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Research article

Selection of artificial intelligence provider via multi-attribute decision-making technique under the model of complex intuitionistic fuzzy rough sets

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Abstract: Choosing an optimal artificial intelligence (AI) provider involves multiple factors, including scalability, cost, performance, and dependability. To ensure that decisions align with organizational objectives, multi-attribute decision-making (MADM) approaches aid in the systematic evaluation and comparison of AI vendors. Therefore, in this article, we propose a MADM technique based on the framework of the complex intuitionistic fuzzy rough model. This approach effectively manages the complex truth grade and complex false grade along with lower and upper approximation. Furthermore, we introduced aggregation operators based on Dombi t-norm and t-conorm, including complex intuitionistic fuzzy rough (CIFR) Dombi weighted averaging (CIFRDWA), CIFR Dombi ordered weighted averaging (CIFRDOWA), CIFR Dombi weighted geometric (CIFRDWG), and CIFR Dombi ordered weighted geometric (CIFRDOWG) operators, which were integrated into our MADM technique. We then demonstrated the application of this technique in a case study on AI provider selection. To highlight its advantages, we compared our proposed method with other approaches, showing its superiority in handling complex decision-making scenarios.

Keywords: artificial intelligence; complex intuitionistic fuzzy rough set; Dombi t-norm and t-conorm; MADM technique

Mathematics Subject Classification: 03E52, 03E72, 94D05

1. Introduction

Artificial intelligence (AI) providers, also known as AI suppliers or AI vendors, are companies that offer a wide range of AI technologies, services, and solutions to help businesses and other users adopt and leverage AI for a variety of purposes. These vendors provide cloud-based AI services, allowing companies to access AI tools and infrastructure from industry leaders such as Amazon Web Services, Google Cloud AI, and Microsoft Azure, without the need for in-house development. Amazon web services offer a suite of AI services such as Amazon SageMaker (a fully managed service that enables the developers to build, train and deploy machine learning models) and Amazon Lex (a service that enables developers to build conversational interfaces using voice and text). Google Cloud AI provides AI and machine learning services like Google Automated Machine Learning, TensorFlow, and natural language tools. Microsoft Azure AI offers AI tools like Azure (machine learning, Cognitive Services (for speech, vision, decision making and language) and Bot services). AI vendors also developed customized AI platforms, AI software, and tools for training, building, and deploying AI models, such as DataRobot (specializes in automated machine learning and helps businesses build and deploy machine learning models quickly), H2O.ai (an open source AI platform that provides machine learning and deep learning capabilities for predictive analysis), C3.ai (provides and AI and internet of things platform for developing and deploying large-scale AI applications, especially in industries like energy, manufacturing, and utilities), and Palantir (it offers data integration and AI analytics platforms that is Palantir Foundry and Gotham for organizations to manage and analyze large datasets). In addition, they offer consulting services to guide businesses in creating effective AI strategies and managing AI implementation, with firms like Accenture (it provides AI consulting, data analytics, and implementation services across industries, including AI-driven automation and cognitive services) and Deloitte AI (offers consulting services to develop AI strategies, AI model development, and intelligent automation for businesses) leading in this area. AI vendors supply specialized products and solutions, including AI-powered automation, predictive analytics, and customer service tools from companies like UiPath (specializes in robotic process automation (RPA) and AI-powered automation tools for businesses to streamline processes), NVIDIA (offers GPUs and AI software for deep learning, machine learning, and AI-powered applications in industries like autonomous vehicles, healthcare, and gaming), and Salesforce Einstein (it provides AI-powered customer relationship management tools to help businesses automate sales, service, and marketing tasks). Some vendors provide industry-oriented solutions, for example, Zebra Medical Vision focuses on artificial intelligence in the healthcare industry and provides radiologists with AI tools for quicker and more accurate diagnosis or Ayasdi develops artificial intelligence services for the financial services industry and provides tools for 'see-through' of money mules and frauds. With advancements in AI hardware from companies like NVIDIA and Intel, many vendors ensure that AI models can operate efficiently at scale. Overall, AI vendors enable businesses to automate processes, improve decision-making, innovate, and stay competitive by offering cutting-edge AI solutions tailored to their needs.

Decision-making is a dynamic and complex process. Daily, every person or organization faces different problems. To solve those problems, they have to make a decision. To make a good and sustainable decision, a decision maker has to study all the available data and information with great care and preparation. One of the most important techniques to deal with uncertain and imprecise data is the MADM technique, in which a professional decision maker or specialist provides the input assessments for each alternative to arrive at the desirable answer. However, expressing the assessment value as a precise number of an attribute can be challenging. To address this problem, Zadeh [1] first proposed the fuzzy set (FS) theory and since this theory has a clear explanation of uncertain information in MADM problems, the MADM is most discussed in FS theory because its influence is seen in almost all scientific disciplines. Moreover, Pawlak [2] proposed a rough set (RS) theory to cope with the uncertainty and ambiguity in data processing and knowledge representation. It means that within this framework it is possible to analyze sketchy data without using the so-called fuzzy logic or probability theory. Rough set theory partitions a set over the basis of relative levels of similarity called an equivalence relation or vague indiscernibility. Rough sets were later generalized into fuzzy rough sets (FRSs) by Dubois and Prade [3] who replaced the binary relations with the fuzzy relations. Instead of this the integration used above employs a notion of truth grades of FS within the context of RS and is more sound. After that, Cornelis proposed the concept of intuitionistic fuzzy rough set (IFRS) by combining intuitionistic fuzzy set (IFS) with rough sets.

1.1. Motivation and contribution

AI solution partners are key players within numerous organizations with the purpose of assisting organizations to enhance performance and innovate using AI solutions across multiple fields. They provide specific solutions where existing applications of Artificial Intelligence are not sufficient to address problems in organizations; they assist organizations with AI strategies; and they ensure the data is correct to ensure proper training of the AI model. AI as a Service is a model that implies that organizations can utilize advanced AI tools, without significant capital investments. Vendors also automate working processes, help in the implementation and support of AI solutions, and obey ethical practices about data protection and equal opportunities. Moreover, they offer training to transition new and existing employees to become more proficient in the application and deployment of AI within organizations to enhance their organizational AI talent and to also develop solutions that meet particular sector demands. As AI vendors, this means that constant investment is made in research and development to ensure that businesses are always updated with the latest technology advancements hence the feature of flexibility and scalability as the business expands. In other words, these partnerships facilitate efficient, affordable and innovative structures in business today in the context of AI. Consequently, selecting the right AI provider transforms into a necessity for various organizations that would like to improve their processes and decision-making frameworks within the current rapidly changing technology landscape. The decision-making process of choosing an AI service provider is a multi-attribute decision making process and the criteria used include cost, experience, reliability and scalability. Moreover, from the background study, it also emerges that the idea of complex intuitionistic fuzzy rough set (CIFRS) stands in a more commanding position than all other structures such as FS, IFS, FRS, and IFRS. The main feature of this theory is that it can process fuzzy information in the form of lower approximation and upper approximation all other concepts such as FS, IFS, FRS, and IFRS cannot process second dimension information in the form of lower and upper approximation. So, when employing MADM techniques, particularly with CIFRSs, the motivation lies in the ability to handle the inherent uncertainty, vagueness, and imprecision in such evaluations. Using CIFRSs offers different advantages given as follows:

- Better handling of uncertainty and vagueness in AI provider evaluations.
- Flexible, nuanced decision-making through complex intuitionistic truth grade and complex intuitionistic false grade.
- Reduced information loss and enhanced accuracy in AI provider ranking.
- Support for complex, multi-attribute evaluations in scalable and adaptive ways.

This motivates current research and causes CIFRS to be a powerful tool for organizations aiming to choose the most suitable AI provider. Therefore, this manuscript contains a technique of MADM under the model of CIFRS. The CIFRS is a suitable technique to manage the two-dimensional information of truth grade and false grade along with the lower and UA. Dombi t-norm (DT-N) and Dombi t-conorms (DT-CN) have been used in many developments and are useful structures. Therefore, in this manuscript, we propose the elementary operating conditions (algebraic sum, algebraic product) based on DT-N and DT-CN, and these new operational rules are used to aggregate the averaging and geometric operators. Moreover, we settle the idea of CIFRDWA, CIFRDOWA, CIFRDWG, and CIFRDOWG operators and discuss their properties using basic operations. By utilizing the CIRSCIFRS rough numbers (CIFRNs), a model is constructed using these AOs to evaluate the MADM process. In Figure 1, we establish the flowchart of the introduced work.

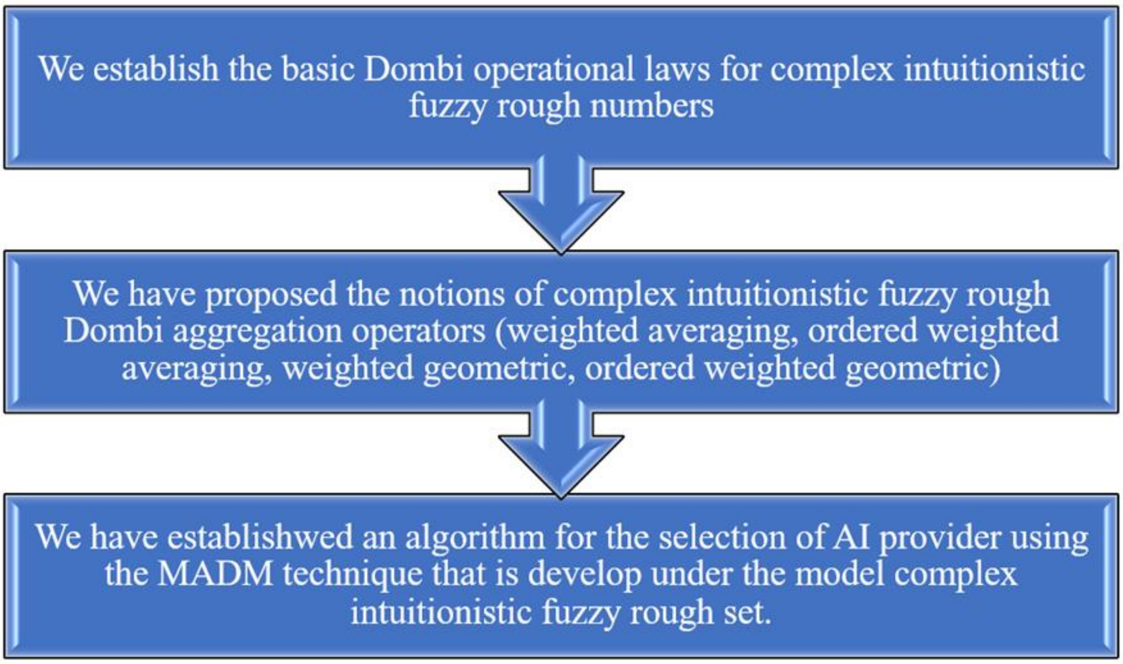

Figure 1. Flowchart of the introduced work.

1.2. Study framework

The rest of the manuscript is sectioned as following: In Section 2, we discuss the background study related to FS theory and RS theory. In Section 3, we address some basic notions of IFS, complex fuzzy set (CFS), FRS, IFRS, and CIFRS and discuss their operational laws. Dombi aggregation operators (AOs) based on CIFR sets and their properties are developed in Section 4. In Section 5, we develop a MADM method under the investigated operators and analyze a case study “Selection of AI provider”. In Section 6, we compare our work with some existing work to reveal the supremacy and advantages. Section 7 of this manuscript contains a conclusion.

2. Literature review

A business or organization that specializes in providing AI solutions, services, or goods to individuals, businesses or other entities is referred to as an AI provider or AI vendor. AI is now an indispensable tool for contemporary enterprises, and AI providers are a significant contributor to its use. These firms provide additionally numerous AI technology, service lines and solution packages, which include various requirements of companies in different industries. These vendors—who provide such applications as computer vision, natural language processing, machine learning and more—are critical to the development, deployment and ongoing maintenance of AI technologies. The impact of AI innovation in a variety of sectors is the rationale for why AI providers are essential. These companies create technology by creating new algorithms, frameworks, and tools to help firms use AI to achieve greater efficiency, productivity, and creativity in carrying out their activities. AI models, APIs, and platforms can be bought off the shelf by the providers; this means that the incorporation of AI features into a solution can be done with little or no understanding of machine learning. Bizzo et al. [4] used AI and clinical decision support for radiologists and patient care. Yu et al. [5] built on this concept of an AI service provider to deliver bundled services to customers. Khaleel et al. [6] proposed workflow scheduling schemes with AI modeling for enhancing dependability and controlling interruption in cloud computing.

In addition, by making these technologies available to a more extensive pool of users, the providers of AI make AI more accessible. AI providers are offering their solutions to researchers and developers as well as to businesses of various sizes to enhance products, optimize processes, and gain valuable information from data. This accessibility helps promote the AI ecosystem’s inclusiveness and hence the uptake. The projects between AI providers and consumers also develop long-term improvement and adaptation to fluctuating demands as well. AI suppliers are critical to ensuring that innovation stays on top of the industry as AI technologies evolve since this ensures that the products supplied to the market are relevant and effective in fulfilling the changing requirements of the market. The goals of selecting an AI supplier are price, performance, reliability and scalability. Consequently, it becomes a MADM issue because decision-makers have to evaluate and assign priority levels to the providers based on these various factors. To enable a logical and well-informed decision to select the best AI provider for a particular situation, MADM approaches assist in quantifying and analyzing the intricate interactions between these features. Wu et al. [7] explain the AI techniques in data science and develop linguistic representations in decision making. Dukyil [8] developed the AI and MCDM approach for a cost-effective Radio Frequency identification tracking management system. Hu et al. [9] developed the governance of AI applications in a business audit via a fusion fuzzy multiple rule-based DM model. Wang et al. [10] give the evaluation of ecological governance in the Yellow River basin based on the Uninorm combination weight and the MULTIMOORA-Borda method.

2.1. Brief review of fuzzy set theory

FS theory is an extension of the crisp set theory. Crisp set theory cannot handle the unclear and vague data but FS can deal with such type of data. According to crisp set theory, if an element is a part of a set then its value is regarded as 1, or does not belong to a set then its value is regarded as 0. FS described the vagueness of an element by a truth grade with a value from the set [0,1]. If an element has a value of 1, it fully belongs to a set, if it has a value of 0, it does not belong to the set. The grading system uses a value between 0 and 1 to indicate how much an element is part of a set. Simply, FS enables elements to have a truth grade that ranges from 0 to 1. The crisp and fuzzy logics are displayed in Figure 2.

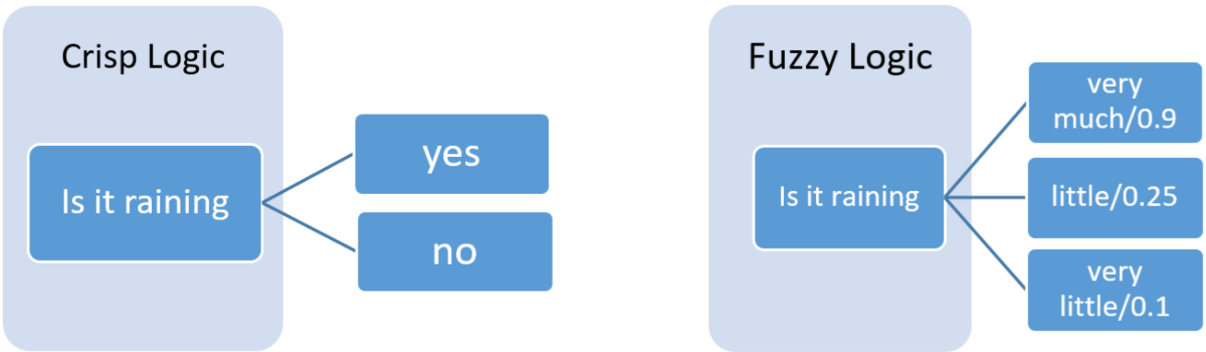

Figure 2. Crisp and fuzzy logics.

In Figure 2, the difference between crisp logic and fuzzy logic is depicted in a weather-related example. On the left side, crisp logic makes binary decisions such as “Is it raining” is answered in a strict yes or no format. There are only two possible outcomes either it is raining, or it is not. On the right side, fuzzy logic is a more nuanced system. Instead of simple yes and no, it assigns degrees of truth to the conditions, such as “very much (0.9) close to true”, “little (0.25) partially true” or “very little (0.1) close to false”. This representation shows how fuzzy models are more realistic than crisp models, which is binary in nature.

FSs provide an efficient framework for modeling and dealing with situations when knowledge is incomplete, unclear, or uncertain. FS theory has many applications in different sectors, including the aerospace, medical, automobile, and energy sectors. Zhuang et al. [11] optimize a sustainable renewable energy portfolio using a multi-tolerance fuzzy programming approach. Wen et al. [12] give an application of fuzzy multi-criteria decision-making methods in civil engineering. Zhang et al. [13] developed a hybrid MADM method for renewable energy portfolio optimization with public participation under uncertainty. Hocine et al. [14] proposed a weighted-additive fuzzy multi-choice goal programming for supporting renewable energy site selection decisions. Shen et al. [15] proposed an outranking approach for multi-attribute group decision-making with interval valued hesitant fuzzy information. Zhuang et al. [16] developed a method for the selection of senior centres using intuitionistic fuzzy MADM. Javed et al. [17] developed a technique of multi-attribute group decision-making with T-spherical fuzzy Dombi power Heronian mean based AOs. Kumar et al. [18] proposed an efficient approach for solving type-2 intuitionistic fuzzy solid transportation problems with their equivalent crisp solid transportation problems. Zhuang et al. [19] propose a MEAN-R decision support system integrating MEAN architecture and R for efficient group decision-making, applied to selecting senior co-living centers. Ye [20] developed multi-attribute group decision making methods with unknown weights in intuitionistic fuzzy settings. Beccali et al. [21] introduced the multi-criteria analysis techniques related to FS methodology and established a decision-making approach in energy planning. Bigaud et al. [22] provide a fuzzy hybrid AI system for the selection of a third-party operations and maintenance provider. Abdullah et al. [23] developed an integrated decision-making framework built on fractional fuzzy sets for the evaluation of AI cloud platforms. Hu et al. [24] developed the key factors for adopting AI-enabled auditing methods by joint operation of fuzzy RS theory and multiple rule-based decision-making techniques.

However, in FS, decision makers have dealt with only the truth grade and cannot deal with the false grade because the false grade also plays an important part in making a decision. To handle this situation, Atanassov [25] originated the thought of IFS to handle the data in the form of two grades, truth grade and false grade, with the requirement that the total of truth grade and false grade be contained within the interval [0,1], which enables better sketch of the deficient and vague data in decision making problems. Since the initiation of IFS, different researchers suggested various concepts of which one of the key concepts is AOs. AOs reduce the set of finite values on the decision-making process to a single value, which was one of the major concerns for specialists on how to obtain a unique outcome from the data collected from various sources. Xu [26] introduced the IF-weighted averaging (IFWA) and IF-ordered WA (IFOWA) AOs in the environment of IFSs. Also, geometric AOs based on IFSs were introduced by Xu and Yager [27]. The graphical technique for ranking accuracy and score function was developed by Ali et al. [28]. He et al. [29,30] initiated the idea of IF neutral averaging operators and geometric interaction averaging operators with application in decision making. Generalized [(IFWA), (IFOWA), (IFHA)] operators were initiated by Zhao et al. [31] and using them in DM. Many AOs have been made in the settings of IFSs such as, IF Einstein WA and geometric operators by Wang and Liu [32], IFDWA and IFDWG operators and application in DM by Seikh and Mandal [33] IF Hamacher WA (IFHWA), IFHOWA, and IFHHA operators by Huang [34], quasi-IF OWA, quasi-IF Choquet OA by Yang and Chen [35]. From the dominant idea of IFSs, much research has been done by different experts in different directions. However, many experts raise the question of what happens when we change the range from [0,1] to complex numbers because in many situations we have been given a piece of data in the arrangement of complex numbers. Remot et al. [36] initiate the dominant paradigm of a CFS in which the truth grade belongs to a complex plane, to address this problem. CFS is an excellent method for defining human opinion in the form of complex grades since it manages two-dimensional data into a single set. Tamir et al. [37] give a new interpretation of complex truth grade. CFS has gained significant attention within the last few years. Many researchers have performed great work in the settings of CFS. Such as, Li and Chiang [38] initiated the idea of complex neuro-FS using CFS, CF geometric AOs, and CF-arithmetic AOs interpreted by Bi et al. [39,40], CF-power AOs by Hu et al. [41].

2.2. Brief review of rough set theory

Pawlak generalized the idea of crisp set theory to handle the uncertain and imprecise data and interpret the dominant notion of RS theory. It has been used as a tool for database mining and knowledge discovery. It is a new era of uncertainty mathematics that is strongly connected to FS theory in its abstract form. In noisy and imprecise data, structural relationships can be found using a RS technique. The complementary extensions of crisp sets are RSs and FSs. RS theory's approximation spaces are sets with many memberships, whereas fuzzy sets focus on partial memberships. RSs and FSs are corresponding generalizations of crisp sets. According to Pawlak's RS, a universal set called the universe of discourse is characterized by two sets called the UA and LA. Later on, Dubois and Prade compared the concept of Pawlak's RS with that of FS and introduced the notion of FRS. They combine the two concepts (FS and RS) of uncertainty and vagueness and provide a more natural way to deal with identical problems. In 2003, Cornelis et al. [42] initiated a new idea of IFRS by merging RS and IFS. It is a more generalized form because it uses the truth grade and false grade in the form of UA and LA. Hence, certain developments in this field have been achieved using the idea of IFRS. Zhou et al. [43] introduce the characterization of IFRSs based on IF implicators. Jane et al. [44] initiated the MADM technique based on the intuitionistic Dombi operators and its application in mutual fund evaluation. Alnoor et al. [45] developed the oil industry benchmarking system and the multi-criteria DM technique for sustainable transportation, which is based on the extension of linear Diophantine FRS. Tan et al. [46] give the idea of granular structures and attribute subset selection based on IFRS. Hussain et al. [47] presented the TOPSIS technique for MCGDM based on the IFR Dombi AOs. Based on the IF coverings Zhang [48] developed the IFR approximation operators. Mehmood et al. [49] delivered the confidence level AOs based on IFRSs and delivered the prioritization and analysis of the factors of Robotics by using the EDAS method based on IFR Yager AOs. Yi et al. [50] introduced the notion of complex fuzzy rough set (CFRS). Emam et al. [51] deduced frank AOs based on CFRS. IFRS is very useful and dominant but it cannot handle the 2nd dimension data into a single set. To overcome this situation, Mehmood et al. [52] propose a concept of CIFRS to tackle both the complex truth grade and complex false grade in the form of UA and LA.

3. Preliminaries

In this segment, we study some basic notions and discuss their fundamental properties. Throughout this article, the universal set is denoted by K.

Atanassov [25] originated the thought of IFS defined as follows:

Definition 1. A IFS $\widetilde{€}$ on a universal set K is presented as

$$\widetilde{€} = \left\{\left(ɨ^*, Ш_{\widetilde{€}}(ɨ^*), И_{\widetilde{€}}(ɨ^*)\right) | ɨ^* \in K\right\},$$

where $Ш_{\widetilde{€}}: K \to [0,1]$ denotes the truth grade and $И_{\widetilde{€}}: K \to [0,1]$ denotes the false grade of every element $ɨ^* \in K$, such that $0 \le Ш_{\widetilde{€}}(ɨ^*) + И_{\widetilde{€}}(ɨ^*) \le 1$.

Tamir et al. [37] introduced the concept of CFS defined as follows:

Definition 2. A CFS $\widetilde{€}$ on a universal set K is presented as

$$\widetilde{€} = \left\{\left(ɨ^*, Ш_{\widetilde{€}}(ɨ^*)\right) | ɨ^* \in K\right\} = \left\{\left(ɨ^*, \sigma_{\widetilde{€}}(ɨ^*) + \iota\rho_{\widetilde{€}}(ɨ^*)\right) | ɨ^* \in K\right\}$$

Where the term $Ш_{\widetilde{€}}(ɨ^*)$ denotes the complex truth grade and $\sigma_{\widetilde{€}}, \rho_{\widetilde{€}} \in [0,1]$.

Dubois and Prade [3] introduced the idea of FRS defined as follows:

Definition 3. For fuzzy approximation space $(K, Ʀ_e)$ and a FS Ñ in K. Then the UA and LA of Ñ w.r.t $(K, Ʀ_e)$ is presented by

$$\overline{\overline{\text{Ꞧ}_e}}(\tilde{\text{N}}) = \left\{ \left(\text{ɨ}^*, \text{Ш}_{\overline{\overline{\text{Ꞧ}_e}}}(\text{ɨ}^*) \right) | \text{ɨ}^* \in K \right\}$$

$$\underline{\underline{\text{Ꞧ}_e}}(\tilde{\text{N}}) = \left\{ \left(\text{ɨ}^*, \text{Ш}_{\underline{\underline{\text{Ꞧ}_e}}}(\text{ɨ}^*) \right) | \text{ɨ}^* \in K \right\}$$

Where,

$$\text{Ш}_{\overline{\overline{\text{Ꞧ}_e}}}(\text{ɨ}^*) = \vee_{\text{ş}^* \in K}[e(\text{ɨ}^*, \text{ş}^*) \wedge \sigma_{\tilde{\text{N}}}(\text{ş}^*)]$$

$$\text{Ш}_{\underline{\underline{\text{Ꞧ}_e}}}(\text{ɨ}^*) = \wedge_{\text{ş}^* \in K}\left[\left(1 - \text{e}(\text{ɨ}^*, \text{ş}^*)\right) \vee \sigma_{\tilde{\text{N}}}(\text{ş}^*)\right]$$

Then, the pair $\text{Ꞧ}_e(\tilde{\text{N}}) = \left(\overline{\overline{\text{Ꞧ}_e}}(\tilde{\text{N}}), \underline{\underline{\text{Ꞧ}_e}}(\tilde{\text{N}}) \right)$ is termed FRS.

Chinram et al. [53] introduced the notion of IFRS defined as follows:

Definition 4. For an IF approximation space (K, R_e^*) and a FS $\tilde{\text{N}}$ in K. Then we define the UA and LA of $\tilde{\text{N}}$ w.r.t (K, R_e^*) denoted and defined as

$$\overline{\overline{\text{Ꞧ}_e}}(\tilde{\text{N}}) = \left\{ \left(\text{ɨ}^*, \text{Ш}_{\overline{\overline{\text{Ꞧ}_e}}}(\text{ɨ}^*), \text{И}_{\overline{\overline{\text{Ꞧ}_e}}}(\text{ɨ}^*) \right) | \text{ɨ}^* \in K \right\}$$

$$\underline{\underline{\text{Ꞧ}_e}}(\tilde{\text{N}}) = \left\{ \left(\text{ɨ}^*, \text{Ш}_{\underline{\underline{\text{Ꞧ}_e}}}(\text{ɨ}^*), \text{И}_{\underline{\underline{\text{Ꞧ}_e}}}(\text{ɨ}^*) \right) | \text{ɨ}^* \in K \right\}$$

Where,

$$\text{Ш}_{\overline{\overline{\text{Ꞧ}_e}}}(\text{ɨ}^*) = \vee_{\text{ş}^* \in K}[e(\text{ɨ}^*, \text{ş}^*) \vee \sigma_{\tilde{\text{N}}}(\text{ş}^*)]$$

$$\text{Ш}_{\underline{\underline{\text{Ꞧ}_e}}}(\text{ɨ}^*) = \wedge_{\text{ş}^* \in K}[\text{e}(\text{ɨ}^*, \text{ş}^*) \wedge \sigma_{\tilde{\text{N}}}(\text{ş}^*)]$$

$$\text{И}_{\overline{\overline{\text{Ꞧ}_e}}}(\text{ɨ}^*) = \wedge_{\text{ş}^* \in K}[g(\text{ɨ}^*, \text{ş}^*) \wedge \tau_{\tilde{\text{N}}}(\text{ş}^*)]$$

$$\text{И}_{\underline{\underline{\text{Ꞧ}_e}}}(\text{ɨ}^*) = \vee_{\text{ş}^* \in K}[g(\text{ɨ}^*, \text{ş}^*) \vee \tau_{\tilde{\text{N}}}(\text{ş}^*)]$$

Where, $0 \le \text{Ш}_{\overline{\overline{\text{Ꞧ}_e}}} + \text{И}_{\overline{\overline{\text{Ꞧ}_e}}} \le 1$, and $0 \le \text{Ш}_{\underline{\underline{\text{Ꞧ}_e}}} + \text{И}_{\underline{\underline{\text{Ꞧ}_e}}} \le 1$. Then the pair $\text{Ꞧ}_e(\tilde{\text{N}}) = \left(\overline{\overline{\text{Ꞧ}_e}}(\tilde{\text{N}}), \underline{\underline{\text{Ꞧ}_e}}(\tilde{\text{N}}) \right) = \left\{ \left(\text{ɨ}^*, < \text{Ш}_{\overline{\overline{\text{Ꞧ}_e}}}(\text{ɨ}^*), \text{И}_{\overline{\overline{\text{Ꞧ}_e}}}(\text{ɨ}^*) >, < \text{Ш}_{\underline{\underline{\text{Ꞧ}_e}}}(\text{ɨ}^*), \text{И}_{\underline{\underline{\text{Ꞧ}_e}}}(\text{ɨ}^*) > \right) \middle| \text{ɨ}^* \in K \right\}$ is called IFRS w.r.t $(K, \text{Ꞧ}_e)$.

Chinram et al. [53] introduced the algebraic operations of IFRS defined as follows:

Definition 5. Suppose $Ɽ_e(\tilde{Ñ}_1) = \left(\overline{\overline{Ɽ_e}}(\tilde{Ñ}_1), \underline{\underline{Ɽ_e}}(\tilde{Ñ}_1)\right)$ and $Ɽ_e(\tilde{Ñ}_2) = \left(\overline{\overline{Ɽ_e}}(\tilde{Ñ}_2), \underline{\underline{Ɽ_e}}(\tilde{Ñ}_2)\right)$ are two intuitionistic fuzzy rough numbers,

i. $Ɽ_e(\tilde{Ñ}_1) \cup Ɽ_e(\tilde{Ñ}_2) = \left(\overline{\overline{Ɽ_e}}(\tilde{Ñ}_1) \cup \overline{\overline{Ɽ_e}}(\tilde{Ñ}_2), \underline{\underline{Ɽ_e}}(\tilde{Ñ}_1) \cup \underline{\underline{Ɽ_e}}(\tilde{Ñ}_2)\right)$;

ii. $Ɽ_e(\tilde{Ñ}_1) \cap Ɽ_e(\tilde{Ñ}_2) = \left(\overline{\overline{Ɽ_e}}(\tilde{Ñ}_1) \cap \overline{\overline{Ɽ_e}}(\tilde{Ñ}_2), \underline{\underline{Ɽ_e}}(\tilde{Ñ}_1) \cap \underline{\underline{Ɽ_e}}(\tilde{Ñ}_2)\right)$;

iii. $Ɽ_e(\tilde{Ñ})^c = \left(\overline{\overline{Ɽ_e}}(\tilde{Ñ})^c, \underline{\underline{Ɽ_e}}(\tilde{Ñ})^c\right)$ where $\overline{\overline{Ɽ_e}}(\tilde{Ñ})^c$ and $\underline{\underline{Ɽ_e}}(\tilde{Ñ})^c$ signifies the complement of $\overline{\overline{Ɽ_e}}(\tilde{Ñ})$ and $\underline{\underline{Ɽ_e}}(\tilde{Ñ})$.

Mahmood et al. [52] introduced the notion of CIFRS defined as follows:

Definition 6. Let K be a universal set and $Ɽ_e$ be a CIF relation over K, and a pair $(K, Ɽ_e)$ define the CIF approximation space. Then, for $\tilde{Ñ} = \{(ꞩ^*, \sigma_{\tilde{Ñ}}(ꞩ^*) + \iota\rho_{\tilde{Ñ}}(ꞩ^*), \tau_{\tilde{Ñ}}(ꞩ^*) + \iota\upsilon_{\tilde{Ñ}}(ꞩ^*)) | ꞩ^* \in K\} \in CIFS(K)$, we describe the UA and LA of $\tilde{Ñ}$ w.r.t $(K, Ɽ_e)$,

$$\overline{\overline{Ɽ_e}}(\tilde{Ñ}) = \left\{\left(ɨ^*, Ш_{\overline{\overline{Ɽ_e}}}(ɨ^*), И_{\overline{\overline{Ɽ_e}}}(ɨ^*)\right) | ɨ^* \in K\right\}$$

$$\underline{\underline{Ɽ_e}}(\tilde{Ñ}) = \left\{\left(ɨ^*, Ш_{\underline{\underline{Ɽ_e}}}(ɨ^*), И_{\underline{\underline{Ɽ_e}}}(ɨ^*)\right) | ɨ^* \in K\right\}$$

Where,

$$Ш_{\overline{\overline{Ɽ_e}}}(ɨ^*) = \vee_{ꞩ^* \in K}[a(ɨ^*, ꞩ^*) \vee \sigma_{\tilde{Ñ}}(ꞩ^*)] + \iota \vee_{ɨ^* \in K}[b(ɨ^*, ꞩ^*) \vee \rho_{\tilde{Ñ}}(ꞩ^*)] = \sigma_{\overline{\overline{Ɽ_e}}} + \iota\rho_{\overline{\overline{Ɽ_e}}}$$

$$Ш_{\underline{\underline{Ɽ_e}}}(ɨ^*) = \wedge_{ꞩ^* \in K}\left[\left(a(ɨ^*, ꞩ^*)\right) \wedge \sigma_{\tilde{Ñ}}(ꞩ^*)\right] + \iota \wedge_{ɨ^* \in K}\left[\left(b(ɨ^*, ꞩ^*)\right) \wedge \rho_{\tilde{Ñ}}(ꞩ^*)\right] = \sigma_{\underline{\underline{Ɽ_e}}} + \iota\rho_{\underline{\underline{Ɽ_e}}}$$

$$И_{\overline{\overline{Ɽ_e}}}(ɨ^*) = \wedge_{ꞩ^* \in K}\left[\left(c(ɨ^*, ꞩ^*)\right) \wedge \tau_{\tilde{Ñ}}(ꞩ^*)\right] + \iota \wedge_{ɨ^* \in K}\left[\left(d(ɨ^*, ꞩ^*)\right) \wedge \upsilon_{\tilde{Ñ}}(ꞩ^*)\right] = \tau_{\overline{\overline{Ɽ_e}}} + \iota\upsilon_{\overline{\overline{Ɽ_e}}}$$

$$И_{\underline{\underline{Ɽ_e}}}(ɨ^*) = \vee_{ꞩ^* \in K}[c(ɨ^*, ꞩ^*) \vee \tau_{\tilde{Ñ}}(ꞩ^*)] + \iota \vee_{ɨ^* \in K}[d(ɨ^*, ꞩ^*) \vee \upsilon_{\tilde{Ñ}}(ꞩ^*)] = \tau_{\underline{\underline{Ɽ_e}}} + \iota\upsilon_{\underline{\underline{Ɽ_e}}}$$

Where, $0 \le \sigma_{\overline{\overline{Ɽ_e}}} + \tau_{\overline{\overline{Ɽ_e}}} \le 1$, $0 \le \sigma_{\underline{\underline{Ɽ_e}}} + \tau_{\underline{\underline{Ɽ_e}}} \le 1$, $0 \le \rho_{\overline{\overline{Ɽ_e}}} + \upsilon_{\overline{\overline{Ɽ_e}}} \le 1$, and $0 \le \rho_{\underline{\underline{Ɽ_e}}} + \upsilon_{\underline{\underline{Ɽ_e}}} \le 1$.

As $\overline{\overline{Ɽ_e}}(\tilde{Ñ})$ and $\underline{\underline{Ɽ_e}}(\tilde{Ñ})$ are CIFRSs. Then, the pair $Ɽ_e(\tilde{Ñ}) = \left(\overline{\overline{Ɽ_e}}(\tilde{Ñ}), \underline{\underline{Ɽ_e}}(\tilde{Ñ})\right) = \left\{\left(ɨ^*, < Ш_{\overline{\overline{Ɽ_e}}}(ɨ^*), И_{\overline{\overline{Ɽ_e}}}(ɨ^*) >, < Ш_{\underline{\underline{Ɽ_e}}}(ɨ^*), И_{\underline{\underline{Ɽ_e}}}(ɨ^*) >\right) \middle| ɨ^* \in K\right\}$ is called CIFRS w.r.t $(K, Ɽ_e)$. For easiness, we will say that $Ɽ_e(\tilde{N}) = \left(\overline{\overline{Ɽ_e}}(\tilde{N}), \underline{\underline{Ɽ_e}}(\tilde{N})\right) = \left(\left(\sigma_{\overline{\overline{Ɽ_e}}} + \iota\rho_{\overline{\overline{Ɽ_e}}}, \tau_{\overline{\overline{Ɽ_e}}} + \iota\upsilon_{\overline{\overline{Ɽ_e}}}\right), \left(\sigma_{\underline{\underline{Ɽ_e}}} + \iota\rho_{\underline{\underline{Ɽ_e}}}, \tau_{\underline{\underline{Ɽ_e}}} + \iota\upsilon_{\underline{\underline{Ɽ_e}}}\right)\right)$ represents CIFRN. In the below Table 1, we explain each letter used in the definition of CIFRS.

Table 1. Explanation of each letter in the definition of CIFRS.

Letters used in the formula	Explanation
$\overline{\overline{Ɽ_e}}(\tilde{N})$	Upper approximation
$Ш_{\overline{\overline{Ɽ_e}}}(ƚ^*)$	Complex truth grade of upper approximation
$И_{\overline{\overline{Ɽ_e}}}(ƚ^*)$	Complex false grade of upper approximation
$\boldsymbol{\sigma}_{\overline{\overline{Ɽ_e}}} + \boldsymbol{\iota\rho}_{\overline{\overline{Ɽ_e}}}$	Real + Imaginary part of truth grade of upper approximation
$\boldsymbol{\tau}_{\overline{\overline{Ɽ_e}}} + \boldsymbol{\iota\upsilon}_{\overline{\overline{Ɽ_e}}}$	Real + Imaginary part of false grade of upper approximation
$\underline{\underline{Ɽ_e}}(\tilde{N})$	Lower approximation
$Ш_{\underline{\underline{Ɽ_e}}}(ƚ^*)$	Complex truth grade of lower approximation
$И_{\underline{\underline{Ɽ_e}}}(ƚ^*)$	Complex false grade of lower approximation
$\boldsymbol{\sigma}_{\underline{\underline{Ɽ_e}}} + \boldsymbol{\iota\rho}_{\underline{\underline{Ɽ_e}}}$	Real + Imaginary part of truth grade of lower approximation
$\boldsymbol{\tau}_{\underline{\underline{Ɽ_e}}} + \boldsymbol{\iota\upsilon}_{\underline{\underline{Ɽ_e}}}$	Real + Imaginary part of false grade of lower approximation

Mahmood et al. [52] introduced the algebraic operations of CIFRS defined as follows:

Definition 7. For two CIFRNs, $Ɽ_e(\tilde{N}_1) = \left(\overline{\overline{Ɽ_e}}(\tilde{N}_1), \underline{\underline{Ɽ_e}}(\tilde{N}_1)\right) = \left(\sigma_{\overline{\overline{Ɽ_e}}} + \iota\rho_{\overline{\overline{Ɽ_e}}}, \tau_{\overline{\overline{Ɽ_e}}} + \iota\upsilon_{\overline{\overline{Ɽ_e}}}, \sigma_{\underline{\underline{Ɽ_e}}} + \iota\rho_{\underline{\underline{Ɽ_e}}}, \tau_{\underline{\underline{Ɽ_e}}} + \iota\upsilon_{\underline{\underline{Ɽ_e}}}\right)$ and $Ɽ_e(\tilde{N}_2) = \left(\overline{\overline{Ɽ_e}}(\tilde{N}_2), \underline{\underline{Ɽ_e}}(\tilde{N}_2)\right) = \left(\sigma_{\overline{\overline{Ɽ_e}}} + \iota\rho_{\overline{\overline{Ɽ_e}}}, \tau_{\overline{\overline{Ɽ_e}}} + \iota\upsilon_{\overline{\overline{Ɽ_e}}}, \sigma_{\underline{\underline{Ɽ_e}}} + \iota\rho_{\underline{\underline{Ɽ_e}}}, \tau_{\underline{\underline{Ɽ_e}}} + \iota\upsilon_{\underline{\underline{Ɽ_e}}}\right)$

1) Complement:

$$Ɽ_e(\tilde{N}_1)^c = \left(\overline{\overline{Ɽ_e}}(\tilde{N}_1)^c, \underline{\underline{Ɽ_e}}(\tilde{N}_1)^c\right) = \left(\tau_{\overline{\overline{Ɽ_e}}} + \iota\upsilon_{\overline{\overline{Ɽ_e}}}, \sigma_{\overline{\overline{Ɽ_e}}} + \iota\rho_{\overline{\overline{Ɽ_e}}}, \tau_{\underline{\underline{Ɽ_e}}} + \iota\upsilon_{\underline{\underline{Ɽ_e}}}, \sigma_{\underline{\underline{Ɽ_e}}} + \iota\rho_{\underline{\underline{Ɽ_e}}}\right)$$

2) Union:

$$Ɽ_e(\tilde{N}_1) \cup Ɽ_e(\tilde{N}_2) = \left(\overline{\overline{Ɽ_e}}(\tilde{N}_1) \cup \overline{\overline{Ɽ_e}}(\tilde{N}_2), \underline{\underline{Ɽ_e}}(\tilde{N}_1) \cup \underline{\underline{Ɽ_e}}(\tilde{N}_2)\right)$$

$$= \begin{pmatrix} \max\left[\sigma_{\overline{\overline{Ɍ_e}}}(\tilde{Ñ}_1), \sigma_{\overline{\overline{Ɍ_e}}}(\tilde{Ñ}_2)\right] + \iota \max\left[\rho_{\overline{\overline{Ɍ_e}}}(\tilde{Ñ}_1), \rho_{\overline{\overline{Ɍ_e}}}(\tilde{Ñ}_2)\right] \\ \min\left[\tau_{\overline{\overline{Ɍ_e}}}(\tilde{Ñ}_1), \tau_{\overline{\overline{Ɍ_e}}}(\tilde{Ñ}_2)\right] + \iota \min\left[\upsilon_{\overline{\overline{Ɍ_e}}}(\tilde{Ñ}_1) + \upsilon_{\overline{\overline{Ɍ_e}}}(\tilde{Ñ}_2)\right] \\ \max\left[\sigma_{\underline{\underline{Ɍ_e}}}(\tilde{Ñ}_1), \sigma_{\underline{\underline{Ɍ_e}}}(\tilde{Ñ}_2)\right] + \iota \max\left[\rho_{\underline{\underline{Ɍ_e}}}(\tilde{Ñ}_1), \rho_{\underline{\underline{Ɍ_e}}}(\tilde{Ñ}_2)\right] \\ \min\left[\tau_{\underline{\underline{Ɍ_e}}}(\tilde{Ñ}_1), \tau_{\underline{\underline{Ɍ_e}}}(\tilde{Ñ}_2)\right] + \iota \min\left[\upsilon_{\underline{\underline{Ɍ_e}}}(\tilde{Ñ}_1) + \upsilon_{\underline{\underline{Ɍ_e}}}(\tilde{Ñ}_2)\right] \end{pmatrix}$$

3) Intersection:

$$Ɍ_e(\tilde{Ñ}_1) \cap Ɍ_e(\tilde{Ñ}_2) = \left(\overline{\overline{Ɍ_e}}(\tilde{Ñ}_1) \cap \overline{\overline{Ɍ_e}}(\tilde{Ñ}_2), \underline{\underline{Ɍ_e}}(\tilde{Ñ}_1) \cap \underline{\underline{Ɍ_e}}(\tilde{Ñ}_2)\right)$$

$$= \begin{pmatrix} \min\left[\sigma_{\overline{\overline{Ɍ_e}}}(\tilde{Ñ}_1), \sigma_{\overline{\overline{Ɍ_e}}}(\tilde{Ñ}_2)\right] + \iota \min\left[\rho_{\overline{\overline{Ɍ_e}}}(\tilde{Ñ}_1), \rho_{\overline{\overline{Ɍ_e}}}(\tilde{Ñ}_2)\right] \\ \max\left[\tau_{\overline{\overline{Ɍ_e}}}(\tilde{Ñ}_1), \tau_{\overline{\overline{Ɍ_e}}}(\tilde{Ñ}_2)\right] + \iota \max\left[\upsilon_{\overline{\overline{Ɍ_e}}}(\tilde{Ñ}_1) + \upsilon_{\overline{\overline{Ɍ_e}}}(\tilde{Ñ}_2)\right] \\ \min\left[\sigma_{\underline{\underline{Ɍ_e}}}(\tilde{Ñ}_1), \sigma_{\underline{\underline{Ɍ_e}}}(\tilde{Ñ}_2)\right] + \iota \min\left[\rho_{\underline{\underline{Ɍ_e}}}(\tilde{Ñ}_1), \rho_{\underline{\underline{Ɍ_e}}}(\tilde{Ñ}_2)\right] \\ \max\left[\tau_{\underline{\underline{Ɍ_e}}}(\tilde{Ñ}_1), \tau_{\underline{\underline{Ɍ_e}}}(\tilde{Ñ}_2)\right] + \iota \max\left[\upsilon_{\underline{\underline{Ɍ_e}}}(\tilde{Ñ}_1) + \upsilon_{\underline{\underline{Ɍ_e}}}(\tilde{Ñ}_2)\right] \end{pmatrix}$$

Mahmood et al. [52] introduced the score and accuracy functions of CIFRS defined as follows:

Definition 8. The score function S_F of a CIFRN $Ɍ_e(\tilde{Ñ}) = \left(\overline{\overline{Ɍ_e}}(\tilde{Ñ}), \underline{\underline{Ɍ_e}}(\tilde{Ñ})\right) = \Big(\sigma_{\overline{\overline{Ɍ_e}}} + \iota\rho_{\overline{\overline{Ɍ_e}}}, \tau_{\overline{\overline{Ɍ_e}}} + \iota\upsilon_{\overline{\overline{Ɍ_e}}}, \sigma_{\underline{\underline{Ɍ_e}}} + \iota\rho_{\underline{\underline{Ɍ_e}}}, \tau_{\underline{\underline{Ɍ_e}}} + \iota\upsilon_{\underline{\underline{Ɍ_e}}}\Big)$ is define as

$$S_F\left(Ɍ_e(\tilde{Ñ})\right) = \frac{1}{8}\left(4 + \sigma_{\overline{\overline{Ɍ_e}}} + \rho_{\overline{\overline{Ɍ_e}}} + \sigma_{\underline{\underline{Ɍ_e}}} + \rho_{\underline{\underline{Ɍ_e}}} - \tau_{\overline{\overline{Ɍ_e}}} - \upsilon_{\overline{\overline{Ɍ_e}}} - \tau_{\underline{\underline{Ɍ_e}}} - \upsilon_{\underline{\underline{Ɍ_e}}}\right), \qquad S_F\left(Ɍ_e(\tilde{Ñ})\right) \in [0,1].$$

The accuracy function A_F of a CIFRN is defined as

$$A_F\left(Ɍ_e(\tilde{Ñ})\right) = \frac{1}{8}\left(\sigma_{\overline{\overline{Ɍ_e}}} + \rho_{\overline{\overline{Ɍ_e}}} + \sigma_{\underline{\underline{Ɍ_e}}} + \rho_{\underline{\underline{Ɍ_e}}} + \tau_{\overline{\overline{Ɍ_e}}} + \upsilon_{\overline{\overline{Ɍ_e}}} + \tau_{\underline{\underline{Ɍ_e}}} + \upsilon_{\underline{\underline{Ɍ_e}}}\right), \qquad A_F\left(Ɍ_e(\tilde{Ñ})\right) \in [0,1].$$

3.1. Dombi operation

In this section, we interpret Dombi [54] operations known as Dombi sum and product, which are special cases of t-norms and t-conorms defined as follows:

Definition 9. Let $ȩ_1$ and $ȩ_2$ be two real numbers with $℃ \geq 1$. Then DT-N and DT-CN are elaborated as

$$\beta_{Dom}(ȩ_1, ȩ_2) = \frac{1}{1 + \left\{\left(\frac{1 - ȩ_1}{ȩ_1}\right)^{℃} + \left(\frac{1 - ȩ_2}{ȩ_2}\right)^{℃}\right\}^{\frac{1}{℃}}}$$

$$\beta'_{Dom}(ȩ_1, ȩ_2) = 1 - \frac{1}{1 + \left\{\left(\frac{ȩ_1}{1 - ȩ_1}\right)^{℃} + \left(\frac{ȩ_2}{1 - ȩ_2}\right)^{℃}\right\}^{\frac{1}{℃}}}$$

3.2. Dombi operations on CIFRNs

In this sub-sequel, we develop the concept of Dombi operation on CIFRNs.

Definition 10. Let $Ɽ_e(\tilde{N}_1) = \left(\overline{\overline{Ɽ_e}}(\tilde{N}_1), \underline{\underline{Ɽ_e}}(\tilde{N}_1)\right) = \left(\sigma_{\overline{\overline{Ɽ_e}}} + \iota\rho_{\overline{\overline{Ɽ_e}}}, \tau_{\overline{\overline{Ɽ_e}}} + \iota\upsilon_{\overline{\overline{Ɽ_e}}}, \sigma_{\underline{\underline{Ɽ_e}}} + \iota\rho_{\underline{\underline{Ɽ_e}}}, \tau_{\underline{\underline{Ɽ_e}}} + \iota\upsilon_{\underline{\underline{Ɽ_e}}}\right)$ and $Ɽ_e(\tilde{N}_2) = \left(\overline{\overline{Ɽ_e}}(\tilde{N}_2), \underline{\underline{Ɽ_e}}(\tilde{N}_2)\right) = \left(\sigma_{\overline{\overline{Ɽ_e}}} + \iota\rho_{\overline{\overline{Ɽ_e}}}, \tau_{\overline{\overline{Ɽ_e}}} + \iota\upsilon_{\overline{\overline{Ɽ_e}}}, \sigma_{\underline{\underline{Ɽ_e}}} + \iota\rho_{\underline{\underline{Ɽ_e}}}, \tau_{\underline{\underline{Ɽ_e}}} + \iota\upsilon_{\underline{\underline{Ɽ_e}}}\right)$ be two CIFRNs, then

1) $Ɽ_e(\tilde{N}_1) \oplus Ɽ_e(\tilde{N}_2) = \left(\overline{\overline{Ɽ_e}}(\tilde{N}_1) \oplus \overline{\overline{Ɽ_e}}(\tilde{N}_2), \underline{\underline{Ɽ_e}}(\tilde{N}_1) \oplus \underline{\underline{Ɽ_e}}(\tilde{N}_2)\right)$

$$= \left(\begin{array}{c}
\left(\begin{array}{c}
1 - \frac{1}{1 + \left\{\left(\frac{\sigma_{\overline{\overline{Ɽ_e}}}(\tilde{N}_1)}{1 - \sigma_{\overline{\overline{Ɽ_e}}}(\tilde{N}_1)}\right)^{℃} + \left(\frac{\sigma_{\overline{\overline{Ɽ_e}}}(\tilde{N}_2)}{1 - \sigma_{\overline{\overline{Ɽ_e}}}(\tilde{N}_2)}\right)^{℃}\right\}^{\frac{1}{℃}}} + \\
\iota\left(1 - \frac{1}{1 + \left\{\left(\frac{\rho_{\overline{\overline{Ɽ_e}}}(\tilde{N}_1)}{1 - \rho_{\overline{\overline{Ɽ_e}}}(\tilde{N}_1)}\right)^{℃} + \left(\frac{\rho_{\overline{\overline{Ɽ_e}}}(\tilde{N}_2)}{1 - \rho_{\overline{\overline{Ɽ_e}}}(\tilde{N}_2)}\right)^{℃}\right\}^{\frac{1}{℃}}}\right), \\
\frac{1}{1 + \left\{\left(\frac{1 - \tau_{\overline{\overline{Ɽ_e}}}(\tilde{N}_1)}{\tau_{\overline{\overline{Ɽ_e}}}(\tilde{N}_1)}\right)^{℃} + \left(\frac{1 - \tau_{\overline{\overline{Ɽ_e}}}(\tilde{N}_2)}{\tau_{\overline{\overline{Ɽ_e}}}(\tilde{N}_2)}\right)^{℃}\right\}^{\frac{1}{℃}}} + \iota\frac{1}{1 + \left\{\left(\frac{1 - \upsilon_{\overline{\overline{Ɽ_e}}}(\tilde{N}_1)}{\upsilon_{\overline{\overline{Ɽ_e}}}(\tilde{N}_1)}\right)^{℃} + \left(\frac{1 - \upsilon_{\overline{\overline{Ɽ_e}}}(\tilde{N}_2)}{\upsilon_{\overline{\overline{Ɽ_e}}}(\tilde{N}_2)}\right)^{℃}\right\}^{\frac{1}{℃}}}
\end{array}\right), \\
\left(\begin{array}{c}
1 - \frac{1}{1 + \left\{\left(\frac{\sigma_{\underline{\underline{Ɽ_e}}}(\tilde{N}_1)}{1 - \sigma_{\underline{\underline{Ɽ_e}}}(\tilde{N}_1)}\right)^{℃} + \left(\frac{\sigma_{\underline{\underline{Ɽ_e}}}(\tilde{N}_2)}{1 - \sigma_{\underline{\underline{Ɽ_e}}}(\tilde{N}_2)}\right)^{℃}\right\}^{\frac{1}{℃}}} \\
+\iota\left(1 - \frac{1}{1 + \left\{\left(\frac{\rho_{\underline{\underline{Ɽ_e}}}(\tilde{N}_1)}{1 - \rho_{\underline{\underline{Ɽ_e}}}(\tilde{N}_1)}\right)^{℃} + \left(\frac{\rho_{\underline{\underline{Ɽ_e}}}(\tilde{N}_2)}{1 - \rho_{\underline{\underline{Ɽ_e}}}(\tilde{N}_2)}\right)^{℃}\right\}^{\frac{1}{℃}}}\right), \\
\frac{1}{1 + \left\{\left(\frac{1 - \tau_{\underline{\underline{Ɽ_e}}}(\tilde{N}_1)}{\tau_{\underline{\underline{Ɽ_e}}}(\tilde{N}_1)}\right)^{℃} + \left(\frac{1 - \tau_{\underline{\underline{Ɽ_e}}}(\tilde{N}_2)}{\tau_{\underline{\underline{Ɽ_e}}}(\tilde{N}_2)}\right)^{℃}\right\}^{\frac{1}{℃}}} + \iota\frac{1}{1 + \left\{\left(\frac{1 - \upsilon_{\underline{\underline{Ɽ_e}}}(\tilde{N}_1)}{\upsilon_{\underline{\underline{Ɽ_e}}}(\tilde{N}_1)}\right)^{℃} + \left(\frac{1 - \upsilon_{\underline{\underline{Ɽ_e}}}(\tilde{N}_2)}{\upsilon_{\underline{\underline{Ɽ_e}}}(\tilde{N}_2)}\right)^{℃}\right\}^{\frac{1}{℃}}}
\end{array}\right)
\end{array}\right)$$

2) $Ꞧ_e(\tilde{N}_1) \otimes Ꞧ_e(\tilde{N}_2) = \left(\overline{\overline{Ꞧ_e}}(\tilde{N}_1) \otimes \overline{\overline{Ꞧ_e}}(\tilde{N}_2), \underline{\underline{Ꞧ_e}}(\tilde{N}_1) \otimes \underline{\underline{Ꞧ_e}}(\tilde{N}_2)\right)$

$$= \left(\begin{array}{c} \left(\begin{array}{c} \frac{1}{1+\left\{\left(\frac{1-\sigma_{\overline{\overline{Ꞧ_e}}}(\tilde{N}_1)}{\sigma_{\overline{\overline{Ꞧ_e}}}(\tilde{N}_1)}\right)^{\text{℃}}+\left(\frac{1-\sigma_{\overline{\overline{Ꞧ_e}}}(\tilde{N}_2)}{\sigma_{\overline{\overline{Ꞧ_e}}}(\tilde{N}_2)}\right)^{\text{℃}}\right\}^{\frac{1}{\text{℃}}}} + \iota \frac{1}{1+\left\{\left(\frac{1-\rho_{\overline{\overline{Ꞧ_e}}}(\tilde{N}_1)}{\rho_{\overline{\overline{Ꞧ_e}}}(\tilde{N}_1)}\right)^{\text{℃}}+\left(\frac{1-\rho_{\overline{\overline{Ꞧ_e}}}(\tilde{N}_2)}{\rho_{\overline{\overline{Ꞧ_e}}}(\tilde{N}_2)}\right)^{\text{℃}}\right\}^{\frac{1}{\text{℃}}}}, \\ 1-\frac{1}{1+\left\{\left(\frac{\tau_{\overline{\overline{Ꞧ_e}}}(\tilde{N}_1)}{1-\tau_{\overline{\overline{Ꞧ_e}}}(\tilde{N}_1)}\right)^{\text{℃}}+\left(\frac{\tau_{\overline{\overline{Ꞧ_e}}}(\tilde{N}_2)}{1-\tau_{\overline{\overline{Ꞧ_e}}}(\tilde{N}_2)}\right)^{\text{℃}}\right\}^{\frac{1}{\text{℃}}}} + \\ \iota\left(1-\frac{1}{1+\left\{\left(\frac{\upsilon_{\overline{\overline{Ꞧ_e}}}(\tilde{N}_1)}{1-\upsilon_{\overline{\overline{Ꞧ_e}}}(\tilde{N}_1)}\right)^{\text{℃}}+\left(\frac{\upsilon_{\overline{\overline{Ꞧ_e}}}(\tilde{N}_2)}{1-\upsilon_{\overline{\overline{Ꞧ_e}}}(\tilde{N}_2)}\right)^{\text{℃}}\right\}^{\frac{1}{\text{℃}}}}\right) \end{array} \right), \\ \left(\begin{array}{c} \frac{1}{1+\left\{\left(\frac{1-\sigma_{\underline{\underline{Ꞧ_e}}}(\tilde{N}_1)}{\sigma_{\underline{\underline{Ꞧ_e}}}(\tilde{N}_1)}\right)^{\text{℃}}+\left(\frac{1-\sigma_{\underline{\underline{Ꞧ_e}}}(\tilde{N}_2)}{\sigma_{\underline{\underline{Ꞧ_e}}}(\tilde{N}_2)}\right)^{\text{℃}}\right\}^{\frac{1}{\text{℃}}}} + \iota \frac{1}{1+\left\{\left(\frac{1-\rho_{\underline{\underline{Ꞧ_e}}}(\tilde{N}_1)}{\rho_{\underline{\underline{Ꞧ_e}}}(\tilde{N}_1)}\right)^{\text{℃}}+\left(\frac{1-\rho_{\underline{\underline{Ꞧ_e}}}(\tilde{N}_2)}{\rho_{\underline{\underline{Ꞧ_e}}}(\tilde{N}_2)}\right)^{\text{℃}}\right\}^{\frac{1}{\text{℃}}}}, \\ 1-\frac{1}{1+\left\{\left(\frac{\tau_{\underline{\underline{Ꞧ_e}}}(\tilde{N}_1)}{1-\tau_{\underline{\underline{Ꞧ_e}}}(\tilde{N}_1)}\right)^{\text{℃}}+\left(\frac{\tau_{\underline{\underline{Ꞧ_e}}}(\tilde{N}_2)}{1-\tau_{\underline{\underline{Ꞧ_e}}}(\tilde{N}_2)}\right)^{\text{℃}}\right\}^{\frac{1}{\text{℃}}}} + \\ \iota\left(1-\frac{1}{1+\left\{\left(\frac{\upsilon_{\underline{\underline{Ꞧ_e}}}(\tilde{N}_1)}{1-\upsilon_{\underline{\underline{Ꞧ_e}}}(\tilde{N}_1)}\right)^{\text{℃}}+\left(\frac{\upsilon_{\underline{\underline{Ꞧ_e}}}(\tilde{N}_2)}{1-\upsilon_{\underline{\underline{Ꞧ_e}}}(\tilde{N}_2)}\right)^{\text{℃}}\right\}^{\frac{1}{\text{℃}}}}\right) \end{array} \right) \end{array} \right)$$

3) $\lambda Ꞧ_e(\tilde{N}) = \left(\lambda\overline{\overline{Ꞧ_e}}(\tilde{N}), \lambda\underline{\underline{Ꞧ_e}}(\tilde{N})\right)$

$$= \left(\left(\begin{array}{c} 1 - \dfrac{1}{1 + \left\{\lambda\left(\dfrac{\sigma_{\overline{\overline{R_e}}}(\tilde{N}_1)}{1 - \sigma_{\overline{\overline{R_e}}}(\tilde{N}_1)}\right)^{℃}\right\}^{\frac{1}{℃}}} + \iota\left(1 - \dfrac{1}{1 + \left\{\lambda\left(\dfrac{\rho_{\overline{\overline{R_e}}}(\tilde{N}_1)}{1 - \rho_{\overline{\overline{R_e}}}(\tilde{N}_1)}\right)^{℃}\right\}^{\frac{1}{℃}}}\right), \\ \dfrac{1}{1 + \left\{\lambda\left(\dfrac{1 - \tau_{\overline{\overline{R_e}}}(\tilde{N}_1)}{\tau_{\overline{\overline{R_e}}}(\tilde{N}_1)}\right)^{℃}\right\}^{\frac{1}{℃}}} + \iota\left(\dfrac{1}{1 + \left\{\lambda\left(\dfrac{1 - \upsilon_{\overline{\overline{R_e}}}(\tilde{N}_1)}{\upsilon_{\overline{\overline{R_e}}}(\tilde{N}_1)}\right)^{℃}\right\}^{\frac{1}{℃}}}\right) \end{array}\right), \left(\begin{array}{c} 1 - \dfrac{1}{1 + \left\{\lambda\left(\dfrac{\sigma_{\underline{\underline{R_e}}}(\tilde{N}_1)}{1 - \sigma_{\underline{\underline{R_e}}}(\tilde{N}_1)}\right)^{℃}\right\}^{\frac{1}{℃}}} + \iota\left(1 - \dfrac{1}{1 + \left\{\lambda\left(\dfrac{\rho_{\underline{\underline{R_e}}}(\tilde{N}_1)}{1 - \rho_{\underline{\underline{R_e}}}(\tilde{N}_1)}\right)^{℃}\right\}^{\frac{1}{℃}}}\right), \\ \dfrac{1}{1 + \left\{\lambda\left(\dfrac{1 - \tau_{\underline{\underline{R_e}}}(\tilde{N}_1)}{\tau_{\underline{\underline{R_e}}}(\tilde{N}_1)}\right)^{℃}\right\}^{\frac{1}{℃}}} + \iota\left(\dfrac{1}{1 + \left\{\lambda\left(\dfrac{1 - \upsilon_{\underline{\underline{R_e}}}(\tilde{N}_1)}{\upsilon_{\underline{\underline{R_e}}}(\tilde{N}_1)}\right)^{℃}\right\}^{\frac{1}{℃}}}\right) \end{array}\right)\right)$$

4) $\left(R_e(\tilde{N})\right)^{\lambda} = \left(\left(\overline{\overline{R_e}}(\tilde{N})\right)^{\lambda}, \left(\underline{\underline{R_e}}(\tilde{N})\right)^{\lambda}\right)$

$$
= \left(\begin{array}{l} \left(\dfrac{1}{1+\left\{\lambda\left(\dfrac{1-\sigma_{\overline{\overline{\text{Ɽ}_e}}}(\tilde{N}_1)}{\sigma_{\overline{\overline{\text{Ɽ}_e}}}(\tilde{N}_1)}\right)^{℃}\right\}^{\frac{1}{℃}}} + \iota\left(\dfrac{1}{1+\left\{\lambda\left(\dfrac{1-\rho_{\overline{\overline{\text{Ɽ}_e}}}(\tilde{N}_1)}{\rho_{\overline{\overline{\text{Ɽ}_e}}}(\tilde{N}_1)}\right)^{℃}\right\}^{\frac{1}{℃}}}\right), \; 1-\dfrac{1}{1+\left\{\lambda\left(\dfrac{\tau_{\overline{\overline{\text{Ɽ}_e}}}(\tilde{N}_1)}{1-\tau_{\overline{\overline{\text{Ɽ}_e}}}(\tilde{N}_1)}\right)^{℃}\right\}^{\frac{1}{℃}}} + \iota\left(1-\dfrac{1}{1+\left\{\lambda\left(\dfrac{\upsilon_{\overline{\overline{\text{Ɽ}_e}}}(\tilde{N}_1)}{1-\upsilon_{\overline{\overline{\text{Ɽ}_e}}}(\tilde{N}_1)}\right)^{℃}\right\}^{\frac{1}{℃}}}\right)\right), \\ \left(\dfrac{1}{1+\left\{\lambda\left(\dfrac{1-\sigma_{\underline{\underline{\text{Ɽ}_e}}}(\tilde{N}_1)}{\sigma_{\underline{\underline{\text{Ɽ}_e}}}(\tilde{N}_1)}\right)^{℃}\right\}^{\frac{1}{℃}}} + \iota\left(\dfrac{1}{1+\left\{\lambda\left(\dfrac{1-\rho_{\underline{\underline{\text{Ɽ}_e}}}(\tilde{N}_1)}{\rho_{\underline{\underline{\text{Ɽ}_e}}}(\tilde{N}_1)}\right)^{℃}\right\}^{\frac{1}{℃}}}\right), \; 1-\dfrac{1}{1+\left\{\lambda\left(\dfrac{\tau_{\underline{\underline{\text{Ɽ}_e}}}(\tilde{N}_1)}{1-\tau_{\underline{\underline{\text{Ɽ}_e}}}(\tilde{N}_1)}\right)^{℃}\right\}^{\frac{1}{℃}}} + \iota\left(1-\dfrac{1}{1+\left\{\lambda\left(\dfrac{\upsilon_{\underline{\underline{\text{Ɽ}_e}}}(\tilde{N}_1)}{1-\upsilon_{\underline{\underline{\text{Ɽ}_e}}}(\tilde{N}_1)}\right)^{℃}\right\}^{\frac{1}{℃}}}\right)\right) \end{array} \right)
$$

4. CIFR Dombi average AOs

In this sequel, we develop the idea of a CIFRDWA operator and discuss several new aggregation operators based on CIFRNs with their properties using Dombi operations.

Definition 11. Suppose $\text{Ɽ}_e(\tilde{N}_j) = \left(\overline{\overline{\text{Ɽ}_e}}(\tilde{N}_j), \underline{\underline{\text{Ɽ}_e}}(\tilde{N}_j)\right)(j = 1,2,3 \ldots, \mathfrak{s})$ be a gathering of CIFRNs and $ß = (ß_1, ß_2, ß_3 \ldots, ß_{\mathfrak{s}})^T$ be the weight vector (WV) with $ß_j \in [0,1]$ such that $\sum_{j=1}^{\mathfrak{s}} ß_j = 1$, then a CIFRDWA operator is defined as

$$
\text{CIFRDWA}\left(\text{Ɽ}_e(\tilde{N}_1), \text{Ɽ}_e(\tilde{N}_2), \ldots, \text{Ɽ}_e(\tilde{N}_{\mathfrak{s}})\right) = \left(\oplus_{j=1}^{\mathfrak{s}} ß_j \overline{\overline{\text{Ɽ}_e}}(\tilde{N}_j), \oplus_{\mathfrak{t}=1}^{\mathfrak{s}} ß_j \underline{\underline{\text{Ɽ}_e}}(\tilde{N}_j)\right)
$$

$$=\left(\left(ß_1\overline{\overline{Ɍ_e}}(\tilde{N}_1)\oplus ß_2\overline{\overline{Ɍ_e}}(\tilde{N}_2)\oplus\ldots..\oplus ß_ꞩ\overline{\overline{Ɍ_e}}(\tilde{N}_ꞩ)\right),\left(ß_1\underline{\underline{Ɍ_e}}(\tilde{N}_1)\oplus ß_2\underline{\underline{Ɍ_e}}(\tilde{N}_2)\oplus\ldots..\oplus ß_ꞩ\underline{\underline{Ɍ_e}}(\tilde{N}_ꞩ)\right)\right)$$

From the above definition, the results for CIFRDWA operator are:

Theorem 1. Using the equation above, we obtain the CIFRNs and

$$\text{CIFRDWA}\left(Ɍ_e(\tilde{N}_1),Ɍ_e(\tilde{N}_2),\ldots,Ɍ_e(\tilde{N}_ꞩ)\right)=\left(\oplus_{j=1}^{ꞩ}ß_j\overline{\overline{Ɍ_e}}(\tilde{N}_j),\oplus_{j=1}^{ꞩ}ß_j\underline{\underline{Ɍ_e}}(\tilde{N}_j)\right)$$

$$=\left(\begin{array}{c}\left(\begin{array}{c}\left(1-\dfrac{1}{1+\left\{\sum_{j=1}^{ꞩ}ß_j\left(\dfrac{\sigma_{\overline{\overline{Ɍ_e}}}(\tilde{N}_j)}{1-\sigma_{\overline{\overline{Ɍ_e}}}(\tilde{N}_j)}\right)^{℃}\right\}^{\frac{1}{℃}}}+\iota\left(1-\dfrac{1}{1+\left\{\sum_{j=1}^{ꞩ}ß_j\left(\dfrac{\rho_{\overline{\overline{Ɍ_e}}}(\tilde{N}_j)}{1-\rho_{\overline{\overline{Ɍ_e}}}(\tilde{N}_j)}\right)^{℃}\right\}^{\frac{1}{℃}}}\right),\right.\\ \left.\dfrac{1}{1+\left\{\sum_{j=1}^{ꞩ}ß_j\left(\dfrac{1-\tau_{\overline{\overline{Ɍ_e}}}(\tilde{N}_j)}{\tau_{\overline{\overline{Ɍ_e}}}(\tilde{N}_j)}\right)^{℃}\right\}^{\frac{1}{℃}}}+\iota\left(\dfrac{1}{1+\left\{\sum_{j=1}^{ꞩ}ß_j\left(\dfrac{1-\upsilon_{\overline{\overline{Ɍ_e}}}(\tilde{N}_j)}{\upsilon_{\overline{\overline{Ɍ_e}}}(\tilde{N}_j)}\right)^{℃}\right\}^{\frac{1}{℃}}}\right)\right)\end{array}\right),\\ \left(\begin{array}{c}\left(1-\dfrac{1}{1+\left\{\sum_{j=1}^{ꞩ}ß_j\left(\dfrac{\sigma_{\underline{\underline{Ɍ_e}}}(\tilde{N}_j)}{1-\sigma_{\underline{\underline{Ɍ_e}}}(\tilde{N}_j)}\right)^{℃}\right\}^{\frac{1}{℃}}}+\iota\left(1-\dfrac{1}{1+\left\{\sum_{j=1}^{ꞩ}ß_j\left(\dfrac{\rho_{\underline{\underline{Ɍ_e}}}(\tilde{N}_j)}{1-\rho_{\underline{\underline{Ɍ_e}}}(\tilde{N}_j)}\right)^{℃}\right\}^{\frac{1}{℃}}}\right),\right.\\ \left.\dfrac{1}{1+\left\{\sum_{j=1}^{ꞩ}ß_j\left(\dfrac{1-\tau_{\underline{\underline{Ɍ_e}}}(\tilde{N}_j)}{\tau_{\underline{\underline{Ɍ_e}}}(\tilde{N}_j)}\right)^{℃}\right\}^{\frac{1}{℃}}}+\iota\left(\dfrac{1}{1+\left\{\sum_{j=1}^{ꞩ}ß_j\left(\dfrac{1-\upsilon_{\underline{\underline{Ɍ_e}}}(\tilde{N}_j)}{\upsilon_{\underline{\underline{Ɍ_e}}}(\tilde{N}_j)}\right)^{℃}\right\}^{\frac{1}{℃}}}\right)\right)\end{array}\right)\end{array}\right)$$

Proof. See Appendix A.

Theorem 2. (Idempotency property) Assume $Ɍ_e(\tilde{N}_j)=\left(\overline{\overline{Ɍ_e}}(\tilde{N}_j),\underline{\underline{Ɍ_e}}(\tilde{N}_j)\right)(j=1,2,3\ldots,ꞩ)$ be a gathering of CIFRNs and $ß = (ß_1, ß_2, ß_3 \ldots, ß_{\mathfrak{s}})^T$ be the WV with $ß_j \in [0,1]$ such that $\sum_{j=1}^{\mathfrak{s}} ß_j = 1$. If $Ꞧ_e(\tilde{Ñ}_j) = Ꞧ_e(\tilde{Ñ})\ \forall\ (j = 1,2,3 \ldots, \mathfrak{s})$, where $Ꞧ_e(\tilde{Ñ}) = \left(\overline{\overline{Ꞧ_e}}(\tilde{Ñ}), \underline{\underline{Ꞧ_e}}(\tilde{Ñ})\right)$ then

$$\text{CIFRDWA}\left(Ꞧ_e(\tilde{Ñ}_1), Ꞧ_e(\tilde{Ñ}_2), \ldots, Ꞧ_e(\tilde{Ñ}_{\mathfrak{s}})\right) = Ꞧ_e(\tilde{Ñ})$$

Proof. See Appendix A.

Theorem 3. (Boundedness property) Assume $Ꞧ_e(\tilde{Ñ}_j) = \left(\left(Ꞧ_e(\tilde{Ñ}_j)\right)^+, \left(Ꞧ_e(\tilde{Ñ}_j)\right)^-\right) (t = 1,2,3 \ldots, \mathfrak{s})$ be a gathering of CIFRNs where $\left(Ꞧ_e(\tilde{Ñ}_j)\right)^+ = \left(\min_j \overline{\overline{Ꞧ_e}}(\tilde{Ñ}_j), \max_j \underline{\underline{Ꞧ_e}}(\tilde{Ñ}_j)\right)$ and $\left(Ꞧ_e(\tilde{Ñ}_j)\right)^- = \left(\max_j \overline{\overline{Ꞧ_e}}(\tilde{Ñ}_j), \min_j \underline{\underline{Ꞧ_e}}(\tilde{Ñ}_j)\right)$, then

$$\left(Ꞧ_e(\tilde{Ñ}_j)\right)^- \leq \text{CIFRDWA}\left(Ꞧ_e(\tilde{Ñ}_1), Ꞧ_e(\tilde{Ñ}_2), \ldots, Ꞧ_e(\tilde{Ñ}_{\mathfrak{s}})\right) \leq \left(Ꞧ_e(\tilde{Ñ}_j)\right)^+$$

Proof. See Appendix A.

Theorem 4. (Monotonicity property) Suppose

$$Ꞧ_e(\tilde{Ñ}_j) = \left(\overline{\overline{Ꞧ_e}}(\tilde{Ñ}_j), \underline{\underline{Ꞧ_e}}(\tilde{Ñ}_j)\right) = \begin{pmatrix} \left(\sigma_{\overline{\overline{Ꞧ_e}}}(\tilde{Ñ}_j) + \iota\rho_{\overline{\overline{Ꞧ_e}}}(\tilde{Ñ}_j), \tau_{\overline{\overline{Ꞧ_e}}}(\tilde{Ñ}_j) + \iota\upsilon_{\overline{\overline{Ꞧ_e}}}(\tilde{Ñ}_j)\right), \\ \left(\sigma_{\underline{\underline{Ꞧ_e}}}(\tilde{Ñ}_j) + \iota\rho_{\underline{\underline{Ꞧ_e}}}(\tilde{Ñ}_j), \tau_{\underline{\underline{Ꞧ_e}}}(\tilde{Ñ}_j) + \iota\upsilon_{\underline{\underline{Ꞧ_e}}}(\tilde{Ñ}_j)\right) \end{pmatrix}$$

and

$$Ꞧ_e(\tilde{Ñ}'_j) = \left(\overline{\overline{Ꞧ_e}}(\tilde{Ñ}'_j), \underline{\underline{Ꞧ_e}}(\tilde{Ñ}'_j)\right) = \begin{pmatrix} \left(\sigma_{\overline{\overline{Ꞧ_e}}}(\tilde{Ñ}'_j) + \iota\rho_{\overline{\overline{Ꞧ_e}}}(\tilde{Ñ}'_j), \tau_{\overline{\overline{Ꞧ_e}}}(\tilde{Ñ}'_j) + \iota\upsilon_{\overline{\overline{Ꞧ_e}}}(\tilde{Ñ}'_j)\right), \\ \left(\sigma_{\underline{\underline{Ꞧ_e}}}(\tilde{Ñ}'_j) + \iota\rho_{\underline{\underline{Ꞧ_e}}}(\tilde{Ñ}'_j), \tau_{\underline{\underline{Ꞧ_e}}}(\tilde{Ñ}'_j) + \iota\upsilon_{\underline{\underline{Ꞧ_e}}}(\tilde{Ñ}'_j)\right) \end{pmatrix} (j = 1, 2, \ldots, \mathfrak{s})$$

be a gathering of two CIFRSs, and $ß = (ß_1, ß_2, \ldots, ß_{\mathfrak{s}})^T$ be the WV with $ß_t \in [0,1]$ and $\sum_{t=1}^{\mathfrak{s}} ß_t = 1$. If $\overline{\overline{Ꞧ_e}}(\tilde{Ñ}_j) \leq \overline{\overline{Ꞧ_e}}(B_t)$, $\underline{\underline{Ꞧ_e}}(\tilde{Ñ}_j) \leq \underline{\underline{Ꞧ_e}}(\tilde{Ñ}'_j)$, then

$$\text{CIFRDWA}\left(Ꞧ_e(\tilde{Ñ}_1), Ꞧ_e(\tilde{Ñ}_2), \ldots, Ꞧ_e(\tilde{Ñ}_{\mathfrak{s}})\right) \leq \text{CIFRDWA}\left(Ꞧ_e(\tilde{Ñ}'_1), Ꞧ_e(\tilde{Ñ}'_2), \ldots, Ꞧ_e(\tilde{Ñ}'_{\mathfrak{s}})\right)$$

Proof. The proof can be followed from theorem 2 and 3.

Definition 12. Let $Ꞧ_e(\tilde{Ñ}_j) = \left(\overline{\overline{Ꞧ_e}}(\tilde{Ñ}_j), \underline{\underline{Ꞧ_e}}(\tilde{Ñ}_j)\right) (j = 1,2,3 \ldots, \mathfrak{s})$ be a gathering of CIFRNs and $ß = (ß_1, ß_2, ß_3 \ldots, ß_{\mathfrak{s}})^T$ be the WV with $ß_j \in [0,1]$ such that $\sum_{j=1}^{\mathfrak{s}} ß_j = 1$. Then, a CIFRDOWA operator is determined as

$$\text{CIFRDOWA}\left(Ꞧ_e(\tilde{N}_1), Ꞧ_e(\tilde{N}_2), \ldots, Ꞧ_e(\tilde{N}_ꞩ)\right) = \left(\oplus_{j=1}^{ꞩ} ß_j \overline{\overline{Ꞧ_e}}\left(\tilde{N}_{ŏ(j)}\right), \oplus_{j=1}^{ꞩ} ß_j \underline{\underline{Ꞧ_e}}\left(\tilde{N}_{ŏ(j)}\right)\right)$$

$$= \left(\left(ß_1 \overline{\overline{Ꞧ_e}}\left(\tilde{N}_{ŏ(1)}\right) \oplus ß_2 \overline{\overline{Ꞧ_e}}\left(\tilde{N}_{ŏ(2)}\right) \oplus \ldots\ldots \oplus ß_ꞩ \overline{\overline{Ꞧ_e}}\left(\tilde{N}_{ŏ(ꞩ)}\right)\right), \left(ß_1 \underline{\underline{Ꞧ_e}}\left(\tilde{N}_{ŏ(1)}\right) \oplus ß_2 \underline{\underline{Ꞧ_e}}\left(\tilde{N}_{ŏ(2)}\right) \oplus \ldots\ldots \oplus ß_ꞩ \underline{\underline{Ꞧ_e}}\left(\tilde{N}_{ŏ(ꞩ)}\right)\right)\right)$$

Where $\left(ŏ(1), ŏ(2), ŏ(3), \ldots, ŏ(ꞩ)\right)$ is a permutation of the largest gathering $Ꞧ_e\left(\tilde{N}_{ŏ(j)}\right) \forall j$.

From the above definition the results for CIFRDOWA operator are:

Theorem 5. Using the equation above, we obtain the CIFRNs and

$$\text{CIFRDOWA}\left(R_e^*(\tilde{N}_1), R_e^*(\tilde{N}_2), \ldots, R_e^*(\tilde{N}_ꞩ)\right) = \left(\oplus_{j=1}^{ꞩ} ß_j \overline{\overline{Ꞧ_e}}\left(\tilde{N}_{ŏ(j)}\right), \oplus_{j=1}^{ꞩ} ß_j \underline{\underline{Ꞧ_e}}\left(\tilde{N}_{ŏ(j)}\right)\right) =$$

$$\left(\begin{array}{l}
\left(\begin{array}{l}
1 - \frac{1}{1 + \left\{\sum_{j=1}^{ꞩ} ß_j \left(\frac{\sigma_{\overline{\overline{Ꞧ_e}}}\left(\tilde{N}_{ŏ(j)}\right)}{1 - \sigma_{\overline{\overline{Ꞧ_e}}}\left(\tilde{N}_{ŏ(j)}\right)}\right)^{℃}\right\}^{\frac{1}{℃}}} + \iota\left(1 - \frac{1}{1 + \left\{\sum_{j=1}^{ꞩ} ß_j \left(\frac{\rho_{\overline{\overline{Ꞧ_e}}}\left(\tilde{N}_{ŏ(j)}\right)}{1 - \rho_{\overline{\overline{Ꞧ_e}}}\left(\tilde{N}_{ŏ(j)}\right)}\right)^{℃}\right\}^{\frac{1}{℃}}}\right), \\
\frac{1}{1 + \left\{\sum_{j=1}^{ꞩ} ß_j \left(\frac{1 - \tau_{\overline{\overline{Ꞧ_e}}}\left(\tilde{N}_{ŏ(j)}\right)}{\tau_{\overline{\overline{Ꞧ_e}}}\left(\tilde{N}_{ŏ(j)}\right)}\right)^{℃}\right\}^{\frac{1}{℃}}} + \iota\left(\frac{1}{1 + \left\{\sum_{j=1}^{ꞩ} ß_j \left(\frac{1 - \upsilon_{\overline{\overline{Ꞧ_e}}}\left(\tilde{N}_{ŏ(j)}\right)}{\upsilon_{\overline{\overline{Ꞧ_e}}}\left(\tilde{N}_{ŏ(j)}\right)}\right)^{℃}\right\}^{\frac{1}{℃}}}\right)
\end{array}\right), \\
\left(\begin{array}{l}
1 - \frac{1}{1 + \left\{\sum_{j=1}^{ꞩ} ß_j \left(\frac{\sigma_{\underline{\underline{Ꞧ_e}}}\left(\tilde{N}_{ŏ(j)}\right)}{1 - \sigma_{\underline{\underline{Ꞧ_e}}}\left(\tilde{N}_{ŏ(j)}\right)}\right)^{℃}\right\}^{\frac{1}{℃}}} + \iota\left(1 - \frac{1}{1 + \left\{\sum_{j=1}^{ꞩ} ß_j \left(\frac{\rho_{\underline{\underline{Ꞧ_e}}}\left(\tilde{N}_{ŏ(j)}\right)}{1 - \rho_{\underline{\underline{Ꞧ_e}}}\left(\tilde{N}_{ŏ(j)}\right)}\right)^{℃}\right\}^{\frac{1}{℃}}}\right), \\
\frac{1}{1 + \left\{\sum_{j=1}^{ꞩ} ß_j \left(\frac{1 - \tau_{\underline{\underline{Ꞧ_e}}}\left(\tilde{N}_{ŏ(j)}\right)}{\tau_{\underline{\underline{Ꞧ_e}}}\left(\tilde{N}_{ŏ(j)}\right)}\right)^{℃}\right\}^{\frac{1}{℃}}} + \iota\left(\frac{1}{1 + \left\{\sum_{j=1}^{ꞩ} ß_j \left(\frac{1 - \upsilon_{\underline{\underline{Ꞧ_e}}}\left(\tilde{N}_{ŏ(j)}\right)}{\upsilon_{\underline{\underline{Ꞧ_e}}}\left(\tilde{N}_{ŏ(j)}\right)}\right)^{℃}\right\}^{\frac{1}{℃}}}\right)
\end{array}\right)
\end{array}\right)$$

Proof. The proof is similar to theorem 1.

Theorem 6. (Idempotency property) Assume $\text{Ʀ}_e(\tilde{\text{N}}_j) = \left(\overline{\overline{\text{Ʀ}_e}}(\tilde{\text{N}}_j), \underline{\underline{\text{Ʀ}_e}}(\tilde{\text{N}}_j)\right)(j = 1,2,3 \dots, \mathfrak{s})$ be a gathering of CIFRNs and $\text{ß} = (\text{ß}_1, \text{ß}_2, \text{ß}_3 \dots, \text{ß}_\mathfrak{s})^T$ be the WV with $\text{ß}_j \in [0, 1]$ such that $\sum_{j=1}^{\mathfrak{s}} \text{ß}_j = 1$. If $\text{Ʀ}_e(\tilde{\text{N}}_j) = \text{Ʀ}_e(\tilde{\text{N}})\ \forall\ (j = 1,2,3 \dots, \mathfrak{s})$, where $\text{Ʀ}_e(\tilde{\text{N}}) = \left(\overline{\overline{\text{Ʀ}_e}}(\tilde{\text{N}}), \underline{\underline{\text{Ʀ}_e}}(\tilde{\text{N}})\right)$ then

$$\text{CIFRDOWA}\left(\text{Ʀ}_e(\tilde{\text{N}}_1), \text{Ʀ}_e(\tilde{\text{N}}_2), \dots, \text{Ʀ}_e(\tilde{\text{N}}_\mathfrak{s})\right) = \text{Ʀ}_e(\tilde{\text{N}})$$

Proof. The proof is similar to theorem 2.

Theorem 7. (Boundedness property) Assume $\text{Ʀ}_e(\tilde{\text{N}}_j) = \left(\left(\text{Ʀ}_e(\tilde{\text{N}}_j)\right)^+, \left(\text{Ʀ}_e(\tilde{\text{N}}_j)\right)^-\right)(t = 1,2,3 \dots, \mathfrak{s})$ be a gathering of CIFRNs where $\left(\text{Ʀ}_e(\tilde{\text{N}}_j)\right)^+ = \left(\min_t \overline{\overline{\text{Ʀ}_e}}(\tilde{\text{N}}_j), \max_t \underline{\underline{\text{Ʀ}_e}}(\tilde{\text{N}}_j)\right)$ and $\left(\text{Ʀ}_e(\tilde{\text{N}}_j)\right)^- = \left(\max_j \overline{\overline{\text{Ʀ}_e}}(\tilde{\text{N}}_j), \min_j \underline{\underline{\text{Ʀ}_e}}(\tilde{\text{N}}_j)\right)$, then

$$\left(\text{Ʀ}_e(\tilde{\text{N}}_j)\right)^- \le \text{CIFRDOWA}\left(\text{Ʀ}_e(\tilde{\text{N}}_1), \text{Ʀ}_e(\tilde{\text{N}}_2), \dots, \text{Ʀ}_e(\tilde{\text{N}}_\mathfrak{s})\right) \le \left(\text{Ʀ}_e(\tilde{\text{N}}_j)\right)^+$$

Proof. The prove is similar to theorem 3.

Theorem 8. (Monotonicity property) Suppose

$$\text{Ʀ}_e(\tilde{\text{N}}_j) = \left(\overline{\overline{\text{Ʀ}_e}}(\tilde{\text{N}}_j), \underline{\underline{\text{Ʀ}_e}}(\tilde{\text{N}}_j)\right) = \begin{pmatrix} \left(\sigma_{\overline{\overline{\text{Ʀ}_e}}}(\tilde{\text{N}}_j) + \iota\rho_{\overline{\overline{\text{Ʀ}_e}}}(\tilde{\text{N}}_j), \tau_{\overline{\overline{\text{Ʀ}_e}}}(\tilde{\text{N}}_j) + \iota\upsilon_{\overline{\overline{\text{Ʀ}_e}}}(\tilde{\text{N}}_j)\right), \\ \left(\sigma_{\underline{\underline{\text{Ʀ}_e}}}(\tilde{\text{N}}_j) + \iota\rho_{\underline{\underline{\text{Ʀ}_e}}}(\tilde{\text{N}}_j), \tau_{\underline{\underline{\text{Ʀ}_e}}}(\tilde{\text{N}}_j) + \iota\upsilon_{\underline{\underline{\text{Ʀ}_e}}}(\tilde{\text{N}}_j)\right) \end{pmatrix}$$

and

$$\text{Ʀ}_e(\tilde{\text{N}}'_j) = \left(\overline{\overline{\text{Ʀ}_e}}(\tilde{\text{N}}'_j), \underline{\underline{\text{Ʀ}_e}}(\tilde{\text{N}}'_j)\right) = \begin{pmatrix} \left(\sigma_{\overline{\overline{\text{Ʀ}_e}}}(\tilde{\text{N}}'_j) + \iota\rho_{\overline{\overline{\text{Ʀ}_e}}}(\tilde{\text{N}}'_j), \tau_{\overline{\overline{\text{Ʀ}_e}}}(\tilde{\text{N}}'_j) + \iota\upsilon_{\overline{\overline{\text{Ʀ}_e}}}(\tilde{\text{N}}'_j)\right), \\ \left(\sigma_{\underline{\underline{\text{Ʀ}_e}}}(\tilde{\text{N}}'_j) + \iota\rho_{\underline{\underline{\text{Ʀ}_e}}}(\tilde{\text{N}}'_j), \tau_{\underline{\underline{\text{Ʀ}_e}}}(\tilde{\text{N}}'_j) + \iota\upsilon_{\underline{\underline{\text{Ʀ}_e}}}(\tilde{\text{N}}'_j)\right) \end{pmatrix}(j = 1, 2, \dots, \mathfrak{s})$$

be a gathering of two CIFRSs, and $\text{ß} = (\text{ß}_1, \text{ß}_2, \dots, \text{ß}_\mathfrak{s})^T$ be the WV with $\text{ß}_\text{t} \in [0, 1]$ and $\sum_{\text{t}=1}^{\mathfrak{s}} \text{ß}_\text{t} = 1$. If $\overline{\overline{\text{Ʀ}_e}}(\tilde{\text{N}}_j) \le \overline{\overline{\text{Ʀ}_e}}(B_t)$, $\underline{\underline{\text{Ʀ}_e}}(\tilde{\text{N}}_j) \le \underline{\underline{\text{Ʀ}_e}}(\tilde{\text{N}}'_j)$, then

$$\text{CIFRDWA}\left(\text{Ʀ}_e(\tilde{\text{N}}_1), \text{Ʀ}_e(\tilde{\text{N}}_2), \dots, \text{Ʀ}_e(\tilde{\text{N}}_\mathfrak{s})\right) \le \text{CIFRDOWA}\left(\text{Ʀ}_e(\tilde{\text{N}}'_1), \text{Ʀ}_e(\tilde{\text{N}}'_2), \dots, \text{Ʀ}_e(\tilde{\text{N}}'_\mathfrak{s})\right)$$

Proof. The proof can be followed from theorem 2 and 3.

4.1. CIFR Dombi geometric AOs

This part includes, the idea of the CIFRDWG operator, and the CIFRDOWG operator based on

CIFRNs, and we discuss their properties using basic operations.

Definition 13. Suppose $Ɍ_e(\tilde{N}_j) = \left(\overline{\overline{Ɍ_e}}(\tilde{N}_j), \underline{\underline{Ɍ_e}}(\tilde{N}_j)\right)(j = 1,2,3 \ldots, ꞩ)$ be a gathering of CIFRNs and $ß = (ß_1, ß_2, ß_3 \ldots, ß_ꞩ)^T$ be the WV with $ß_j \in [0, 1]$ such that $\sum_{j=1}^{ꞩ} ß_j = 1$, then a CIFRDWG operator is defined as

$$\text{CIFRDWG}\left(Ɍ_e(\tilde{N}_1), Ɍ_e(\tilde{N}_2), \ldots, Ɍ_e(\tilde{N}_ꞩ)\right) = \left(\otimes_{j=1}^{ꞩ} ß_j \overline{\overline{Ɍ_e}}(\tilde{N}_j), \otimes_{t=1}^{ꞩ} ß_j\, \underline{\underline{Ɍ_e}}(\tilde{N}_j)\right)$$

$$= \left(\left(ß_1 \overline{\overline{Ɍ_e}}(\tilde{N}_1) \otimes ß_2 \overline{\overline{Ɍ_e}}(\tilde{N}_2) \otimes \ldots\ldots \otimes ß_ꞩ \overline{\overline{Ɍ_e}}(\tilde{N}_ꞩ)\right), \left(ß_1 \underline{\underline{Ɍ_e}}(\tilde{N}_1) \otimes ß_2 \underline{\underline{Ɍ_e}}(\tilde{N}_2) \otimes \ldots\ldots \otimes ß_ꞩ \underline{\underline{Ɍ_e}}(\tilde{N}_ꞩ)\right)\right)$$

The following are the outcomes for the CIFRDWG operator based on the definition above.

Theorem 9. Using the equation above, we obtain the CIFRNs and

$$\text{CIFRDWG}\left(Ɍ_e(\tilde{N}_1), Ɍ_e(\tilde{N}_2), \ldots, Ɍ_e(\tilde{N}_ꞩ)\right) = \left(\otimes_{j=1}^{ꞩ} ß_j \overline{\overline{Ɍ_e}}(\tilde{N}_j), \otimes_{j=1}^{ꞩ} ß_j\, \underline{\underline{Ɍ_e}}(\tilde{N}_j)\right)$$

$$= \left(\begin{array}{l} \left(\begin{array}{l} \frac{1}{1+\left\{\sum_{j=1}^{ꞩ} ß_j \left(\frac{1-\sigma_{\overline{\overline{Ɍ_e}}}(\tilde{N}_j)}{\sigma_{\overline{\overline{Ɍ_e}}}(\tilde{N}_j)}\right)^{℃}\right\}^{\frac{1}{℃}}} + \iota\left(\frac{1}{1+\left\{\sum_{j=1}^{ꞩ} ß_j \left(\frac{1-\rho_{\overline{\overline{Ɍ_e}}}(\tilde{N}_j)}{\rho_{\overline{\overline{Ɍ_e}}}(\tilde{N}_j)}\right)^{℃}\right\}^{\frac{1}{℃}}}\right), \\ 1-\frac{1}{1+\left\{\sum_{j=1}^{ꞩ} ß_j \left(\frac{\tau_{\overline{\overline{Ɍ_e}}}(\tilde{N}_j)}{1-\tau_{\overline{\overline{Ɍ_e}}}(\tilde{N}_j)}\right)^{℃}\right\}^{\frac{1}{℃}}} + \iota\left(1-\frac{1}{1+\left\{\sum_{j=1}^{ꞩ} ß_j \left(\frac{\upsilon_{\overline{\overline{Ɍ_e}}}(\tilde{N}_j)}{1-\upsilon_{\overline{\overline{Ɍ_e}}}(\tilde{N}_j)}\right)^{℃}\right\}^{\frac{1}{℃}}}\right) \end{array}\right), \\ \left(\begin{array}{l} \frac{1}{1+\left\{\sum_{j=1}^{ꞩ} ß_j \left(\frac{1-\sigma_{\underline{\underline{Ɍ_e}}}(\tilde{N}_j)}{\sigma_{\underline{\underline{Ɍ_e}}}(\tilde{N}_j)}\right)^{℃}\right\}^{\frac{1}{℃}}} + \iota\left(\frac{1}{1+\left\{\sum_{j=1}^{ꞩ} ß_j \left(\frac{1-\rho_{\underline{\underline{Ɍ_e}}}(\tilde{N}_j)}{\rho_{\underline{\underline{Ɍ_e}}}(\tilde{N}_j)}\right)^{℃}\right\}^{\frac{1}{℃}}}\right), \\ 1-\frac{1}{1+\left\{\sum_{j=1}^{ꞩ} ß_j \left(\frac{\tau_{\underline{\underline{Ɍ_e}}}(\tilde{N}_j)}{1-\tau_{\underline{\underline{Ɍ_e}}}(\tilde{N}_j)}\right)^{℃}\right\}^{\frac{1}{℃}}} + \iota\left(1-\frac{1}{1+\left\{\sum_{j=1}^{ꞩ} ß_j \left(\frac{\upsilon_{\underline{\underline{Ɍ_e}}}(\tilde{N}_j)}{1-\upsilon_{\underline{\underline{Ɍ_e}}}(\tilde{N}_j)}\right)^{℃}\right\}^{\frac{1}{℃}}}\right) \end{array}\right) \end{array}\right)$$

Proof. See Appendix A.

Theorem 10. (Idempotency property) Assume $\text{Ɽ}_e(\tilde{\text{N}}_j) = \left(\overline{\overline{\text{Ɽ}_e}}(\tilde{\text{N}}_j), \underline{\underline{\text{Ɽ}_e}}(\tilde{\text{N}}_j)\right) (j = 1,2,3 \ldots, \mathfrak{s})$ be a gathering of CIFRNs and $\text{ß} = (\text{ß}_1, \text{ß}_2, \text{ß}_3 \ldots, \text{ß}_\mathfrak{s})^T$ be the WV with $\text{ß}_j \in [0,1]$ such that $\sum_{j=1}^{\mathfrak{s}} \text{ß}_j = 1$. If $\text{Ɽ}_e(\tilde{\text{N}}_j) = \text{Ɽ}_e(\tilde{\text{N}}) \; \forall \; (j = 1,2,3 \ldots, \mathfrak{s})$, where $\text{Ɽ}_e(\tilde{\text{N}}) = \left(\overline{\overline{\text{Ɽ}_e}}(\tilde{\text{N}}), \underline{\underline{\text{Ɽ}_e}}(\tilde{\text{N}})\right)$ then

$$\text{CIFRDWG}\left(\text{Ɽ}_e(\tilde{\text{N}}_1), \text{Ɽ}_e(\tilde{\text{N}}_2), \ldots, \text{Ɽ}_e(\tilde{\text{N}}_\mathfrak{s})\right) = \text{Ɽ}_e(\tilde{\text{N}})$$

Theorem 11. (boundedness property) Assume $\text{Ɽ}_e(\tilde{\text{N}}_j) = \left(\left(\text{Ɽ}_e(\tilde{\text{N}}_j)\right)^+, \left(\text{Ɽ}_e(\tilde{\text{N}}_j)\right)^-\right) (t = 1,2,3 \ldots, \mathfrak{s})$ be a gathering of CIFRNs where $\left(\text{Ɽ}_e(\tilde{\text{N}}_j)\right)^+ = \left(\min_{\text{t}} \overline{\overline{\text{Ɽ}_e}}(\tilde{\text{N}}_j), \max_{\text{t}} \underline{\underline{\text{Ɽ}_e}}(\tilde{\text{N}}_j)\right)$ and $\left(\text{Ɽ}_e(\tilde{\text{N}}_j)\right)^- = \left(\max_j \overline{\overline{\text{Ɽ}_e}}(\tilde{\text{N}}_j), \min_j \underline{\underline{\text{Ɽ}_e}}(\tilde{\text{N}}_j)\right)$, then

$$\left(\text{Ɽ}_e(\tilde{\text{N}}_j)\right)^- \leq \text{CIFRDWG}\left(\text{Ɽ}_e(\tilde{\text{N}}_1), \text{Ɽ}_e(\tilde{\text{N}}_2), \ldots, \text{Ɽ}_e(\tilde{\text{N}}_\mathfrak{s})\right) \leq \left(\text{Ɽ}_e(\tilde{\text{N}}_j)\right)^+$$

Theorem 12. (Monotonicity property) Let

$$\text{Ɽ}_e(\tilde{\text{N}}_j) = \left(\overline{\overline{\text{Ɽ}_e}}(\tilde{\text{N}}_j), \underline{\underline{\text{Ɽ}_e}}(\tilde{\text{N}}_j)\right) = \begin{pmatrix} \left(\sigma_{\overline{\overline{\text{Ɽ}_e}}}(\tilde{\text{N}}_j) + \iota\rho_{\overline{\overline{\text{Ɽ}_e}}}(\tilde{\text{N}}_j), \tau_{\overline{\overline{\text{Ɽ}_e}}}(\tilde{\text{N}}_j) + \iota\upsilon_{\overline{\overline{\text{Ɽ}_e}}}(\tilde{\text{N}}_j)\right), \\ \left(\sigma_{\underline{\underline{\text{Ɽ}_e}}}(\tilde{\text{N}}_j) + \iota\rho_{\underline{\underline{\text{Ɽ}_e}}}(\tilde{\text{N}}_j), \tau_{\underline{\underline{\text{Ɽ}_e}}}(\tilde{\text{N}}_j) + \iota\upsilon_{\underline{\underline{\text{Ɽ}_e}}}(\tilde{\text{N}}_j)\right) \end{pmatrix}$$

and

$$\text{Ɽ}_e(\tilde{\text{N}}'_j) = \left(\overline{\overline{\text{Ɽ}_e}}(\tilde{\text{N}}'_j), \underline{\underline{\text{Ɽ}_e}}(\tilde{\text{N}}'_j)\right) = \begin{pmatrix} \left(\sigma_{\overline{\overline{\text{Ɽ}_e}}}(\tilde{\text{N}}'_j) + \iota\rho_{\overline{\overline{\text{Ɽ}_e}}}(\tilde{\text{N}}'_j), \tau_{\overline{\overline{\text{Ɽ}_e}}}(\tilde{\text{N}}'_j) + \iota\upsilon_{\overline{\overline{\text{Ɽ}_e}}}(\tilde{\text{N}}'_j)\right), \\ \left(\sigma_{\underline{\underline{\text{Ɽ}_e}}}(\tilde{\text{N}}'_j) + \iota\rho_{\underline{\underline{\text{Ɽ}_e}}}(\tilde{\text{N}}'_j), \tau_{\underline{\underline{\text{Ɽ}_e}}}(\tilde{\text{N}}'_j) + \iota\upsilon_{\underline{\underline{\text{Ɽ}_e}}}(\tilde{\text{N}}'_j)\right) \end{pmatrix} (j = 1, 2, \ldots, \mathfrak{s})$$

be a gathering of two CIFRSs, and $\text{ß} = (\text{ß}_1, \text{ß}_2, \ldots, \text{ß}_\mathfrak{s})^T$ be the WV with $\text{ß}_\text{t} \in [0,1]$ and $\sum_{\text{t}=1}^{\mathfrak{s}} \text{ß}_\text{t} = 1$. If $\overline{\overline{\text{Ɽ}_e}}(\tilde{\text{N}}_j) \leq \overline{\overline{\text{Ɽ}_e}}(B_t)$, $\underline{\underline{\text{Ɽ}_e}}(\tilde{\text{N}}_j) \leq \underline{\underline{\text{Ɽ}_e}}(\tilde{\text{N}}'_j)$, then

$$\text{CIFRDWG}\left(\text{Ɽ}_e(\tilde{\text{N}}_1), \text{Ɽ}_e(\tilde{\text{N}}_2), \ldots, \text{Ɽ}_e(\tilde{\text{N}}_\mathfrak{s})\right) \leq \text{CIFRDWG}\left(\text{Ɽ}_e(\tilde{\text{N}}'_1), \text{Ɽ}_e(\tilde{\text{N}}'_2), \ldots, \text{Ɽ}_e(\tilde{\text{N}}'_\mathfrak{s})\right)$$

Definition 14. Suppose $\text{Ɽ}_e(\tilde{\text{N}}_j) = \left(\overline{\overline{\text{Ɽ}_e}}(\tilde{\text{N}}_j), \underline{\underline{\text{Ɽ}_e}}(\tilde{\text{N}}_j)\right) (j = 1,2,3 \ldots, \mathfrak{s})$ be a gathering of CIFRNs and $\text{ß} = (\text{ß}_1, \text{ß}_2, \text{ß}_3 \ldots, \text{ß}_\mathfrak{s})^T$ be the WV with $\text{ß}_j \in [0,1]$ such that $\sum_{j=1}^{\mathfrak{s}} \text{ß}_j = 1$. Then, a CIFRDOWG operator is determined as

$$\text{CIFRDOWG}\left(\text{Ɽ}_e(\tilde{\text{N}}_1), \text{Ɽ}_e(\tilde{\text{N}}_2), \ldots, \text{Ɽ}_e(\tilde{\text{N}}_\mathfrak{s})\right) = \left(\otimes_{j=1}^{\mathfrak{s}} \text{ß}_j \overline{\overline{\text{Ɽ}_e}}(\tilde{\text{N}}_{\breve{o}(j)}), \otimes_{j=1}^{\mathfrak{s}} \text{ß}_j \underline{\underline{\text{Ɽ}_e}}(\tilde{\text{N}}_{\breve{o}(j)})\right)$$

$$= \left(\begin{array}{c}\left(ß_1\overline{\overline{\text{Ꞧ}_e}}(\tilde{\text{Ñ}}_{\breve{o}(1)})\otimes ß_2\overline{\overline{\text{Ꞧ}_e}}(\tilde{\text{Ñ}}_{\breve{o}(2)})\otimes \ldots\ldots \otimes ß_{ꞩ}\overline{\overline{\text{Ꞧ}_e}}(\tilde{\text{Ñ}}_{\breve{o}(ꞩ)})\right), \\ \left(ß_1\underline{\underline{\text{Ꞧ}_e}}(\tilde{\text{Ñ}}_{\breve{o}(1)})\otimes ß_2\underline{\underline{\text{Ꞧ}_e}}(\tilde{\text{Ñ}}_{\breve{o}(2)})\otimes \ldots\ldots \otimes ß_{ꞩ}\underline{\underline{\text{Ꞧ}_e}}(\tilde{\text{Ñ}}_{\breve{o}(ꞩ)})\right)\end{array}\right)$$

Where $(\breve{o}(1), \breve{o}(2), \breve{o}(3), \ldots, \breve{o}(ꞩ))$ is a permutation of the largest gathering $\text{Ꞧ}_e(\tilde{\text{Ñ}}_{\breve{o}(j)})\ \forall\, j$.

The following are the outcomes for the CIFRDWG operator based on the definition above.

Theorem 13. Using the equation above, we obtain the CIFRNs and

$$\text{CIFRDOWG}\left(R_e^*(\tilde{\text{Ñ}}_1), R_e^*(\tilde{\text{Ñ}}_2), \ldots, R_e^*(\tilde{\text{Ñ}}_{ꞩ})\right) = \left(\otimes_{j=1}^{ꞩ} ß_j\overline{\overline{\text{Ꞧ}_e}}(\tilde{\text{Ñ}}_{\breve{o}(j)}), \otimes_{j=1}^{ꞩ} ß_j\,\underline{\underline{\text{Ꞧ}_e}}(\tilde{\text{Ñ}}_{\breve{o}(j)})\right)$$

$$= \left(\begin{array}{c}\left(\begin{array}{c}\frac{1}{1+\left\{\sum_{j=1}^{ꞩ} ß_j\left(\frac{1-\sigma_{\overline{\overline{\text{Ꞧ}_e}}}(\tilde{\text{Ñ}}_{\breve{o}(j)})}{\sigma_{\overline{\overline{\text{Ꞧ}_e}}}(\tilde{\text{Ñ}}_{\breve{o}(j)})}\right)^{℃}\right\}^{\frac{1}{℃}}} + \iota\left(\frac{1}{1+\left\{\sum_{j=1}^{ꞩ} ß_j\left(\frac{1-\rho_{\overline{\overline{\text{Ꞧ}_e}}}(\tilde{\text{Ñ}}_{\breve{o}(j)})}{\rho_{\overline{\overline{\text{Ꞧ}_e}}}(\tilde{\text{Ñ}}_{\breve{o}(j)})}\right)^{℃}\right\}^{\frac{1}{℃}}}\right), \\ 1-\frac{1}{1+\left\{\sum_{j=1}^{ꞩ} ß_j\left(\frac{\tau_{\overline{\overline{\text{Ꞧ}_e}}}(\tilde{\text{Ñ}}_{\breve{o}(j)})}{1-\tau_{\overline{\overline{\text{Ꞧ}_e}}}(\tilde{\text{Ñ}}_{\breve{o}(j)})}\right)^{℃}\right\}^{\frac{1}{℃}}} + \\ \iota\left(1-\frac{1}{1+\left\{\sum_{j=1}^{ꞩ} ß_j\left(\frac{\upsilon_{\overline{\overline{\text{Ꞧ}_e}}}(\tilde{\text{Ñ}}_{\breve{o}(j)})}{1-\upsilon_{\overline{\overline{\text{Ꞧ}_e}}}(\tilde{\text{Ñ}}_{\breve{o}(j)})}\right)^{℃}\right\}^{\frac{1}{℃}}}\right)\end{array}\right), \\ \left(\begin{array}{c}\frac{1}{1+\left\{\sum_{j=1}^{ꞩ} ß_j\left(\frac{1-\sigma_{\underline{\underline{\text{Ꞧ}_e}}}(\tilde{\text{Ñ}}_{\breve{o}(j)})}{\sigma_{\underline{\underline{\text{Ꞧ}_e}}}(\tilde{\text{Ñ}}_{\breve{o}(j)})}\right)^{℃}\right\}^{\frac{1}{℃}}} + \iota\left(\frac{1}{1+\left\{\sum_{j=1}^{ꞩ} ß_j\left(\frac{1-\rho_{\underline{\underline{\text{Ꞧ}_e}}}(\tilde{\text{Ñ}}_{\breve{o}(j)})}{\rho_{\underline{\underline{\text{Ꞧ}_e}}}(\tilde{\text{Ñ}}_{\breve{o}(j)})}\right)^{℃}\right\}^{\frac{1}{℃}}}\right), \\ 1-\frac{1}{1+\left\{\sum_{j=1}^{ꞩ} ß_j\left(\frac{\tau_{\underline{\underline{\text{Ꞧ}_e}}}(\tilde{\text{Ñ}}_{\breve{o}(j)})}{1-\tau_{\underline{\underline{\text{Ꞧ}_e}}}(\tilde{\text{Ñ}}_{\breve{o}(j)})}\right)^{℃}\right\}^{\frac{1}{℃}}} + \\ \iota\left(1-\frac{1}{1+\left\{\sum_{j=1}^{ꞩ} ß_j\left(\frac{\upsilon_{\underline{\underline{\text{Ꞧ}_e}}}(\tilde{\text{Ñ}}_{\breve{o}(j)})}{1-\upsilon_{\underline{\underline{\text{Ꞧ}_e}}}(\tilde{\text{Ñ}}_{\breve{o}(j)})}\right)^{℃}\right\}^{\frac{1}{℃}}}\right)\end{array}\right)\end{array}\right)$$

Theorem 14. (Idempotency property) Assume $Ʀ_e(\tilde{Ñ}_j) = \left(\overline{\overline{Ʀ_e}}(\tilde{Ñ}_j), \underline{\underline{Ʀ_e}}(\tilde{Ñ}_j)\right) (j = 1,2,3 \ldots, \mathfrak{s})$ be a gathering of CIFRNs and $ß = (ß_1, ß_2, ß_3 \ldots, ß_\mathfrak{s})^T$ be the WV with $ß_j \in [0,1]$ such that $\sum_{j=1}^{\mathfrak{s}} ß_j = 1$. If $Ʀ_e(\tilde{Ñ}_j) = Ʀ_e(\tilde{Ñ})\ \forall\ (j = 1,2,3 \ldots, \mathfrak{s})$, where $Ʀ_e(\tilde{Ñ}) = \left(\overline{\overline{Ʀ_e}}(\tilde{Ñ}), \underline{\underline{Ʀ_e}}(\tilde{Ñ})\right)$ then

$$\text{CIFRDOWG}\left(Ʀ_e(\tilde{Ñ}_1), Ʀ_e(\tilde{Ñ}_2), \ldots, Ʀ_e(\tilde{Ñ}_\mathfrak{s})\right) = Ʀ_e(\tilde{Ñ})$$

Theorem 15. (Boundedness property) Assume $Ʀ_e(\tilde{Ñ}_j) = \left(\left(Ʀ_e(\tilde{Ñ}_j)\right)^+, \left(Ʀ_e(\tilde{Ñ}_j)\right)^-\right) (t = 1,2,3 \ldots, \mathfrak{s})$ be a gathering of CIFRNs where $\left(Ʀ_e(\tilde{Ñ}_j)\right)^+ = \left(\min_{t} \overline{\overline{Ʀ_e}}(\tilde{Ñ}_j), \max_{t} \underline{\underline{Ʀ_e}}(\tilde{Ñ}_j)\right)$ and $\left(Ʀ_e(\tilde{Ñ}_j)\right)^- = \left(\max_{j} \overline{\overline{Ʀ_e}}(\tilde{Ñ}_j), \min_{j} \underline{\underline{Ʀ_e}}(\tilde{Ñ}_j)\right)$, then

$$\left(Ʀ_e(\tilde{Ñ}_j)\right)^- \leq \text{CIFRDOWG}\left(Ʀ_e(\tilde{Ñ}_1), Ʀ_e(\tilde{Ñ}_2), \ldots, Ʀ_e(\tilde{Ñ}_\mathfrak{s})\right) \leq \left(Ʀ_e(\tilde{Ñ}_j)\right)^+$$

Theorem 16. (Monotonicity property**)** Let

$$Ʀ_e(\tilde{Ñ}_j) = \left(\overline{\overline{Ʀ_e}}(\tilde{Ñ}_j), \underline{\underline{Ʀ_e}}(\tilde{Ñ}_j)\right) = \begin{pmatrix} \left(\sigma_{\overline{\overline{Ʀ_e}}}(\tilde{Ñ}_j) + \iota\rho_{\overline{\overline{Ʀ_e}}}(\tilde{Ñ}_j), \tau_{\overline{\overline{Ʀ_e}}}(\tilde{Ñ}_j) + \iota\upsilon_{\overline{\overline{Ʀ_e}}}(\tilde{Ñ}_j)\right), \\ \left(\sigma_{\underline{\underline{Ʀ_e}}}(\tilde{Ñ}_j) + \iota\rho_{\underline{\underline{Ʀ_e}}}(\tilde{Ñ}_j), \tau_{\underline{\underline{Ʀ_e}}}(\tilde{Ñ}_j) + \iota\upsilon_{\underline{\underline{Ʀ_e}}}(\tilde{Ñ}_j)\right) \end{pmatrix}$$

and

$$Ʀ_e(\tilde{Ñ}'_j) = \left(\overline{\overline{Ʀ_e}}(\tilde{Ñ}'_j), \underline{\underline{Ʀ_e}}(\tilde{Ñ}'_j)\right) = \begin{pmatrix} \left(\sigma_{\overline{\overline{Ʀ_e}}}(\tilde{Ñ}'_j) + \iota\rho_{\overline{\overline{Ʀ_e}}}(\tilde{Ñ}'_j), \tau_{\overline{\overline{Ʀ_e}}}(\tilde{Ñ}'_j) + \iota\upsilon_{\overline{\overline{Ʀ_e}}}(\tilde{Ñ}'_j)\right), \\ \left(\sigma_{\underline{\underline{Ʀ_e}}}(\tilde{Ñ}'_j) + \iota\rho_{\underline{\underline{Ʀ_e}}}(\tilde{Ñ}'_j), \tau_{\underline{\underline{Ʀ_e}}}(\tilde{Ñ}'_j) + \iota\upsilon_{\underline{\underline{Ʀ_e}}}(\tilde{Ñ}'_j)\right) \end{pmatrix} (j = 1, 2, \ldots, \mathfrak{s})$$

be a gathering of two CIFRSs, and $ß = (ß_1, ß_2, \ldots, ß_\mathfrak{s})^T$ be the WV with $ß_t \in [0,1]$ and $\sum_{t=1}^{\mathfrak{s}} ß_t = 1$. If $\overline{\overline{Ʀ_e}}(\tilde{Ñ}_j) \leq \overline{\overline{Ʀ_e}}(B_t)$, $\underline{\underline{Ʀ_e}}(\tilde{Ñ}_j) \leq \underline{\underline{Ʀ_e}}(\tilde{Ñ}'_j)$, then

$$\text{CIFRDOWG}\left(Ʀ_e(\tilde{Ñ}_1), Ʀ_e(\tilde{Ñ}_2), \ldots, Ʀ_e(\tilde{Ñ}_\mathfrak{s})\right) \leq \text{CIFRDOWG}\left(Ʀ_e(\tilde{Ñ}'_1), Ʀ_e(\tilde{Ñ}'_2), \ldots, Ʀ_e(\tilde{Ñ}'_\mathfrak{s})\right)$$

5. A new approach of MADM in the framework of CIFR information

In this section, we discuss the MADM procedure under the investigated operators for determining the beneficial alternative from the class of alternatives.

Suppose $Ç_s = \{Ç_1, Ç_2, \ldots, Ç_m\}$ be a set of alternatives, $Ḅ_r = \{Ḅ_1, Ḅ_2, \ldots, Ḅ_n\}$ be a set of attributes and $ώ = (ώ_1, ώ_2, \ldots, ώ_n)^T$ be the WV of attributes with $ώ_r \in [0,1]$, $r = (1,2,3, \ldots, n)$ and $\sum_{r=1}^{n} ώ_r = 1$. Based on the deduced attributes, the decision maker will calculate the evaluated vales of the alternatives. These assessed values are found in the CIFRNs structure, which is, $Ḿ =$

$(\ddot{\Omega}_{sr})_{m\times n} = \left(\left(\overline{\overline{Ш}}_{sr}, \overline{\overline{И}}_{sr}\right), \left(\underline{\underline{Ш}}_{sr}, \underline{\underline{И}}_{sr}\right)\right)_{m\times n} = \begin{pmatrix} \left(\overline{\overline{\sigma}}_{sr} + i\overline{\overline{\rho}}_{sr}, \overline{\overline{\tau}}_{sr} + i\overline{\overline{\upsilon}}_{sr}\right), \\ \left(\underline{\underline{\sigma}}_{sr} + i\underline{\underline{\rho}}_{sr}, \underline{\underline{\tau}}_{sr} + i\underline{\underline{\upsilon}}_{sr}\right) \end{pmatrix}_{m\times n}$ is a CIFR decision matrix. To handle the MADM problem, we interpret the beneath algorithm.

5.1. Algorithm

To solve MADM dilemma, we use the CIFRDWA and CIFRDWG operators to create an algorithm in the framework of CIFRSs.

Stage-1: Using the CIFRDWA operator and CIFRDWG operator to the CIFR information provided in the matrix $\acute{M}$, find all the aggregated values of alternatives $Ç_s = \{Ç_1, Ç_2, \dots, Ç_m\}$.

Stage-2: the combined outcomes score values are examined by definition 8.

Stage-3: In this stage, use the score values to rank the alternatives and attain the optimum alternative.

Stage-4: End.

5.2. Case study

One of the methodologies and applications mentioned in the given references is Multiple Criteria Decision-Making (MADM) which is appropriate to the case of choosing an AI services provider. Chen et al. [55] showed that AHP, a MADM technique, can be used in construction topics and thus, it can be used in the selection of an AI service provider. This supports the company's approach of assessing the providers based on the multiple attributes of the service. Zhuang and Fu [56] have also conducted empirical studies on the housing preference structures, which is the same way that the company intends to structure the preferences for the AI services by advocating on the cost, expertise, reliability, and scalability. Its focus on non-paradigmatic transitions between metropoles suggests that it may be useful to examine regional differences, which might be useful when assessing the skills and efficiency of AI suppliers. Last, Chi et al. [57] used post-training analysis to budgeting criteria for unmanned aerial system design which is similar to the company's requirement of ranking attributes in decision making. Another feature that has also been looked at in the course of this study is sustainability, and there are other factors such as scalability, which should also be considered when choosing an appropriate AI provider. Together these references provide practical support for one of the company's strategic activities, namely, rational decision-making; moreover, these references stress that decision-making in the choice of business partners and technologies should take into account several factors.

To improve its business operations, the corporation is currently choosing an AI services supplier. Four important factors are evaluated during the DM process: Cost, expertise, reliability, and scalability. All are used to determine if the given AI service provider is suitable for the specific needs of the business.

1) $Ḅ_1$: **Cost:** The cost attribute means the money that is required to acquire and maintain the AI services. The organization needs to find an AI services provider who can offer quality solutions at reasonable market price.
2) $Ḅ_2$: **Expertise:** This is important when assessing the effectiveness of the AI services provider with regards to delivering unique solutions in AI technologies. Here, the organization is keen on

selecting a provider who has experienced professionals, skilled, and knowledgeable about the organization's sector.

3) $Ḅ_{3}$: **Reliability:** This placed so much reliance on the AI services provider who has to deliver reliable and dependable services. The supplier that is capable of meeting the set time lines, meeting the service delivery standards, and providing reliable support to the organization is considered very valuable.
4) $Ḅ_{4}$: **Scalability:** As for a number of weaknesses, one of them has to do with scalability: since the business is expected to expand and expand in the future, the possibility of expansion must be present. For the successful implementation of AI technology, the service provider selected needs to be able to scale up its solutions to accommodate the dynamics of the business.

The Artificial Intelligence Service Providers that have been identified include $Ç_{1}$, $Ç_{2}$, $Ç_{3}$, and $Ç_{4}$. In order to make the correct decision in regard to the needs and the strategic position of the company, every of the options will be evaluated according to the characteristics described above.

This statement outlines the structure for the company's MADM analysis through which the company will assess and rank the AI service providers based on aforementioned qualities in order to make the correct decision. The weights for each attribute are $(0.2, 0.3, 0.4, 0.1)$. Below Table 2 represents CIFRNs.

Table 2. Complex intuitionistic fuzzy rough numbers.

	$Ḅ_{1}$	$Ḅ_{2}$	$Ḅ_{3}$	$Ḅ_{4}$
$Ç_{1}$	$\left(\begin{pmatrix}(0.4+\iota 0.2),\\(0.5+\iota 0.3)\end{pmatrix},\begin{pmatrix}(0.6+\iota 0.2),\\(0.1+\iota 0.4)\end{pmatrix}\right)$	$\left(\begin{pmatrix}(0.2+\iota 0.3),\\(0.2+\iota 0.6)\end{pmatrix},\begin{pmatrix}(0.3+\iota 0.1),\\(0.1+\iota 0.5)\end{pmatrix}\right)$	$\left(\begin{pmatrix}(0.4+\iota 0.4),\\(0.2+\iota 0.5)\end{pmatrix},\begin{pmatrix}(0.4+\iota 0.3),\\(0.2+\iota 0.5)\end{pmatrix}\right)$	$\left(\begin{pmatrix}(0.3+\iota 0.5),\\(0.2+\iota 0.3)\end{pmatrix},\begin{pmatrix}(0.3+\iota 0.6),\\(0.2+\iota 0.3)\end{pmatrix}\right)$
$Ç_{2}$	$\left(\begin{pmatrix}(0.7+\iota 0.2),\\(0.2+\iota 0.6)\end{pmatrix},\begin{pmatrix}(0.3+\iota 0.4),\\(0.5+\iota 0.2)\end{pmatrix}\right)$	$\left(\begin{pmatrix}(0.4+\iota 0.3),\\(0.2+\iota 0.6)\end{pmatrix},\begin{pmatrix}(0.3+\iota 0.3),\\(0.2+\iota 0.6)\end{pmatrix}\right)$	$\left(\begin{pmatrix}(0.3+\iota 0.5),\\(0.5+\iota 0.3)\end{pmatrix},\begin{pmatrix}(0.4+\iota 0.3),\\(0.3+\iota 0.5)\end{pmatrix}\right)$	$\left(\begin{pmatrix}(0.5+\iota 0.2),\\(0.4+\iota 0.3)\end{pmatrix},\begin{pmatrix}(0.1+\iota 0.7),\\(0.6+\iota 0.2)\end{pmatrix}\right)$
$Ç_{3}$	$\left(\begin{pmatrix}(0.2+\iota 0.5),\\(0.7+\iota 0.1)\end{pmatrix},\begin{pmatrix}(0.7+\iota 0.2),\\(0.1+\iota 0.3)\end{pmatrix}\right)$	$\left(\begin{pmatrix}(0.2+\iota 0.4),\\(0.5+\iota 0.1)\end{pmatrix},\begin{pmatrix}(0.4+\iota 0.1),\\(0.3+\iota 0.7)\end{pmatrix}\right)$	$\left(\begin{pmatrix}(0.5+\iota 0.3),\\(0.2+\iota 0.4)\end{pmatrix},\begin{pmatrix}(0.2+\iota 0.3),\\(0.3+\iota 0.6)\end{pmatrix}\right)$	$\left(\begin{pmatrix}(0.6+\iota 0.3),\\(0.4+\iota 0.3)\end{pmatrix},\begin{pmatrix}(0.2+\iota 0.5),\\(0.6+\iota 0.2)\end{pmatrix}\right)$
$Ç_{4}$	$\left(\begin{pmatrix}(0.2+\iota 0.2),\\(0.4+\iota 0.6)\end{pmatrix},\begin{pmatrix}(0.4+\iota 0.3),\\(0.4+\iota 0.2)\end{pmatrix}\right)$	$\left(\begin{pmatrix}(0.5+\iota 0.4),\\(0.3+\iota 0.4)\end{pmatrix},\begin{pmatrix}(0.5+\iota 0.2),\\(0.4+\iota 0.1)\end{pmatrix}\right)$	$\left(\begin{pmatrix}(0.3+\iota 0.6),\\(0.2+\iota 0.3)\end{pmatrix},\begin{pmatrix}(0.1+\iota 0.3),\\(0.2+\iota 0.6)\end{pmatrix}\right)$	$\left(\begin{pmatrix}(0.1+\iota 0.8),\\(0.9+\iota 0.1)\end{pmatrix},\begin{pmatrix}(0.8+\iota 0.1),\\(0.2+\iota 0.4)\end{pmatrix}\right)$

Stage 1: Using the CIFRDWA operators to determine the values, $Ç_{s} = \{Ç_{1}, Ç_{2}, \ldots, Ç_{m}\}$

$$Ç_{1} = \begin{pmatrix}\left((0.1318 + i0.1472), (0.2325 + i0.5309)\right),\\ \left((0.2596 + i0.1732), (0.0453 + i0.6070)\right)\end{pmatrix}$$

$$Ç_{2} = \begin{pmatrix}\left((0.4110 + i0.1915), (0.1956 + i0.7228)\right),\\ \left((0.1192 + i0.2758), (0.2839 + i0.2884)\right)\end{pmatrix}$$

$$Ç_3 = \begin{pmatrix} ((0.2470 + i0.1753), (0.7807 + i0.0464)), \\ ((0.3852 + i0.0866), (0.1006 + i0.4215)) \end{pmatrix}$$

$$Ç_4 = \begin{pmatrix} ((0.1622 + i0.5695), (0.4896 + i0.1841)), \\ ((0.4992 + i0.0611), (0.4232 + i0.06728)) \end{pmatrix}$$

Stage 2: In $S_F(Ç_{\mathscr{s}})(\mathscr{s} = 1,\ 2,\ 3,\ 4)$, the obtained score values are

$$S_F(Ç_1) = 0.412, S_F(Ç_2) = 0.4383, S_F(Ç_3) = 0.4431, S_F(Ç_4) = 0.5159$$

Stage 3: Using the following score values $S_F(Ç_{\mathscr{s}})(\mathscr{s} = 1, 2, 3, 4)$ of the total CIFRNs, rank the values $\breve{A}_{\mathscr{s}}(\mathscr{s} = 1, 2, 3, 4)$:

$$Ç_4 > Ç_3 > Ç_2 > Ç_1.$$

$Ç_4$ is selected as the best.
Stage 4: End.

If we use the CIFRDWG operator then the result of above problem is follows:
Stage 1: Using the CIFRDWG operators to determine the values, $Ç_{\mathscr{s}} = \{Ç_1, Ç_2, \dots, Ç_m\}$.

$$Ç_1 = \begin{pmatrix} ((0.2565 + i0.2884), (0.1111 + i0.3610)), \\ ((0.4687 + i0.0676), (0.0183 + i0.2875)) \end{pmatrix}$$

$$Ç_2 = \begin{pmatrix} ((0.7113 + i0.2371), (0.1921 + i0.3782)), \\ ((0.1559 + i0.4876), (0.2054 + i0.3535)) \end{pmatrix}$$

$$Ç_3 = \begin{pmatrix} ((0.1991 + i0.5848), (0.4216 + i0.0918)), \\ ((0.4638 + i0.0675), (0.1511 + i0.5629)) \end{pmatrix}$$

$$Ç_4 = \begin{pmatrix} ((0.1470 + i0.3400), (0.8052 + i0.2475)), \\ ((0.7256 + i0.1250), (0.1124 + i0.3244)) \end{pmatrix}$$

Stage 2: In $S_F(Ç_{\mathscr{s}})(\mathscr{s} = 1, 2, 3, 4)$**,** the obtained score values of are

$$S_F(Ç_1) = 0.5379, S_F(Ç_2) = 0.5578, S_F(Ç_3) = 0.5109, S_F(Ç_4) = 0.4810$$

Stage 3: Using the following score values $S_F(Ç_{\mathscr{s}})(\mathscr{s} = 1, 2, 3, 4)$ of the total CIFRNs, rank the values $\breve{A}_{\mathscr{s}}(\mathscr{s} = 1, 2, 3, 4)$:

$$Ç_2 > Ç_1 > Ç_3 > Ç_4.$$

$Ç_2$ is selected as the best.
Stage 4: End

6. Comparative analysis

To demonstrate the value and significance of our proposed work, we present a comparative study between the proposed operators and existing ones. Comparison plays a vital role in assessing the accuracy and superiority of any newly introduced work, as it allows for a clearer understanding of the strengths, limitations, and applicability of various approaches. Through such comparisons, we aim to highlight the improvements offered by our methods in terms of accuracy, computational efficiency, and decision-making effectiveness. Hence, we investigate and compare decision making techniques from previously published approaches on FRSs and IFRSs with our proposed methods. Specifically, we evaluate our work against the methods presented by Xu [26], Xu and Yager [27], Seikh and Mandal [33], Hussain et al. [47], Yi et al. [50], Emam et al. [51], Chinram et al. [53], and Yahya et al. [58]. These methods were chosen for comparison due to their relevance and impact in the field of FRS based decision making, and because they represent a broad range of AOs and decision-making frameworks. Each of these methods introduces unique approaches to handling uncertainty and imprecision in decision making problems, making them ideal for a comprehensive comparison with our newly proposed Dombi AOs. A detailed comparison of the performance, properties, and advantages of these methods is presented in Table 3, which underscores the superiority and practical utility of our approach in solving complex decision-making problems.

Table 3. A Comparison of suggested and current work.

Methods	Score values of Alternatives				Rankings
	$\c{C}_1$	$\c{C}_2$	$\c{C}_3$	$\c{C}_4$	
Xu [26] method	NO				NO
Xu and Yager [27] method	NO				NO
Seikh and Mandal [33] method	NO				NO
Hussain et al. [47] method	NO				NO
Yi et al. [50] method	NO				NO
Emam et al. [51] method	NO				NO
Chinram et al. [53] method	NO				NO
Yahya et al. [58] method	NO				NO
CIFRDWA AOs (Proposed)	0.4357	0.4797	0.4452	0.5091	$\c{C}_4 > \c{C}_2 > \c{C}_3 > \c{C}_1$
CIFRDWG AOs (Proposed)	0.5379	0.5986	0.4794	0.4492	$\c{C}_2 > \c{C}_1 > \c{C}_3 > \c{C}_4$

Table 3 reveals the characteristic analysis of the delivered notions with all the existing literature above. We compare our work with IFWA by Xu [26] and IFWG by Xu and Yager [27] and observe that it can use the information that contains truth grade and false grade while the delivered notions rely on complex intuitionistic fuzzy rough information. Because of this, IFWA and IFWG cannot handle that type of information. Seikh and Mandal [33] utilize the IF Dombi AOs and can deal with the set of possible truth grade and false grade and cannot handle our developed structure data, such as roughness with complex data. If we compare our work with Hussain et al. [47], Chinram et al. [53], and Yahya et al. [58]. We observed that these methods can deal with IF information with roughness but it cannot handle the 2nd dimension information. Although Yi et al.'s [50] method can deal with complex information with roughness and based on this information it can aggregate the existing structures like FS and FRS but fails to aggregate our proposed structure. In the same way, Emam et al. [51] deduced frank AOs based on

CFRS, which can aggregate the complex information with roughness but it should be noted that it is unable to resolve the rough information in the form of truth and false aspects of any data whose values existed in the 2nd dimension. Therefore, none of the established theories solve the given information in Table 3. Hence, it is clear that our established work is more effective and can aggregate the information of FS, RS, IFS, CIFS, and IFRS.

7. Conclusions

With the advancements in FS theory, researchers have been exploring new structures to derive valuable insights. CIFRSs, which is generalized from the pioneering concept of IFRS for dealing with complex uncertainties. The ranges of values are extended to the unit circle in the complex plane for both truth grade and false grade along with the UA and LA. When choosing an AI service supplier, there are several factors to consider, including scalability, cost, performance, and dependability. Selecting the best suppliers according to these attributes is a MADM problem related to AI providers containing extra fuzzy information along with roughness. Therefore, based on CIFRSs and Dombi operational laws, here in this manuscript, we have established basic Dombi operational laws for CIFRNs, based on these operations we delivered the Dombi AOs such as CIFRDWA, CIFRDOWA, CIFRDWG, and CIFRDOWG operators. To show the effectiveness, superiority, and practicality of our approach, we use these notions to deliver a MADM technique and analyze a case study "Selection of AI provider". We compare our work with other work in order to reveal the supremacy and advantages.

7.1. Advantages

Using CIFRSs provides different key advantages, such as better handling of uncertainty and vagueness in AI provider evaluations. It enables flexible, nuanced decision-making through the use of complex intuitionistic truth grade and complex intuitionistic false grade, reducing information loss and enhanced accuracy in ranking AI providers. Additionally, CIFRSs support complex, multi-attribute evaluations in a scalable and adaptive ways. It makes CIFRS a powerful tool for organizations aiming to choose the most suitable AI provider. This technique is a suitable for managing the two-dimensional information of truth grade and false grade along with the lower and UA.

7.2. Limitations

This work has certain limitations and gaps: For example, when a decision maker encounters the information that the sum of the truth grade and false grade exceeds the range, then the basic condition of CIFRS fails. Furthermore, if the decision makers consider the picture fuzzy information that include the abstinence grade, then CIFRS also fail to handle this type of information.

7.3. Future work

Our long-term goal is to extend this theory to additional structures like bipolar complex fuzzy sets [59,60] and other structures including decision-making techniques [61–63].

Author contributions

Tahir Mehmood: Conceptualization, Supervision, Investigation, Methodology, Validation; Ahmad Idrees: Original draft preparation, Visualization, Formal analysis, Writing review and editing, Validation; Majed Albaity: Writing and review, Validation, Investigation, Formal analysis, visualization; Ubaid ur Rehman: Methodology, Investigation, Writing and review, Validation. All authors have read and approved the final version of the manuscript for publication.

Conflict of interest

The authors declare no conflict of interest.

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Appendix A

Proof of Theorem 1. Using a well-known mathematical induction (MI) technique, we demonstrate the preceding equation by assuming that, for $\mathfrak{s} = 2$, we obtain

$$\mathrm{CIFRDWA}\left(Ꞃ_e(\tilde{Ñ}_1), Ꞃ_e(\tilde{Ñ}_2)\right) = \left(\oplus_{j=1}^{2} ß_j \overline{\overline{Ꞃ_e}}(\tilde{Ñ}_j), \oplus_{j=1}^{2} ß_j \underline{\underline{Ꞃ_e}}(\tilde{Ñ}_j)\right)$$

$$= \left(ß_1 \overline{\overline{Ꞃ_e}}(\tilde{Ñ}_1) \oplus ß_2 \overline{\overline{Ꞃ_e}}(\tilde{Ñ}_2), ß_1 \underline{\underline{Ꞃ_e}}(\tilde{Ñ}_1) \oplus ß_2 \underline{\underline{Ꞃ_e}}(\tilde{Ñ}_2)\right)$$

$$
= \left(\left(\frac{ 1 - \frac{1}{1 + \left\{ \text{ß}_1 \left(\frac{\sigma_{\overline{\overline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_1)}{1 - \sigma_{\overline{\overline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_1)} \right)^{\text{℃}} + \text{ß}_2 \left(\frac{\sigma_{\overline{\overline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_2)}{1 - \sigma_{\overline{\overline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_2)} \right)^{\text{℃}} \right\}^{\frac{1}{\text{℃}}}} + \iota \left(1 - \frac{1}{1 + \left\{ \text{ß}_1 \left(\frac{\rho_{\overline{\overline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_1)}{1 - \rho_{\overline{\overline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_1)} \right)^{\text{℃}} + \text{ß}_2 \left(\frac{\rho_{\overline{\overline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_2)}{1 - \rho_{\overline{\overline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_2)} \right)^{\text{℃}} \right\}^{\frac{1}{\text{℃}}}} \right), }{ \frac{1}{1 + \left\{ \text{ß}_1 \left(\frac{1 - \tau_{\overline{\overline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_1)}{\tau_{\overline{\overline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_1)} \right)^{\text{℃}} + \text{ß}_2 \left(\frac{1 - \tau_{\overline{\overline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_2)}{\tau_{\overline{\overline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_2)} \right)^{\text{℃}} \right\}^{\frac{1}{\text{℃}}}} + \iota \frac{1}{1 + \left\{ \text{ß}_1 \left(\frac{1 - \upsilon_{\overline{\overline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_1)}{\upsilon_{\overline{\overline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_1)} \right)^{\text{℃}} + \text{ß}_2 \left(\frac{1 - \upsilon_{\overline{\overline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_2)}{\upsilon_{\overline{\overline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_2)} \right)^{\text{℃}} \right\}^{\frac{1}{\text{℃}}}} } \right), \left(\frac{ 1 - \frac{1}{1 + \left\{ \text{ß}_1 \left(\frac{\sigma_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_1)}{1 - \sigma_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_1)} \right)^{\text{℃}} + \text{ß}_2 \left(\frac{\sigma_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_2)}{1 - \sigma_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_2)} \right)^{\text{℃}} \right\}^{\frac{1}{\text{℃}}}} + \iota \left(1 - \frac{1}{1 + \left\{ \text{ß}_1 \left(\frac{\rho_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_1)}{1 - \rho_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_1)} \right)^{\text{℃}} + \text{ß}_2 \left(\frac{\rho_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_2)}{1 - \rho_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_2)} \right)^{\text{℃}} \right\}^{\frac{1}{\text{℃}}}} \right), }{ \frac{1}{1 + \left\{ \text{ß}_1 \left(\frac{1 - \tau_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_1)}{\tau_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_1)} \right)^{\text{℃}} + \text{ß}_2 \left(\frac{1 - \tau_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_2)}{\tau_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_2)} \right)^{\text{℃}} \right\}^{\frac{1}{\text{℃}}}} + \iota \frac{1}{1 + \left\{ \text{ß}_1 \left(\frac{1 - \upsilon_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_1)}{\upsilon_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_1)} \right)^{\text{℃}} + \text{ß}_2 \left(\frac{1 - \upsilon_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_2)}{\upsilon_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_2)} \right)^{\text{℃}} \right\}^{\frac{1}{\text{℃}}}} } \right) \right)
$$

$$= \left(\begin{matrix} \left(\begin{matrix} 1 - \frac{1}{1 + \left\{ \sum_{j=1}^{2} ß_j \left(\frac{\sigma_{\overline{\overline{Ꞧ_e}}}(\tilde{Ñ}_j)}{1 - \sigma_{\overline{\overline{Ꞧ_e}}}(\tilde{Ñ}_j)} \right)^{℃} \right\}^{\frac{1}{℃}}} + \iota \left(1 - \frac{1}{1 + \left\{ \sum_{j=1}^{2} ß_j \left(\frac{\rho_{\overline{\overline{Ꞧ_e}}}(\tilde{Ñ}_j)}{1 - \rho_{\overline{\overline{Ꞧ_e}}}(\tilde{Ñ}_j)} \right)^{℃} \right\}^{\frac{1}{℃}}} \right), \\ \frac{1}{1 + \left\{ \sum_{j=1}^{2} ß_j \left(\frac{1 - \tau_{\overline{\overline{Ꞧ_e}}}(\tilde{Ñ}_j)}{\tau_{\overline{\overline{Ꞧ_e}}}(\tilde{Ñ}_j)} \right)^{℃} \right\}^{\frac{1}{℃}}} + \iota \left(\frac{1}{1 + \left\{ \sum_{j=1}^{2} ß_j \left(\frac{1 - \upsilon_{\overline{\overline{Ꞧ_e}}}(\tilde{Ñ}_j)}{\upsilon_{\overline{\overline{Ꞧ_e}}}(\tilde{Ñ}_j)} \right)^{℃} \right\}^{\frac{1}{℃}}} \right) \end{matrix} \right), \\ \left(\begin{matrix} 1 - \frac{1}{1 + \left\{ \sum_{j=1}^{2} ß_j \left(\frac{\sigma_{\underline{\underline{Ꞧ_e}}}(\tilde{Ñ}_j)}{1 - \sigma_{\underline{\underline{Ꞧ_e}}}(\tilde{Ñ}_j)} \right)^{℃} \right\}^{\frac{1}{℃}}} + \iota \left(1 - \frac{1}{1 + \left\{ \sum_{j=1}^{2} ß_j \left(\frac{\rho_{\underline{\underline{Ꞧ_e}}}(\tilde{Ñ}_j)}{1 - \rho_{\underline{\underline{Ꞧ_e}}}(\tilde{Ñ}_j)} \right)^{℃} \right\}^{\frac{1}{℃}}} \right), \\ \frac{1}{1 + \left\{ \sum_{j=1}^{2} ß_j \left(\frac{1 - \tau_{\underline{\underline{Ꞧ_e}}}(\tilde{Ñ}_j)}{\tau_{\underline{\underline{Ꞧ_e}}}(\tilde{Ñ}_j)} \right)^{℃} \right\}^{\frac{1}{℃}}} + \iota \left(\frac{1}{1 + \left\{ \sum_{j=1}^{2} ß_j \left(\frac{1 - \upsilon_{\underline{\underline{Ꞧ_e}}}(\tilde{Ñ}_j)}{\upsilon_{\underline{\underline{Ꞧ_e}}}(\tilde{Ñ}_j)} \right)^{℃} \right\}^{\frac{1}{℃}}} \right) \end{matrix} \right) \end{matrix} \right)$$

The result is true for $\mathfrak{s} = 2$. Next, suppose that it is true for $\mathfrak{s} = Ķ$.

$$\text{CIFRDWA}\left(Ꞧ_e(\tilde{Ñ}_1), Ꞧ_e(\tilde{Ñ}_2), \ldots, Ꞧ_e(\tilde{Ñ}_{\mathfrak{s}})\right) = \left(\oplus_{j=1}^{Ķ} ß_j \overline{\overline{Ꞧ_e}}(\tilde{Ñ}_j), \oplus_{j=1}^{Ķ} ß_j \underline{\underline{Ꞧ_e}}(\tilde{Ñ}_j) \right)$$

$$= \left(\begin{array}{l} \left(\begin{array}{l} 1 - \dfrac{1}{1 + \left\{ \sum_{j=1}^{Ⱪ} ß_j \left(\dfrac{\sigma_{\overline{\overline{Ɽ_e}}}(Ñ_j)}{1 - \sigma_{\overline{\overline{Ɽ_e}}}(Ñ_j)} \right)^{℃} \right\}^{\frac{1}{℃}}} + \iota \left(1 - \dfrac{1}{1 + \left\{ \sum_{j=1}^{Ⱪ} ß_j \left(\dfrac{\rho_{\overline{\overline{Ɽ_e}}}(Ñ_j)}{1 - \rho_{\overline{\overline{Ɽ_e}}}(Ñ_j)} \right)^{℃} \right\}^{\frac{1}{℃}}} \right), \\ \dfrac{1}{1 + \left\{ \sum_{j=1}^{Ⱪ} ß_j \left(\dfrac{1 - \tau_{\overline{\overline{Ɽ_e}}}(Ñ_j)}{\tau_{\overline{\overline{Ɽ_e}}}(Ñ_j)} \right)^{℃} \right\}^{\frac{1}{℃}}} + \iota \left(\dfrac{1}{1 + \left\{ \sum_{j=1}^{Ⱪ} ß_j \left(\dfrac{1 - \upsilon_{\overline{\overline{Ɽ_e}}}(Ñ_j)}{\upsilon_{\overline{\overline{Ɽ_e}}}(Ñ_j)} \right)^{℃} \right\}^{\frac{1}{℃}}} \right) \end{array} \right), \\ \left(\begin{array}{l} 1 - \dfrac{1}{1 + \left\{ \sum_{j=1}^{Ⱪ} ß_j \left(\dfrac{\sigma_{\underline{\underline{Ɽ_e}}}(Ñ_j)}{1 - \sigma_{\underline{\underline{Ɽ_e}}}(Ñ_j)} \right)^{℃} \right\}^{\frac{1}{℃}}} + \iota \left(1 - \dfrac{1}{1 + \left\{ \sum_{j=1}^{Ⱪ} ß_j \left(\dfrac{\rho_{\underline{\underline{Ɽ_e}}}(Ñ_j)}{1 - \rho_{\underline{\underline{Ɽ_e}}}(Ñ_j)} \right)^{℃} \right\}^{\frac{1}{℃}}} \right), \\ \dfrac{1}{1 + \left\{ \sum_{j=1}^{Ⱪ} ß_j \left(\dfrac{1 - \tau_{\underline{\underline{Ɽ_e}}}(Ñ_j)}{\tau_{\underline{\underline{Ɽ_e}}}(Ñ_j)} \right)^{℃} \right\}^{\frac{1}{℃}}} + \iota \left(\dfrac{1}{1 + \left\{ \sum_{j=1}^{Ⱪ} ß_j \left(\dfrac{1 - \upsilon_{\underline{\underline{Ɽ_e}}}(Ñ_j)}{\upsilon_{\underline{\underline{Ɽ_e}}}(Ñ_j)} \right)^{℃} \right\}^{\frac{1}{℃}}} \right) \end{array} \right) \end{array} \right)$$

Further, we have to prove that it is true for $\mathfrak{s} = Ⱪ + 1$, we have

$$\mathrm{CIFRDWA}\left(Ɽ_e(Ñ_1), Ɽ_e(Ñ_2), \ldots, Ɽ_e(Ñ_{\mathfrak{s}})\right) = \left(\bigoplus_{j=1}^{Ⱪ+1} ß_j \overline{\overline{Ɽ_e}}(Ñ_j), \bigoplus_{j=1}^{Ⱪ+1} ß_j \underline{\underline{Ɽ_e}}(Ñ_j) \right)$$

$$
= \left(\begin{array}{l} \left(\begin{array}{l} 1 - \dfrac{1}{1 + \left\{ \sum_{j=1}^{\mathrm{K}} ß_j \left(\dfrac{\sigma_{\overline{\overline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)}{1 - \sigma_{\overline{\overline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)} \right)^{\text{℃}} \right\}^{\frac{1}{\text{℃}}}} + \iota \left(1 - \dfrac{1}{1 + \left\{ \sum_{j=1}^{\mathrm{K}} ß_j \left(\dfrac{\rho_{\overline{\overline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)}{1 - \rho_{\overline{\overline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)} \right)^{\text{℃}} \right\}^{\frac{1}{\text{℃}}}} \right), \\ \dfrac{1}{1 + \left\{ \sum_{j=1}^{\mathrm{K}} ß_j \left(\dfrac{1 - \tau_{\overline{\overline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)}{\tau_{\overline{\overline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)} \right)^{\text{℃}} \right\}^{\frac{1}{\text{℃}}}} + \iota \left(\dfrac{1}{1 + \left\{ \sum_{j=1}^{\mathrm{K}} ß_j \left(\dfrac{1 - \upsilon_{\overline{\overline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)}{\upsilon_{\overline{\overline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)} \right)^{\text{℃}} \right\}^{\frac{1}{\text{℃}}}} \right) \end{array} \right), \\ \left(\begin{array}{l} 1 - \dfrac{1}{1 + \left\{ \sum_{j=1}^{\mathrm{K}} ß_j \left(\dfrac{\sigma_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)}{1 - \sigma_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)} \right)^{\text{℃}} \right\}^{\frac{1}{\text{℃}}}} + \iota \left(1 - \dfrac{1}{1 + \left\{ \sum_{j=1}^{\mathrm{K}} ß_j \left(\dfrac{\rho_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)}{1 - \rho_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)} \right)^{\text{℃}} \right\}^{\frac{1}{\text{℃}}}} \right), \\ \dfrac{1}{1 + \left\{ \sum_{j=1}^{\mathrm{K}} ß_j \left(\dfrac{1 - \tau_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)}{\tau_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)} \right)^{\text{℃}} \right\}^{\frac{1}{\text{℃}}}} + \iota \left(\dfrac{1}{1 + \left\{ \sum_{j=1}^{\mathrm{K}} ß_j \left(\dfrac{1 - \upsilon_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)}{\upsilon_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)} \right)^{\text{℃}} \right\}^{\frac{1}{\text{℃}}}} \right) \end{array} \right) \end{array} \right)
$$

$$\oplus\left(\begin{pmatrix}\left(1-\frac{1}{1+\left\{\text{ß}_{\text{Ķ}+1}\left(\frac{\sigma_{\overline{\overline{\text{Ŗ}_e}}}(\tilde{\text{N}}_{\text{Ķ}+1})}{1-\sigma_{\overline{\overline{\text{Ŗ}_e}}}(\tilde{\text{N}}_{\text{Ķ}+1})}\right)^{℃}\right\}^{\frac{1}{℃}}}+\iota\left(1-\frac{1}{1+\left\{\text{ß}_{\text{Ķ}+1}\left(\frac{\rho_{\overline{\overline{\text{Ŗ}_e}}}(\tilde{\text{N}}_{\text{Ķ}+1})}{1-\rho_{\overline{\overline{\text{Ŗ}_e}}}(\tilde{\text{N}}_{\text{Ķ}+1})}\right)^{℃}\right\}^{\frac{1}{℃}}}\right)\right), \\ \frac{1}{1+\left\{\text{ß}_{\text{Ķ}+1}\left(\frac{1-\tau_{\overline{\overline{\text{Ŗ}_e}}}(\tilde{\text{N}}_{\text{Ķ}+1})}{\tau_{\overline{\overline{\text{Ŗ}_e}}}(\tilde{\text{N}}_{\text{Ķ}+1})}\right)^{℃}\right\}^{\frac{1}{℃}}}+\iota\left(\frac{1}{1+\left\{\text{ß}_{\text{Ķ}+1}\left(\frac{1-\upsilon_{\overline{\overline{\text{Ŗ}_e}}}(\tilde{\text{N}}_{\text{Ķ}+1})}{\upsilon_{\overline{\overline{\text{Ŗ}_e}}}(\tilde{\text{N}}_{\text{Ķ}+1})}\right)^{℃}\right\}^{\frac{1}{℃}}}\right)\end{pmatrix}, \begin{pmatrix}\left(1-\frac{1}{1+\left\{\text{ß}_{\text{Ķ}+1}\left(\frac{\sigma_{\underline{\underline{\text{Ŗ}_e}}}(\tilde{\text{N}}_{\text{Ķ}+1})}{1-\sigma_{\underline{\underline{\text{Ŗ}_e}}}(\tilde{\text{N}}_{\text{Ķ}+1})}\right)^{℃}\right\}^{\frac{1}{℃}}}+\iota\left(1-\frac{1}{1+\left\{\text{ß}_{\text{Ķ}+1}\left(\frac{\rho_{\underline{\underline{\text{Ŗ}_e}}}(\tilde{\text{N}}_{\text{Ķ}+1})}{1-\rho_{\underline{\underline{\text{Ŗ}_e}}}(\tilde{\text{N}}_{\text{Ķ}+1})}\right)^{℃}\right\}^{\frac{1}{℃}}}\right)\right), \\ \frac{1}{1+\left\{\text{ß}_{\text{Ķ}+1}\left(\frac{1-\tau_{\underline{\underline{\text{Ŗ}_e}}}(\tilde{\text{N}}_{\text{Ķ}+1})}{\tau_{\underline{\underline{\text{Ŗ}_e}}}(\tilde{\text{N}}_{\text{Ķ}+1})}\right)^{℃}\right\}^{\frac{1}{℃}}}+\iota\left(\frac{1}{1+\left\{\text{ß}_{\text{Ķ}+1}\left(\frac{1-\upsilon_{\underline{\underline{\text{Ŗ}_e}}}(\tilde{\text{N}}_{\text{Ķ}+1})}{\upsilon_{\underline{\underline{\text{Ŗ}_e}}}(\tilde{\text{N}}_{\text{Ķ}+1})}\right)^{℃}\right\}^{\frac{1}{℃}}}\right)\end{pmatrix}\right)$$

$$= \left(\left(\begin{array}{c} \left(1 - \frac{1}{1 + \left\{ \sum_{j=1}^{\text{Ķ}+1} \text{ß}_j \left(\frac{\sigma_{\overline{\overline{\text{Ɽ}_e}}}(\tilde{\text{N}}_j)}{1 - \sigma_{\overline{\overline{\text{Ɽ}_e}}}(\tilde{\text{N}}_j)} \right)^{℃} \right\}^{\frac{1}{℃}}} + \iota \left(1 - \frac{1}{1 + \left\{ \sum_{j=1}^{\text{Ķ}+1} \text{ß}_j \left(\frac{\rho_{\overline{\overline{\text{Ɽ}_e}}}(\tilde{\text{N}}_j)}{1 - \rho_{\overline{\overline{\text{Ɽ}_e}}}(\tilde{\text{N}}_j)} \right)^{℃} \right\}^{\frac{1}{℃}}} \right) \right), \\ \left(\frac{1}{1 + \left\{ \sum_{j=1}^{\text{Ķ}+1} \text{ß}_j \left(\frac{1 - \tau_{\overline{\overline{\text{Ɽ}_e}}}(\tilde{\text{N}}_j)}{\tau_{\overline{\overline{\text{Ɽ}_e}}}(\tilde{\text{N}}_j)} \right)^{℃} \right\}^{\frac{1}{℃}}} + \iota \left(\frac{1}{1 + \left\{ \sum_{j=1}^{\text{Ķ}+1} \text{ß}_j \left(\frac{1 - \upsilon_{\overline{\overline{\text{Ɽ}_e}}}(\tilde{\text{N}}_j)}{\upsilon_{\overline{\overline{\text{Ɽ}_e}}}(\tilde{\text{N}}_j)} \right)^{℃} \right\}^{\frac{1}{℃}}} \right) \right) \end{array} \right), \left(\begin{array}{c} \left(1 - \frac{1}{1 + \left\{ \sum_{j=1}^{\text{Ķ}+1} \text{ß}_j \left(\frac{\sigma_{\underline{\underline{\text{Ɽ}_e}}}(\tilde{\text{N}}_j)}{1 - \sigma_{\underline{\underline{\text{Ɽ}_e}}}(\tilde{\text{N}}_j)} \right)^{℃} \right\}^{\frac{1}{℃}}} + \iota \left(1 - \frac{1}{1 + \left\{ \sum_{j=1}^{\text{Ķ}+1} \text{ß}_j \left(\frac{\rho_{\underline{\underline{\text{Ɽ}_e}}}(\tilde{\text{N}}_j)}{1 - \rho_{\underline{\underline{\text{Ɽ}_e}}}(\tilde{\text{N}}_j)} \right)^{℃} \right\}^{\frac{1}{℃}}} \right) \right), \\ \left(\frac{1}{1 + \left\{ \sum_{j=1}^{\text{Ķ}+1} \text{ß}_j \left(\frac{1 - \tau_{\underline{\underline{\text{Ɽ}_e}}}(\tilde{\text{N}}_j)}{\tau_{\underline{\underline{\text{Ɽ}_e}}}(\tilde{\text{N}}_j)} \right)^{℃} \right\}^{\frac{1}{℃}}} + \iota \left(\frac{1}{1 + \left\{ \sum_{j=1}^{\text{Ķ}+1} \text{ß}_j \left(\frac{1 - \upsilon_{\underline{\underline{\text{Ɽ}_e}}}(\tilde{\text{N}}_j)}{\upsilon_{\underline{\underline{\text{Ɽ}_e}}}(\tilde{\text{N}}_j)} \right)^{℃} \right\}^{\frac{1}{℃}}} \right) \right) \end{array} \right) \right)$$

Hence, it is also hold for $\mathfrak{s} = \text{Ķ} + 1$. Thus, by MI, it is hold for all $\mathfrak{s} \geq 0$.

As a result, $\overline{\overline{\text{Ɽ}_e}}(\tilde{\text{N}}_t)$ and $\underline{\underline{\text{Ɽ}_e}}(\tilde{\text{N}}_t)$ are CIFRNs according to the preceding theorem. Accordingly, $\bigoplus_{\mathfrak{t}=1}^{\mathfrak{s}} \text{ß}_{\mathfrak{t}} \overline{\overline{\text{Ɽ}_e}}(\tilde{\text{N}}_t)$ and $\bigoplus_{\mathfrak{t}=1}^{\mathfrak{s}} \text{ß}_{\mathfrak{t}} \underline{\underline{\text{Ɽ}_e}}(\tilde{\text{N}}_t)$ are also CIFRNs. Hence, CIFRDWA is CIFRN as well.

Proof of Theorem 2. Let $\text{Ɽ}_e(\tilde{\text{N}}_j) = \text{Ɽ}_e(\tilde{\text{N}})\ \forall\ (j = 1,2,3 \ldots, \mathfrak{s})$ then

$$\text{CIFRDWA}\left(\text{Ɽ}_e(\tilde{\text{N}}_1), \text{Ɽ}_e(\tilde{\text{N}}_2), \ldots, \text{Ɽ}_e(\tilde{\text{N}}_{\mathfrak{s}})\right) = \left(\bigoplus_{j=1}^{\mathfrak{s}} \text{ß}_j \overline{\overline{\text{Ɽ}_e}}(\tilde{\text{N}}_j), \bigoplus_{j=1}^{\mathfrak{s}} \text{ß}_j \underline{\underline{\text{Ɽ}_e}}(\tilde{\text{N}}_j) \right)$$

$$= \left(\begin{array}{l}\left(\begin{array}{l}1-\frac{1}{1+\left\{\sum_{j=1}^{ꞩ} ß_j\left(\frac{\sigma_{\overline{\overline{Ꞧ_e}}}(\tilde{Ñ}_j)}{1-\sigma_{\overline{\overline{Ꞧ_e}}}(\tilde{Ñ}_j)}\right)^{℃}\right\}^{\frac{1}{℃}}}+\iota\left(1-\frac{1}{1+\left\{\sum_{j=1}^{ꞩ} ß_j\left(\frac{\rho_{\overline{\overline{Ꞧ_e}}}(\tilde{Ñ}_j)}{1-\rho_{\overline{\overline{Ꞧ_e}}}(\tilde{Ñ}_j)}\right)^{℃}\right\}^{\frac{1}{℃}}}\right), \\ \frac{1}{1+\left\{\sum_{j=1}^{ꞩ} ß_j\left(\frac{1-\tau_{\overline{\overline{Ꞧ_e}}}(\tilde{Ñ}_j)}{\tau_{\overline{\overline{Ꞧ_e}}}(\tilde{Ñ}_j)}\right)^{℃}\right\}^{\frac{1}{℃}}}+\iota\left(\frac{1}{1+\left\{\sum_{j=1}^{ꞩ} ß_j\left(\frac{1-\upsilon_{\overline{\overline{Ꞧ_e}}}(\tilde{Ñ}_j)}{\upsilon_{\overline{\overline{Ꞧ_e}}}(\tilde{Ñ}_j)}\right)^{℃}\right\}^{\frac{1}{℃}}}\right)\end{array}\right), \\ \left(\begin{array}{l}1-\frac{1}{1+\left\{\sum_{j=1}^{ꞩ} ß_j\left(\frac{\sigma_{\underline{\underline{Ꞧ_e}}}(\tilde{Ñ}_j)}{1-\sigma_{\underline{\underline{Ꞧ_e}}}(\tilde{Ñ}_j)}\right)^{℃}\right\}^{\frac{1}{℃}}}+\iota\left(1-\frac{1}{1+\left\{\sum_{j=1}^{ꞩ} ß_j\left(\frac{\rho_{\underline{\underline{Ꞧ_e}}}(\tilde{Ñ}_j)}{1-\rho_{\underline{\underline{Ꞧ_e}}}(\tilde{Ñ}_j)}\right)^{℃}\right\}^{\frac{1}{℃}}}\right), \\ \frac{1}{1+\left\{\sum_{j=1}^{ꞩ} ß_j\left(\frac{1-\tau_{\underline{\underline{Ꞧ_e}}}(\tilde{Ñ}_j)}{\tau_{\underline{\underline{Ꞧ_e}}}(\tilde{Ñ}_j)}\right)^{℃}\right\}^{\frac{1}{℃}}}+\iota\left(\frac{1}{1+\left\{\sum_{j=1}^{ꞩ} ß_j\left(\frac{1-\upsilon_{\underline{\underline{Ꞧ_e}}}(\tilde{Ñ}_j)}{\upsilon_{\underline{\underline{Ꞧ_e}}}(\tilde{Ñ}_j)}\right)^{℃}\right\}^{\frac{1}{℃}}}\right)\end{array}\right)\end{array}\right)$$

For all $j, Ꞧ_e(\tilde{Ñ}_t) = Ꞧ_e(\tilde{Ñ})$. Therefore,

$$= \left(\begin{pmatrix} \left(1 - \frac{1}{1 + \left\{ \left(\frac{\sigma_{\overline{\overline{Ɍ_e}}}(\tilde{N})}{1 - \sigma_{\overline{\overline{Ɍ_e}}}(\tilde{N})} \right)^{℃} \right\}^{\frac{1}{℃}}} + \iota \left(1 - \frac{1}{1 + \left\{ \left(\frac{\rho_{\overline{\overline{Ɍ_e}}}(\tilde{N})}{1 - \rho_{\overline{\overline{Ɍ_e}}}(\tilde{N})} \right)^{℃} \right\}^{\frac{1}{℃}}} \right) \right), \\ \frac{1}{1 + \left\{ \left(\frac{1 - \tau_{\overline{\overline{Ɍ_e}}}(\tilde{N})}{\tau_{\overline{\overline{Ɍ_e}}}(\tilde{N})} \right)^{℃} \right\}^{\frac{1}{℃}}} + \iota \left(\frac{1}{1 + \left\{ \left(\frac{1 - \upsilon_{\overline{\overline{Ɍ_e}}}(\tilde{N})}{\upsilon_{\overline{\overline{Ɍ_e}}}(\tilde{N})} \right)^{℃} \right\}^{\frac{1}{℃}}} \right) \end{pmatrix}, \begin{pmatrix} \left(1 - \frac{1}{1 + \left\{ \left(\frac{\sigma_{\underline{\underline{Ɍ_e}}}(\tilde{N})}{1 - \sigma_{\underline{\underline{Ɍ_e}}}(\tilde{N})} \right)^{℃} \right\}^{\frac{1}{℃}}} + \iota \left(1 - \frac{1}{1 + \left\{ \left(\frac{\rho_{\underline{\underline{Ɍ_e}}}(\tilde{N}_1)}{1 - \rho_{\underline{\underline{Ɍ_e}}}(\tilde{N}_1)} \right)^{℃} \right\}^{\frac{1}{℃}}} \right) \right), \\ \frac{1}{1 + \left\{ \left(\frac{1 - \tau_{\underline{\underline{Ɍ_e}}}(\tilde{N})}{\tau_{\underline{\underline{Ɍ_e}}}(\tilde{N})} \right)^{℃} \right\}^{\frac{1}{℃}}} + \iota \left(\frac{1}{1 + \left\{ \left(\frac{1 - \upsilon_{\underline{\underline{Ɍ_e}}}(\tilde{N})}{\upsilon_{\underline{\underline{Ɍ_e}}}(\tilde{N})} \right)^{℃} \right\}^{\frac{1}{℃}}} \right) \end{pmatrix} \right)$$

$$= \left(\overline{\overline{Ɍ_e}}(\tilde{N}), \underline{\underline{Ɍ_e}}(\tilde{N}) \right) = Ɍ_e(\tilde{N}).$$

Proof of Theorem 3. As

$$\left(Ɍ_e(\tilde{N}_j) \right)^{+} = \begin{bmatrix} \left(\left(\max_j \sigma_{\overline{\overline{Ɍ_e}}}(\tilde{N}_j) + \iota \max_j \rho_{\overline{\overline{Ɍ_e}}}(\tilde{N}_j) \right), \left(\min_t \tau_{\overline{\overline{Ɍ_e}}}(\tilde{N}_j) + \iota \min_j \upsilon_{\overline{\overline{Ɍ_e}}}(\tilde{N}_j) \right) \right), \\ \left(\left(\max_j \sigma_{\underline{\underline{Ɍ_e}}}(\tilde{N}_j) + \iota \max_j \rho_{\underline{\underline{Ɍ_e}}}(\tilde{N}_j) \right), \left(\min_t \tau_{\underline{\underline{Ɍ_e}}}(\tilde{N}_j) + \iota \min_j \upsilon_{\underline{\underline{Ɍ_e}}}(\tilde{N}_j) \right) \right) \end{bmatrix}$$

and

$$\left(\text{Ř}_e(\tilde{N}_j)\right)^- = \left[\begin{matrix}\left(\left(\min_j \sigma_{\overline{\overline{\text{Ř}_e}}}(\tilde{N}_j) + \iota \min_j \rho_{\overline{\overline{\text{Ř}_e}}}(\tilde{N}_j)\right), \left(\max_j \tau_{\overline{\overline{\text{Ř}_e}}}(\tilde{N}_j) + \iota \max_j \upsilon_{\overline{\overline{\text{Ř}_e}}}(\tilde{N}_j)\right)\right), \\ \left(\left(\min_j \sigma_{\underline{\underline{\text{Ř}_e}}}(\tilde{N}_j) + \iota \min_j \rho_{\underline{\underline{\text{Ř}_e}}}(\tilde{N}_j)\right), \left(\max_j \tau_{\underline{\underline{\text{Ř}_e}}}(\tilde{N}_j) + \iota \max_j \upsilon_{\underline{\underline{\text{Ř}_e}}}(\tilde{N}_j)\right)\right)\end{matrix}\right]$$

Since for every $j = 1,2,3 \ldots \text{ꞩ}$, we have

$$\min_j \left\{\sigma_{\underline{\underline{\text{Ř}_e}}}(\tilde{N}_j)\right\} \leq \sigma_{\underline{\underline{\text{Ř}_e}}}(\tilde{N}_j) \leq \max_j \left\{\sigma_{\underline{\underline{\text{Ř}_e}}}(\tilde{N}_j)\right\}$$

$$\left(\frac{\min_j \left\{\sigma_{\underline{\underline{\text{Ř}_e}}}(\tilde{N}_j)\right\}}{1 - \min_j \left\{\sigma_{\underline{\underline{\text{Ř}_e}}}(\tilde{N}_j)\right\}}\right) \leq \frac{\sigma_{\underline{\underline{\text{Ř}_e}}}(\tilde{N}_j)}{1 - \sigma_{\underline{\underline{\text{Ř}_e}}}(\tilde{N}_j)} \leq \left(\frac{\max_j \left\{\sigma_{\underline{\underline{\text{Ř}_e}}}(\tilde{N}_j)\right\}}{1 - \max_j \left\{\sigma_{\underline{\underline{\text{Ř}_e}}}(\tilde{N}_j)\right\}}\right)$$

$$1 + \left(\frac{\min_j \left\{\sigma_{\underline{\underline{\text{Ř}_e}}}(\tilde{N}_j)\right\}}{1 - \min_j \left\{\sigma_{\underline{\underline{\text{Ř}_e}}}(\tilde{N}_j)\right\}}\right) \leq 1 + \frac{\sigma_{\underline{\underline{\text{Ř}_e}}}(\tilde{N}_j)}{1 - \sigma_{\underline{\underline{\text{Ř}_e}}}(\tilde{N}_j)} \leq 1 + \left(\frac{\max_j \left\{\sigma_{\underline{\underline{\text{Ř}_e}}}(\tilde{N}_j)\right\}}{1 - \max_j \left\{\sigma_{\underline{\underline{\text{Ř}_e}}}(\tilde{N}_j)\right\}}\right)$$

$$\frac{1}{1 + \left(\frac{\min_j \left\{\sigma_{\underline{\underline{\text{Ř}_e}}}(\tilde{N}_j)\right\}}{1 - \min_j \left\{\sigma_{\underline{\underline{\text{Ř}_e}}}(\tilde{N}_j)\right\}}\right)} \leq \frac{1}{1 + \frac{\sigma_{\underline{\underline{\text{Ř}_e}}}(\tilde{N}_j)}{1 - \sigma_{\underline{\underline{\text{Ř}_e}}}(\tilde{N}_j)}} \leq \frac{1}{1 + \left(\frac{\max_j \left\{\sigma_{\underline{\underline{\text{Ř}_e}}}(\tilde{N}_j)\right\}}{1 - \max_j \left\{\sigma_{\underline{\underline{\text{Ř}_e}}}(\tilde{N}_j)\right\}}\right)}$$

$$1 - \frac{1}{1 + \left(\sum_{j=1}^{\text{ꞩ}} \text{ß}_j \left(\frac{\min_j \left\{\sigma_{\underline{\underline{\text{Ř}_e}}}(\tilde{N}_j)\right\}}{1 - \min_j \left\{\sigma_{\underline{\underline{\text{Ř}_e}}}(\tilde{N}_j)\right\}}\right)^{℃}\right)^{\frac{1}{℃}}} \leq 1 - \frac{1}{1 + \left(\sum_{j=1}^{\text{ꞩ}} \text{ß}_j \left(\frac{\sigma_{\underline{\underline{\text{Ř}_e}}}(\tilde{N}_j)}{1 - \sigma_{\underline{\underline{\text{Ř}_e}}}(\tilde{N}_j)}\right)^{℃}\right)^{\frac{1}{℃}}}$$

$$\leq 1 - \frac{1}{1 + \left(\sum_{j=1}^{\text{ꞩ}} \text{ß}_j \left(\frac{\max_j \left\{\sigma_{\underline{\underline{\text{Ř}_e}}}(\tilde{N}_j)\right\}}{1 - \max_j \left\{\sigma_{\underline{\underline{\text{Ř}_e}}}(\tilde{N}_j)\right\}}\right)^{℃}\right)^{\frac{1}{℃}}}$$

$$1-\frac{1}{1+\left(\frac{\min\limits_{j}\left\{\sigma_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)\right\}}{1-\min\limits_{j}\left\{\sigma_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)\right\}}\right)} \leq 1-\frac{1}{1+\left(\sum_{j=1}^{\mathfrak{s}} ß_j\left(\frac{\sigma_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)}{1-\sigma_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)}\right)^{℃}\right)^{\frac{1}{℃}}}$$

$$\leq 1-\frac{1}{1+\left(\frac{\max\limits_{j}\left\{\sigma_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)\right\}}{1-\max\limits_{j}\left\{\sigma_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)\right\}}\right)}$$

Hence,

$$\min_{j}\left\{\sigma_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)\right\} \leq 1-\frac{1}{1+\left(\sum_{j=1}^{\mathfrak{s}} ß_j\left(\frac{\sigma_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)}{1-\sigma_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)}\right)^{℃}\right)^{\frac{1}{℃}}} \leq \max_{j}\left\{\sigma_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)\right\} \tag{1}$$

Similarly,

$$\min_{j}\left\{\rho_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)\right\} \leq 1-\frac{1}{1+\left(\sum_{j=1}^{\mathfrak{s}} ß_j\left(\frac{\rho_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)}{1-\rho_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)}\right)^{℃}\right)^{\frac{1}{℃}}} \leq \max_{j}\left\{\rho_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)\right\} \tag{2}$$

Now, for every $t = 1,2,3 \ldots \mathfrak{s}$, we have

$$\min_{j}\left\{\tau_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)\right\} \leq \tau_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j) \leq \max_{j}\left\{\tau_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)\right\}$$

$$1-\min_{j}\left\{\tau_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)\right\} \geq 1-\tau_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j) \geq 1-\max_{j}\left\{\tau_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)\right\}$$

$$1+\frac{1-\min\limits_{j}\left\{\tau_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)\right\}}{\min\limits_{j}\left\{\tau_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)\right\}} \geq 1=\frac{1-\tau_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)}{\tau_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)} \geq 1+\frac{1-\max\limits_{j}\left\{\tau_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)\right\}}{\max\limits_{j}\left\{\tau_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)\right\}}$$

This implies that

$$1+\left(\sum_{j=1}^{\mathfrak{s}} ß_j\left(\frac{1-\min_j\left\{\tau_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)\right\}}{\min_j\left\{\tau_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)\right\}}\right)^{℃}\right)^{\frac{1}{℃}} \geq 1+\left(\sum_{j=1}^{\mathfrak{s}} ß_j\left(\frac{1-\tau_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)}{\tau_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)}\right)^{℃}\right)^{\frac{1}{℃}}$$

$$\geq 1+\left(\sum_{j=1}^{\mathfrak{s}} ß_j\left(\frac{1-\max_j\left\{\tau_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)\right\}}{\max_j\left\{\tau_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)\right\}}\right)^{℃}\right)^{\frac{1}{℃}}$$

$$\frac{1}{1+\left(\sum_{j=1}^{\mathfrak{s}} ß_j\left(\frac{1-\min_j\left\{\tau_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)\right\}}{\min_j\left\{\tau_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)\right\}}\right)^{℃}\right)^{\frac{1}{℃}}} \leq \frac{1}{1+\left(\sum_{j=1}^{\mathfrak{s}} ß_j\left(\frac{1-\tau_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)}{\tau_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)}\right)^{℃}\right)^{\frac{1}{℃}}}$$

$$\leq \frac{1}{1+\left(\sum_{j=1}^{\mathfrak{s}} ß_j\left(\frac{1-\max_j\left\{\tau_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)\right\}}{\max_j\left\{\tau_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)\right\}}\right)^{℃}\right)^{\frac{1}{℃}}}$$

$$\frac{1}{1+\frac{1-\min_j\left\{\tau_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)\right\}}{\min_j\left\{\tau_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)\right\}}} \leq \frac{1}{1+\left(\sum_{j=1}^{\mathfrak{s}} ß_j\left(\frac{1-\tau_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)}{\tau_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)}\right)^{℃}\right)^{\frac{1}{℃}}} \leq \frac{1}{1+\frac{1-\max_j\left\{\tau_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)\right\}}{\max_j\left\{\tau_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)\right\}}}$$

$$\min_j\left\{\tau_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)\right\} \leq \frac{1}{1+\left(\sum_{j=1}^{\mathfrak{s}} ß_j\left(\frac{1-\tau_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)}{\tau_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)}\right)^{℃}\right)^{\frac{1}{℃}}} \leq \max_j\left\{\tau_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)\right\} \tag{3}$$

Similarly,

$$\min_j\left\{\upsilon_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)\right\} \leq \frac{1}{1+\left(\sum_{j=1}^{\mathfrak{s}} ß_j\left(\frac{1-\upsilon_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)}{\upsilon_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)}\right)^{℃}\right)^{\frac{1}{℃}}} \leq \max_j\left\{\upsilon_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_j)\right\} \tag{4}$$

Similarly, we can show that

$$\min_j \left\{\sigma_{\overline{\overline{Ȓ_e}}}\left(\tilde{Ñ}_j\right)\right\} \leq 1 - \frac{1}{1+\left(\sum_{j=1}^{ꞩ} ß_j \left(\frac{\sigma_{\overline{\overline{Ȓ_e}}}\left(\tilde{Ñ}_j\right)}{1-\sigma_{\overline{\overline{Ȓ_e}}}\left(\tilde{Ñ}_j\right)}\right)^{℃}\right)^{\frac{1}{℃}}} \leq \max_j \left\{\sigma_{\overline{\overline{Ȓ_e}}}\left(\tilde{Ñ}_j\right)\right\} \tag{5}$$

$$\min_j \left\{\rho_{\overline{\overline{Ȓ_e}}}\left(\tilde{Ñ}_j\right)\right\} \leq 1 - \frac{1}{1+\left(\sum_{j=1}^{ꞩ} ß_j \left(\frac{\rho_{\overline{\overline{Ȓ_e}}}\left(\tilde{Ñ}_j\right)}{1-\rho_{\overline{\overline{Ȓ_e}}}\left(\tilde{Ñ}_j\right)}\right)^{℃}\right)^{\frac{1}{℃}}} \leq \max_j \left\{\rho_{\overline{\overline{Ȓ_e}}}\left(\tilde{Ñ}_j\right)\right\} \tag{6}$$

$$\min_j \left\{\tau_{\overline{\overline{Ȓ_e}}}\left(\tilde{Ñ}_j\right)\right\} \leq \frac{1}{1+\left(\sum_{j=1}^{ꞩ} ß_j \left(\frac{1-\tau_{\overline{\overline{Ȓ_e}}}\left(\tilde{Ñ}_j\right)}{\tau_{\overline{\overline{Ȓ_e}}}\left(\tilde{Ñ}_j\right)}\right)^{℃}\right)^{\frac{1}{℃}}} \leq \max_j \left\{\tau_{\overline{\overline{Ȓ_e}}}\left(\tilde{Ñ}_j\right)\right\} \tag{7}$$

$$\min_j \left\{\upsilon_{\overline{\overline{Ȓ_e}}}\left(\tilde{Ñ}_j\right)\right\} \leq \frac{1}{1+\left(\sum_{j=1}^{ꞩ} ß_j \left(\frac{1-\upsilon_{\overline{\overline{Ȓ_e}}}\left(\tilde{Ñ}_j\right)}{\upsilon_{\overline{\overline{Ȓ_e}}}\left(\tilde{Ñ}_j\right)}\right)^{℃}\right)^{\frac{1}{℃}}} \leq \max_j \left\{\upsilon_{\overline{\overline{Ȓ_e}}}\left(\tilde{Ñ}_j\right)\right\} \tag{8}$$

From Eqs (1)–(8) we have

$$\left(Ȓ_e\left(\tilde{Ñ}_j\right)\right)^{-} \leq \text{CIFRDWA}\left(Ȓ_e(\tilde{Ñ}_1), Ȓ_e(\tilde{Ñ}_2), \ldots, Ȓ_e(\tilde{Ñ}_{ꞩ})\right) \leq \left(Ȓ_e\left(\tilde{Ñ}_j\right)\right)^{+}.$$

Proof of Theorem 9. Using a well-known mathematical induction (MI) technique, we demonstrate the preceding equation by assuming that, for $ꞩ = 2$, we obtain

$$\text{CIFRDWG}\left(Ȓ_e(\tilde{Ñ}_1), Ȓ_e(\tilde{Ñ}_2)\right) = \left(\otimes_{j=1}^{2} ß_j \overline{\overline{Ȓ_e}}\left(\tilde{Ñ}_j\right), \otimes_{j=1}^{2} ß_j \underline{\underline{Ȓ_e}}\left(\tilde{Ñ}_j\right)\right)$$

$$= \left(ß_1 \overline{\overline{Ȓ_e}}(\tilde{Ñ}_1) \otimes ß_2 \overline{\overline{Ȓ_e}}(\tilde{Ñ}_2), ß_1 \underline{\underline{Ȓ_e}}(\tilde{Ñ}_1) \otimes ß_2 \underline{\underline{Ȓ_e}}(\tilde{Ñ}_2)\right)$$

$$
=\left(\begin{array}{l}\left(\begin{array}{c}\dfrac{1}{1+\left\{ß_1\left(\dfrac{1-\sigma_{\overline{\overline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_1)}{\sigma_{\overline{\overline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_1)}\right)^{℃}+ß_2\left(\dfrac{1-\sigma_{\overline{\overline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_2)}{\sigma_{\overline{\overline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_2)}\right)^{℃}\right\}^{\frac{1}{℃}}}\\ +\iota\left(\dfrac{1}{1+\left\{ß_1\left(\dfrac{1-\rho_{\overline{\overline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_1)}{\rho_{\overline{\overline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_1)}\right)^{℃}+ß_2\left(\dfrac{1-\rho_{\overline{\overline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_2)}{\rho_{\overline{\overline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_2)}\right)^{℃}\right\}^{\frac{1}{℃}}}\right),\\ 1-\dfrac{1}{1+\left\{ß_1\left(\dfrac{\tau_{\overline{\overline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_1)}{1-\tau_{\overline{\overline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_1)}\right)^{℃}+ß_2\left(\dfrac{\tau_{\overline{\overline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_2)}{1-\tau_{\overline{\overline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_2)}\right)^{℃}\right\}^{\frac{1}{℃}}}\\ +\iota\left(1-\dfrac{1}{1+\left\{ß_1\left(\dfrac{\upsilon_{\overline{\overline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_1)}{1-\upsilon_{\overline{\overline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_1)}\right)^{℃}+ß_2\left(\dfrac{\upsilon_{\overline{\overline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_2)}{1-\upsilon_{\overline{\overline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_2)}\right)^{℃}\right\}^{\frac{1}{℃}}}\right)\end{array}\right),\\ \left(\begin{array}{c}\dfrac{1}{1+\left\{ß_1\left(\dfrac{1-\sigma_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_1)}{\sigma_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_1)}\right)^{℃}+ß_2\left(\dfrac{1-\sigma_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_2)}{\sigma_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_2)}\right)^{℃}\right\}^{\frac{1}{℃}}}\\ +\iota\left(\dfrac{1}{1+\left\{ß_1\left(\dfrac{1-\rho_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_1)}{\rho_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_1)}\right)^{℃}+ß_2\left(\dfrac{1-\rho_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_2)}{\rho_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_2)}\right)^{℃}\right\}^{\frac{1}{℃}}}\right),\\ 1-\dfrac{1}{1+\left\{ß_1\left(\dfrac{\tau_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_1)}{1-\tau_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_1)}\right)^{℃}+ß_2\left(\dfrac{\tau_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_2)}{1-\tau_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_2)}\right)^{℃}\right\}^{\frac{1}{℃}}}\\ +\iota\left(1-\dfrac{1}{1+\left\{ß_1\left(\dfrac{\upsilon_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_1)}{1-\upsilon_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_1)}\right)^{℃}+ß_2\left(\dfrac{\upsilon_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_2)}{1-\upsilon_{\underline{\underline{\mathrm{R}_e}}}(\tilde{\mathrm{N}}_2)}\right)^{℃}\right\}^{\frac{1}{℃}}}\right)\end{array}\right)\end{array}\right)
$$

$$
= \left(\left(\begin{array}{c} \dfrac{1}{1+\left\{\sum_{j=1}^{2} ß_j \left(\dfrac{1-\sigma_{\overline{\overline{R_e}}}(\tilde{N}_j)}{\sigma_{\overline{\overline{R_e}}}(\tilde{N}_j)} \right)^{℃} \right\}^{\frac{1}{℃}}} + \iota \left(\dfrac{1}{1+\left\{\sum_{j=1}^{2} ß_j \left(\dfrac{1-\rho_{\overline{\overline{R_e}}}(\tilde{N}_j)}{\rho_{\overline{\overline{R_e}}}(\tilde{N}_j)} \right)^{℃} \right\}^{\frac{1}{℃}}} \right), \\ 1-\dfrac{1}{1+\left\{\sum_{j=1}^{2} ß_j \left(\dfrac{\tau_{\overline{\overline{R_e}}}(\tilde{N}_j)}{1-\tau_{\overline{\overline{R_e}}}(\tilde{N}_j)} \right)^{℃} \right\}^{\frac{1}{℃}}} + \iota \left(1-\dfrac{1}{1+\left\{\sum_{j=1}^{2} ß_j \left(\dfrac{\upsilon_{\overline{\overline{R_e}}}(\tilde{N}_j)}{1-\upsilon_{\overline{\overline{R_e}}}(\tilde{N}_j)} \right)^{℃} \right\}^{\frac{1}{℃}}} \right) \end{array} \right), \left(\begin{array}{c} \dfrac{1}{1+\left\{\sum_{j=1}^{2} ß_j \left(\dfrac{1-\sigma_{\underline{\underline{R_e}}}(\tilde{N}_j)}{\sigma_{\underline{\underline{R_e}}}(\tilde{N}_j)} \right)^{℃} \right\}^{\frac{1}{℃}}} + \iota \left(\dfrac{1}{1+\left\{\sum_{j=1}^{2} ß_j \left(\dfrac{1-\rho_{\underline{\underline{R_e}}}(\tilde{N}_j)}{\rho_{\underline{\underline{R_e}}}(\tilde{N}_j)} \right)^{℃} \right\}^{\frac{1}{℃}}} \right), \\ 1-\dfrac{1}{1+\left\{\sum_{j=1}^{2} ß_j \left(\dfrac{\tau_{\underline{\underline{R_e}}}(\tilde{N}_j)}{1-\tau_{\underline{\underline{R_e}}}(\tilde{N}_j)} \right)^{℃} \right\}^{\frac{1}{℃}}} + \iota \left(1-\dfrac{1}{1+\left\{\sum_{j=1}^{2} ß_j \left(\dfrac{\upsilon_{\underline{\underline{R_e}}}(\tilde{N}_j)}{1-\upsilon_{\underline{\underline{R_e}}}(\tilde{N}_j)} \right)^{℃} \right\}^{\frac{1}{℃}}} \right) \end{array} \right) \right)
$$

The result is true for $\mathfrak{s} = 2$. Next, suppose that it is true for $\mathfrak{s} = Ķ$.

$$
\mathrm{CIFRDWG}\left(R_e(\tilde{N}_1), R_e(\tilde{N}_2), \ldots, R_e(\tilde{N}_{\mathfrak{s}})\right) = \left(\otimes_{j=1}^{Ķ} ß_j \overline{\overline{R_e}}(\tilde{N}_j), \otimes_{j=1}^{Ķ} ß_j \underline{\underline{R_e}}(\tilde{N}_j) \right)
$$

$$= \left(\begin{pmatrix} \frac{1}{1+\left\{\sum_{j=1}^{Ķ} ß_j \left(\frac{1-\sigma_{\overline{\overline{Ɽ_e}}}(\tilde{Ñ}_j)}{\sigma_{\overline{\overline{Ɽ_e}}}(\tilde{Ñ}_j)}\right)^{℃}\right\}^{\frac{1}{℃}}} + \iota\left(\frac{1}{1+\left\{\sum_{j=1}^{Ķ} ß_j \left(\frac{1-\rho_{\overline{\overline{Ɽ_e}}}(\tilde{Ñ}_j)}{\rho_{\overline{\overline{Ɽ_e}}}(\tilde{Ñ}_j)}\right)^{℃}\right\}^{\frac{1}{℃}}}\right), \\ 1-\frac{1}{1+\left\{\sum_{j=1}^{Ķ} ß_j \left(\frac{\tau_{\overline{\overline{Ɽ_e}}}(\tilde{Ñ}_j)}{1-\tau_{\overline{\overline{Ɽ_e}}}(\tilde{Ñ}_j)}\right)^{℃}\right\}^{\frac{1}{℃}}} + \iota\left(1-\frac{1}{1+\left\{\sum_{j=1}^{Ķ} ß_j \left(\frac{\upsilon_{\overline{\overline{Ɽ_e}}}(\tilde{Ñ}_j)}{1-\upsilon_{\overline{\overline{Ɽ_e}}}(\tilde{Ñ}_j)}\right)^{℃}\right\}^{\frac{1}{℃}}}\right) \end{pmatrix}, \begin{pmatrix} \frac{1}{1+\left\{\sum_{j=1}^{Ķ} ß_j \left(\frac{1-\sigma_{\underline{\underline{Ɽ_e}}}(\tilde{Ñ}_j)}{\sigma_{\underline{\underline{Ɽ_e}}}(\tilde{Ñ}_j)}\right)^{℃}\right\}^{\frac{1}{℃}}} + \iota\left(\frac{1}{1+\left\{\sum_{j=1}^{Ķ} ß_j \left(\frac{1-\rho_{\underline{\underline{Ɽ_e}}}(\tilde{Ñ}_j)}{\rho_{\underline{\underline{Ɽ_e}}}(\tilde{Ñ}_j)}\right)^{℃}\right\}^{\frac{1}{℃}}}\right), \\ 1-\frac{1}{1+\left\{\sum_{j=1}^{Ķ} ß_j \left(\frac{\tau_{\underline{\underline{Ɽ_e}}}(\tilde{Ñ}_j)}{1-\tau_{\underline{\underline{Ɽ_e}}}(\tilde{Ñ}_j)}\right)^{℃}\right\}^{\frac{1}{℃}}} + \iota\left(1-\frac{1}{1+\left\{\sum_{j=1}^{Ķ} ß_j \left(\frac{\upsilon_{\underline{\underline{Ɽ_e}}}(\tilde{Ñ}_j)}{1-\upsilon_{\underline{\underline{Ɽ_e}}}(\tilde{Ñ}_j)}\right)^{℃}\right\}^{\frac{1}{℃}}}\right) \end{pmatrix}\right)$$

Further, we have to prove that it is true for $\mathfrak{s} = Ķ + 1$, we have

$$\text{CIFRDWG}\left(Ɽ_e(\tilde{Ñ}_1), Ɽ_e(\tilde{Ñ}_2), \ldots, Ɽ_e(\tilde{Ñ}_{\mathfrak{s}})\right) = \left(\otimes_{j=1}^{Ķ+1} ß_j \overline{\overline{Ɽ_e}}(\tilde{Ñ}_j), \otimes_{j=1}^{Ķ+1} ß_j \underline{\underline{Ɽ_e}}(\tilde{Ñ}_j)\right)$$

$$
= \left(\begin{array}{l}
\left(\begin{array}{l}
\dfrac{1}{1+\left\{\sum_{j=1}^{\text{Ḵ}} \text{ß}_j \left(\dfrac{1-\sigma_{\overline{\overline{\text{Ꞧ}_e}}}(\tilde{\text{N}}_j)}{\sigma_{\overline{\overline{\text{Ꞧ}_e}}}(\tilde{\text{N}}_j)}\right)^{℃}\right\}^{\frac{1}{℃}}} + \iota \left(\dfrac{1}{1+\left\{\sum_{j=1}^{\text{Ḵ}} \text{ß}_j \left(\dfrac{1-\rho_{\overline{\overline{\text{Ꞧ}_e}}}(\tilde{\text{N}}_j)}{\rho_{\overline{\overline{\text{Ꞧ}_e}}}(\tilde{\text{N}}_j)}\right)^{℃}\right\}^{\frac{1}{℃}}} \right), \\
1-\dfrac{1}{1+\left\{\sum_{j=1}^{\text{Ḵ}} \text{ß}_j \left(\dfrac{\tau_{\overline{\overline{\text{Ꞧ}_e}}}(\tilde{\text{N}}_j)}{1-\tau_{\overline{\overline{\text{Ꞧ}_e}}}(\tilde{\text{N}}_j)}\right)^{℃}\right\}^{\frac{1}{℃}}} + \iota \left(1-\dfrac{1}{1+\left\{\sum_{j=1}^{\text{Ḵ}} \text{ß}_j \left(\dfrac{\upsilon_{\overline{\overline{\text{Ꞧ}_e}}}(\tilde{\text{N}}_j)}{1-\upsilon_{\overline{\overline{\text{Ꞧ}_e}}}(\tilde{\text{N}}_j)}\right)^{℃}\right\}^{\frac{1}{℃}}} \right)
\end{array} \right), \\
\left(\begin{array}{l}
\dfrac{1}{1+\left\{\sum_{j=1}^{\text{Ḵ}} \text{ß}_j \left(\dfrac{1-\sigma_{\underline{\underline{\text{Ꞧ}_e}}}(\tilde{\text{N}}_j)}{\sigma_{\underline{\underline{\text{Ꞧ}_e}}}(\tilde{\text{N}}_j)}\right)^{℃}\right\}^{\frac{1}{℃}}} + \iota \left(\dfrac{1}{1+\left\{\sum_{j=1}^{\text{Ḵ}} \text{ß}_j \left(\dfrac{1-\rho_{\underline{\underline{\text{Ꞧ}_e}}}(\tilde{\text{N}}_j)}{\rho_{\underline{\underline{\text{Ꞧ}_e}}}(\tilde{\text{N}}_j)}\right)^{℃}\right\}^{\frac{1}{℃}}} \right), \\
1-\dfrac{1}{1+\left\{\sum_{j=1}^{\text{Ḵ}} \text{ß}_j \left(\dfrac{\tau_{\underline{\underline{\text{Ꞧ}_e}}}(\tilde{\text{N}}_j)}{1-\tau_{\underline{\underline{\text{Ꞧ}_e}}}(\tilde{\text{N}}_j)}\right)^{℃}\right\}^{\frac{1}{℃}}} + \iota \left(1-\dfrac{1}{1+\left\{\sum_{j=1}^{\text{Ḵ}} \text{ß}_j \left(\dfrac{\upsilon_{\underline{\underline{\text{Ꞧ}_e}}}(\tilde{\text{N}}_j)}{1-\upsilon_{\underline{\underline{\text{Ꞧ}_e}}}(\tilde{\text{N}}_j)}\right)^{℃}\right\}^{\frac{1}{℃}}} \right)
\end{array} \right)
\end{array} \right)
$$

$$
\otimes\left(\begin{array}{c}\left(\begin{array}{c}\dfrac{1}{1+\left\{ß_{Ķ+1}\left(\dfrac{1-\sigma_{\overline{\overline{Ŗ_e}}}\left(\tilde{N}_{Ķ+1}\right)}{\sigma_{\overline{\overline{Ŗ_e}}}\left(\tilde{N}_{Ķ+1}\right)}\right)^{℃}\right\}^{\frac{1}{℃}}}+\iota\left(\dfrac{1}{1+\left\{ß_{Ķ+1}\left(\dfrac{1-\rho_{\overline{\overline{Ŗ_e}}}\left(\tilde{N}_{Ķ+1}\right)}{\rho_{\overline{\overline{Ŗ_e}}}\left(\tilde{N}_{Ķ+1}\right)}\right)^{℃}\right\}^{\frac{1}{℃}}}\right), \\ 1-\dfrac{1}{1+\left\{ß_{Ķ+1}\left(\dfrac{\tau_{\overline{\overline{Ŗ_e}}}\left(\tilde{N}_{Ķ+1}\right)}{1-\tau_{\overline{\overline{Ŗ_e}}}\left(\tilde{N}_{Ķ+1}\right)}\right)^{℃}\right\}^{\frac{1}{℃}}}+\iota\left(1-\dfrac{1}{1+\left\{ß_{Ķ+1}\left(\dfrac{\upsilon_{\overline{\overline{Ŗ_e}}}\left(\tilde{N}_{Ķ+1}\right)}{1-\upsilon_{\overline{\overline{Ŗ_e}}}\left(\tilde{N}_{Ķ+1}\right)}\right)^{℃}\right\}^{\frac{1}{℃}}}\right)\end{array}\right), \\ \left(\begin{array}{c}\dfrac{1}{1+\left\{ß_{Ķ+1}\left(\dfrac{1-\sigma_{\underline{\underline{Ŗ_e}}}\left(\tilde{N}_{Ķ+1}\right)}{\sigma_{\underline{\underline{Ŗ_e}}}\left(\tilde{N}_{Ķ+1}\right)}\right)^{℃}\right\}^{\frac{1}{℃}}}+\iota\left(\dfrac{1}{1+\left\{ß_{Ķ+1}\left(\dfrac{1-\rho_{\underline{\underline{Ŗ_e}}}\left(\tilde{N}_{Ķ+1}\right)}{\rho_{\underline{\underline{Ŗ_e}}}\left(\tilde{N}_{Ķ+1}\right)}\right)^{℃}\right\}^{\frac{1}{℃}}}\right), \\ 1-\dfrac{1}{1+\left\{ß_{Ķ+1}\left(\dfrac{\tau_{\underline{\underline{Ŗ_e}}}\left(\tilde{N}_{Ķ+1}\right)}{1-\tau_{\underline{\underline{Ŗ_e}}}\left(\tilde{N}_{Ķ+1}\right)}\right)^{℃}\right\}^{\frac{1}{℃}}}+\iota\left(1-\dfrac{1}{1+\left\{ß_{Ķ+1}\left(\dfrac{\upsilon_{\underline{\underline{Ŗ_e}}}\left(\tilde{N}_{Ķ+1}\right)}{1-\upsilon_{\underline{\underline{Ŗ_e}}}\left(\tilde{N}_{Ķ+1}\right)}\right)^{℃}\right\}^{\frac{1}{℃}}}\right)\end{array}\right)\end{array}\right)
$$

$$
= \left(\left(\begin{array}{l}
\dfrac{1}{1+\left\{\sum_{j=1}^{\text{Ķ}+1} \text{ß}_j \left(\dfrac{1-\sigma_{\overline{\overline{\mathrm{R}}}_e}(\tilde{N}_j)}{\sigma_{\overline{\overline{\mathrm{R}}}_e}(\tilde{N}_j)}\right)^{\text{℃}}\right\}^{\frac{1}{\text{℃}}}} + \iota\left(\dfrac{1}{1+\left\{\sum_{j=1}^{\text{Ķ}+1} \text{ß}_j \left(\dfrac{1-\rho_{\overline{\overline{\mathrm{R}}}_e}(\tilde{N}_j)}{\rho_{\overline{\overline{\mathrm{R}}}_e}(\tilde{N}_j)}\right)^{\text{℃}}\right\}^{\frac{1}{\text{℃}}}}\right), \\
1-\dfrac{1}{1+\left\{\sum_{j=1}^{\text{Ķ}+1} \text{ß}_j \left(\dfrac{\tau_{\overline{\overline{\mathrm{R}}}_e}(\tilde{N}_j)}{1-\tau_{\overline{\overline{\mathrm{R}}}_e}(\tilde{N}_j)}\right)^{\text{℃}}\right\}^{\frac{1}{\text{℃}}}} + \iota\left(1-\dfrac{1}{1+\left\{\sum_{j=1}^{\text{Ķ}+1} \text{ß}_j \left(\dfrac{\upsilon_{\overline{\overline{\mathrm{R}}}_e}(\tilde{N}_j)}{1-\upsilon_{\overline{\overline{\mathrm{R}}}_e}(\tilde{N}_j)}\right)^{\text{℃}}\right\}^{\frac{1}{\text{℃}}}}\right)
\end{array} \right), \right.
$$

$$
\left. \left(\begin{array}{l}
\dfrac{1}{1+\left\{\sum_{j=1}^{\text{Ķ}+1} \text{ß}_j \left(\dfrac{1-\sigma_{\underline{\underline{\mathrm{R}}}_e}(\tilde{N}_j)}{\sigma_{\underline{\underline{\mathrm{R}}}_e}(\tilde{N}_j)}\right)^{\text{℃}}\right\}^{\frac{1}{\text{℃}}}} + \iota\left(\dfrac{1}{1+\left\{\sum_{j=1}^{\text{Ķ}+1} \text{ß}_j \left(\dfrac{1-\rho_{\underline{\underline{\mathrm{R}}}_e}(\tilde{N}_j)}{\rho_{\underline{\underline{\mathrm{R}}}_e}(\tilde{N}_j)}\right)^{\text{℃}}\right\}^{\frac{1}{\text{℃}}}}\right), \\
1-\dfrac{1}{1+\left\{\sum_{j=1}^{\text{Ķ}+1} \text{ß}_j \left(\dfrac{\tau_{\underline{\underline{\mathrm{R}}}_e}(\tilde{N}_j)}{1-\tau_{\underline{\underline{\mathrm{R}}}_e}(\tilde{N}_j)}\right)^{\text{℃}}\right\}^{\frac{1}{\text{℃}}}} + \iota\left(1-\dfrac{1}{1+\left\{\sum_{j=1}^{\text{Ķ}+1} \text{ß}_j \left(\dfrac{\upsilon_{\underline{\underline{\mathrm{R}}}_e}(\tilde{N}_j)}{1-\upsilon_{\underline{\underline{\mathrm{R}}}_e}(\tilde{N}_j)}\right)^{\text{℃}}\right\}^{\frac{1}{\text{℃}}}}\right)
\end{array} \right) \right)
$$

Hence, it is also held for $\text{ꞩ} = \text{Ķ}+1$. Thus, by MI, it is held $\forall\, \text{ꞩ} \geq 0$.

As a result, $\overline{\overline{\mathrm{R}}}_e(\tilde{N}_t)$ and $\underline{\underline{\mathrm{R}}}_e(\tilde{N}_t)$ are CIFRNs according to the preceding theorem. Accordingly, $\bigoplus_{t=1}^{\text{ꞩ}} \text{ß}_t \overline{\overline{\mathrm{R}}}_e(\tilde{N}_t)$ and $\bigoplus_{t=1}^{\text{ꞩ}} \text{ß}_t\, \underline{\underline{\mathrm{R}}}_e(\tilde{N}_t)$ are also CIFRNs. Hence, CIFRDWA is CIFRN as well.