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Research article

Strategy evolution of a novel cooperative game model induced by reward feedback and a time delay

Haowen Gong[1], Huijun Xiang[1], Yifei Wang[1,*], Huaijin Gao[2,*] and Xinzhu Meng[1]

[1] College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao 266590, China

[2] School of Mathematics and Statistics, Weifang University, Weifang 261061, China

* **Correspondence:** Email: wangyifei9907@163.com, gaohuaijin@wfu.edu.cn.

Abstract: Rewarding cooperators and punishing defectors are effective measures for promoting cooperation in evolutionary game theory. Given that previous models treated rewards as constants, this does not reflect real-world dynamics changes. Therefore, this paper focused on the classical payoff matrix and examined the dynamic variable rewards affected by cooperation and defection strategies, as well as the impact of time delays. First, for the system without a time delay, we analyzed the existence and stability of numerous equilibrium points and explored transcritical bifurcations under various conditions. Second, for the time-delay system, we discussed a series of delayed dynamical behaviors including Hopf bifurcation, period, and the stability and direction of bifurcation. Finally, the changes of cooperation strategy were observed by numerical simulation, and some interesting results were obtained: (i) Under certain circumstances, even if the reward given to the cooperators reaches the maximum, the proportion of cooperators is still zero, which means that increasing rewards does not always promote cooperation. (ii) The initial state can affect the choice of cooperation strategy and defection strategy. (iii) The increase of the time delay makes the stable equilibrium point disappear and forms a stable limit cycle.

Keywords: evolutionary game theory; replicator dynamics; reward feedback; time delay; Hopf bifurcation
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1. Introduction

In natural and social systems, an individual's strategy is often shaped by complex game processes that influence not only personal outcomes but also the evolutionary trajectory of group behavior [1–4]. The dynamic equilibrium between cooperation and defection significantly impacts the system's long-term stability. However, maintaining cooperation is challenging under limited resources and competitive pressure, particularly when defection yields short-term gains [5–8]. Researchers are focusing on mechanisms such as rewards to promote cooperation or punishments to deter defection [9, 10]. Understanding the dynamic balance between cooperation and defection, and the factors that affect this balance, is crucial for developing effective incentive policies and sustaining stable cooperative relationships. This issue is both a theoretical and practical challenge with broad implications for economics [11], sociology [12], and biology [13].

In the past, most of the classical game theory simply considered the change of strategy, but now more and more researchers are beginning to address the influence of the surrounding environment on strategy. Strategy and environment change and interact with each other [14, 15]. Individual strategies within a group can alter the environmental state, which in turn affects subsequent strategy choices. Environmental feedback is a crucial factor influencing participant behavior and the evolution of group dynamics. For instance, plants' nitrogen fixation strategies can change soil nitrogen content [16], and over time, this nitrogen content influences plant selection strategies. In microbial communities, collaborators produce enzymes to decompose nutrients for microorganisms. Conversely, changes in nutrient availability affect whether microorganisms choose to collaborate [17]. Positive environmental feedback following a decision increases the likelihood of that decision being repeated, thereby raising the group's adoption rate of the strategy. Rewarding cooperators is one of the most direct ways to promote individual selection and cooperation in the population [18,19]. Li et al. explored the influence of reciprocal rewards on cooperative evolution in the dilemma of voluntary society by adding the third strategy, the loner strategy [20]. Besides, it is also interesting to study different game models in the complex network structure [21, 22]. In previous studies, the reward for collaborators was treated as a constant. However, in real life, cooperative reward can change with the change of collaborators and betrayers in the population. It has been verified that dynamic reward is more beneficial to cooperation than fixed reward in the game of space public goods [23].

Time delay plays a significant and complex role in systems [24, 25]. It not only influences the response speed and stability of the system but also plays a crucial role in its dynamic behavior and long-term regulation [26–29]. It takes time to complete many biological processes, which leads to time lag. Biological systems often experience various types of delays, such as latency delays [30] and growth delays [31]. Despite numerous studies attempting to reveal the dynamic characteristics and stability conditions of cooperation and defection strategies in game models, most models assume immediate payoff feedback, neglecting the delay effects present in many real-world scenarios. Tao and Wang incorporated time delay into an evolutionary game model with two strategies, finding that it influences the stability of equilibria. They analyzed the stability conditions for both systems with and without time delay [32]. Khalifa et al. studied how discrete and distributed delays affect evolutionary stable strategies. They found that evolutionary stable strategies are asymptotically stable regardless of the rate parameters under an exponential delay distribution [33].

Therefore, this paper not only considers that the reward intensity can be affected by the strategic proportion, but also further explores the role of time delay in the feedback game system through the replicator equation [34,35]. In section 2, a feedback game system is constructed for the classical payoff matrix, and the existence and stability of the equilibrium points of the system without time delay and the conditions for the system to experience transcritical bifurcation are analyzed. In section 3, we consider the payoff delay and find the critical delay of Hopf bifurcation in time-delay systems. In section 4, we explore the related properties of hopf bifurcation. Numerical simulation and biological significance analysis of strategic dynamics are exhibited in section 5. The sixth section summarizes the paper and presents future prospects.

2. Model and results without time delay

This section investigates an infinitely mixed group, in which the payoff matrix of cooperative strategy and defective strategy is the classical payoff matrix as follows

$$\begin{array}{c|cc} & C & D \\ \hline C & a & b \\ D & c & d \end{array} \tag{2.1}$$

where non-negative parameters a and b represent the payoffs of the collaborator meeting the collaborator and the betrayer, respectively. Non-negative parameters c and d represent the payoffs of the betrayer meeting the collaborator and the betrayer, respectively.

To incentivize cooperators to enhance their cooperative efforts, they will be rewarded with an additional amount r on top of their original payoff. This adjustment modifies the payoff matrix to

$$\begin{array}{c|cc} & C & D \\ \hline C & a+r & b+r \\ D & c & d \end{array}. \tag{2.2}$$

Let the proportion of cooperators among the participants be $x(t)$, and the proportion of defectors be $y(t)$. According to the matrix (2.2), the payoff equations for cooperators and defectors are given by

$$\begin{cases} \pi_C(t) = (a+r)x(t) + (b+r)y(t), \\ \pi_D(t) = cx(t) + dy(t). \end{cases} \tag{2.3}$$

Then according to the replicator equation, we have

$$\begin{cases} \dot{x}(t) = x(t)\left(\pi_C(t) - \bar{f}(t)\right), \\ \dot{y}(t) = y(t)\left(\pi_D(t) - \bar{f}(t)\right), \end{cases} \tag{2.4}$$

where $\bar{f}(t) = \pi_C(t)\,x(t) + \pi_D(t)\,y(t)$.

Here, we assume that the reward intensity r is a variable that changes over time, ranging from a minimum value of 0 to a maximum value of m. To more effectively promote cooperation, the reward for cooperators is influenced by the proportions of defectors and cooperators within the population. Specifically, the presence of more defectors increases the reward for cooperators, while a higher proportion of cooperators decreases the reward. In other words, as the number of defectors rises, the reward for cooperators increases. Therefore the dynamic equation for reward intensity is described by

$$\dot{r}(t) = r(t)(m - r(t))(u_1 y(t) - u_2 x(t)), \tag{2.5}$$

where u_1 and u_2 represent the growth rates of defectors and cooperators, respectively. Letting $\beta = \frac{u_1}{u_2}$, we have

$$\dot{r}(t) = r(t)(m - r(t))(\beta y(t) - x(t)). \tag{2.6}$$

In the above equations, $\pi_C(t)$ and $\pi_D(t)$ represent the payoffs of the group that chooses to cooperate and defect in the population, respectively, and $\bar{f}(t)$ represents the average payoff. We eliminate $y(t) = 1 - x(t)$, substitute Eq (2.4) into the above equation, and combining with the reward intensity Eq (2.6), where ϵ indicates the relative speed between the influence reward intensity and the strategy. The inequality $\epsilon < 1$ ($\epsilon > 1$) means that the strategy evolves faster (slower) than the reward intensity. So we obtain a high-order evolutionary game system with reward feedback and the classical payoff matrix as follows:

$$\begin{cases} \dot{x}(t) = x(t)(1 - x(t))((a - b - c + d)x(t) + b + r(t) - d), \\ \dot{r}(t) = \epsilon r(t)(m - r(t))(\beta - (\beta + 1)x(t)). \end{cases} \tag{2.7}$$

2.1. Existence and stability of numerous equilibrium points

The system (2.7) may have seven equilibrium points: $E_1(0,0)$, $E_2(0,m)$, $E_3(1,m)$, $E_4(1,0)$, $E_5(x_5,0)$, $E_6(x_6,m)$, and $E_7(x_7,r_7)$, where $x_5 = \frac{d-b}{a-b-c+d}$, $x_6 = \frac{d-b-m}{a-b-c+d}$, $x_7 = \frac{\beta}{\beta+1}$, and $r_7 = \frac{d-b-\beta a+\beta c}{\beta+1}$. Obviously, E_1, E_2, E_3, and E_4 always exist.

Define

$$\begin{aligned} &(\boldsymbol{H_1})\ \ (a-c)(b-d) < 0, \\ &(\boldsymbol{H_2})\ \ (a-c+m)(b-d+m) < 0, \\ &(\boldsymbol{H_3})\ \ d-b-\beta a+\beta c > 0, \\ &(\boldsymbol{H_4})\ \ m\beta+m-d+b+\beta a-\beta c > 0, \\ &(\boldsymbol{H_5})\ \ b-d+m < 0, \\ &(\boldsymbol{H_6})\ \ c-a < 0, \\ &(\boldsymbol{H_7})\ \ a-b-c+d < 0, \end{aligned}$$

where x_5, x_6, and $x_7 \in (0,1)$ and $r_7 \in (0,m)$ in the game.

For the equilibrium point E_5, we have $\frac{d-b}{a-b-c+d} \in (0,1)$. Thus, E_5 exists when $(a-c)(b-d) < 0$, that is, H_1 holds.

For the equilibrium point E_6, we have $\frac{d-b-m}{a-b-c+d} \in (0,1)$. Thus, E_6 exists when $(a-c+m)(b-d+m) < 0$, that is, H_2 holds.

For the equilibrium point E_7, given $\frac{\beta}{\beta+1} \in (0,1)$ and $\frac{d-b-\beta a+\beta c}{\beta+1} \in (0,m)$, we obtain $0 < d-b-\beta a+\beta c < m(\beta+1)$, that is, H_3 and H_4 hold.

Thus the existence of equilibria is summarized as follows:

Lemma 2.1. *(1) The boundary equilibria E_1, E_2, E_3, and E_4 always exist.*
(2) $E_5(x_5,0)$ exists when H_1 holds.
(3) $E_6(x_6,m)$ exists when H_2 holds.
(4) $E_7(x_7,r_7)$ exists when H_3 and H_4 hold.

Next, we determine the stability conditions for the above seven equilibria. The Jacobian matrix of system (2.7) is known to be

$$\begin{pmatrix} -3(a-b-c+d)x^2 + 2(a-2b-c+2d-r)x + b - d + r & -x(x-1) \\ -\epsilon r(m-r)(\beta+1) & \epsilon(m-2r)[\beta - x(\beta+1)] \end{pmatrix}. \tag{2.8}$$

The stability of the equilibrium points can be assessed by examining the real parts of the eigenvalues of the Jacobian matrix evaluated at those equilibrium points.

The following theorem is utilized to elucidate the local stability of system (2.7).

Theorem 2.1. *(1) The equilibrium E_1 is always unstable.*
(2) E_2 is locally asymptotically stable when H_5 is true.
(3) E_3 is always unstable.
(4) E_4 is locally asymptotically stable when H_6 is true.
(5) E_5 is locally asymptotically stable when H_3 is not true and H_7 is true.
(6) E_6 is locally asymptotically stable when H_4 is not true and H_7 is true.
(7) E_7 is locally asymptotically stable when H_7 is true.

Proof. (1) The Jacobian matrix at E_1 is

$$J_1 = \begin{pmatrix} b-d & 0 \\ 0 & \epsilon m\beta \end{pmatrix}.$$

We obtain $\lambda_{11} = b - d$ and $\lambda_{12} = \epsilon m\beta$. It is easy to see that E_1 is unstable since $\lambda_{12} > 0$.

(2) The matrix (2.9) at the equilibrium point E_2 is

$$J_2 = \begin{pmatrix} b-d+m & 0 \\ 0 & -\epsilon m\beta \end{pmatrix}.$$

Given that $\lambda_{21} = b - d + m$ and $\lambda_{22} = -\epsilon m\beta$, E_2 is locally asymptotically stable when H_5 is satisfied.

(3) The matrix (2.9) at the equilibrium point E_3 is

$$J_3 = \begin{pmatrix} c-a-m & 0 \\ 0 & \epsilon m \end{pmatrix}.$$

We obtain $\lambda_{31} = c - a - m$ and $\lambda_{32} = \epsilon m > 0$. So E_3 is unstable.

(4) The matrix (2.9) at E_4 is

$$J_4 = \begin{pmatrix} c-a & 0 \\ 0 & -\epsilon m \end{pmatrix}.$$

It is easy to see that $\lambda_{41} = c - a$ and $\lambda_{42} = -\epsilon m < 0$. Therefore, E_4 is locally asymptotically stable when H_6 is true.

(5) The matrix (2.9) at E_5 is

$$J_5 = \begin{pmatrix} j_{51} & j_{52} \\ 0 & j_{53} \end{pmatrix},$$

where $j_{51} = -\frac{(d-b)(c-a)}{a-b-c+d}$, $j_{52} = -\frac{(d-b)(c-a)}{(a-b-c+d)^2}$, and $j_{53} = \frac{\epsilon m(d-b-\beta a+\beta c)}{b+c-a-d}$.
Thus, given that $\lambda_{51} = j_{51}$ and $\lambda_{52} = j_{53}$, the equilibrium E_5 is locally asymptotically stable when condition H_7 is true and H_3 is not.

(6) For system (2.7) at E_6, the Jacobian matrix (2.9) is

$$J_6 = \begin{pmatrix} j_{61} & j_{62} \\ 0 & j_{63} \end{pmatrix},$$

where $j_{61} = -\frac{(d-b-m)(c-a-m)}{a-b-c+d}$, $j_{62} = -\frac{(d-b-m)(c-a-m)}{(a-b-c+d)^2}$, and $j_{63} = -\frac{\epsilon m(d-b-\beta a+\beta c-m\beta-m)}{b+c-a-d}$. Thus, given that $\lambda_{61} = j_{61}$ and $\lambda_{62} = j_{63}$, the equilibrium point E_6 is locally asymptotically stable when H_7 is true and H_4 is not.

(7) The Jacobian matrix of sysytem (2.7) at E_7 is

$$J_7 = \begin{pmatrix} j_{71} & j_{72} \\ j_{73} & 0 \end{pmatrix},$$

where $j_{71} = \frac{(-a-b+c+d-2r)\beta^2+(2a-3b-2c+3d-r)}{(\beta+1)^2})$, $j_{72} = \frac{\beta}{(\beta+1)^2}$, and $j_{73} = -\epsilon\frac{(d-b-\beta a+\beta c)(m\beta+m-d+b+\beta a-\beta c)}{\beta+1}$. One has

$$\mathrm{Det}(J_7) = -\frac{\epsilon(d-b-\beta a+\beta c)(d-b+m+\beta a-\beta c+\beta m)}{\beta+1},$$

$$\mathrm{Tr}(J_7) = \frac{(a-b-c+d)\beta}{(\beta+1)^2}.$$

If $\mathrm{Det}(J_7) > 0$ and $\mathrm{tr}(J_7) < 0$, at this time, $a-b-c+d < 0$, that is, H_7 is true. Therefore, E_7 is locally asymptotically stable when H_7 is true. □

In summary, Table 1 provides a recap of the information discussed above.

Table 1. Existence and stability of equilibria.

Equilibria	Existence conditions	Stability conditions
E_1	always	unstable
E_2	always	H_5
E_3	always	unstable
E_4	always	H_6
E_5	H_1	H_7 and not H_3
E_6	H_2	H_7 and not H_4
E_7	H_3 and H_4	H_7

Thus the bistability of the equilibrium points can be obtained.

Theorem 2.2. *When conditions H_5 and H_6 are both true, the equilibrium points E_2 and E_4 are stable.*

2.2. Bifurcation analysis

Next, we use the Sotomayor theorem to explore the bifurcation scenarios of system (2.7) [36].

Theorem 2.3. *(1) When the cooperator meets the defector, the payoff of the cooperator is $b = b^* = d - m$, and a transcritical bifurcation occurs at E_2.*
(2) When a cooperator meets the cooperator, the payoff of the cooperator is $b \neq d$ and $a = a^* = c$, and a transcritical bifurcation occurs at E_4.
(3) When the ratio of the defective strategy promotion reward to the cooperator inhibition reward is $\beta = \beta^* = \frac{d-b}{a-c}$, a transcritical bifurcation occurs at E_5.
(4) When $\beta = \beta^{**} = \frac{d-b-m}{a-c+m}$, a transcritical bifurcation occurs at E_6.

Proof. (1) The Jacobian matrix J_2 has eigenvalues $\lambda_{21} = b - d + m$ and $\lambda_{22} = -\epsilon m\beta$. Let $\lambda_{21} = 0$, and we get $b = b^* = d - m$, and at this time $\lambda_{22} < 0$. In this case, the eigenvectors corresponding to the zero eigenvalue for J_2 and J_2^T are

$$V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad W_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

We rewrite Eq (2.7) as

$$F = \begin{pmatrix} S \\ Q \end{pmatrix} = \begin{pmatrix} x(1-x)((a-b-c+d)x+b+r-d) \\ \epsilon r(m-r)(\beta-(\beta+1)x) \end{pmatrix}. \tag{2.9}$$

Then we get

$$F_b(E_2, b^*) = \begin{pmatrix} x(1-x)^2 \\ 0 \end{pmatrix}\Bigg|_{E_2} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$DF_b(E_2, b^*) = \begin{pmatrix} x(2x-2)+(x-1)^2 & 0 \\ 0 & 0 \end{pmatrix}\Bigg|_{E_2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$D^2F_b(E_2, b^*)(V_1, V_1) = \begin{pmatrix} 2(a-c+m) \\ 0 \end{pmatrix}.$$

Further, we have

$$\begin{cases} W_1^T F_b(E_2, b^*) = 0, \\ W_1^T DF_b(E_2, b^*)V_1 = 1 \neq 0, \\ W_1^T D^2F_b(E_2, b^*)(V_1, V_1) = 2(a-c+m) \neq 0. \end{cases}$$

Therefore, when $b = b^* = d - m$, a transcritical bifurcation occurs at E_2.

(2) According to the the Jacobian matrix J_4, it has eigenvalues $\lambda_{41} = c - a$ and $\lambda_{42} = -\epsilon m$. Let $\lambda_{41} = 0$, and we get $a = a^* = c$, that is $\lambda_2 < 0$. The eigenvectors corresponding to the eigenvalue 0 for J_4 and J_4^T are

$$V_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad W_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Thus one has

$$F_a(E_4, a^*) = \begin{pmatrix} S_a \\ Q_a \end{pmatrix} = \begin{pmatrix} x^2(1-x) \\ 0 \end{pmatrix}\Bigg|_{E_4} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$DF_a(E_4, a^*) = \begin{pmatrix} \frac{\partial F_a}{\partial x} & \frac{\partial F_a}{\partial r} \end{pmatrix} = \begin{pmatrix} -x(2x-2)+x^2 & 0 \\ 0 & 0 \end{pmatrix}\Bigg|_{E_4} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$D^2F_a(E_4, a^*)(V_2, V_2) = \begin{pmatrix} 2(b-d) \\ 0 \end{pmatrix}.$$

Further, we have

$$\begin{cases} W_2^T F_a(E_4, a^*) = 0, \\ W_2^T DF_a(E_4, a^*)V_2 = -1 \neq 0, \\ W_2^T D^2F_a(E_4, a^*)(V_2, V_2) = 2(b-d) \neq 0. \end{cases}$$

Therefore, when $b \neq d$ and $a = a^* = c$, a transcritical bifurcation occurs at E_4.

(3) The Jacobian matrix J_5 has eigenvalues $\lambda_{51} = \frac{\epsilon m(d-b-\beta a+\beta c)}{b+c-a-d}$ and $\lambda_{52} = -\frac{(d-b)(c-a)}{a-b-c+d}$. Let $\lambda_{51} = 0$, that is $d - b - \beta a + \beta c = 0$, and we get $\beta = \beta^* = \frac{d-b}{a-c}$. At this time, $\lambda_2 < 0$. The eigenvectors corresponding to the eigenvalue 0 for J_5 and J_5^T are

$$V_3 = \begin{pmatrix} -\frac{1}{a-b-c+d} \\ 1 \end{pmatrix}, \quad W_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Thus we have

$$F_\beta(E_5, \beta^*) = \begin{pmatrix} S_\beta \\ Q_\beta \end{pmatrix} = \begin{pmatrix} 0 \\ -\epsilon r(m-r)(x-1) \end{pmatrix}\Bigg|_{E_5} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$DF_\beta(E_5, \beta^*) = \begin{pmatrix} 0 & 0 \\ -\epsilon r(m-r) & \epsilon r(x-1) - \epsilon(m-r)(x-1) \end{pmatrix}\Bigg|_{E_5} = \begin{pmatrix} 0 & 0 \\ 0 & -\epsilon m(x-1) \end{pmatrix},$$

and

$$D^2 F_\beta(E_5, \beta^*)(V_3, V_3) = \begin{pmatrix} L \\ \frac{2\epsilon m}{a-c} \end{pmatrix},$$

where $L = -\frac{2(x+1)}{a-b-c+d} + \frac{2(a-2b-c+2d)}{(a-b-c+d)^2}$.

Further, we have

$$\begin{cases} W_3^T F_\beta(E_5, \beta^*) = 0, \\ W_3^T DF_\beta(E_5, \beta^*) V_3 = -\epsilon m \frac{c-a}{a-b-c+d} \neq 0, \\ W_3^T D^2 F_\beta(E_5, \beta^*)(V_3, V_3) = \frac{2\epsilon m}{a-c} \neq 0. \end{cases}$$

Hence, when $\beta = \beta^* = \frac{d-b}{a-c}$, a transcritical bifurcation occurs at E_5.

(4) The Jacobian matrix at E_6 has eigenvalues $\lambda_{61} = -\frac{\epsilon m(d-b-\beta a+\beta c-m\beta-m)}{b+c-a-d}$ and $\lambda_{62} = -\frac{(d-b-m)(c-a-m)}{a-b-c+d}$. Let $\lambda_{61} = 0$, that is $d - b - \beta a + \beta c - m\beta - m = 0$, and we get $\beta = \beta^{**} = \frac{d-b-m}{a-c+m}$. At this time, $\lambda_{62} < 0$. The eigenvectors corresponding to the zero eigenvalue for J_6 and J_6^T are

$$V_4 = \begin{pmatrix} -\frac{1}{a-b-c+d} \\ 1 \end{pmatrix}, \quad W_4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then one has

$$F_\beta(E_6, \beta^{**}) = \begin{pmatrix} S_\beta \\ Q_\beta \end{pmatrix} = \begin{pmatrix} 0 \\ -\epsilon r(m-r)(x-1) \end{pmatrix}\Bigg|_{E_6} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and

$$DF_\beta(E_6, \beta^{**}) = \begin{pmatrix} \frac{\partial F_\beta}{\partial x} & \frac{\partial F_\beta}{\partial r} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -\epsilon r(m-r) & \epsilon r(x-1) - \epsilon(m-r)(x-1) \end{pmatrix}\Bigg|_{E_6} = \begin{pmatrix} 0 & 0 \\ 0 & \epsilon m(x-1) \end{pmatrix}.$$

Similarly, the calculation of $D^2 F_\beta(E_6, \beta^{**})$ follows the same method as above. Substituting the values of E_6 and β^{**} yields

$$D^2 F_\beta(E_6, \beta^{**})(V_4, V_4) = \begin{pmatrix} 0 \\ -\frac{2\epsilon m}{a-c+m} \end{pmatrix}.$$

We have

$$\begin{cases} W_4^T F_\beta(E_6,\beta^{**}) = 0, \\ W_4^T DF_\beta(E_6,\beta^{**})V_4 = \epsilon m \frac{c-a-m}{a-b-c+d} \neq 0, \\ W_4^T D^2F_\beta(E_6,\beta^{**})(V_4,V_4) = -\frac{2\epsilon m}{a-c+m} \neq 0. \end{cases}$$

Therefore, when $\beta = \beta^{**} = \frac{d-b-m}{a-c+m}$, a transcritical bifurcation occurs at E_6. □

3. Model and results with time delay

In the real world, games are not instantaneous and involve time delays, such as a payoff time delay and feedback time delay. This section introduces the concept of a payoff time delay, denoted as τ, which represents the time required for players to realize revenue during the game. In other words, a player's income at time t depends on the proportion of players at time $(t-\tau)$. Consequently, the average expected payoffs for cooperation and defection strategies are, respectively,

$$\begin{cases} \pi_C^d(t) = (a + r(t))\, x(t-\tau) + (b + r(t))y(t-\tau), \\ \pi_D^d(t) = cx(t-\tau) + dy(t-\tau). \end{cases} \tag{3.1}$$

Based on the replicator equation, the evolutionary game system with reward feedback and a time delay is described by

$$\begin{cases} \dot{x}(t) = x(t)(1-x(t))((a-b-c+d)x(t-\tau) + b + r(t) - d), \\ \dot{r}(t) = \epsilon r(t)(m - r(t))(\beta - (\beta+1)x(t)). \end{cases} \tag{3.2}$$

Theorem 3.1. *For the time-delay system (3.2),*
(1) when $\tau \in (0, \tau_k)$, system (3.2) is stable at E_7,
(2) when $\tau \in (\tau_k, +\infty)$, system (3.2) is unstable at E_7,
(3) when $\tau = \tau_k$, system (3.2) incurs a Hopf bifurcation at E_7.

Proof. Linearizing system (3.2) at the equilibrium point E_7, we obtain

$$\begin{cases} \dot{x}(t) = m_1 x(t) + m_2(t-\tau) + m_3 r(t), \\ \dot{r}(t) = m_4 x(t) + m_5 r(t), \end{cases} \tag{3.3}$$

where

$$\begin{cases} m_1 = (1-2x_7)((a-b-c+d)x_7 + b - d + r_7), \\ m_2 = x_7(1-x_7)(a-b-c+d), \\ m_3 = x_7(1-x_7), \\ m_4 = -\epsilon r_7(m-r_7)(\beta+1), \\ m_5 = \epsilon(m-2r_7)(\beta-(\beta+1)x_7). \end{cases}$$

From the second equation of the system (3.2), we obtain

$$x(t) = \frac{1}{m_4} r'(t) - \frac{m_5}{m_4} r(t). \tag{3.4}$$

Differentiating the above equation can get

$$x'(t) = \frac{1}{m_4}r''(t) - \frac{m_5}{m_4}r'(t). \tag{3.5}$$

Substituting Eqs (3.4) and (3.5) into the first equation of the system (3.2), one has

$$r''(t) - m_5 r'(t) = m_1 r'(t) - m_1 m_5 r(t) + m_2 r'(t-\tau) - m_2 m_5 r(t-\tau). \tag{3.6}$$

Letting $r = e^{\lambda t}$ and substituting into the above equation, the following characteristic equation can be obtained:

$$\lambda^2 - (m_1 + m_5 + m_2 e^{-\lambda\tau})\lambda + m_2 m_5 e^{-\lambda\tau} + m_1 m_5 - m_3 m_4 = 0. \tag{3.7}$$

Let

$$s_1 = m_1 + m_5,\ \ s_2 = m_1 m_5 - m_3 m_4,\ \ s_3 = m_2 m_5.$$

Then we obtain the simplified form of the characteristic equation:

$$\lambda^2 - (s_1 + m_2 e^{-\lambda\tau})\lambda + s_3 e^{-\lambda\tau} + s_2 = 0. \tag{3.8}$$

Let $\lambda = i\omega$, and then Eq (3.8) become

$$-\omega^2 - s_1\omega i - m_2\omega i e^{-i\omega\tau} + s_3 e^{-i\omega\tau} + s_2 = 0. \tag{3.9}$$

Substituting $e^{-i\omega\tau} = \cos(\omega\tau) - i\sin(\omega\tau)$ into Eq (3.9), one has

$$-\omega^2 - s_1\omega i - m_2\omega i\cos(\omega\tau) - m_2\omega\sin(\omega\tau) + s_3\cos(\omega\tau) - s_3 i\sin(\omega\tau) + s_2 = 0.$$

Then we obtain

$$\begin{cases} s_3\cos(\omega\tau) - m_2\omega\sin(\omega\tau) = \omega^2 - s_2, \\ s_3\sin(\omega\tau) + m_2\omega\cos(\omega\tau) = -s_1\omega. \end{cases} \tag{3.10}$$

Taking the square sum of the two equations in the above system yields

$$\omega^4 + (s_1^2 - 2s_2 - m_2^2)\omega^2 + s_2^2 - s_3^2 = 0. \tag{3.11}$$

Let $\bar{\omega} = \omega^2$, and substituting into the above equation, we obtain

$$\bar{\omega}^2 + (s_1^2 - 2s_2 - m_2^2)\bar{\omega} + s_2^2 - s_3^2 = 0. \tag{3.12}$$

If $s_2^2 - s_3^2 < 0$ is true, then Eq (3.12) has at least one positive root $\bar{\omega}$,

$$\bar{\omega} = \frac{2s_2 - s_1^2 + m_2^2 + \sqrt{(-2s_2 + s_1^2 - m_2^2)^2 - 4(s_2^2 - s_3^2)}}{2}. \tag{3.13}$$

According to the above equation, one has

$$\omega = \sqrt{\frac{2s_2 - s_1^2 + m_2^2 + \sqrt{(-2s_2 + s_1^2 - m_2^2)^2 - 4(s_2^2 - s_3^2)}}{2}}. \tag{3.14}$$

Furthermore, from the system of Eq (3.10), we have

$$(s_3^2 + m_2^2\omega^2)\cos(\omega\tau) = s_3\omega^2 - s_2 s_3 - s_1 m_2\omega^2. \tag{3.15}$$

Then the critical time delay is

$$\tau_k = \frac{1}{\omega}\arccos\left(\frac{s_3\omega^2 - s_2 s_3 - s_1 m_2\omega^2}{s_3^2 + m_2^2\omega^2}\right) + \frac{2k\pi}{\omega}, k = 0, 1, 2, ... \tag{3.16}$$

□

The following lemma proves that τ satisfies the transversal condition of bifurcation.

Lemma 3.1. *If* $s_2^2 - s_3^2 < 0$ *holds, then*

$$\left[\frac{\mathrm{d}Re(\lambda)}{\mathrm{d}\tau}\right]_{\tau=\tau_k} > 0, \quad k = 1, 2, \ldots$$

Proof. According to Eq (3.8), we have

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\tau} = \frac{(\lambda s_3 - \lambda^2 m_2)e^{-\lambda\tau}}{2\lambda - s_1 - (m_2 + s_3\tau - m_2\lambda\tau)e^{-\lambda\tau}}.$$

Then

$$\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1} = \frac{2\lambda - s_1 - m_2 e^{-\lambda\tau}}{(\lambda s_3 - \lambda^2 m_2)e^{-\lambda\tau}} - \frac{\tau}{\lambda}.$$

Substituting $\lambda = i\omega$ into the above equation results in

$$\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1}|_{\lambda=i\omega} = \frac{2i\omega - s_1 - m_2(\cos(\omega\tau) - i\sin(\omega\tau))}{(i\omega s_3 + \omega^2 m_2)(\cos(\omega\tau) - i\sin(\omega\tau))} - \frac{\tau}{\omega i}.$$

Therefore, its real part is

$$\mathrm{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1}|_{\lambda=i\omega} = \frac{2\omega^2 - 2s_2 + s_1^2 - m_2^2}{s_3^2 + \omega^2 m_2^2}.$$

Substituting the value from Eq (3.16) into the above equation leads to

$$\mathrm{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1}_{\tau=\tau_k} = \frac{\sqrt{(-2s_2 + s_1^2 - m_2^2)^2 - 4(s_2^2 - s_3^2)}}{s_3^2 + \omega^2 m_2^2} > 0.$$

Then

$$\mathrm{sign}\left\{\frac{\mathrm{dRe}(\lambda)}{\mathrm{d}\tau}_{\tau=\tau_k}\right\} = \mathrm{sign}\left\{(\tfrac{\mathrm{d}\lambda}{\mathrm{d}\tau})_{\tau=\tau_k}\right\} > 0.$$

□

4. Stability and direction of Hopf bifurcation

Next, we explore various properties of Hopf bifurcation including direction, stability, and period changes [37].

Theorem 4.1. *For system (3.2) with time delay,*
(1) the Hopf bifurcation is supercritical (subcritical) when $\iota > 0$ $(\iota < 0)$,
(2) the bifurcating periodic solution is stable (unstable) if $\chi < 0$ $(\chi > 0)$,
(3) when $\varkappa > 0$ $(\varkappa < 0)$, *the period increases (decreases).*
The values of ι, χ, *and* $\varkappa$ *are given by Eq (4.2) below.*

Proof. Let $\dot{x}(t) = x(t) - x_7$, $\dot{r}(t) = r(t) - r_7$, and $\tau = \tau_0 + \upsilon \in \mathbb{R}$. System (3.2) is rewritten as

$$\dot{Z}(t) = L_\upsilon(Z_t) + S(\upsilon, Z_t), \tag{4.1}$$

where $Z(t) = (x(t), r(t))^T \in \mathbb{R}^2$, $L_\upsilon : C \to \mathbb{R}$, and $S : \mathbb{R} \times C \to \mathbb{R}$ are given by $L_\upsilon(\varphi) = (\tau_0 + \upsilon)J_7\varphi(-1)$, $S(\upsilon, \varphi) = (\tau_0 + \upsilon)(S_1, S_2)^T$, and $\varphi = (\varphi_1, \varphi_2)^T$. S_1 and S_2 are

$$\begin{aligned} S_1 &= g_1\varphi_1(0)\varphi_1(-1) + g_2\varphi_1^2(0)\varphi_1(-1), \\ S_2 &= 0, \end{aligned}$$

where

$$\begin{aligned} g_1 &= (1 - x_7)(a - b - c + d), \\ g_2 &= -(a - b - c + d). \end{aligned}$$

We have a bounded variational function $\kappa(\gamma, \upsilon), (\gamma \in [0, 1])$, according to the Riesz representation theorem, satisfying

$$L_\upsilon(\varphi) = \int_{-1}^{0} \lceil \kappa(\gamma, \upsilon)\varphi(\gamma), \quad \varphi \in C\left([-1, 0], \mathbb{R}^2\right).$$

Next

$$L_\upsilon(\varphi) = -\tau J_7\eta(\gamma + 1),$$

and

$$\eta(\gamma) = \begin{cases} 1, & \gamma \in [-1, 0), \\ 0, & \gamma = 0. \end{cases}$$

Define

$$P(\upsilon)\varphi = \begin{cases} \frac{\mathrm{d}\varphi(\gamma)}{\mathrm{d}\gamma}, & \gamma \in [-1, 0), \\ \int_{-1}^{0} \mathrm{d}\kappa(q, \upsilon)\varphi(q), & \gamma = 0, \end{cases}$$

and

$$Q(\upsilon)\varphi = \begin{cases} 0, & \gamma \in [-1, 0), \\ S(\upsilon, \varphi), & \gamma = 0. \end{cases}$$

System (4.1) is rewritten as

$$\dot{Z}(t) = P(\upsilon)Z_t + Q(\upsilon)Z_t,$$

where $Z_t(\gamma) = Z(t+\gamma) \in C, \gamma \in [-1, 0]$. For $\phi \in C([0,1], R^{2*})$, we get

$$P^*\phi(q) = -\begin{cases} \frac{d\phi(q)}{d\gamma}, & q \in (0,1], \\ \int_{-1}^{0} d\kappa^T(t,0)\phi(-1), & q = 0, \end{cases}$$

and

$$\langle \phi(\gamma), \varphi(\gamma) \rangle = \bar{\phi}(0)\varphi(0) - \int_{-1}^{0}\int_{0}^{\gamma} \bar{\phi}(\delta - \gamma) d\kappa(\gamma)\varphi(\delta) d\delta,$$

where $\kappa(\gamma) = \kappa(\gamma, 0)$. $P = P(0)$ and P^* are adjoint operators. Since for eigenvalue $i\omega_0\tau_0$, the eigenvector of $P(0)$ is $v(\gamma) = (1, v_7)^T e^{i\omega_0\tau_0\gamma}$ and the eigenvector of P^* is $v^*(q) = U(1, v_7^*)e^{i\omega_0\tau_0 q}$ when the eigenvalue is $-i\omega_0\tau_0$, we have $\langle v^*(q), v(p) \rangle = 1$ and

$$\begin{aligned} v_7 &= \frac{m_4}{i\omega_0 - m_5}, \\ v_7^* &= -\frac{m_3}{i\omega_0 + m_5}, \\ U &= [1 + \bar{v}_7 v_7^* + m_2\tau_0 e^{i\omega_0\tau_0}]^{-1}. \end{aligned}$$

One has

$$\begin{aligned} g_{20} &= 2\overline{U}\tau_0\left(g_1 e^{-i\omega_0\tau_0}\right), \\ g_{11} &= \overline{U}\tau_0\left(g_1\left(e^{i\omega_0\tau_0} + e^{-i\omega_0\tau_0}\right)\right), \\ g_{02} &= 2\overline{U}\tau_0\left(g_1 e^{i\omega_0\tau_0}\right), \\ g_{21} &= 2\overline{U}\tau_0\Bigg[g_1\left(W_{11}^{(1)}(-1) + \frac{W_{20}^{(1)}(-1)}{2}e^{-2i\omega_0\tau_0} + \frac{W_{20}^{(1)}(0)}{2}e^{i\omega_0\tau_0} + W_{11}^{(1)}(0)e^{-i\omega_0\tau_0}\right) \\ &\quad + g_2\left(e^{i\omega_0\tau_0} + 2e^{-i\omega_0\tau_0}\right)\Bigg], \end{aligned}$$

and

$$\begin{aligned} W_{20}(\psi) &= \frac{ig_{20}p(0)e^{i\omega_0\tau_0\psi}}{\omega_0\tau_0} + \frac{i\overline{g}_{02}\overline{p}(0)e^{-i\omega_0\tau_0\psi}}{3\omega_0\tau_0} + \Omega_1 e^{2i\omega_0\tau_0\psi}, \\ W_{11}(\psi) &= \frac{-ig_{11}p(0)e^{i\omega_0\tau_0\psi}}{\omega_0\tau_0} + \frac{i\overline{g}_{11}\overline{p}(0)e^{-i\omega_0\tau_0\psi}}{\omega_0\tau_0} + \Omega_2. \end{aligned}$$

The values of Ω_1 and Ω_2 are given by

$$\begin{aligned} \Omega_1 &= 2\begin{pmatrix} 2i\omega_0 - m_1 - m_3 e^{-2i\omega_0\tau_0} & -m_3 \\ -m_4 & 2i\omega_0 - m_5 \end{pmatrix}^{-1}\begin{pmatrix} g_1 e^{-i\omega_0\tau_0} \\ 0 \end{pmatrix}, \\ \Omega_2 &= -\begin{pmatrix} m_1 + m_2 & m_3 \\ m_4 & m_5 \end{pmatrix}^{-1}\begin{pmatrix} g_1(e^{i\omega_0\tau_0} + e^{-i\omega_0\tau_0}) \\ 0 \end{pmatrix}. \end{aligned}$$

Therefore, we define

$$
\begin{aligned}
N_1(0) &= \left(g_{11}g_{20} - 2\left|g_{11}\right|^2 - \frac{\left|g_{02}\right|^2}{3}\right)\frac{i}{2\omega_0\tau_0} + \frac{1}{2}g_{21},\\
\iota &= -\frac{\mathrm{Re}[N_1(0)]}{\mathrm{Re}[\lambda'(\tau_0)]},\\
\chi &= 2\mathrm{Re}[N_1(0)],\\
\varkappa &= -\frac{\mathrm{Im}[N_1(0)] + \iota \mathrm{Im}[\lambda'(\tau_0)]}{\omega_0\tau_0}.
\end{aligned}
\tag{4.2}
$$

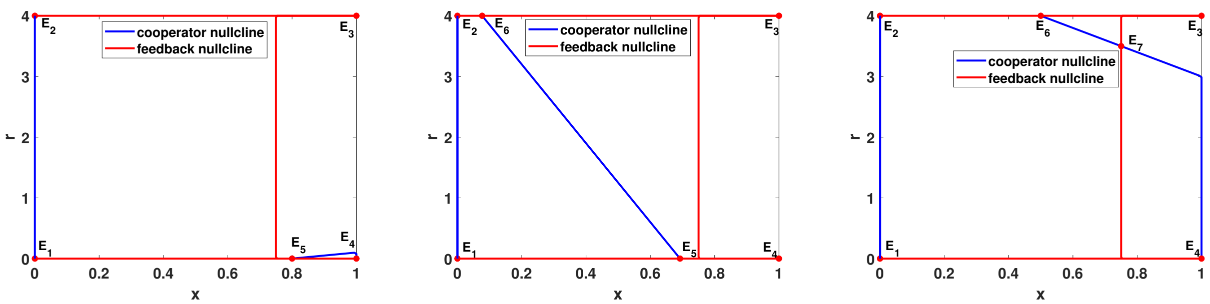
The proof of this theorem is complete. □

5. Numerical simulation and analysis

This section uses numerical simulations to confirm the previous theoretical derivations, and analyzes its biological significance.

5.1. Non-delay system

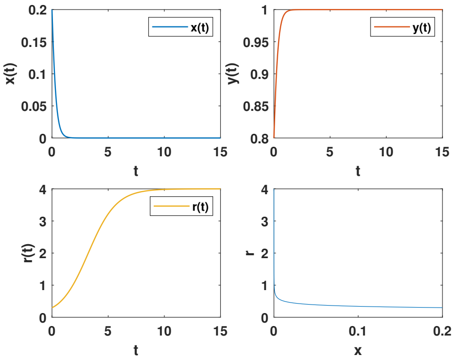
Figure 1 exhibits the equilibrium points of system (2.7) under varying parameter settings. The equilibrium points E_1, E_2, E_3, and E_4 consistently exist across all parameter configurations. Specifically, the parameters depicted in Figure 1(a) satisfy the conditions necessary for the existence of E_5. Figure 1(b) demonstrates the presence of both E_5 and E_6, while Figure 1(c) reveals the existence of E_6 and E_7.



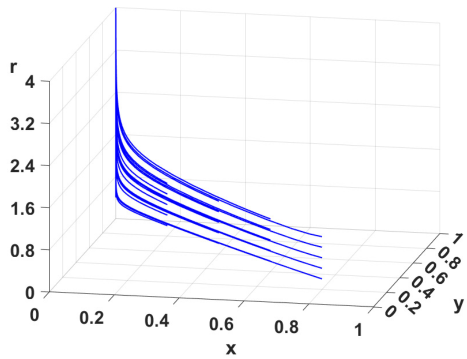
(a) Existence of E_5 (b) Existence of E_5 and E_6 (c) Existence of E_6 and E_7

Figure 1. Existence of equilibrium points. Parameters: (a) $a = 0.9$, $b = 1.2$, $c = 1$, $d = 0.8$, $\epsilon = 0.1$, $\beta = 3$, $m = 4$; (b) $a = 3$, $b = 0.5$, $c = 1$, $d = 5$, $\epsilon = 0.1$, $\beta = 3$, $m = 4$; (c) $a = 2$, $b = 1$, $c = 5$, $d = 6$, $\epsilon = 0.1$, $\beta = 3$, $m = 4$.

Subsequent figures, Figures 2–6, depict the stabilization of system (2.7) at the equilibria E_2, E_3, E_4, E_5, E_6, and E_7, respectively. Irrespective of the initial conditions, the system consistently converges to a stable equilibrium point, in accordance with Theorem 2.1. The analysis reveals that the stable values of cooperation strategy and reward intensity vary with different parameter settings. By tuning these parameters, one can achieve the desired proportions of cooperation strategy and reward intensity. It shows five different situations: defectors and reward intensity dominate, cooperators exist independently and stably, cooperators and defectors coexist, and reward intensity reaches the maximum, and defectors, cooperators, and reward intensity coexist. When the reward intensity reaches the maximum, there is a phenomenon that no player chooses to cooperate, as shown in Figure 2. Notably, when the reward intensity is minimized, the cooperation strategy can potentially reach its maximum, as illustrated in Figure 3. Similarly, Figure 6 shows that cooperators, defectors, and reward intensity can reach a stable coexistence state. This is conducive to maintaining population diversity.

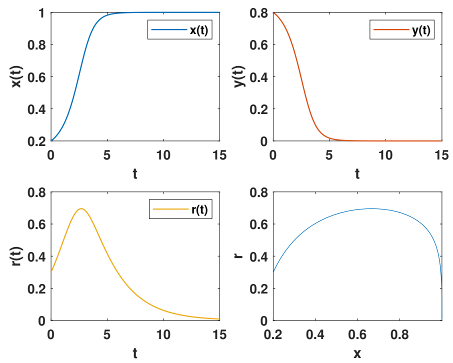


(a) Time series diagrams and phase diagram

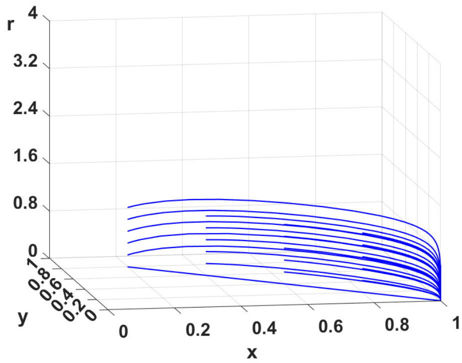


(b) Three-dimensional diagram

Figure 2. The stable equilibrium E_2 for $a = 0.8$, $b = 0.6$, $c = 2$, $d = 5$, $\epsilon = 0.1$, $\beta = 2$, and $m = 4$.

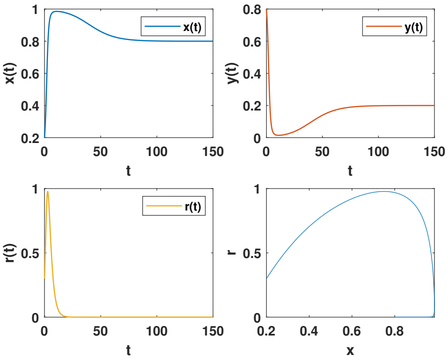


(a) Time series diagrams and phase diagram

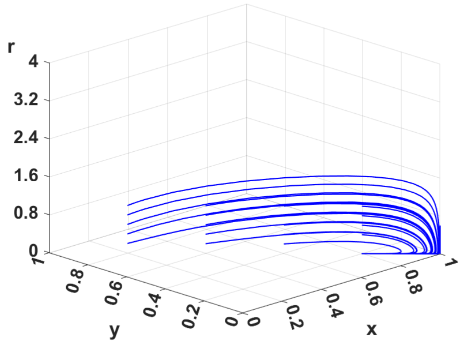


(b) Three-dimensional diagram

Figure 3. The stable equilibrium E_4 for $a = 2$, $b = 0.6$, $c = 1$, $d = 0.8$, $\epsilon = 0.1$, $\beta = 2$, and $m = 4$.

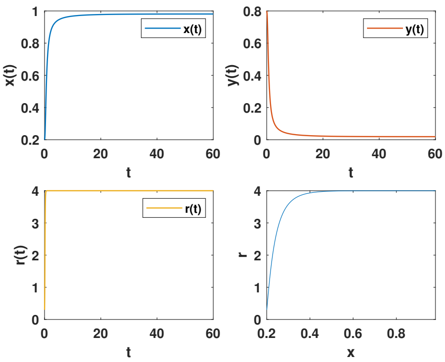


(a) Time series diagrams and phase diagram

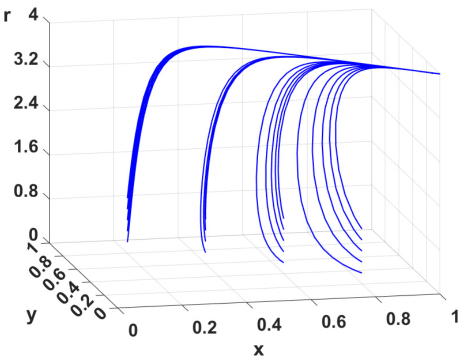


(b) Three-dimensional diagram

Figure 4. The stable equilibrium E_5 with the same parameters as those in Figure 1(a).

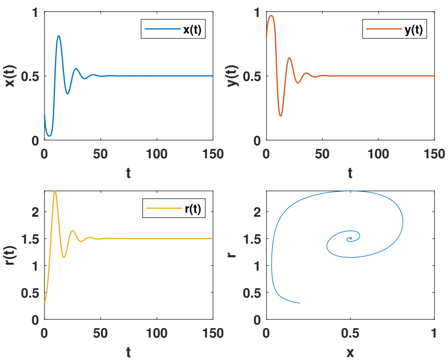


(a) Time series diagrams and phase diagram

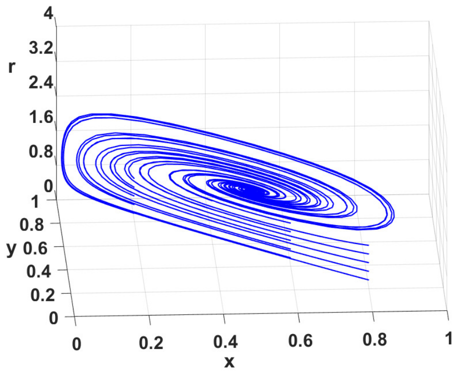


(b) Three-dimensional diagram

Figure 5. The stable equilibrium E_6 for $a = 0.9$, $b = 2$, $c = 5$, $d = 0.8$, $\epsilon = 0.1$, $\beta = 53$, and $m = 4$.



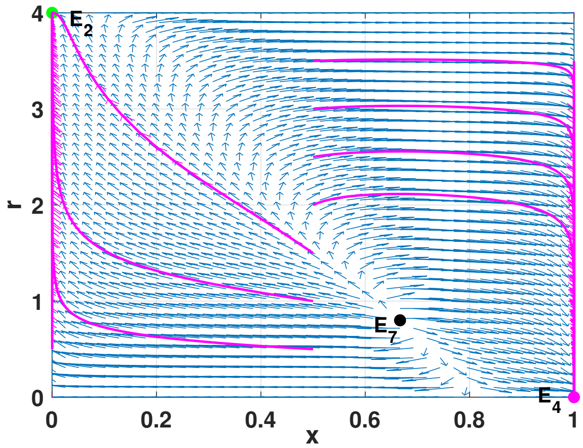
(a) Time series diagrams and phase diagram



(b) Three-dimensional diagram

Figure 6. The stability of E_7 for $a = 2$, $b = 1$, $c = 4$, $d = 2$, $\epsilon = 0.1$, $\beta = 1$, and $m = 4$.

Figure 7 illustrates the bistable phenomenon of equilibrium points E_2 and E_4. We observe that the final stable equilibrium point of system (2.7) is influenced by the initial states. Different initial values result in the system converging to various equilibrium points. At this stage, the maximum value of the cooperator is influenced by the initial condition of the number of collaborators and reward intensity. This indicates that even when the reward intensity reaches its maximum, the proportion of the cooperation strategy can still be zero due to the different initial values.

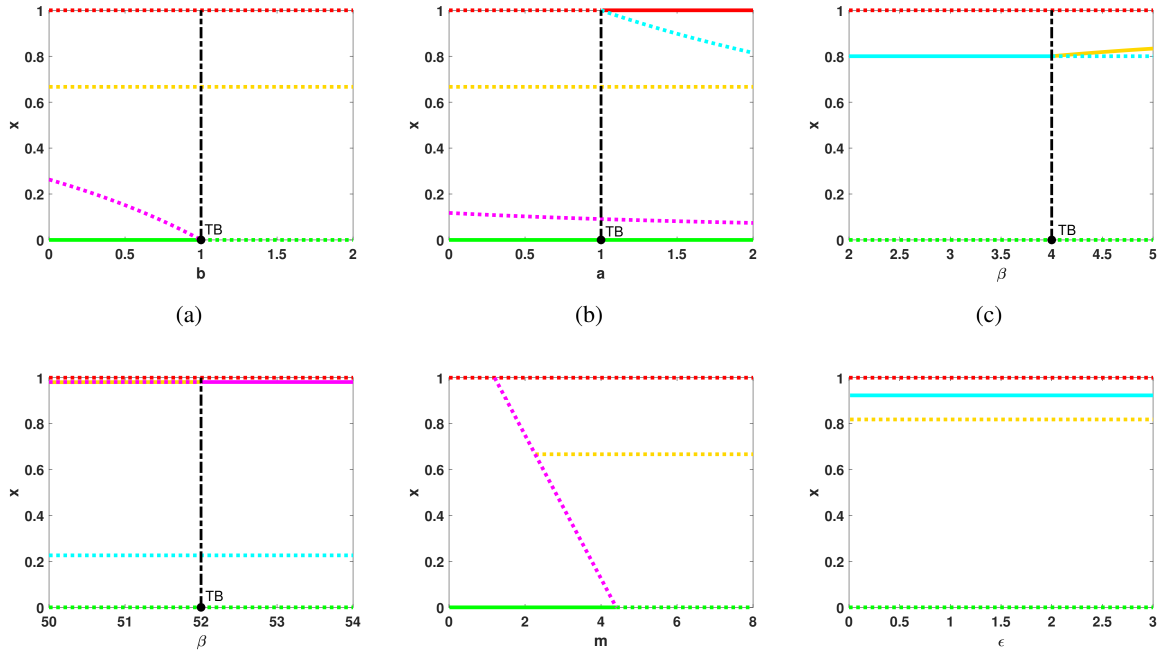


(a) The stability of E_2 and E_4

Figure 7. The bistable diagram of E_2 and E_4 for $a = 2$, $b = 0.6$, $c = 1$, $d = 5$, $\epsilon = 0.1$, $\beta = 2$, and $m = 4$.

Theorem 2.3 is illustrated in Figure 8. Figure 8 provides a clear depiction of the state of each equilibrium point. In Figure 8(a), the equilibrium point E_2 transitions from stable to unstable. Figure 8(b) shows that the equilibrium point E_4 initially exhibits instability but becomes stable subsequently; additionally, a bistable phenomenon involving equilibrium points E_2 and E_4 is observed following the transcritical bifurcation. Figure 8(c) reveals that the previously stable equilibrium E_5 becomes unstable, giving rise to a new stable equilibrium E_6. Finally, (d) demonstrates that the stable equilibrium E_6 transitions to instability, while a new stable internal equilibrium E_7 emerges. These observations highlight that variations in parameters are intricately linked to the system's final stable state and influence the dynamics of cooperative strategies.

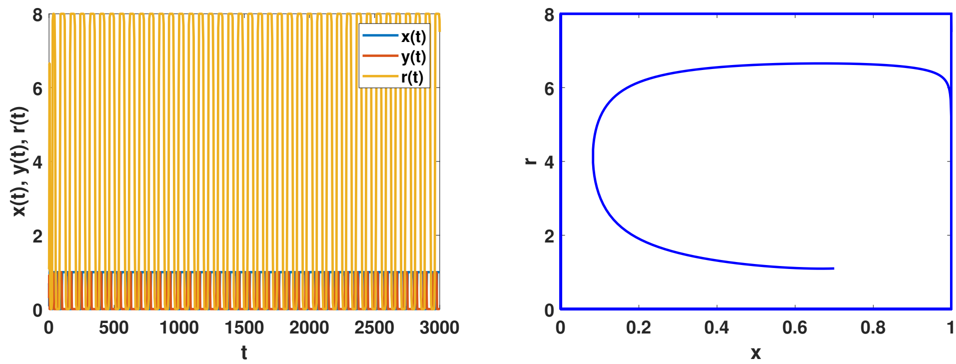
Fix the parameter values in Figure 2 and change the maximum value of the reward intensity, as shown in Figure 8(e). At this time, all the equilibrium points in the system (2.7) are no longer stable. The reward intensity oscillates between the highest and lowest intensity, while the players oscillate between complete cooperation and complete defection, which forms an evolutionary oscillation dynamic as shown in Figure 9. The influence of different relative speeds, ϵ, on the system is shown in Figure 8(f). When ϵ changes from 0 to 3, the equilibrium points of the system are not affected, and the original stability is maintained. This suggests that, in this case, changes in relative speed do not impact the player's choice of strategy or the intensity of rewards.



(a) (b) (c)

(d) (e) (f)

Figure 8. Parameter variation diagram. The parameters of (a) and (e) are the same as those of Figure 2, (b) and (c) are the same as those of Figures 3 and 4, and the parameters of (d) and (f) are the same as those of Figure 5. E_2, E_4, E_5, E_6, and E_7 are represented by the green, red, cyan, magenta, and yellow lines, respectively. Solid lines show stable equilibria, whereas dotted lines denote unstable equilibrium points.



(a) (b)

Figure 9. Time series diagram and phase diagram for $a = 2$, $b = 0.6$, $c = 1$, $d = 0.8$, $\epsilon = 0.1$, $\beta = 2$, and $m = 8$.

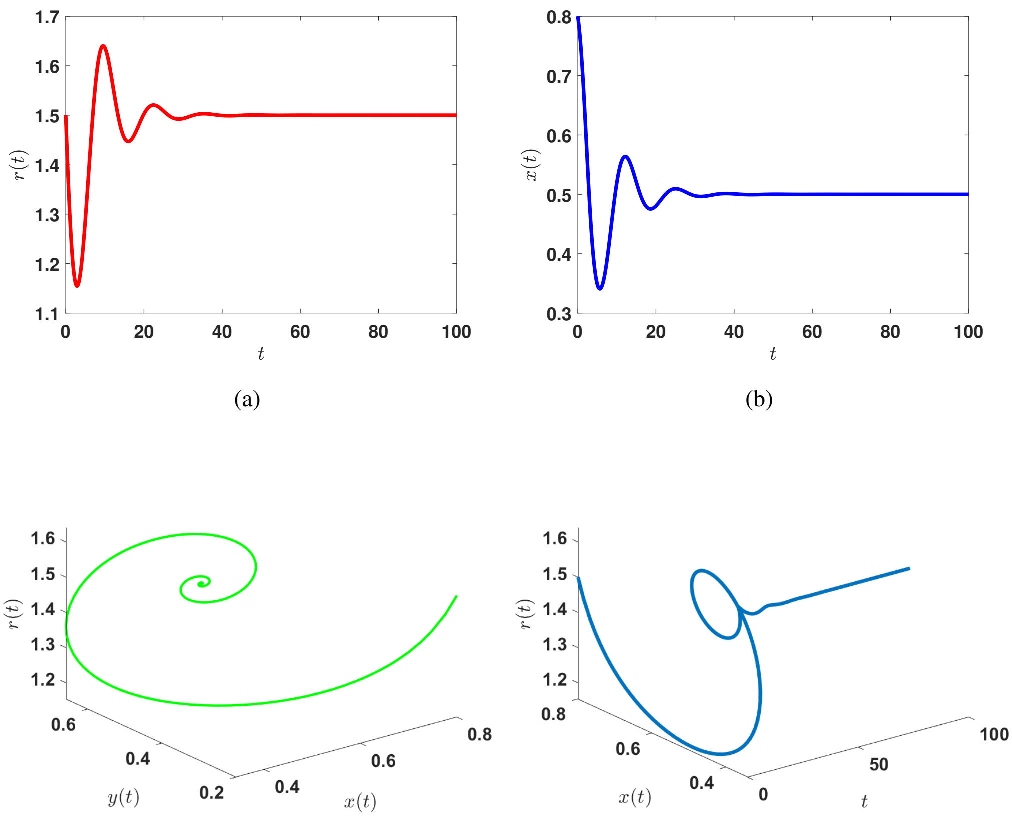
5.2. Time-delay system

We continue to study the dynamic behavior of the time-delay system (3.2) by selecting the parameters in Figure 6. Then the internal equilibrium point is $E_7\,(0.5, 1.5)$, and the following results can be obtained through calculation.

$$\omega_0 = 0.5757,\ \tau_0 = 2.7285,$$
$$N_1(0) = -0.3076 - 1.9775i,\ \frac{d(\mathrm{Re}\,\lambda)}{d\tau} = 0.0773 > 0,$$
$$\iota = 3.9812,\ \chi = -0.6152,\ \varkappa = 1.3442.$$

Therefore, Hopf bifurcation is supercritical, and the periodic solution of the bifurcation is stable and the period increases.

Regarding the system (3.2), we first examine the behavior of the system under a small time delay, as illustrated in Figure 10. When the time delay is minimal, the ratio of reward intensity to cooperation strategy stabilizes following a minor perturbation. At this stage, the time-delay system remains stable at E_7. Additionally, as evidenced by Figure 10(c), the system ultimately approaches a stable state.



(a) (b)

(c) (d)

Figure 10. E_7 is stable when $\tau = 1$. The parameters of the following figures are the same as those of Figure 6.

An oscillatory phenomenon emerges as the time delay gradually increases as depicted in Figure 11. At this point, the variables $x(t)$ and $r(t)$ begin to oscillate periodically rather than remaining stable at a single value, leading to the instability of the previously stable internal equilibrium point E_7. Subsequently, a stable limit cycle is observed in Figure 11(c). Simultaneously, the increase in time delay causes the stable strategy to begin oscillating, leading to a Hopf bifurcation at the critical time delay. After the Hopf bifurcation, the proportion of the cooperative strategy and reward intensity oscillate between the blue and the red lines, as illustrated in Figure 12. This shows that the time delay induces the switching of different stable states of the system.

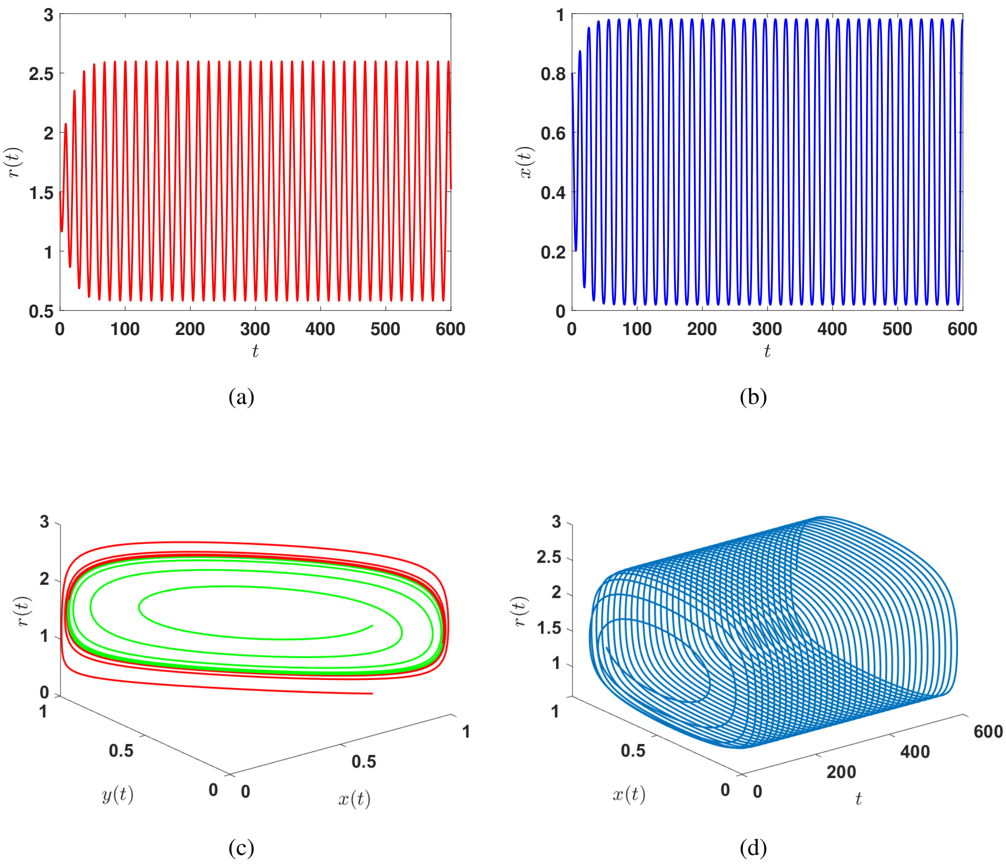


Figure 11. E_7 is unstable and there is stable periodic oscillation when $\tau = 4$.

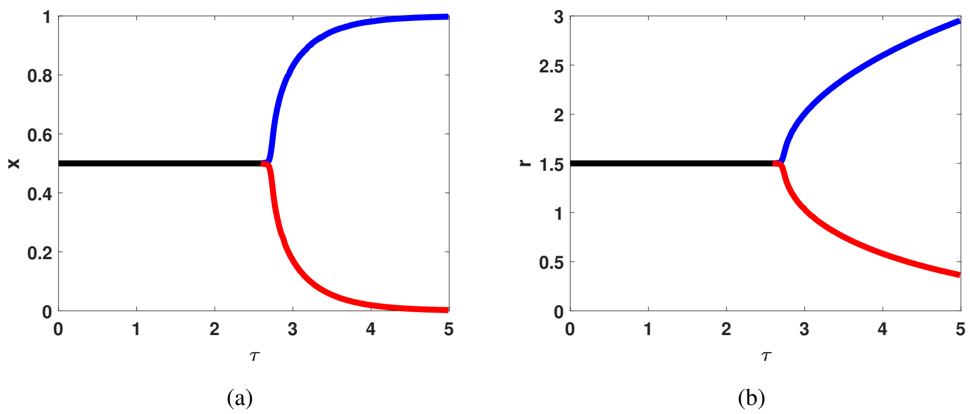


Figure 12. Bifurcation diagram of x and r with respect to τ when $\tau_0 = 2.7285$.

6. Conclusions

Building upon the classical evolutionary game theory framework, this paper introduces the concepts of reward feedback and time delay. To facilitate a more nuanced application of reward feedback for enhancing cooperation, the model posits that the proportion of the defection strategy promotes a modest reward for cooperators, while a higher proportion of the cooperative strategy suppresses the reward intensity. Under certain conditions, even when the reward intensity reaches its maximum value, the cooperation strategy may still yield a minimum value of zero. Additionally, in systems without a time delay, bistability is observed, and the attainment of the maximum cooperation strategy is contingent upon the initial values of the cooperative strategies and reward intensity. Specifically, variations in initial conditions lead to differing final states of stability for the cooperation strategy. In the system incorporating a time delay, the previously stable equilibrium point becomes unstable as the time delay increases. At the critical time delay, Hopf bifurcation occurs, causing oscillations in both strategies and the reward intensity rather than maintaining a stable constant value. The delayed system yields a series of delayed dynamical behaviors including Hopf bifurcation, period, stability, and direction of bifurcation. This shows that the time delay induces the switching of different stable states of the system.

With the advancement of feedback evolutionary game theory, exploring various feedback mechanisms and enhancing the evolutionary game model remain key areas of future investigation [38, 39]. Cheng et al. have examined the evolutionary game system within spatial and temporal contexts, uncovering numerous interesting phenomena [40]. However, time delay emerges as a critical factor in spatiotemporal games, warranting further research in this area [41, 42]. This aspect represents a significant direction for our future studies.

Author contributions

Writing-original draft preparation, Haowen Gong and Huijun Xiang; writing-review and editing, Haowen Gong and Yifei Wang; visualization, Haowen Gong and Yifei Wang; supervision, Huaijin Gao and Xinzhu Meng; project administration, Xinzhu Meng; and funding acquisition, Xinzhu Meng. All authors have read and agreed to the published version of the manuscript.

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Conflict of interest

The authors declare no conflicts of interest.

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