limit functions obtained by the scheme $\delta_{\alpha,\theta,m}$, using the continuous function (left) given in (6.13) and discontinuous function (right) described in (6.14), for specific value of $\alpha = 1/8$.

Example 6.2. In this example, we give a comparison of the behavior of the scheme $\delta_{\alpha,0,2}$ to the Gibbs phenomenon for different values of α. For this, we consider the initial data from a discontinuous function $f(x)$

$$f(x) = \begin{cases} \sin(\pi x), & \text{if } 0 \leq x \leq 0.5, \\ -\sin(\pi x), & \text{if } 0.5 < x < 1. \end{cases} \tag{6.15}$$

In Figure 5, the black dotted lines depict the original control polygon, while the black solid circles represent the initial control points. The solid blue lines illustrate the behavior of the scheme $\delta_{\alpha,0,2}$. Figure 5a shows the limit curve produced by $\delta_{\alpha,0,2}$ at $\alpha = -\frac{1}{5}$ and the Figure 5b exhibits the limit curve generated by $\delta_{\alpha,0,2}$ at $\alpha = \frac{3}{7}$. As seen in Figure 5a, the scheme $\delta_{\alpha,0,2}$ fails to avoid the Gibbs phenomenon when the value of α a does not satisfy the constraints outlined in Theorem 6.1.



(a) $\alpha = -\frac{1}{5}$ (b) $\alpha = \frac{3}{7}$

Figure 5. Gibbs phenomenon: The limit curves obtained by the scheme $\delta_{\alpha,0,2}$, using the discontinuous function given in (6.15), at different values of α.

7. Parametric impact of $\delta_{\alpha,\theta,m}$

In this section, we deal with the visual performance of the proposed schemes $\delta_{\alpha,0,2}$, $\delta_{\alpha,1,3}$, $\delta_{\alpha,2,4}$, $\delta_{\alpha,3,5}$, and $\delta_{\alpha,4,6}$. The proposed schemes offer higher continuity C^{θ} and create smooth limit curves. Hence, it provides great flexibility for the designers to create smooth curves according to their requirements. We present some figures which graphically demonstrate the smoothness of the proposed schemes. These figures are created by applying the proposed schemes to different closed polygons.

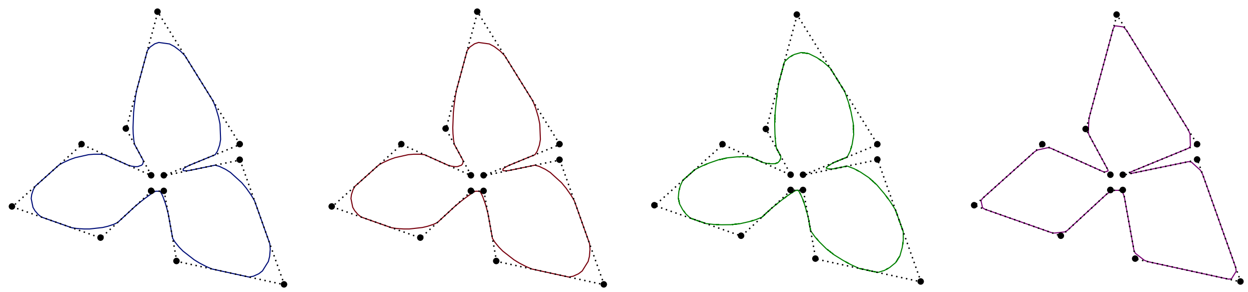
Example 7.1. In this example, we showcase the approximating behavior of $\delta_{\alpha,0,2}$ on a closed control polygon for various α values. Figure 6a illustrates this behavior for $\alpha = \frac{3}{32}$. Figure 6b–d corresponds to $\alpha = \frac{1}{4}$, $\frac{5}{16}$, and $\frac{19}{64}$, respectively.

Example 7.2. In this example, we present the approximating behavior of $\delta_{\alpha,1,3}$ on a closed control polygon for different values of α. Figure 7a illustrates the behavior of $\delta_{\alpha,1,3}$ when $\alpha = \frac{1}{6}$. Figure 7b–d corresponds to $\alpha = \frac{1}{4}$, $\frac{1}{8}$, and $\frac{1}{12}$, respectively.

Example 7.3. In this example, we express the approximating behavior of $\delta_{\alpha,2,4}$ on an open polygon for different values of α. Figure 8a depicts the behavior of $\delta_{\alpha,2,4}$ when $\alpha = \frac{1}{8}$. Figure 8b–d illustrates the behaviors of $\delta_{\alpha,2,4}$ for $\alpha = \frac{5}{16}$, $\frac{13}{32}$, and $\frac{19}{64}$, respectively.

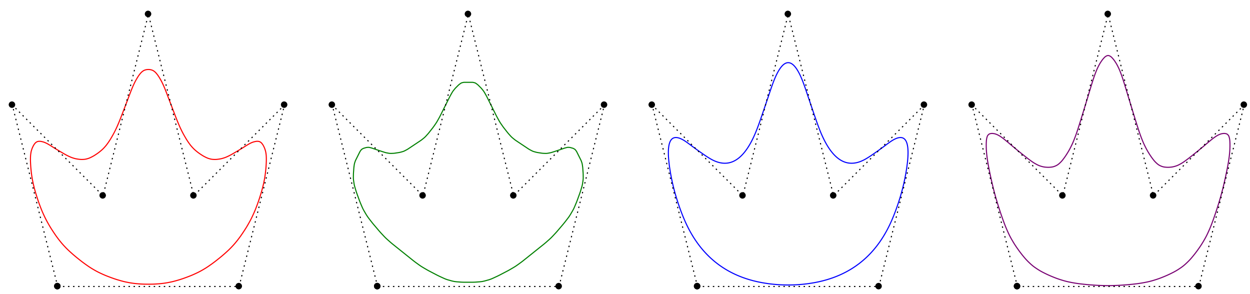
Example 7.4. In this example, we present the approximating behavior of $\delta_{\alpha,3,5}$ on a closed control polygon for different values of α. Figure 9a depicts the behavior of $\delta_{\alpha,3,5}$ when $\alpha = \frac{1}{8}$. Figure 9b–d illustrates the behavior of $\delta_{\alpha,3,5}$ for $\alpha = \frac{5}{16}, \frac{13}{32}$, and $-\frac{5}{32}$, respectively.

Example 7.5. In this example, we examine the approximating behavior of $\delta_{\alpha,4,6}$ on a closed control polygon across various values of α. Figure 10a illustrates the behavior of $\delta_{\alpha,4,6}$ when $\alpha = \frac{1}{8}$. Figure 10b–d represents the behaviors of $\delta_{\alpha,4,6}$ for $\alpha = \frac{5}{16}, \frac{13}{32}$, and $-\frac{5}{32}$, respectively.



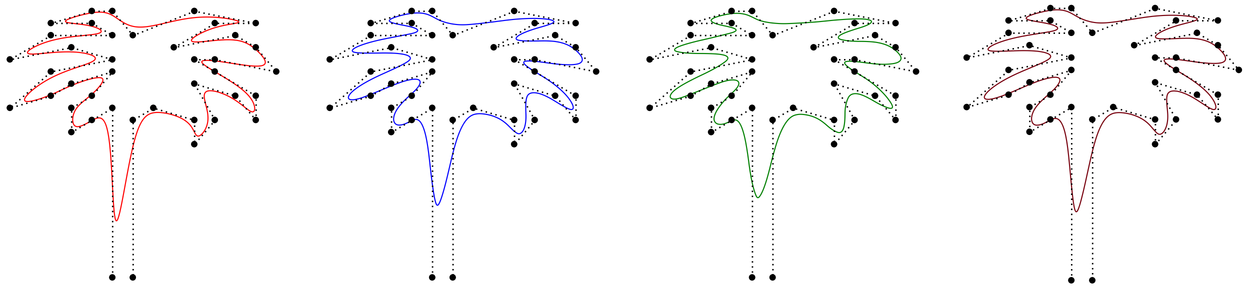
(*a*) $\alpha = 3/32$ (*b*) $\alpha = 1/4$ (*c*) $\alpha = 5/16$ (*d*) $\alpha = 19/64$

Figure 6. Behavior of limit curves generated by $\delta_{\alpha,0,2}$ for the different values of α. The dotted line represents the original closed polygon and the solid circles represent the initial control points.



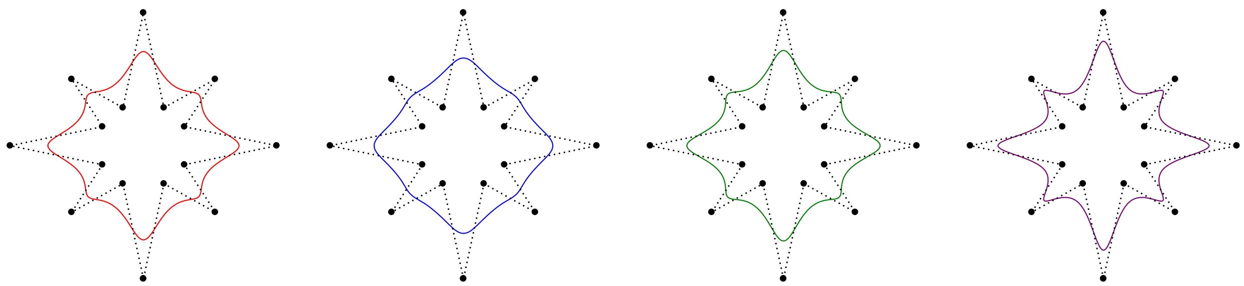
(*a*) $\alpha = 1/6$ (*b*) $\alpha = 1/4$ (*c*) $\alpha = 1/8$ (*d*) $\alpha = 1/12$

Figure 7. Behavior of limit curves generated by $\delta_{\alpha,1,3}$ for the different values of α. The dotted line represents the original closed polygon and the solid circles represent the initial control points.



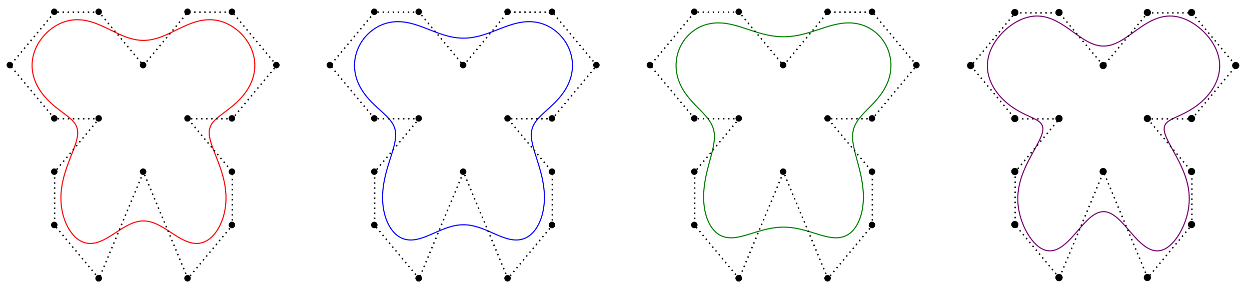
(*a*) $\alpha = 1/8$ (*b*) $\alpha = 5/16$ (*c*) $\alpha = 13/32$ (*d*) $\alpha = 19/64$

Figure 8. Behavior of limit curves generated by $\delta_{\alpha,2,4}$ for the different values of α. The dotted line represents the original closed polygon and the solid circles represent the initial control points.



(*a*) $\alpha = 1/8$ (*b*) $\alpha = 5/16$ (*c*) $\alpha = 13/32$ (*d*) $\alpha = -5/32$

Figure 9. Behavior of limit curves generated by $\delta_{\alpha,3,5}$ for the different values of α. The dotted line represents the original closed polygon and the solid circles represent the initial control points.



(*a*) $\alpha = 1/8$ (*b*) $\alpha = 5/16$ (*c*) $\alpha = 13/32$ (*d*) $\alpha = -5/32$

Figure 10. Behavior of limit curves generated by $\delta_{\alpha,4,6}$ for the different values of α. The dotted line represents the original closed polygon and the solid circles represent the initial control points.

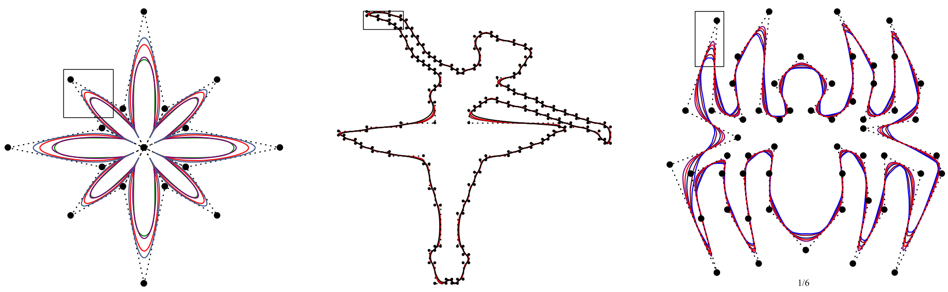
8. Comparison of $\delta_{\alpha,\theta,m}$

In this section, we provide a comparative analysis of our proposed schemes and discuss that the proposed schemes coincide with existing schemes in Table 5.

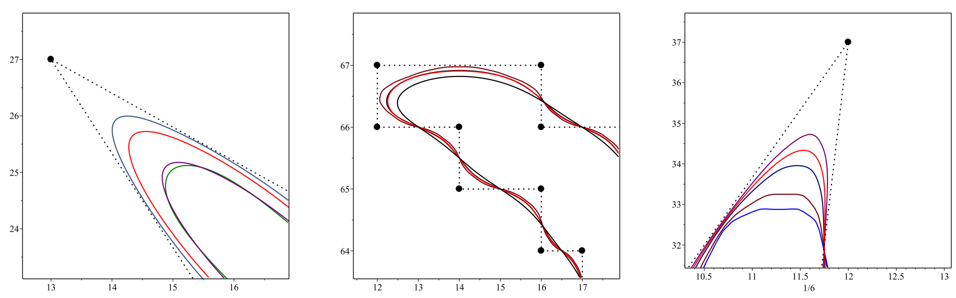
Table 5. Comparison of $\delta_{\alpha,\theta,m}$, where AO stands for approximation order.

m-point	Scheme	Continuity	Support	AO	Coincide with
2	$\delta_{\alpha,0,2}$	C^0	7/3	2	scheme [25]
3	$\delta_{\alpha,1,3}$	C^1	10/3	2	
4	$\delta_{\alpha,2,4}$	C^2	13/3	2	
5	$\delta_{\alpha,3,5}$	C^3	16/3	4	scheme [26]
6	$\delta_{\alpha,4,6}$	C^4	19/3	4	scheme [20]
$\vdots$	$\vdots$	$\vdots$			
m	$\delta_{\alpha,\theta,m}$	C^θ			

Figure 11 illustrates the comparison of limit curves produced by $\delta_{\alpha,\theta,m}$ for different values of α.



(a) *Comparison* (b) *Comparison* (c) *Comparison*



(d) *Magnifying factor* (e) *Magnifying factor* (f) *Magnifying factor*

Figure 11. Comparison of limit curves generated by $\delta_{\alpha,\theta,m}$ for the different values of α. (a): $\alpha = \frac{13}{32}(blue), \frac{1}{8}(red), -\frac{1}{16}(purple)$, *and* $-\frac{5}{32}$ $(green)$, (b): $\alpha = \frac{13}{32}(brown), \frac{1}{8}(navyblue), -\frac{1}{16}(red)$, *and* $-\frac{5}{32}$ $(black)$, (c): $\alpha = \frac{13}{32}(purple), \frac{1}{8}(red), -\frac{1}{16}(navyblue), -\frac{5}{32}$ $(brown)$, *and* $-\frac{6}{25}(blue)$. (d)-(f) illustrate the magnification factor. The dotted line represents the original closed polygon and the solid circles represent the initial control points.

9. Conclusions

In this paper, we have introduced a family of m-point quaternary approximating subdivision schemes that offer flexibility and enhanced control through the use of a shape control parameter and two additional parameters determining smoothness. Our analysis has demonstrated the versatility of these schemes in generating smooth curves, with the ability to adjust the smoothness and shape of the limit curves by varying the parameters. The study thoroughly examined the polynomial generation and reproduction capabilities of the proposed schemes, establishing their readiness in practical applications.

Furthermore, we addressed the Gibbs phenomenon, both theoretically and graphically, showcasing the ability of our schemes to minimize unwanted oscillations near discontinuities. A comparative study with existing schemes has confirmed the effectiveness of the proposed approach, particularly in terms of improving smoothness and control. These findings suggest that our family of m-point quaternary subdivision schemes is well-suited for applications in computer graphics and geometric modeling, where smoothness and flexibility are critical.

Author contributions

Reem K. Alhefthi1: Formal analysis, Software, funding acquisition; Pakeeza Ashraf: Validation, Methodology, Writing–review & editing; Ayesha Abid: Resources, Writing–review & editing; Shahram Rezapour, Abdul Ghaffar: Software, Writing–review & editing; Mustafa Inc: Writing–review & editing, supervision. All authors have read and agreed to the published version of the manuscript.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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