Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(11): 33062–33086.
DOI:10.3934/math.20241580
Received: 18 September 2024
Revised: 06 November 2024
Accepted: 15 November 2024
Published: 21 November 2024

Research article

Mean-variance investment and risk control strategies for a dynamic contagion process with diffusion

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Abstract: This paper explored an investment and risk control issue within a contagious financial market, specifically focusing on a mean-variance (MV) framework for an insurer. The market's risky assets were depicted via a jump-diffusion model, featuring jumps due to a multivariate dynamic contagion process with diffusion (DCPD). The process enveloped several popular processes, including the Hawkes process with exponentially decaying intensity, the Cox process with Poisson shot-noise intensity, and the Cox process with Cox-Ingersoll-Ross (CIR) intensity. The model distinguished between externally excited jumps, indicative of exogenous influences, modeled by the Cox process, and internally excited jumps, representing endogenous factors captured by the Hawkes process. Given an expected terminal wealth, the insurer seeked to minimize the variance of terminal wealth by adjusting the issuance volume of policies and investing the surplus in the financial market. In order to address this MV problem, we employed a suite of mathematical techniques, including the stochastic maximum principle (SMP), backward stochastic differential equations (BSDEs), and linear-quadratic (LQ) control techniques. These methodologies facilitated the derivation of both the efficient strategy and the efficient frontier. The presentation of the results in a semi-closed form was governed by a nonlocal partial differential equation (PDE). For empirical validation and demonstration of our methodology's efficacy, we provided a series of numerical examples.

Keywords: investment and risk control strategy; mean-variance problem; dynamic contagion process with diffusion; backward stochastic differential equation; stochastic maximum principle
Mathematics Subject Classification: 91G05, 91G10, 93E20

1. Introduction

In recent years, asset prices have experienced occasional and persistent jumps due to a variety of factors, such as public sentiment and coronavirus disease. The Lévy jump-diffusion models stand out as the most popular frameworks incorporating jump components, primarily propelled by a Lévy process.

These models have gained widespread acclaim across various financial domains, including applications such as option pricing, term structure analysis, and credit risk modeling. However, empirical research has consistently shown that when an asset, or a class of assets, experiences a jump of price, it is often followed by additional jumps within a brief timeframe. These subsequent fluctuations occur not only in the same asset or class but also across different assets or classes, leading to the contagion or clustering effects within the market. Aït-Sahalia et al. [1] noted that jumps within the standard Lévy jump-diffusion model are rare events, thus falling short of adequately explaining the clustering that occurs during large price movements.

Various models and methods have been developed to elucidate the clustering effects observed in jump phenomena. The Hawkes process, originally introduced by Hawkes [11], emerges as a particularly compelling example. The Hawkes process, featuring simultaneous jumps in both the point process and its intensity, is characterized by a stochastic intensity process determined by the process's history. Consequently, the occurrence for an event enhances the likelihood for subsequent events, enabling effective modeling of contagion effects within the finance and insurance sectors. Shen and Zou [20] captured the contagion and clustering effects observed within the financial market by using a multivariate Hawkes process. They derive the mean-variance (MV) optimal solution through the application of the stochastic maximum principle (SMP) and the theory of backward stochastic differential equations (BSDEs). Aït-Sahalia and Hurd [2] tackled the intricate issue of optimizing investment and consumption strategies within frameworks where asset prices are modulated by multivariate Hawkes processes. A closed-form solution of the log-utility investor was achieved by using the Hamilton-Jacobi-Bellman (HJB) equation method. Swishchuk et al. [22] delved into how Hawkes processes affect optimal investment strategies for insurers in incomplete markets, applying asset-liability management approaches. In a similar vein, Liu et al. [12] investigated an optimization challenge for households in a market influenced by contagion, considering a portfolio that includes a life insurance product, a type of risk-free asset and multiple types of risk assets. Notable works in the financial field that apply the Hawkes process to modeling include Azizpour et al. [3], Chavez-Demoulin and McGill [7], and Embrechts et al. [10], along with the references cited therein.

Based on the foundational principles of the Hawkes process, Dassios and Zhao [8] defined a new point process, designated as the dynamic contagion process (DCP). This model enriches the traditional Hawkes process and the Cox process with Poisson shot-noise intensity by integrating both internally and externally excited jumps. Such a comprehensive framework could effectively model the dynamic contagion effects arising from both internal and external factors within a system. In recent research, Cao et al. [6] investigated the optimization problem of reinsurance and investment using DCP for time-inconsistent MV criterion. Closed-form solution was derived through the application of the extended HJB equation approach. Subsequently, Dassios and Zhao [9] enhanced the DCP by introducing an additional independent diffusion. This led to the proposal of a DCP with diffusion (DCPD), which introduces a Brownian motion to simulate white noise in markets, making it more applicable to the finance and insurance sectors. Pasricha and Selvamuthu [14] employed a Markov modulated DCPD to price the synthetic collateralized debt obligations, demonstrating its effectiveness in the field of credit risk management. Wu et al. [23] applied the DCPD to model the claim process, addressing an optimization problem of reinsurance and investment for MV criterion. This progression of research underscores the versatility and utility of the DCPD model in tackling complex financial problems.

It is widely acknowledged that for insurers, beyond engaging in investment activities in financial markets, precisely controlling the number of policies sold to avoid excessive underwriting risk is of paramount importance. Therefore, the examination of insurers' optimization strategies of investment and risk control is particularly crucial. Zou and Cadenillas [25] investigated this issue with the objective for maximizing expected utility, assuming negative correlation between financial and insurance markets and providing the specific form of the optimal strategy through the martingale method. Bo and Wang [5] employed the HJB equation method to derive the optimal solution with a power utility function. Bo et al. [4] extended this issue to scenarios that include regime switching and default risk. Shen and Zou [19] addressed a similar issue for the MV criterion by formulating a time-consistent problem with a deterministic auxiliary process to obtain the optimal strategy and value function in closed form. For other studies, one can consult the works of Peng and Wang [16], Peng et al. [15], Shen and Yin [18], Zhou et al. [24], and many others.

This paper explores the integration of investment and risk control strategies for an MV insurer who is allowed to participate in a contagion risky financial market. Using multidimensional DCPD to model the price jumps of risky assets can capture both internally and externally excited jumps, thereby effectively reflecting the financial market's clustering and contagion effects. In the insurance market, we use a general jump process to describe the unit liabilities (risks) of the insurer. The insurer seeks to minimize the variance of terminal wealth, under the condition of a given expected terminal wealth, by adjusting the issuance volume of policies (i.e., risk control strategy) and investing the surplus in the financial market (i.e., investment strategy). Utilizing the SMP, BSDEs, and linear-quadratic (LQ) control techniques, we express the efficient strategy and the efficient frontier for the MV problem through the solution of a nonlocal partial differential equation (PDE). We also provide illustrative numerical examples to demonstrate the economic characteristics for the efficient frontier.

The recently published study by Shen and Zou [20] explored an MV portfolio selection problem within a contagious financial market. Our work presents distinct differences from Shen and Zou [20] in at least four significant aspects. First, Shen and Zou [20] describe the prices of risky assets using a multivariate Hawkes process, while our model employs a multivariate DCPD, introducing a more complex challenge to the MV problem. Second, the induced BSDE is different due to different market dynamics. Notably, our paper introduces an additional diffusion term related to the Brownian motion, necessitating substantial effort to demonstrate its uniform integrability as a martingale—a critical step for the proof of Lemma 3.1. Third, whereas the intensity process in Shen and Zou [20] enables moment estimates through the standard stochastic differential equation (SDE) theory because it satisfies the Lipschitz and linear growth conditions, our scenario requires a novel approach. The special structure of our intensity process allows for the derivation of moment estimates via fundamental inequalities for local martingales, diverging from the methodology in Shen and Zou [20]. Finally, our investigation extends to a combined market of finance and insurance, where the dynamics of the insurance sector can be modulated through policy issuance volume adjustments, offering a comprehensive view of both markets' interplay.

The following is an explanation of other sections in this article: Section 2 introduces the DCPD and the MV problem within a combined market. Section 3 derives a candidate solution by utilizing the SMP and the BSDE theory. Section 4 identifies the efficient strategy and the efficient frontier for the MV problem by using the completing the square technique from LQ theory. Section 5 provides numerical analyses, and the article concludes with a summary in Section 6.

2. Preliminaries

2.1. General setting

Consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ adhering to the usual conditions of right-continuity and completeness. Let $[0, T]$ denote a finite time interval, where the terminal time is $T < \infty$. In the subsequent discussion, it is assumed that all processes reside within this space. Under the probability measure $\mathbb{P}$, the operators $\mathbb{E}[\cdot]$ and $\mathbb{V}\mathrm{ar}[\cdot]$ represent the expectation and variance, respectively. Additionally, $\mathbb{E}[\cdot|\mathcal{F}_t]$ is simplified to $\mathbb{E}_t[\cdot]$. We now define five independent stochastic processes as follows:

- An n-dimensional standard Brownian motion denoted by

$$W := (W_1(t), W_2(t), \ldots, W_n(t))^\top{}_{t\in[0,T]}.$$

- A d-dimensional standard Brownian motion denoted by

$$B := (B_1(t), B_2(t), \ldots, B_d(t))^\top{}_{t\in[0,T]}.$$

- An m-dimensional càdlàg (right-continuous with left limits) point process denoted by

$$N := (N_1(t), N_2(t), \ldots, N_m(t))^\top{}_{t\in[0,T]}.$$

- An e-dimensional Poisson process denoted by

$$M := (M_1(t), M_2(t), \ldots, M_e(t))^\top{}_{t\in[0,T]}.$$

- A one-dimensional Poisson random measure $\gamma(\cdot, \cdot)$.

Define $\lambda := \{(\lambda_1(t), \lambda_2(t), \ldots, \lambda_m(t))^\top\}_{t\in[0,T]}$ as the intensity process of N, in which the instantaneous intensity of N_l is represented by $\lambda_l(t)$ for $l = 1, 2, \ldots, m$ at each time t within $[0, T]$. We consider λ in its càglàd (left continuous with right limits) form, expressed as

$$\lambda(t) = \lim_{\Delta t\to 0} \frac{\mathbb{E}_t\left[N(t + \Delta t) - N(t)\right]}{\Delta t}, \quad t \in [0, T].$$

This paper utilizes the DCPD, as introduced in Dassios and Zhao [9], to model the point process N. Specifically, the intensity process λ_l is expressed as

$$\mathrm{d}\lambda_l(t) = \alpha_l(\lambda_{l\infty} - \lambda_l(t))\mathrm{d}t + \sum_{i=1}^{d} \beta_{li} \sqrt{\lambda_l(t)}\mathrm{d}B_i(t) + \sum_{i=1}^{m} \zeta_{li}\mathrm{d}N_i(t) + \sum_{i=1}^{e} \delta_{li}\mathrm{d}M_i(t), \tag{2.1}$$

where $\lambda_l(0) = \lambda_{l0} > 0$, all coefficients in (2.1) are assumed to be constants, and $\alpha_l > 0$, $\lambda_{l\infty} \geq 0, \beta_{li} \geq 0$, $\zeta_{li} \geq 0$, and $\delta_{li} \geq 0$. The equation indicates that the intensity process λ_l mean reverts to its long-term level $\lambda_{l\infty}$ with a reversion rate of α_l and exhibits volatility of $\beta_{li} \sqrt{\lambda_l(t)}$. Additionally, it increases by ζ_{li} when process N_i experiences a jump and by δ_{li} when process M_i does.

Remark 2.1. *As demonstrated by* (2.1)*, the DCPD serves as an expansive generalization, incorporating the Hawkes process with exponentially decaying intensity, the Cox process with Poisson shot-noise intensity, and the Cox process with Cox-Ingersoll-Ross (CIR) intensity. This framework is especially potent for modeling risky assets due to its dual capacity for capturing the clustering phenomenon. On one hand, it demonstrates internal-excitation: an event in N_l at a given time t raises its own immediate intensity $\lambda_l(t)$ through ζ_{ll} (self-excitation), while simultaneously increasing the immediate intensities $\lambda_i(t)$ for alternative processes N_i through ζ_{il} (cross-excitation), for $i \neq l$ and $i = 1, 2, \dots, m$. On the other hand, it showcases external-excitation, with a surge in the external Poisson process M_l amplifying the jump intensity $\lambda(t)$ of N instantly by $\delta_{(l)}$, with $\delta_{(l)}$ signifying the lth column in the matrix δ. This combined mechanism of internal and external excitation augments the probability of observing increased jumps within a brief duration and across various components, effectively elucidating the phenomenon of jumping clusters within the financial market.*

The vector form of the intensity process $\lambda(t)$ is represented as

$$\mathrm{d}\lambda(t) = \alpha(\lambda_\infty - \lambda(t))\mathrm{d}t + \mathrm{Diag}[\sqrt{\lambda(t)}]\beta\mathrm{d}B(t) + \zeta\mathrm{d}N(t) + \delta\mathrm{d}M(t), \tag{2.2}$$

where

$$\begin{aligned} \lambda(0) = \lambda_0 &:= (\lambda_{10}, \lambda_{20}, \dots, \lambda_{m0})^\top, \quad \lambda_\infty := (\lambda_{1\infty}, \lambda_{2\infty}, \dots, \lambda_{m\infty})^\top, \\ \alpha &:= \mathrm{Diag}[(\alpha_1, \alpha_2, \dots, \alpha_m)^\top], \quad \mathrm{Diag}[\sqrt{\lambda(t)}] = \mathrm{Diag}[(\sqrt{\lambda_1(t)}, \sqrt{\lambda_2(t)}, \dots, \sqrt{\lambda_m(t)})^\top], \\ \beta &:= [\beta_{li}]_{m\times d}, \quad \zeta := [\zeta_{li}]_{m\times m}, \quad \delta := [\delta_{li}]_{m\times e}. \end{aligned}$$

Additionally, three compensated processes are defined: $\widetilde{N} := \{(\widetilde{N}_1(t), \widetilde{N}_2(t), \dots, \widetilde{N}_m(t))^\top\}_{t\in[0,T]}$ from N, expressed as

$$\widetilde{N}(t) := N(t) - \int_0^t \lambda(s)\mathrm{d}s, \quad t \in [0, T],$$

$\widetilde{M} := \{(\widetilde{M}_1(t), \widetilde{M}_2(t), \dots, \widetilde{M}_e(t))^\top\}_{t\in[0,T]}$ from M, represented as

$$\widetilde{M}(t) := M(t) - \rho t, \quad t \in [0, T],$$

and $\widetilde{\gamma}(\cdot,\cdot)$ from $\gamma(\cdot,\cdot)$, formulated as

$$\widetilde{\gamma}(\mathrm{d}t, \mathrm{d}z) := \gamma(\mathrm{d}t, \mathrm{d}z) - \nu^\gamma(\mathrm{d}z)\mathrm{d}t, \quad t \in [0, T],$$

where $\nu^\gamma(\cdot)$ is the Lévy density of jump sizes of the random measure $\gamma(\cdot,\cdot)$ and the vector $\rho := (\rho_1, \rho_2, \dots, \rho_e)^\top$ represents the intensity vector of M.

2.2. *The market model*

This research considers an insurer operating within a market that integrates both financial and insurance components. The financial sector includes a risk-free asset alongside k types of risky assets. The price of the risk-free asset is defined as

$$\mathrm{d}S_0(t) = rS_0(t)\mathrm{d}t, \quad S_0(0) = 1, \tag{2.3}$$

where the interest rate $r > 0$ is a constant. Define the prices of risky assets as

$$dS_i(t) = S_i(t-)\Big[\mu_i dt + \sum_{j=1}^{n} \sigma_{ij} dW_j(t) + \sum_{l=1}^{m} J_{il}\Big(Z_l(t)dN_l(t) - \mathbb{E}[Z_l(t)]\lambda_l(t)dt\Big)\Big], \quad i = 1, 2, \ldots, k, \tag{2.4}$$

in which the initial value of risky assets is represented as $s_{i0} = S_i(0) > 0$ and N_l is the lth entry of the m-dimensional DCPD whose intensity process λ is given by (2.2). Define μ as the expected return rate vector and σ as the volatility matrix, by

$$\mu := (\mu_1, \mu_2, \ldots, \mu_k)^\top, \quad \sigma := [\sigma_{ij}]_{k\times n}.$$

Additionally, define b as the risk premium vector and ξ as the jump size matrix, where

$$b := (\mu_1 - r, \mu_2 - r, \ldots, \mu_k - r)^\top, \quad \xi(Z(t)) := [J_{il}Z_l(t)]_{k\times m}, \qquad \forall t \in [0, T].$$

In the market model (2.4), several parameters are constants: the drift coefficient μ_i, the volatility rate σ_{ij}, which are always positive, and the scaling factor J_{il}, which falls within the interval $[0, 1]$. This holds for all $i = 1, 2, \ldots, k$, $j = 1, 2, \ldots, n$, and $l = 1, 2, \ldots, m$. Define a sequence of independent and identically distributed (i.i.d.) random variables $Z_l = Z_l(t)_{t\in[0,T]}$ to represent the jump sizes within the range $(-1, \infty)$. Additionally, $\nu_l^N(\cdot)$ serves as the common probability measure for Z_l, possessing a finite second moment with $l = 1, 2, \ldots, m$. It is easy to deduce that under the condition of $J_{il} \in [0, 1]$ and $Z_l \in (-1, \infty)$, the asset price $S_i(t)$ remains strictly positive for all $i = 1, 2, \ldots, k$. Let us denote $\nu^N := (\nu_1^N, \nu_2^N, \ldots, \nu_m^N)^\top$. As usual, we assume the variance-covariance matrix $\sigma\sigma^\top$ is positive definite, i.e., $\sigma\sigma^\top \geq \varepsilon I_k$, for some constants ε, and I_k is the k-dimensional identity matrix. Furthermore, we suppose that $\{Z_l\}_{l=1,2,\ldots,m}$, W, B, N, M, and γ are independent of each other, with an augmented filtration denoted by $\mathbb{F}$.

Based on the aforementioned model setup, it follows from Lemma 2.1 in Shen and Zou [19] that the variance-covariance matrix is generalized as

$$\Sigma(t) := \sigma\sigma^\top + \int_{(-1,\infty)^m} \xi(z)\,\mathrm{Diag}[\lambda(t) \bullet \nu^N(dz)]\xi(z)^\top, \tag{2.5}$$

which is positive definite. Additionally, the operator $\bullet$ is defined by

$$\lambda(t) \bullet \nu^N(dz) = (\lambda_1(t)\nu_1^N(dz_1), \lambda_2(t)\nu_2^N(dz_2), \ldots, \lambda_m(t)\nu_m^N(dz_m))^\top.$$

Remark 2.2. *Both internal and external excitation characteristics from the DCPD are reflected in the financial market model (2.4). The model exhibits contagion, meaning that a jump in one risky asset, triggered by internal or external factors, will enhance the intensity process of jumping for all risky assets. This implies that more jumps are likely to occur in the foreseeable future, thereby generating contagion and clustering effects.*

Furthermore, by employing the standard notation for random measures (refer to Øksendal and Sulem [13]), it follows that

$$\int_0^t J_{il}(Z_l(t)dN_l(t) - \mathbb{E}[Z_l(t)]\lambda_l(t)dt) = \int_0^t \int_{(-1,\infty)} J_{il}z_l\widetilde{N}_l(dt, dz_l), \quad \forall t \in [0, T], l = 1, 2, \ldots m,$$

where

$$\widetilde{N}_l(\mathrm{d}t, \mathrm{d}z_l) := N_l(\mathrm{d}t, \mathrm{d}z_l) - \lambda_l(t)\nu_l^N(\mathrm{d}z_l)dt$$

is the compensated random measure generated from the DCPD. Define

$$\widetilde{N}(\mathrm{d}t, \mathrm{d}z) := \left(\widetilde{N}_1(\mathrm{d}t, \mathrm{d}z_1), \widetilde{N}_2(\mathrm{d}t, \mathrm{d}z_2), \ldots, \widetilde{N}_m(\mathrm{d}t, \mathrm{d}z_m)\right)^\top.$$

Then, the vector form of the market model (2.4) can be inferred as

$$\mathrm{d}S(t) = \mathrm{Diag}[S(t-)]\Big(\mu\mathrm{d}t + \sigma\mathrm{d}W(t) + \int_{(-1,\infty)^m} \xi(z)\widetilde{N}(\mathrm{d}t, \mathrm{d}z)\Big).$$

In the insurance market, we define the unit liabilities (risk) of the insurer, which are represented by the jump process $R = \{R(t)\}_{t\in[0,T]}$, where

$$\mathrm{d}R(t) = \psi\mathrm{d}t + \int_{\mathbb{R}} \phi(t, z)\widetilde{\gamma}(\mathrm{d}t, \mathrm{d}z). \tag{2.6}$$

Here, $\psi \geq 0$ and $\phi(t, z) > 0$ with all t and z. In addition, we assume that $\phi(t, \cdot) = \phi(\cdot)$ is homogeneous and deterministic with

$$\int_{\mathbb{R}} \phi(z)\nu^\gamma(dz) < +\infty \quad \text{and} \quad \int_{\mathbb{R}} \phi^2(z)\nu^\gamma(dz) < +\infty.$$

In the context of the financial market, we use $\pi_i(t)$ to represent the investment amount in the ith risky asset at time t for $i = 1, 2, \ldots, k$, and consider the situation where an insurer adopts an investment strategy denoted as $\pi := \{\pi_1(t), \pi_2(t), \ldots, \pi_k(t)\}^\top_{t\in[0,T]}$. Additionally, in the insurance market, $\ell := \{\ell(t)\}_{t\in[0,T]}$ is defined as the risk control (liability) strategy implemented by the insurer, representing the underwriting liabilities at time t. The notation θ is defined as the unit premium rate for unit liabilities (risk) R, in which $\theta > 0$. Thus, the gains of the insurance business follow the formula $\ell(t)(\theta\mathrm{d}t - \mathrm{d}R(t))$ for any fixed ℓ.

Remark 2.3. *In the discussion above regarding the combined market* (2.3)*,* (2.4)*, and* (2.6)*, we assume that the model parameters (r, μ, θ, and ψ) are constants. Notice that all subsequent analyses and results remain valid if these parameters are represented through bounded and deterministic processes. Furthermore, when the deterministic function $\phi(t, \cdot)$ is not time-homogeneous, replacing the constant unit premium rate θ with a time-dependent rate $\theta(t)$ becomes necessary. In this case, it is essential to assume that $\theta(t)$ remains positive and bounded for all t.*

Let $u = (\pi, \ell)$ denote the shorthand notation for the insurer's investment and risk control strategy. The wealth process, denoted as $X(t) := X^u(t)$, has dynamics represented as follows:

$$\begin{aligned}\mathrm{d}X(t) &= \frac{X(t) - \sum_{i=1}^k \pi_i(t)}{S_0(t)}\mathrm{d}S_0(t) + \sum_{i=1}^k \frac{\pi_i(t)}{S_i(t)}\mathrm{d}S_i(t) + \ell(t)(\theta\mathrm{d}t - \mathrm{d}R(t)) \\ &= (rX(t) + \pi(t)^\top b + \ell(t)a)\,\mathrm{d}t + \pi(t)^\top\sigma\mathrm{d}W(t) + \int_{(-1,\infty)^m} \pi(t)^\top\xi(z)\widetilde{N}(\mathrm{d}t, \mathrm{d}z) \\ &\quad - \int_{\mathbb{R}} \ell(t)\phi(z)\widetilde{\gamma}(\mathrm{d}t, \mathrm{d}z), \quad t \in [0, T],\end{aligned} \tag{2.7}$$

where the insurer's initial wealth is defined as $x_0 = X(0) > 0$ and $a := \theta - \psi$.

In the remainder of this paper, the following notations are used:

- $\mathcal{S}^2_{\mathcal{F}}(0,T;\mathbb{R})$: the set of $\mathbb{R}$-valued, $\mathbb{F}$-adapted, càdlàg processes $\{\varphi(t)\}_{t\in[0,T]}$ such that

$$\mathbb{E}\Big[\sup_{t\in[0,T]}|\varphi(t)|^2\Big]<\infty;$$

- $\mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^k)$: the set of $\mathbb{R}^k$-valued, $\mathbb{F}$-predictable processes $\{\varphi(t)\}_{t\in[0,T]}$ such that

$$\mathbb{E}\Big[\int_0^T|\varphi(t)|^2\mathrm{d}t\Big]<\infty;$$

- $\mathcal{L}^{2,N}_{\mathcal{F}}(0,T;\mathbb{R}^m)$: the set of $\mathbb{R}^m$-valued, $\mathbb{F}$-predictable processes $\{\varphi(t,\cdot)_{t\in[0,T]}\}$ such that

$$\mathbb{E}\Big[\sum_{l=1}^m\int_0^T\int_{(-1,\infty)}|\varphi_l(t,z_l)|^2\lambda_l(t)\nu_l^N(\mathrm{d}z_l)\mathrm{d}t\Big]<\infty;$$

- $\mathcal{L}^{2,\gamma}_{\mathcal{F}}(0,T;\mathbb{R})$: the set of $\mathbb{R}$-valued, $\mathbb{F}$-predictable processes $\{\varphi(t,\cdot)_{t\in[0,T]}\}$ such that

$$\mathbb{E}\Big[\int_0^T\int_{\mathbb{R}}|\varphi(t,z)|^2\nu^\gamma(\mathrm{d}z)\mathrm{d}t\Big]<\infty.$$

As commonly required in the literature, we must impose specific integrability conditions on the strategy u. These conditions are detailed below.

Definition 2.1. *An investment and risk control strategy u is considered admissible if (1) $\pi\in\mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^k)$ and $\xi^\top\pi\in\mathcal{L}^{2,N}_{\mathcal{F}}(0,T;\mathbb{R}^m)$; (2) $\ell\in\mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R})$. The set of all admissible strategies is denoted as $\mathcal{A}$.*

Remark 2.4. *When the admissible set is defined, the condition $\ell\geq 0$ is not imposed. In other words, ℓ is allowed to be less than 0. For $\ell\geq 0$, ℓ represents the amount of liabilities the insurer decides to undertake in the insurance business, while θ denotes the premium rate the insurer receives from underwriting policies against the risk R. The modeling approach is based on Chapter 6 in Stein [21], inspired by the American international group case during the 2007-2008 financial crisis. When $\ell<0$, the gains of the insurance business can be reformulated as $-(-\ell(t))[\theta\mathrm{d}t-\mathrm{d}R(t)]$. In this scenario, $-\ell$ indicates the number of reinsurance policies purchased by the insurer, θ is the reinsurance premium rate, and R represents the risk borne by the reinsurer. Thus, $\theta\mathrm{d}t-\mathrm{d}R(t)$ constitutes the price of each reinsurance policy. Similar assumptions can be found in Shen and Zou [19].*

Remark 2.5. *According to standard SDE theory, the SDE (2.7) admits a unique strong solution $X\in\mathcal{S}^2_{\mathcal{F}}(0,T;\mathbb{R})$ with any $u\in\mathcal{A}$. Therefore, both $\mathbb{E}[X(T)]$ and $\mathbb{E}[X^2(T)]$ exist, which in turn ensures that the variance $\mathbb{V}\mathrm{ar}[X(T)]$ is also well-defined. This makes the forthcoming discussion on the MV problem meaningful.*

2.3. The problem

We now progress to formulating of the MV problem, which is the central focus of this study. The insurer aims to address the MV problem as outlined below:

$$\begin{cases}\min\limits_{u\in\mathcal{A}}\mathcal{J}(x_0,\lambda_0;u):=\min\limits_{u\in\mathcal{A}}\mathbb{E}[(X(T)-\kappa)^2],\\ \text{subject to}\begin{cases}\mathbb{E}[X(T)]=\kappa,\\ X\ \textit{is the solution to}\ (2.7),\end{cases}\end{cases}\tag{2.8}$$

where x_0 and λ_0 represent the initial values of the wealth for insurer and the intensity process λ, respectively. A solution u^* of the aforementioned problem is called an optimal (investment and risk control) strategy, also referred to as an efficient strategy. In order to preclude trivial scenarios, let us impose the condition $\kappa \geq x_0 e^{rT}$.

Since problem (2.8) is a convex optimization problem constrained by $\mathbb{E}[X(T)] = \kappa$, the Lagrange multiplier method can be applied. Define

$$\mathcal{J}_1(x_0, \lambda_0; u, \vartheta) := \mathbb{E}[(X(T) - \kappa)^2] + 2\vartheta(\mathbb{E}[X(T)] - \kappa) = \mathbb{E}[(X(T) - (\kappa - \vartheta))^2] - \vartheta^2,$$

wherein the range of the Lagrange multiplier ϑ is $\mathbb{R}$. The MV problem, as presented in (2.8), can be transformed into a maximum-minimum problem according to the principles of Lagrangian duality theory, expressed as

$$\begin{cases} \max\limits_{\vartheta \in \mathbb{R}} \min\limits_{u \in \mathcal{A}} \mathcal{J}_1(x_0, \lambda_0; u, \vartheta) = \max\limits_{\vartheta \in \mathbb{R}} \min\limits_{u \in \mathcal{A}} \{\mathbb{E}[(X(T) - (\kappa - \vartheta))^2] - \vartheta^2\}, \\ \text{subject to } X \text{ as in (2.7).} \end{cases} \tag{2.9}$$

Therefore, two steps are involved in solving problem (2.9). Initially, the internal minimization problem is solved for each fixed Lagrange multiplier ϑ. Subsequently, the optimal solution of the external maximization problem is determined. Therefore, following the first step, the internal unconstrained minimization problem is expressed as

$$\begin{cases} \min\limits_{u \in \mathcal{A}} \mathcal{J}_2(x_0, \lambda_0; u, c) := \min\limits_{u \in \mathcal{A}} \mathbb{E}[(X(T) - c)^2], \\ \text{subject to } X \text{ as in (2.7),} \end{cases} \tag{2.10}$$

where the range of the free parameter c is $\mathbb{R}$. Let us denote by u_c^* the optimal solution to problem (2.10). Upon solving this problem, we can address problem (2.9) by examining

$$\max_{\vartheta \in \mathbb{R}} \mathcal{J}_3(x_0, \lambda_0; \vartheta) := \max_{\vartheta \in \mathbb{R}} \left\{ \mathcal{J}_2(x_0, \lambda_0; u_c^*, \kappa - \vartheta) - \vartheta^2 \right\}. \tag{2.11}$$

Therefore, the efficient strategy of the MV problem (2.8) is denoted as $u^* := u_c^*|_{c=\kappa-\vartheta^*}$, where ϑ^* is the solution to problem (2.11). Under the condition $\kappa \geq x_0 e^{rT}$ and the strategy u^*, the terminal wealth is defined as $X^*(T)$. Then, the efficient frontier appears as $(\mathbb{E}[(X^*(T) - \kappa)^2], \kappa)$.

3. Stochastic maximum principle

This section addresses the solution of the MV problem (2.8) by focusing on the unconstrained minimization problem (2.10). This problem is tackled through the application of the SMP. Given that the standard SMP provides only a necessary condition for optimality, the strategy u_c^* defined in (3.26) and (3.27) emerges as a potential optimal solution for problem (2.10). Nevertheless, the unbounded nature of the intensity process λ suggests that strategies determined by the SMP are not directly applicable for confirming their optimality. Consequently, the verification of this will be postponed to the next section, where the technique of completing the square in LQ control theory will be employed.

First of all, the process $\widehat{X} := \{\widehat{X}(t)\}_{t \in [0,T]}$ is defined as

$$\widehat{X}(t) := X(t) - ce^{-r(T-t)}. \tag{3.1}$$

According to the Itô's formula, one can infer from the above equation that

$$\begin{aligned} \mathrm{d}\widehat{X}(t) =& \Big(r\widehat{X}(t) + \pi(t)^\top b + \ell(t)a\Big)\,\mathrm{d}t + \pi(t)^\top \sigma \mathrm{d}W(t) \\ &+ \int_{(-1,\infty)^m} \pi(t)^\top \xi(z)\widetilde{N}(\mathrm{d}t, \mathrm{d}z) - \int_{\mathbb{R}} \ell(t)\phi(z)\widetilde{\gamma}(\mathrm{d}t, \mathrm{d}z), \end{aligned} \tag{3.2}$$

with $\widehat{X}(0) = x_0 - ce^{-rT}$ representing the initial value. Thus, we can simplify problem (2.10) into the form of

$$\min_{u\in\mathcal{A}} \mathcal{J}_2(x_0, \lambda_0; u, c) = \min_{u\in\mathcal{A}} \mathbb{E}[\widehat{X}(T)^2]. \tag{3.3}$$

Based on the state process $\widehat{X}$ described in (3.2), we can derive the Hamiltonian for problem (2.10) or, equivalently, (3.3), as follows:

$$\begin{aligned} \mathcal{H}(t, x, \pi, \ell, p, q, v, w, s, \eta) :=& \,(rx + \pi^\top b + \ell a)\, p + \pi^\top \sigma q - \int_{\mathbb{R}} \ell\phi(z)s(t, z)\nu^\gamma(\mathrm{d}z) \\ &+ \int_{(-1,\infty)^m} \pi^\top \xi(z)\, \mathrm{Diag}[\lambda(t) \bullet \nu^N(\mathrm{d}z)]w(t, z). \end{aligned}$$

Here, the 6-tuple (p, q, v, w, s, η) is referred to as the adjoint process. It should be noted that $p := \{p(t)\}_{t\in[0,T]} \in \mathbb{R}$, $q := \{q(t)\}_{t\in[0,T]} \in \mathbb{R}^n$, $v := \{v(t)\}_{t\in[0,T]} \in \mathbb{R}^d$, $w := \{w(t, z)\}_{(t,z)\in[0,T]\times(-1,\infty)^m} \in \mathbb{R}^m$, $s := \{s(t, z)\}_{(t,z)\in[0,T]\times\mathbb{R}} \in \mathbb{R}$, and $\eta := \{\eta(t)\}_{t\in[0,T]} \in \mathbb{R}^e$. According to standard control theory, the adjoint process (p, q, v, w, s, η) satisfies the following adjoint equation:

$$\begin{cases} \mathrm{d}p(t) = -rp(t)\mathrm{d}t + q(t)^\top \mathrm{d}W(t) + v(t)^\top \mathrm{d}B(t) + \displaystyle\int_{(-1,\infty)^m} w(t, z)^\top \widetilde{N}(\mathrm{d}t, \mathrm{d}z) \\ \qquad\qquad + \displaystyle\int_{\mathbb{R}} s(t, z)\widetilde{\gamma}(\mathrm{d}t, \mathrm{d}z) + \eta(t)^\top \mathrm{d}\widetilde{M}(t), \\ p(T) = 2\widehat{X}(T) = 2(X(T) - c). \end{cases} \tag{3.4}$$

It is important to point out that in (3.4), the drift term is determined by

$$-\frac{\partial}{\partial x}\mathcal{H}(t, \widehat{X}(t), \pi(t), \ell(t), p(t), q(t), v(t), w(t, \cdot), s(t, \cdot), \eta(t)) = -rp(t).$$

An ansatz for p is proposed to address the adjoint Eq (3.4), expressed as

$$p(t) = Y(t)\widehat{X}(t), \tag{3.5}$$

in which Y is defined as the solution to the follwoing BSDE:

$$\begin{cases} \mathrm{d}Y(t) = -f(t, Y(t), P(t), Q(t), G(t))\mathrm{d}t + P(t)^\top \mathrm{d}B(t) + Q(t)^\top \mathrm{d}\widetilde{N}(t) + G(t)^\top \mathrm{d}\widetilde{M}(t), \\ Y(T) = 2. \end{cases} \tag{3.6}$$

The driver f is currently unknown in (3.6), but will be determined at a later stage. Employing (3.2) and (3.6) and utilizing Itô's formula to $p(t) = Y(t)\widehat{X}(t)$, yields

$$\mathrm{d}p(t) = \widehat{X}(t-)\mathrm{d}Y(t) + Y(t-)\mathrm{d}\widehat{X}(t) + \mathrm{d}[Y(t), \widehat{X}(t)]$$

$$
\begin{aligned}
=&\Big[-f(t,Y(t),P(t),Q(t),G(t))\widehat{X}(t)+Y(t)\big(r\widehat{X}(t)+\pi(t)^{\top}b+\ell(t)a\big)\\
&+\sum_{i=1}^{k}\sum_{l=1}^{m}\int_{(-1,\infty)}\pi_i(t)\xi_{il}(z_l)Q_l(t)\lambda_l(t)\nu_l^N(\mathrm{d}z_l)\Big]\mathrm{d}t\\
&+Y(t)\pi(t)^{\top}\sigma\mathrm{d}W(t)+\widehat{X}(t)P(t)^{\top}\mathrm{d}B(t)\\
&+\sum_{l=1}^{m}\int_{(-1,\infty)}\Big[\widehat{X}(t-)Q_l(t)+\sum_{i=1}^{k}\pi_i(t)\xi_{il}(z_l)\big(Y(t-)+Q_l(t)\big)\Big]\widetilde{N}_l(\mathrm{d}t,\mathrm{d}z_l)\\
&-\int_{\mathbb{R}}Y(t-)\ell(t)\phi(z)\widetilde{\gamma}(\mathrm{d}t,\mathrm{d}z)+\widehat{X}(t-)G(t)^{\top}\mathrm{d}\widetilde{M}(t).
\end{aligned}\tag{3.7}
$$

By analyzing the dynamics equation for p, as described in (3.4) and (3.7), one can derive

$$
\begin{aligned}
-rY(t)\widehat{X}(t)=&-f(t,Y(t),P(t),Q(t),G(t))\widehat{X}(t)+Y(t)\big(r\widehat{X}(t)+\pi(t)^{\top}b+\ell(t)a\big)\\
&+\sum_{i=1}^{k}\sum_{l=1}^{m}\int_{(-1,\infty)}\pi_i(t)\xi_{il}(z_l)Q_l(t)\lambda_l(t)\nu_l^N(\mathrm{d}z_l),
\end{aligned}\tag{3.8}
$$

$$q(t)=\sigma^{\top}\pi(t)Y(t),\tag{3.9}$$

$$v(t)=P(t)\widehat{X}(t),\tag{3.10}$$

$$w_l(t,z_l)=\widehat{X}(t-)Q_l(t)+\sum_{i=1}^{k}\pi_i(t)\xi_{il}(z_l)\big(Y(t-)+Q_l(t)\big),\tag{3.11}$$

$$s(t,z)=-\ell(t)\phi(z)Y(t-),\tag{3.12}$$

$$\eta(t)=G(t)\widehat{X}(t-).\tag{3.13}$$

By the SMP, a candidate optimal solution is identified as

$$\pi_c^*(t)=\arg\max_{\pi\in\mathbb{R}^k}\mathcal{H}\left(t,\widehat{X}^*(t),\pi,\ell,p^*(t),q^*(t),v^*(t),w^*(t,\cdot),s^*(t,\cdot),\eta^*(t)\right),$$

$$\ell_c^*(t)=\arg\max_{\ell\in\mathbb{R}}\mathcal{H}\left(t,\widehat{X}^*(t),\pi,\ell,p^*(t),q^*(t),v^*(t),w^*(t,\cdot),s^*(t,\cdot),\eta^*(t)\right),$$

where under the strategy $u_c^*=(\pi_c^*,\ell_c^*)$, the state process is defined as $\widehat{X}^*$, while the adjoint process is denoted by $(p^*,q^*,v^*,w^*,s^*,\eta^*)$. Additionally, the subscript c represents the parameter of problem (2.10). Then, u_c^* is derived by using the first-order condition

$$p(t)b+\sigma q(t)+\int_{(-1,\infty)^m}\xi(z)\,\mathrm{Diag}[\lambda(t)\bullet\nu^N(\mathrm{d}z)]w(t,z)=0,$$

$$p(t)a-\int_{\mathbb{R}}\phi(z)s(t,z)\nu^{\gamma}(\mathrm{d}z)=0.$$

Using the above equation and replacing it with (3.5), (3.9), (3.11), and (3.12), it can be inferred that

$$\pi_c^*(t)=-\Gamma(t)^{-1}\mathcal{Z}(t)\widehat{X}^*(t-),\tag{3.14}$$

$$\ell_c^*(t)=-a\Big(\int_{\mathbb{R}}\phi(z)^2\nu^{\gamma}(\mathrm{d}z)\Big)^{-1}\widehat{X}^*(t-),\tag{3.15}$$

where

$$\Gamma(t) := Y(t-)\Big(\sigma\sigma^\top + \int_{(-1,\infty)^m} \xi(z)\mathrm{Diag}[\lambda(t) \bullet \nu^N(\mathrm{d}z)]\xi(z)^\top\Big) + \int_{(-1,\infty)^m} \xi(z)\,\mathrm{Diag}[Q(t) \bullet \lambda(t) \bullet \nu^N(\mathrm{d}z)]\xi(z)^\top, \tag{3.16}$$

$$\mathcal{Z}(t) := bY(t-) + \int_{(-1,\infty)^m} \xi(z)\,\mathrm{Diag}[\lambda(t) \bullet \nu^N(\mathrm{d}z)]Q(t). \tag{3.17}$$

By substituting π_c^* and ℓ_c^* as specified in (3.14) and (3.15) back into (3.8), we obtain the driver f of the BSDE described in (3.6), which is represented as

$$f(t, Y(t), P(t), Q(t), G(t))) = 2rY(t) - \mathcal{Z}(t)^\top\Gamma(t)^{-1}\mathcal{Z}(t) - a^2\Big(\int_{\mathbb{R}} \phi(z)^2\nu^\gamma(\mathrm{d}z)\Big)^{-1} Y(t). \tag{3.18}$$

To solve the solution (Y, P, Q, G) for (3.6), we propose the following ansatz:

$$Y(t) = 2e^{2r(T-t)+g(t,\lambda(t))}, \tag{3.19}$$

where $g(t, \lambda(t))$ is a bounded function that remains to be determined.

Utilizing Itô's formula to (3.19), it follows that

$$\begin{aligned}\mathrm{d}Y(t) = Y(t)\Big[&-2r + \frac{\partial g}{\partial t}(t, \lambda(t)) + \frac{\partial g}{\partial \lambda}(t, \lambda(t))^\top\alpha(\lambda_\infty - \lambda(t)) \\ &+ \frac{1}{2}\sum_{l=1}^{m}\sum_{i=1}^{m}\sum_{j=1}^{d}\Big(\frac{\partial g}{\partial \lambda_l}(t, \lambda(t))\frac{\partial g}{\partial \lambda_i}(t, \lambda(t)) + \frac{\partial^2 g}{\partial \lambda_l \partial \lambda_i}(t, \lambda(t))\Big)\sqrt{\lambda_l(t)}\sqrt{\lambda_i(t)}\beta_{lj}\beta_{ij} \\ &+ \sum_{l=1}^{m}\lambda_l(t)\Big(e^{g(t,\lambda(t)+\zeta_{(l)})-g(t,\lambda(t))} - 1\Big) + \sum_{i=1}^{e}\rho_i\Big(e^{g(t,\lambda(t)+\delta_{(i)})-g(t,\lambda(t))} - 1\Big)\Big]\mathrm{d}t \\ &+ Y(t)\frac{\partial g}{\partial \lambda}(t, \lambda(t))^\top\mathrm{Diag}[\sqrt{\lambda(t)}]\beta\mathrm{d}B(t) + Y(t-)\sum_{l=1}^{m}\Big(e^{g(t,\lambda(t)+\zeta_{(l)})-g(t,\lambda(t))} - 1\Big)\mathrm{d}\widetilde{N}_l(t) \\ &+ Y(t-)\sum_{i=1}^{e}\Big(e^{g(t,\lambda(t)+\delta_{(i)})-g(t,\lambda(t))} - 1\Big)\mathrm{d}\widetilde{M}_i(t),\end{aligned} \tag{3.20}$$

in which the lth column of the matrix ζ is denoted by $\zeta_{(l)}$ for $l = 1, 2, \ldots, m$, and, similarly, the ith column of the matrix δ is represented by $\delta_{(i)}$ for $i = 1, 2, \ldots, e$, that is,

$$\zeta_{(l)} := (\zeta_{1l}, \zeta_{2l}, \ldots, \zeta_{ml})^\top, \quad \delta_{(i)} := (\delta_{1i}, \delta_{2i}, \ldots, \delta_{mi})^\top.$$

For notation simplification, the following are introduced:

$$U(t, \lambda) : = \Big(e^{g(t,\lambda+\zeta_{(1)})-g(t,\lambda)} - 1, \ldots, e^{g(t,\lambda+\zeta_{(m)})-g(t,\lambda)} - 1\Big)^\top, \tag{3.21}$$

$$V(t, \lambda) : = \Big(e^{g(t,\lambda+\delta_{(1)})-g(t,\lambda)} - 1, \ldots, e^{g(t,\lambda+\delta_{(e)})-g(t,\lambda)} - 1\Big)^\top, \tag{3.22}$$

$$\widehat{\Gamma}(t, \lambda) : = \frac{\Gamma(t)}{Y(t-)} = \sigma\sigma^\top + \int_{(-1,\infty)^m} \xi(z)\mathrm{Diag}[(U(t, \lambda) + \mathbf{1}_m) \bullet \lambda \bullet \nu^N(\mathrm{d}z)]\xi(z)^\top, \tag{3.23}$$

$$\widehat{\mathcal{Z}}(t,\lambda) : = \frac{\mathcal{Z}(t)}{Y(t-)} = b + \int_{(-1,\infty)^m} \xi(z)\,\mathrm{Diag}[\lambda \bullet \nu^N(\mathrm{d}z)]U(t,\lambda), \tag{3.24}$$

$$\Psi := \int_{\mathbb{R}} \phi(z)^2 \nu^\gamma(\mathrm{d}z), \tag{3.25}$$

where $\mathbf{1}_m$ denotes the m-dimensional vector consisting entirely of 1. Notably, given the ansatz for Y in (3.19), both $\widehat{\Gamma}$ and $\widehat{\mathcal{Z}}$ are defined as functions solely of t and λ, indicating they are independent from Y, P, Q, and G. From these definitions, it becomes evident that

$$Q(t) = Y(t)U(t,\lambda(t)),$$
$$G(t) = Y(t)V(t,\lambda(t)).$$

Additionally, by rewriting π_c^* and ℓ_c^* as presented in (3.14) and (3.15), one has

$$\pi_c^*(t) = -\widehat{\Gamma}(t,\lambda(t))^{-1}\widehat{\mathcal{Z}}(t,\lambda(t))\widehat{X}^*(t-), \tag{3.26}$$

$$\ell_c^*(t) = -a\Psi^{-1}\widehat{X}^*(t-), \tag{3.27}$$

and by rewriting f as presented in (3.18), we obtain

$$f(t,Y(t),P(t),Q(t),G(t))) = Y(t)\Big[2r - \widehat{\mathcal{Z}}(t,\lambda(t))^\top\widehat{\Gamma}(t,\lambda(t))^{-1}\widehat{\mathcal{Z}}(t,\lambda(t)) - a^2\Psi^{-1}\Big]. \tag{3.28}$$

Through comparing the drift terms of (3.20) and (3.28), one can readily derive the following non-local PDE governing g:

$$\begin{cases} \dfrac{\partial g}{\partial t}(t,\lambda) + \dfrac{\partial g}{\partial \lambda}(t,\lambda)^\top \alpha(\lambda_\infty - \lambda) + U(t,\lambda)^\top\lambda + V(t,\lambda)^\top\rho \\ \quad +\dfrac{1}{2}\displaystyle\sum_{l=1}^m\sum_{i=1}^m\sum_{j=1}^d \Big(\dfrac{\partial g}{\partial \lambda_l}(t,\lambda)\dfrac{\partial g}{\partial \lambda_i}(t,\lambda) + \dfrac{\partial^2 g}{\partial\lambda_l\partial\lambda_i}(t,\lambda)\Big)\sqrt{\lambda_l}\sqrt{\lambda_i}\,\beta_{lj}\beta_{ij} \\ \quad = \widehat{\mathcal{Z}}(t,\lambda)^\top\widehat{\Gamma}(t,\lambda)^{-1}\widehat{\mathcal{Z}}(t,\lambda) + a^2\Psi^{-1}, \\ g(T,\cdot) = 0. \end{cases} \tag{3.29}$$

Moreover, the BSDE (3.20) can be reformulated as

$$\begin{aligned} \mathrm{d}Y(t) = &- Y(t)\Big[2r - \widehat{\mathcal{Z}}(t,\lambda(t))^\top\widehat{\Gamma}(t,\lambda(t))^{-1}\widehat{\mathcal{Z}}(t,\lambda(t)) - a^2\Psi^{-1}\Big]\mathrm{d}t \\ &+ Y(t)\frac{\partial g}{\partial \lambda}(t,\lambda(t))^\top\mathrm{Diag}[\sqrt{\lambda(t)}]\beta\mathrm{d}B(t) \\ &+ Y(t-)U(t,\lambda(t))^\top\mathrm{d}\widetilde{N}(t) + Y(t-)V(t,\lambda(t))^\top\mathrm{d}\widetilde{M}(t). \end{aligned} \tag{3.30}$$

Remark 3.1. *Notably, in comparison with the MV problem involving the Hawkes process as described in Shen and Zou [20], the introduction of DCPD in this paper significantly complicates the resolution of the BSDE* (3.30). *This complexity arises from the additional diffusion term* $Y(t)\frac{\partial g}{\partial\lambda}(t,\lambda(t))^\top \mathit{Diag}[\sqrt{\lambda(t)}]\beta\mathrm{d}B(t)$ *and the jump term* $Y(t-)V(t,\lambda(t))^\top\mathrm{d}\widetilde{M}(t)$ *featured in* (3.30). *The diffusion term, in particular, requires considerable effort to demonstrate that it is a uniformly integrable martingale. To address this, we initially assume that g is a bounded function when proposing a solution for* $Y(t)$ *in* (3.19)*, and we establish its strict upper and lower bounds in Lemma 3.1 below. Furthermore, the Eq* (3.29) *that governs g in this paper is more complex, incorporating second-order derivative terms of g, which adds to the challenge of finding a solution. These elements distinctly differentiate this work from that of Shen and Zou [20].*

We now present a technical result concerning the strict upper and lower bounds of the function g, which solves (3.29).

Lemma 3.1. *If $g(\cdot,\cdot)$ is solution to (3.29), then the following holds:*

$$0 \le e^{g(t,\lambda(t))} \le 1, \quad \forall t \in [0,T], \quad \text{and} \quad 0 \le e^{g(0,\lambda_0)} < 1,$$

in which $\lambda(\cdot)$ represents the intensity process of the DCPD as described in (2.2).

Proof. Integrating from t to T on both sides of the Eq (3.30) yields

$$\begin{aligned} Y(T) - Y(t) &= \int_t^T Y(s)\Big[-2r + \widehat{\mathcal{Z}}(s,\lambda(s))^\top \widehat{\Gamma}(s,\lambda(s))^{-1}\widehat{\mathcal{Z}}(s,\lambda(s)) + a^2\Psi^{-1}\Big]\mathrm{d}s \\ &\quad + \int_t^T Y(s)\frac{\partial g}{\partial \lambda}(s,\lambda(s))^\top \mathrm{Diag}[\sqrt{\lambda(s)}]\beta \mathrm{d}B(s) + \int_t^T Y(s-)U(s,\lambda(s))^\top \mathrm{d}\widetilde{N}(s) \\ &\quad + \int_t^T Y(s-)V(s,\lambda(s))^\top \mathrm{d}\widetilde{M}(s). \end{aligned}$$

Since g is assumed to be a bounded function, the processes Y, U, and V are also bounded. Furthermore, by following the approach used in the proof of Lemma 2.1 in Shen and Zou [20], it is established that $\widehat{\Gamma}(t,\lambda(t)) \ge \varepsilon I_k$. Consequently, given that the distribution function $\nu_l^N(\cdot)$, for $l = 1,\ldots,m$, has a finite second moment, we conclude that

$$\mathbb{E}\Big[\Big|\int_t^T Y(s)\widehat{\mathcal{Z}}(s,\lambda(s))^\top \widehat{\Gamma}(s,\lambda(s))^{-1}\widehat{\mathcal{Z}}(s,\lambda(s))\mathrm{d}s\Big|\Big] \le K\Big\{1 + \mathbb{E}\Big[\int_t^T |\lambda(s)|^2 \mathrm{d}s\Big]\Big\} < +\infty,$$

where K is defined as a positive constant. Additionally, under the condition $\mathbb{E}[\sup_{s\in[0,T]} |\lambda(s)|^2] < \infty$ (see (3.34)), the last inequality holds.

From Theorem 3.7 of Dassios and Zhao [9], it holds that $\mathbb{E}[N(t)] < \infty$. Therefore,

$$\begin{aligned} &\mathbb{E}\Big[\Big|\int_t^T Y(s-)U(s,\lambda(s))^\top \mathrm{d}\widetilde{N}(s)\Big|\Big] \\ &= \mathbb{E}\Big[\Big|\int_t^T Y(s-)U(s,\lambda(s))^\top (\mathrm{d}N(s) - \lambda(s)\mathrm{d}s)\Big|\Big] \\ &\le K\Big\{\mathbb{E}[N(T) - N(t)] + \mathbb{E}\Big[\int_t^T |\lambda(s)|\mathrm{d}s\Big]\Big\} \\ &\le K\Big\{1 + \mathbb{E}\Big[\sup_{s\in[0,T]} |\lambda(s)|^2\Big]\Big\} \\ &< +\infty. \end{aligned}$$

Similarly, one has

$$\mathbb{E}\Big[\Big|\int_t^T Y(s-)V(s,\lambda(s))^\top \mathrm{d}\widetilde{M}(s)\Big|\Big] < +\infty.$$

Therefore,

$$\mathbb{E}\Big[\Big|\int_t^T Y(s)\frac{\partial g}{\partial \lambda}(s,\lambda(s))^\top \mathrm{Diag}[\sqrt{\lambda(s)}]\beta \mathrm{d}B(s)\Big|\Big]$$

$$
\begin{aligned}
&= \mathbb{E}\Big[\Big|2 - Y(t) + \int_t^T Y(s)\Big[2r - \widehat{\mathcal{Z}}(s,\lambda(s))^\top \widehat{\Gamma}(s,\lambda(s))^{-1}\widehat{\mathcal{Z}}(s,\lambda(s)) - a^2\Psi^{-1}\Big]\mathrm{d}s \\
&\quad - \int_t^T Y(s-)U(s,\lambda(s))^\top \mathrm{d}\widetilde{N}(s) - \int_t^T Y(s-)V(s,\lambda(s))^\top \mathrm{d}\widetilde{M}(s)\Big|\Big] \\
&< +\infty.
\end{aligned} \tag{3.31}
$$

This implies that the stochastic integral with respect to the Brownian motion B is a uniformly integrable martingale. Additionally, since Y, U, and V are bounded, the jump components in (3.30) are also martingales. Hence, by the dynamics of Y as presented in (3.30), we can readily deduce that $\{e^{2rt}Y(t)\}_{t\in[0,T]}$ is a sub-martingale, and then for any $t \in [0,T]$,

$$
e^{2rt}Y(t) \le \mathbb{E}_t\Big[e^{2rT}Y(T)\Big].
$$

Noticing $Y(T) = 2$ and (3.19), it can be inferred that

$$
2e^{2r(T-t)} \cdot e^{g(t,\lambda(t))} = Y(t) \le 2e^{2r(T-t)},
$$

thus,

$$
0 \le e^{g(t,\lambda(t))} \le 1, \quad \forall t \in [0,T]. \tag{3.32}
$$

Next, by utilizing Itô's formula to $e^{g(t,\lambda(t))}$ and substituting from (3.29), we find that

$$
\begin{aligned}
\mathrm{d}e^{g(t,\lambda(t))} =& e^{g(t,\lambda(t))}\Big[\frac{\partial g}{\partial t}(t,\lambda(t)) + \frac{\partial g}{\partial \lambda}(t,\lambda(t))^\top \alpha(\lambda_\infty - \lambda(t)) + U(t,\lambda(t))^\top\lambda(t) + V(t,\lambda(t))^\top\rho \\
&+ \frac{1}{2}\sum_{l=1}^m\sum_{i=1}^m\sum_{j=1}^d\Big(\frac{\partial g}{\partial \lambda_l}(t,\lambda(t))\frac{\partial g}{\partial \lambda_i}(t,\lambda(t)) + \frac{\partial^2 g}{\partial\lambda_l\partial\lambda_i}(t,\lambda(t))\Big)\sqrt{\lambda_l(t)}\sqrt{\lambda_i(t)}\beta_{lj}\beta_{ij}\Big]\mathrm{d}t \\
&+ e^{g(t,\lambda(t))}\frac{\partial g}{\partial\lambda}(t,\lambda(t))^\top \mathrm{Diag}[\sqrt{\lambda(t)}]\beta\mathrm{d}B(t) \\
&+ e^{g(t,\lambda(t))}U(t,\lambda(t))^\top\mathrm{d}\widetilde{N}(t) + e^{g(t,\lambda(t))}V(t,\lambda(t))^\top\mathrm{d}\widetilde{M}(t) \\
=& e^{g(t,\lambda(t))}\Big[\widehat{\mathcal{Z}}(t,\lambda(t))^\top\widehat{\Gamma}(t,\lambda(t))^{-1}\widehat{\mathcal{Z}}(t,\lambda(t)) + a^2\Psi^{-1}\Big]\mathrm{d}t \\
&+ e^{g(t,\lambda(t))}\frac{\partial g}{\partial\lambda}(t,\lambda(t))^\top \mathrm{Diag}[\sqrt{\lambda(t)}]\beta\mathrm{d}B(t) \\
&+ e^{g(t,\lambda(t))}U(t,\lambda(t))^\top\mathrm{d}\widetilde{N}(t) + e^{g(t,\lambda(t))}V(t,\lambda(t))^\top\mathrm{d}\widetilde{M}(t).
\end{aligned}
$$

Based on prior analysis, the stochastic integrals are determined to be martingales. When we integrate both sides of the above equation over the interval from t_1 to t_2 and condition on t_1, assuming $0 \le t_1 \le t_2 \le T$, it follows that

$$
\begin{aligned}
0 \le& \mathbb{E}_{t_1}[e^{g(t_2,\lambda(t_2))}] - e^{g(t_1,\lambda(t_1))} \\
=& \mathbb{E}_{t_1}\Big\{\int_{t_1}^{t_2} e^{g(s,\lambda(s))}\Big[\widehat{\mathcal{Z}}(s,\lambda(s))^\top\Gamma(s,\lambda(s))^{-1}\widehat{\mathcal{Z}}(s,\lambda(s)) + a^2\Psi^{-1}\Big]\mathrm{d}s\Big\} \le 1.
\end{aligned} \tag{3.33}
$$

Thus, $\mathbb{E}_t[e^{g(t,\lambda(t))}] \ge e^{g(0,\lambda(0))}$ holds true for any $t \in [0,T]$. If $e^{g(0,\lambda(0))} = 1$, it then follows from (3.32) that $e^{g(t,\lambda(t))} \equiv 1$. In that case, from (3.33), we obtain that $a = 0$ and

$$
\widehat{\mathcal{Z}}(t,\lambda(t)) = b - \int_{(-1,\infty)^m}\xi(z)\,\mathrm{Diag}[\lambda(t)\bullet\nu^N(\mathrm{d}z)]\mathbf{1}_m = \mathbf{0}_k, \quad \mathbb{P}\text{-a.s.}, \quad \text{a.e. } t \in [0,T],
$$

where $\mathbf{0}_k$ denotes the k-dimensional vector consisting entirely of 0. Given that λ is a stochastic process, it is impossible for this to hold across all t. Consequently, it must be true that $e^{g(0,\lambda(0))} = e^{g(0,\lambda_0)} < 1$. □

The solvability results for (3.29) are outlined below.

Lemma 3.2. *The nonlocal PDE given by (3.29) possesses a unique solution.*

Proof. Given that the SDE (2.2) for the DCPD does not satisfy the Lipschitz condition, the moment estimates from standard SDE theory cannot be directly applied to λ. Consequently, our proof deviates somewhat from that of Lemma 3.2 in Shen and Zou [19]. Here, we detail only the modifications made to the proof. To simplify the presentation and prevent the explosion of the results, we will limit our consideration to the case where $m = d = e = 1$. Extending these results to multidimensional cases is straightforward. Unless specified otherwise, the symbol C will represent positive constants, which can vary from one line to the next for subsequent estimates.

We first express (2.2) in the following integral form:

$$\begin{aligned}\lambda(t) =&\lambda_0 e^{(\zeta-\alpha)t} + \frac{\alpha\lambda_\infty}{\alpha-\zeta}\left(1 - e^{(\zeta-\alpha)t}\right) + \int_0^t e^{(\alpha-\zeta)(s-t)}\sqrt{\lambda(s)}\beta \mathrm{d}B(s)\\ &+ \int_0^t e^{(\alpha-\zeta)(s-t)}\zeta \mathrm{d}\widetilde{N}(s) + \int_0^t e^{(\alpha-\zeta)(s-t)}\delta \mathrm{d}M(s).\end{aligned}$$

From the Burkholder-Davis-Gundy (B-D-G) inequality, it holds that

$$\begin{aligned}\mathbb{E}\Big[\sup_{t\in[0,T]}|\lambda(t)|^4\Big] &\le C\Big\{1 + \mathbb{E}\Big[\sup_{t\in[0,T]}\Big|\int_0^t\sqrt{\lambda(s)}\mathrm{d}B(s)\Big|^4 + \sup_{t\in[0,T]}|\widetilde{N}(t)|^4 + |M(T)|^4\Big]\Big\}\\ &\le C\Big\{1 + \mathbb{E}\Big[\Big(\int_0^T\lambda(s)\mathrm{d}s\Big)^2 + |N(T)|^2\Big]\Big\}\\ &\le C\Big\{1 + \mathbb{E}\Big[\sup_{t\in[0,T]}|\lambda(t)|^2\cdot T^2\Big]\Big\}\\ &= C\Big\{1 + \mathbb{E}\Big[2\frac{1}{\varpi}\sup_{t\in[0,T]}|\lambda(t)|^2\cdot\frac{\varpi}{2}T^2\Big]\Big\}\\ &\le C\Big\{1 + \mathbb{E}\Big[\frac{1}{\varpi^2}\sup_{t\in[0,T]}|\lambda(t)|^4\Big]\Big\},\end{aligned}$$

where we have used the fact that the quadratic variation process for $\widetilde{N}$ is N, Theorem 3.8 of Dassios and Zhao [9] to derive that $N(T)$ has a finite second moment, and the fundamental inequality $2ab \le a^2 + b^2$. By choosing the constant $\varpi = \sqrt{2C}$, one easily gets

$$\mathbb{E}\Big[\sup_{t\in[0,T]}|\lambda(t)|^4\Big] < +\infty. \tag{3.34}$$

The remainder of the proof closely follows the arguments presented in Appendix B of Shen and Zou [20], which we do not replicate here. □

Noticing that the intensity process of the DCPD, represented as λ, is unbounded, confirming the optimality of u_c^* is not straightforward. Considering the linearity of the wealth SDE given by (2.7) and the fact that the objective function $\mathcal{J}_2$ is quadratic, we can naturally infer that problem (2.10) is an LQ control problem. According to LQ theory, it is nature to hypothesize that $Y\widehat{X}^2$ behaves as a submartingale for any control strategy $u \in \mathcal{A}$ and as a martingale specifically under the optimal control u_c^*. Assuming this conjecture to be true, we can immediately infer

$$\frac{1}{2}\mathbb{E}\Big[Y(T)(\widehat{X}^*(T))^2\Big] = \frac{1}{2}Y(0)(\widehat{X}^*(0))^2 \leq \frac{1}{2}\mathbb{E}\Big[Y(T)(\widehat{X}(T))^2\Big],$$

where, under the "optimal" strategy u_c^*, the wealth process is defined as $\widehat{X}^*$. Consequently, observing that (3.1), (3.19), and $Y(T) = 2$, it follows that

$$\mathcal{J}_2(x_0, \lambda_0; u^*, c) = \min_{u\in\mathcal{A}} \frac{1}{2}\mathbb{E}\Big[Y(T)(\widehat{X}(T))^2\Big] = e^{2rT+g(0,\lambda_0)}(x_0 - ce^{-rT})^2. \tag{3.35}$$

After determining the value function for problem (2.10) using (3.35), we can infer the solution ϑ^* for problem (2.11), which is a straightforward quadratic problem involving the variable ϑ.

4. Efficient strategy and efficient frontier

This section presents a comprehensive solution of the MV problem (2.8), detailing both the efficient strategy and the efficient frontier in Theorem 4.1. We employ completing the square technique from LQ control theory to confirm the optimality for the obtained candidate strategies. Additionally, Theorem 4.1 also affirms the admissibility of the strategy.

Theorem 4.1. *The efficient strategy $u^*(t) = \{(\pi^*(t), \ell^*(t))\}_{t\in[0,T]}$ of the MV problem (2.8) is determined by*

$$\pi^*(t) = -\widehat{\Gamma}(t, \lambda(t))^{-1}\widehat{\mathcal{Z}}(t, \lambda(t))\left(X^*(t-) - (\kappa - \vartheta^*)e^{-r(T-t)}\right), \tag{4.1}$$

$$\ell^*(t) = -a\Psi^{-1}\left(X^*(t-) - (\kappa - \vartheta^*)e^{-r(T-t)}\right), \tag{4.2}$$

and the efficient frontier is represented as

$$\mathbb{V}\text{ar}[X^*(T)] = \frac{e^{g(0,\lambda_0)}}{1 - e^{g(0,\lambda_0)}}\left(x_0e^{rT} - \kappa\right)^2, \quad \kappa \geq x_0e^{rT}, \tag{4.3}$$

where ϑ^* is the Lagrange multiplier, expressed as

$$\vartheta^* = \frac{e^{g(0,\lambda_0)}}{1 - e^{g(0,\lambda_0)}}\left(x_0e^{rT} - \kappa\right). \tag{4.4}$$

Here, $g(\cdot,\cdot)$ represents the solution of (3.29), $\widehat{\Gamma}(t,\lambda(t))$, $\widehat{\mathcal{Z}}(t,\lambda(t))$, and Ψ are defined by (3.23)–(3.25) respectively, and X^* (referred to as the optimal wealth process) is the solution of SDE (2.7) with the efficient strategy $u^* = (\pi^*, \ell^*)$.

Proof. We first derive u_c^* by utilizing the completing the square technique. To this end, $\widehat{X}^2$ and $Y\widehat{X}^2$ are deduced utilizing the Itô's formula and represented as

$$\mathrm{d}\widehat{X}(t)^2 = \Big(2r\widehat{X}(t)^2 + 2\widehat{X}(t)\pi(t)^\top b + 2a\widehat{X}(t)\ell(t) + \pi(t)^\top\sigma\sigma^\top\pi(t) + \int_{\mathbb{R}} \ell(t)^2\phi(z)^2\nu^\gamma(\mathrm{d}z)$$

$$+\int_{(-1,\infty)^m}\pi(t)^\top\xi(z)\mathrm{Diag}[\lambda(t)\bullet\nu^N(\mathrm{d}z)]\xi(z)^\top\pi(t)\Big)\mathrm{d}t+2\widehat{X}(t)\pi(t)^\top\sigma\mathrm{d}W(t)$$
$$+\int_{(-1,\infty)^m}2\widehat{X}(t-)\pi(t)^\top\xi(z)\widetilde{N}(\mathrm{d}t,\mathrm{d}z)+\int_{(-1,\infty)^m}\pi(t)^\top\xi(z)\mathrm{Diag}[\widetilde{N}(\mathrm{d}t,\mathrm{d}z)]\xi(z)^\top\pi(t)$$
$$+\int_{\mathbb{R}}\Big(\ell(t)^2\phi(z)^2-2\widehat{X}(t-)\ell(t)\phi(z)\Big)\widetilde{\gamma}(\mathrm{d}t,\mathrm{d}z),$$

and

$$\begin{aligned}\mathrm{d}Y(t)\widehat{X}(t)^2=&Y(t)\Big(\mathcal{U}(t,\lambda(t))^\top\widehat{\Gamma}(t,\lambda(t))\mathcal{U}(t,\lambda(t))+\Psi\mathcal{V}(t)^2\Big)\mathrm{d}t\\&+2Y(t)\widehat{X}(t)\pi(t)^\top\sigma\mathrm{d}W(t)+Y(t)\widehat{X}(t)^2\frac{\partial g}{\partial\lambda}(t,\lambda(t))^\top\mathrm{Diag}[\sqrt{\lambda(t)}]\beta\mathrm{d}B(t)\\&+\int_{(-1,\infty)^m}Y(t-)\pi(t)^\top\xi(z)\mathrm{Diag}[(U(t,\lambda(t))+\mathbf{1}_m)\bullet\widetilde{N}(\mathrm{d}t,\mathrm{d}z)]\xi(z)^\top\pi(t)\\&+\int_{(-1,\infty)^m}Y(t-)\widehat{X}(t-)\Big(2\pi(t)^\top\xi(z)\mathrm{Diag}[(U(t,\lambda(t))+\mathbf{1}_m)]+\widehat{X}(t-)U(t,\lambda(t))^\top\Big)\widetilde{N}(\mathrm{d}t,\mathrm{d}z)\\&+\int_{\mathbb{R}}Y(t-)\Big(\ell(t)^2\phi(z)^2-2\widehat{X}(t-)\ell(t)\phi(z)\Big)\widetilde{\gamma}(\mathrm{d}t,\mathrm{d}z)+Y(t-)\widehat{X}(t-)^2V(t,\lambda(t))^\top\mathrm{d}\widetilde{M}(t),\end{aligned}\quad(4.5)$$

where

$$\begin{aligned}\mathcal{U}(t,\lambda(t))=&\pi(t)+\widehat{\Gamma}(t,\lambda(t))^{-1}\widehat{\mathcal{Z}}(t,\lambda(t))\widehat{X}(t),\\\mathcal{V}(t)=&\ell(t)+a\Psi^{-1}\widehat{X}(t).\end{aligned}$$

Define $\{\tau_i\}_{i=1,2,...}$ as the stopping time, represented by

$$\tau_i:=\inf\Big\{t\geq 0;\ |\widehat{X}(t)|>i\Big\},\quad i=1,2,\ldots.$$

It is apparent that $\tau_i\uparrow\infty$ and $(T\wedge\tau_i)\uparrow T$ as $i\to\infty$. Integrating Eq (4.5) from 0 to $T\wedge\tau_i$ and taking expectations, in accordance with (3.31) and the localization technique, ultimately leads to

$$\begin{aligned}&\frac{1}{2}\mathbb{E}[Y(T\wedge\tau_i)(X(T\wedge\tau_i)-c)^2]-\frac{1}{2}Y(0)(x_0-ce^{-rT})^2\\=&\frac{1}{2}\mathbb{E}\Big[\int_0^{T\wedge\tau_i}Y(t)\Big(\mathcal{U}(t,\lambda(t))^\top\widehat{\Gamma}(t,\lambda(t))\mathcal{U}(t,\lambda(t))+\Psi\mathcal{V}(t)^2\Big)\mathrm{d}t\Big],\end{aligned}\quad(4.6)$$

where the integrability conditions outlined in Definition 2.1 have been used. Since $X(\cdot)\in\mathcal{S}^2_{\mathcal{F}}(0,T;\mathbb{R})$ and $Y(\cdot)$ is bounded, the integrand on the above equation's righthand side is nonnegative. Then, by employing the dominated convergence theorem and monotone convergence theorem to both sides of (4.6), and progressively letting i tend to ∞, it follows that

$$\mathbb{E}[(X(T)-c)^2]-\frac{1}{2}Y(0)\left(x_0-ce^{-rT}\right)^2=\frac{1}{2}\mathbb{E}\Big[\int_0^T Y(t)\Big(\mathcal{U}(t,\lambda(t))^\top\widehat{\Gamma}(t,\lambda(t))\mathcal{U}(t,\lambda(t))+\Psi\mathcal{V}(t)^2\Big)\mathrm{d}t\Big].$$

Note that since $Y(t)>0$, $\Psi>0$, and $\widehat{\Gamma}(t,\lambda(t))$ is positive definite, the optimal control can be obtained by making $\mathcal{U}(t,\lambda(t))$ and $\mathcal{V}(t)$ equal to 0, as represented by

$$\pi^*_c(t)=-\widehat{\Gamma}(t,\lambda(t))^{-1}\widehat{\mathcal{Z}}(t,\lambda(t))\left(X^*(t-)-ce^{-r(T-t)}\right),\quad(4.7)$$

$$\ell_c^*(t) = -a\Psi^{-1}\left(X^*(t-) - ce^{-r(T-t)}\right), \tag{4.8}$$

which aligns with the expressions provided in (3.26) and (3.27).

Note that in Section 3, the π_c^* and ℓ_c^* derived from the SMP are represented by (3.26) and (3.27), which constitute only a necessary condition for optimality. π_c^* and ℓ_c^*, as outlined in (4.7) and (4.8), are confirmed as the optimal solutions for problem (2.10) through applying the completing the square technique, thereby constituting a sufficient condition for optimality.

We next address problem (2.11) to determine ϑ^*, thereby identifying the efficient strategy and the efficient frontier. It should be noted that problem (2.11) aims to maximize $\mathcal{J}_2(x_0, \lambda_0; u_c^*, \kappa - \vartheta) - \vartheta^2$ across all $\vartheta \in \mathbb{R}$, where $c = \kappa - \vartheta$ and $\mathcal{J}_2$ denotes the value function for problem (2.10) defined by

$$\mathcal{J}_2(x_0, \lambda_0; u_c^*, \kappa - \vartheta) = \mathbb{E}[(X^*(T) - c)^2] = \frac{1}{2}Y(0)\left(x_0 - ce^{-rT}\right)^2.$$

It is clear that we can determine ϑ^* through

$$\vartheta^* = \arg\max_{\vartheta\in\mathbb{R}} \left\{e^{2rT+g(0,\lambda_0)}\left(x_0 - (\kappa - \vartheta)e^{-rT}\right)^2 - \vartheta^2\right\} = \frac{e^{g(0,\lambda_0)}}{1 - e^{g(0,\lambda_0)}}\left(x_0 e^{rT} - \kappa\right),$$

where the sufficiency of the first-order condition is confirmed by Lemma 3.1, thus establishing that ϑ^* represents the optimal solution for problem (2.11).

Upon determining ϑ^* as specified in (4.4), it is easy to deduce the efficient strategy u^* from the condition $u^* = u^*_{\kappa-\vartheta^*}$, utilizing the formulations for u_c^* presented in (4.7) and (4.8), which coincides with (4.1) and (4.2). Ultimately, applying the Lagrangian duality theorem enables us to infer the efficient frontier as detailed within (4.3) by

$$\mathbb{V}\text{ar}[X^*(T)] = \mathbb{E}\big[(X^*(T) - (\kappa - \vartheta^*))^2\big] - (\vartheta^*)^2. \tag{4.9}$$

To complete the proof, we need to verify that u^* given by (4.1) and (4.2) is admissible. Using (4.9) and the condition $e^{g(0,\lambda_0)} < 1$ derived from Lemma 3.1, it can be deduced that

$$\mathbb{E}[(X^*(T) - (\kappa - \vartheta^*))^2] = \mathbb{V}\text{ar}[X^*(T)] + (\vartheta^*)^2 = \frac{e^{g(0,\lambda_0)}}{(1 - e^{g(0,\lambda_0)})^2}(x_0 e^{rT} - \kappa)^2 < \infty,$$

which means that $\widehat{X}^*(T) = X^*(T) - (\kappa - \vartheta^*)$ is square integrable. Based on the dynamic equation for $\widehat{X}^*$ in (3.2), it can be demonstrated that $(\widehat{X}^*, \pi^*, \ell^*)$ satisfies the following BSDE:

$$\begin{cases} \mathrm{d}y(t) = (ry(t) + \pi^*(t)^\top b + \ell^*(t)a)\mathrm{d}t + \pi^*(t)^\top\sigma\mathrm{d}W(t) \\ \qquad\qquad + \int_{(-1,\infty)^m} \pi^*(t)^\top\xi(z)\widetilde{N}(\mathrm{d}t, \mathrm{d}z) - \int_{\mathbb{R}} \ell^*(t)\phi(z)\widetilde{\gamma}(\mathrm{d}t, \mathrm{d}z), \\ y(T) = \widehat{X}^*(T), \quad t \in [0, T]. \end{cases} \tag{4.10}$$

Introduce the following new notations:

$$\varsigma(t) := \sigma^\top\pi^*(t), \quad \varrho(t, z) := \xi(z)^\top\pi^*(t), \quad \varkappa(t, z) := -\phi(z)\ell^*(t), \tag{4.11}$$

from which we obtain that

$$\pi^*(t) = (\sigma\sigma^\top)^{-1}\sigma\varsigma(t) \quad \text{and} \quad \ell^*(t) = -\Psi^{-1}\int_{\mathbb{R}} \varkappa(t, z)\nu^\gamma(\mathrm{d}z). \tag{4.12}$$

Substituting (4.11) and (4.12) into (4.10), it follows that

$$(y(t), \varsigma(t), \varrho(t,z), \varkappa(t,z)) = (\widehat{X}^*(t), \sigma^\top \pi^*(t), \xi(z)^\top \pi^*(t), -\phi(z)\ell^*(t))$$

is the solution for the BSDE below:

$$\begin{cases} \mathrm{d}y(t) = \Big(ry(t) + b^\top(\sigma\sigma^\top)^{-1}\sigma\varsigma(t) - a\Psi^{-1}\int_{\mathbb{R}} \varkappa(t,z)\nu^\gamma(\mathrm{d}z)\Big)\mathrm{d}t + \varsigma(t)^\top \mathrm{d}W(t) \\ \qquad\qquad + \int_{(-1,\infty)^m} \varrho(t,z)^\top \widetilde{N}(\mathrm{d}t,\mathrm{d}z) + \int_{\mathbb{R}} \varkappa(t,z)\widetilde{\gamma}(\mathrm{d}t,\mathrm{d}z), \\ y(T) = \widehat{X}^*(T), \quad t \in [0,T]. \end{cases} \tag{4.13}$$

In particular, (4.13) represents a linear BSDE characterized by a square integrable terminal condition and a Lipschitz driver as defined by the Definition 2.1 in Quenez and Sulem [17]. Consequently, according to Theorem 2.3 of Quenez and Sulem [17], (4.13) admits a unique solution satisfying

$$\mathbb{E}\Big[\int_0^T |\varsigma(t)|^2 \mathrm{d}t + \sum_{l=1}^m \int_0^T \int_{(-1,\infty)} |\varrho_l(t,z_l)|^2 \lambda_l(t)\nu_l^N(\mathrm{d}z_l)\mathrm{d}t + \int_0^T \int_{\mathbb{R}} |\varkappa(t,z)|^2 \nu^\gamma(\mathrm{d}z)\mathrm{d}t\Big] < +\infty. \tag{4.14}$$

By applying (4.11) and (4.12) once again, we obtain that

$$\pi^* \in \mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^k), \quad \xi^\top \pi^* \in \mathcal{L}^{2,N}_{\mathcal{F}}(0,T;\mathbb{R}^m), \quad \text{and} \quad \ell^* \in \mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}).$$

The proof is completed. □

5. Numerical analysis

This section is dedicated to conducting numerical analysis for demonstrating the theoretical results outlined in Theorem 4.1. Our numerical examination focuses on a univariate case where $n = d = m = e = k = 1$, indicating the presence of a single risky asset, with jumps represented through a one-dimensional DCPD. In this example, model parameters are assigned following Shen and Zou [19] and Dassios and Zhao [8], as delineated in Table 1.

Table 1. Setup of model parameters.

r	μ	σ	$\mathbb{E}[Z]$	$\mathbb{E}[Z^2]$	α	β	ζ	δ
0.02	0.09	0.2	-0.02	0.06	1	1	1.2	10

5.1. Sensitivity analysis

This subsection explores the sensitivity for the efficient frontier within the DCPD, as introduced in (4.3) of Theorem 4.1. Our objective is to understand how variations in the parameters of the DCPD's intensity process influence the efficient frontier. During the subsequent analysis, each parameter is individually adjusted to various levels while keeping all other parameters fixed, as detailed in Table 1.

The initial phase of our analysis delves into the effects of the initial value, λ_0, and the mean-reversion level, λ_∞, on the efficient frontier, as illustrated in Figure 1. The left panel of Figure 1 evaluates three distinct values of λ_0: 0.5, 0.9, and 1.5. Graphical examination indicates a deterioration in the efficient frontier as λ_0 increases. This is because an increase in λ_0 will cause the intensity process to increase, thereby worsening the efficient frontier. The right panel focuses on three levels of λ_∞: 0.5, 0.9, and 2. The graph shows that an increase in λ_∞ is disadvantageous for the MV insurer. To comprehend this result, observe that a higher λ_∞ leads to a higher average intensity λ as λ mean-reverts to λ_∞.

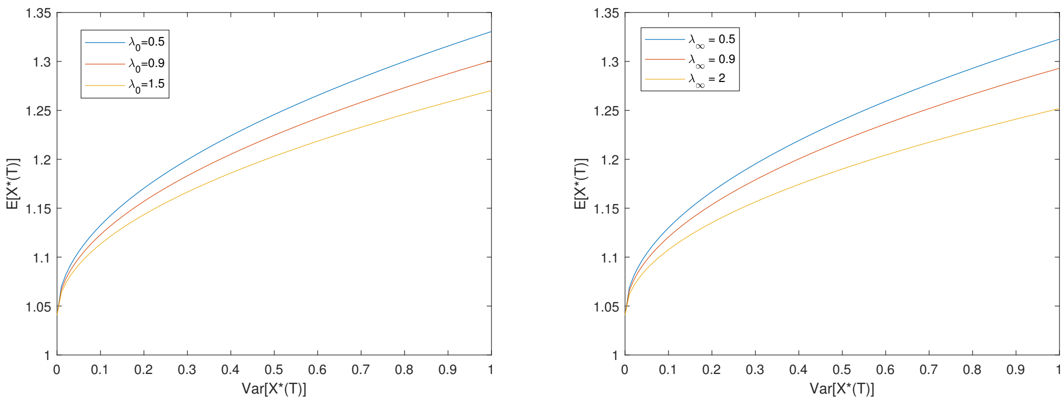

Figure 1. The impact of λ_0 and λ_∞.

Our subsequent analysis examines the influence of α and β, where α indicates the mean-reversion speed of the intensity process, and β represents the volatility coefficient of the diffusion part of the intensity process. On the left part of Figure 2, three levels of α are explored: 1, 2, and 4. An increase in α accelerates the mean-reversion for the intensity process to λ_∞, thereby improving the efficient frontier. This improvement occurs because a larger α enables the intensity process λ to recover quickly from an abnormal state to λ_∞, thereby reducing the uncertainty associated with the risky assets. On the right part of Figure 2, three levels of β are explored: 0.5, 1, and 1.5. It is readily observed that β has positive effects, meaning the efficient frontier decreases as β increases. This implies that an increase in the volatility coefficient within the intensity diffusion could improve the efficient frontier. This result is consistent with the findings in Dassios and Zhao [9].

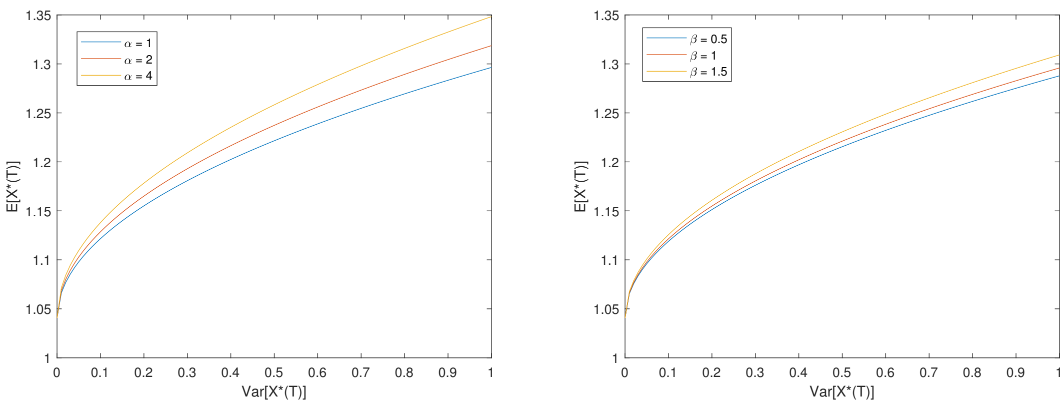

Figure 2. The impact of α and β.

Our final analysis delves into the impact of ζ and δ, where ζ denotes the internal-excitation factor and δ signifies the external-excitation factor, as defined in (2.2). The results are illustrated in Figure 3.

On its left part, three levels of ζ are assessed: 0.5, 1.2, and 2. The figure clearly shows a deterioration of the efficient frontier with an increase in ζ. Similarly, on the right part, where δ is examined at the levels of 2, 4, and 10, an increase in δ also leads to a worsening of the efficient frontier. This is because a larger ζ or δ results in a greater increase in λ following a jump in N or M, which in turn triggers more frequent jumps in the near future, thereby worsening the efficient frontier.

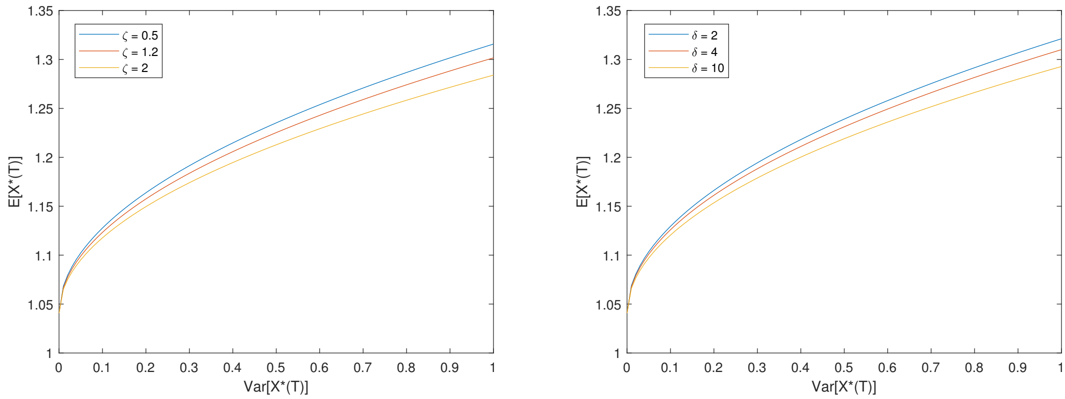

Figure 3. The impact of ζ and δ.

5.2. Comparisons with the Hawkes and DCP models

In the final subsection of the numerical analysis, a comparative analysis is conducted encompassing the Hawkes, DCP, and DCPD models. In the DCPD model, setting $\beta = 0$ and $\delta = 0$ transitions the model to the Hawkes process, marked by the absence of diffusion and external-excitation jumps. Additionally, setting $\beta = 0$ converts the model into the DCP, characterized by the exclusion of diffusion.

By observing Figure 4, it is found that under the parameters set in this paper, the efficient frontier of the Hawkes model is superior to that of the DCP model. This is because the DCP model includes external-excitation characteristics, which increase the intensity process when external events occur, thereby triggering more jumps in the foreseeable future and worsening the efficient frontier. Compared to the DCP model, the DCPD model includes an additional diffusion term. As concluded in the previous section, the additional diffusion term has a positive effect on the efficient frontier, thereby improving it.

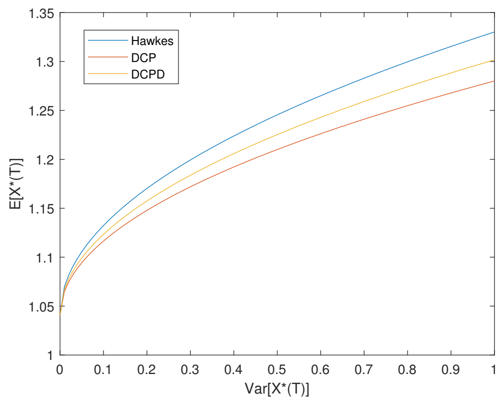

Figure 4. Comparisons among the Hawkes, DCP, and DCPD models.

6. Conclusions

In this paper, we explore investment and risk control strategies within a contagious financial market from an MV perspective for insurers. By adopting a jump-diffusion model that incorporates a multivariate DCPD, we manage to distinguish between externally and internally excited jumps, allowing for a nuanced risk assessment. Our paper uses advanced mathematical techniques, including the SMP, BSDEs, and LQ control, to deduce the efficient strategy and the efficient frontier within a semi-closed form solution. Through extensive numerical simulations, we validate our model and demonstrate its practical applicability in enhancing risk control and investment decision-making for insurers in the face of market contagions. For further research, one may consider the situation where financial market models and insurance risk processes are correlated. Additionally, another promising direction is to extend the present study into a time-inconsistent MV investment and risk control (reinsurance) problem.

Author contributions

Xiuxian Chen: Led the project design, prepared the initial draft, created the figures, and contributed to manuscript revision; Dan Zhu: Participated in project design and contributed to manuscript revision; Zhongyang Sun: Conceptualized and supervised the project, secured funding, and contributed to manuscript revision. All authors have reviewed and approved the final version of the manuscript for publication.

Acknowledgments

This research was supported by the National Natural Science Foundation of China (Grant Nos. 12371472, 12301605, and 11901344), the Shandong Provincial Natural Science Foundation (Grant No. ZR2023MA033), the Youth Innovation Team of Shandong Universities (Grant No. 2022KJ174), and the Funded Project from the Shandong Provincial Higher Education Curriculum Ideological and Political Education Research Center (Grant No. SZ2023014).

Conflict of interest

The authors declare that they have no competing interests.

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