Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(11): 33023–33061.
DOI: 10.3934/math.20241579
Received: 19 September 2024
Revised: 26 October 2024
Accepted: 06 November 2024
Published: 21 November 2024

Research article

The inverse uncertainty distribution of the solutions to a class of higher-order uncertain differential equations

Zeman Wang[1], **Zhong Liu**[1], **Zikun Han**[2], **Xiuying Guo**[1] **and Qiubao Wang**[1,2,*]

[1] Department of Mathematics and Physics, Shijiazhuang Tiedao University, Shijiazhuang 050043, China

[2] Department of Engineering Mechanics, Shijiazhuang Tiedao University, Shijiazhuang 050043, China

* **Correspondence:** Email: wangqiubao12@sina.com.

Abstract: In this paper, we study the higher-order uncertain differential equations (UDEs) as defined by Kaixi Zhang [11], mainly focus on the second-order case. We propose a pivotal condition (monotonicity in some sense, see more details in Section 3), introduce the concept of α-paths of UDEs, and demonstrate its properties. Based on this, we derive the inverse uncertainty distribution of the solution. Finally, we present numerical examples to substantiate the rationality of the condition.

Keywords: higher-order uncertain differential equations; inverse uncertainty distribution; α-paths
Mathematics Subject Classification: 47E05, 34A12

1. Introduction

Events with known frequencies of occurrence are classified as random, while those with unknown frequencies are termed uncertain [1]. With the rapid advancement of science and technology, a multitude of uncertain factors have emerged in real life, rendering the phenomena of uncertainty in the objective world undeniable. Consequently, scholars have begun to incorporate these uncertain factors into the establishment of mathematical models, leading to the research and development of uncertainty theory.

Integrating uncertain factors into differential equations results in the formation of UDEs, a type of differential equation established by Liu in 2007 [1], designed to describe the dynamics of uncertain phenomena. Yao and Chen provided an effective formula for calculating the inverse uncertainty distribution of the solutions to UDEs, known as the Yao-Chen formula [2].

The Yao-Chen formula yields a family of solutions to ordinary differential equations, denoted as α-paths, and it has been indicated [2] that these α-paths represent the inverse uncertainty distribution of the solutions. Therefore, to determine the inverse uncertainty distribution of the solution, we need to solve a family of ordinary differential equations to obtain the α-paths. For first-order scalar UDE, the inverse uncertainty distribution of the solutions can be determined using the Yao-Chen formula. Building on this foundation, research has been conducted in many fields, including finance [3], optimal control [4], population growth [5], pharmacokinetics [6, 7], epidemiology [8], and heat conduction [9, 10]. However, in many practical contexts, first-order scalar UDEs may not fully capture the complexity of real-world scenarios. Often, more intricate cases emerge, such as higher-order, fractional-order, functional differential equations, etc. Thus, investigating theories concerning those UDEs is of significant importance. This paper primarily focuses on the inverse uncertainty distribution problem of a class of higher-order UDEs.

Zhang [11] has rigorously defined the concept of multiple integrals and higher-order derivatives of uncertain processes, establishing the framework for higher-order UDEs. These contributions have addressed previous theoretical deficiencies in higher-order UDEs and provided new analytical methods for solving more complex uncertain system problems. However, the definition of the α-path, which is crucial for determining the inverse uncertainty distribution of UDEs' solutions, was not provided. Therefore, in this article, we define the α-path for second-order UDEs and a class of higher-order UDEs, and determine the corresponding inverse uncertainty distributions for the solutions of these types of UDEs. This will enable a more comprehensive understanding and analysis of higher-order uncertain processes, offering new perspectives and tools for the theoretical development and practical application of uncertain calculus.

To determine the inverse uncertainty distribution of the solutions to UDEs, it is essential to identify the α-paths. The formation of α-paths must satisfy fundamental conditions, primarily concerning the monotonicity with respect to α. That is, at any given time t, the value of X_t^α on an α-path should monotonically increase with respect to α, as illustrated in Figure 1(a). It is impermissible to encounter a scenario as depicted in Figure 1(c) and Figure 2(b). The main focus of this paper is to study under what conditions the α-paths of UDEs behave like the one in Figure 1(a), rather than exhibiting the situations shown in (b) and (c).

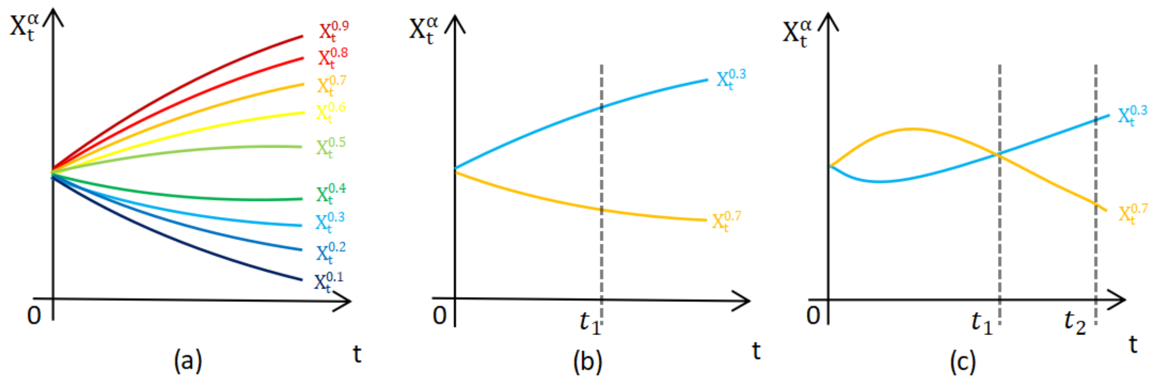

Figure 1. Schematic diagram of α-paths.

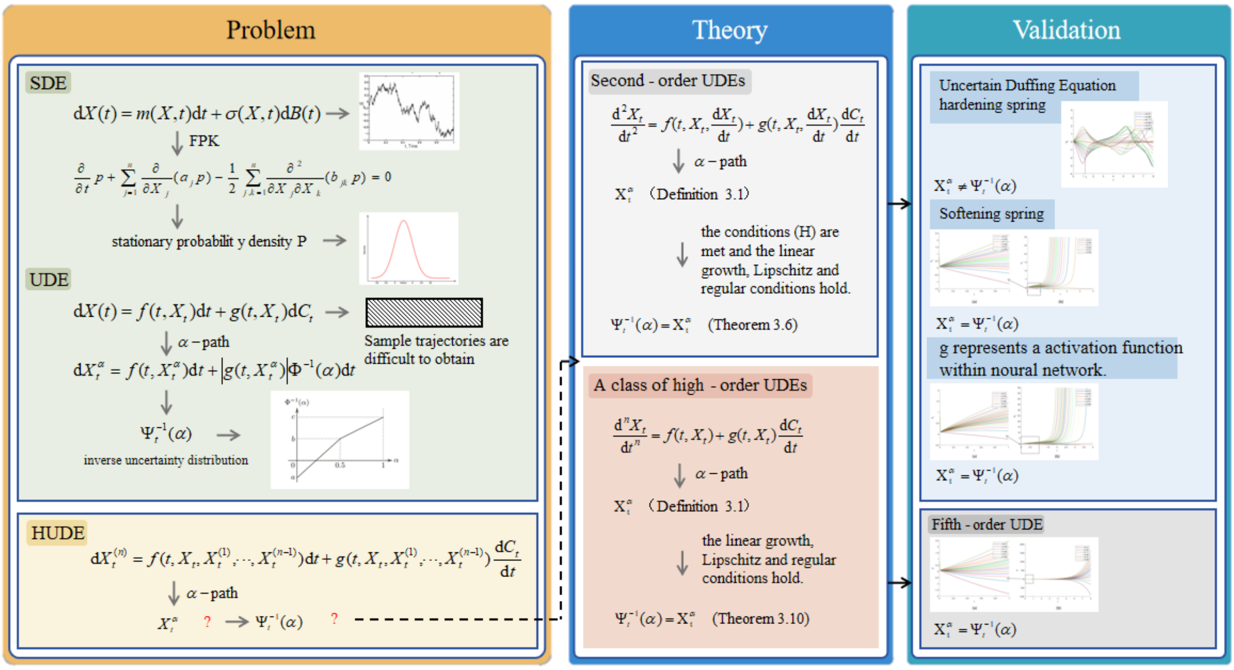

Figure 2. Analysis diagram of the inverse uncertainty distribution in UDEs.

The rest of this paper is organized as follows: In Section 2, we review related concepts and conclusions of uncertainty theory. In Section 3, we define the α-paths for some second-order UDEs and a class of higher-order UDEs, deriving the inverse uncertainty distribution corresponding to the solutions. In Section 4, we provide positive and negative instances of the theorem in Section 3. In Section 5, we give a brief summary of this paper. Figure 2 illustrates the conceptual framework of this manuscript, encapsulating the core ideas and interconnections presented in the paper.

2. Preliminary

In this section, we introduce some basic concepts and theorems about uncertain processes and uncertain calculus.

Definition 2.1. *(Liu [12]) Let Γ be a non-empty set, let Λ be a σ-algebra over Γ, and let $\mathcal{M}$ be an uncertain measure. Then the triplet $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ is called an uncertainty space.*

Theorem 2.1. *[13] (Measure Inversion Theorem) Let ξ be an uncertain variable with uncertainty distribution Φ. Then for any real number x, we have*

$$\mathcal{M}\{\xi \leq x\} = \Phi(x), \mathcal{M}\{\xi > x\} = 1 - \Phi(x).$$

Definition 2.2. *(Liu [14]) An uncertain process C_t is said to be a Liu process if*

(i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous,

(ii) C_t has stationary and independent increments,

(iii) every increment C_{s+t}-C_s is a normal uncertain variable with an expected value of 0 and variance t^2.

Definition 2.3. *(Yao-Chen [2]) Let α be a number between 0 and 1. An uncertain differential equation*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t$$

is said to have an α-path X_t^α if it solves the corresponding ordinary differential equation

$$\mathrm{d}X_t^\alpha = f(t, X_t^\alpha)\mathrm{d}t + |g(t, X_t^\alpha)|\Phi^{-1}(\alpha)\mathrm{d}t$$

where $\Phi^{-1}(\alpha)$ is the inverse standard normal uncertainty distribution, i.e.,

$$\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}.$$

Definition 2.4. *(Liu [12]) Suppose f and g are continuous functions. An uncertain differential equation*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t$$

is said to satisfy the regular condition if

$$g(t, X_t) > 0, \forall t \geq 0.$$

Theorem 2.2. *(Liu [12]) The uncertain differential equation*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t$$

has a unique solution if the functions $f(t, x)$ and $g(t, x)$ satisfy the linear growth condition

$$|f(t, x)| + |g(t, x)| \leq L(1 + |x|), \forall x \in \Re, t \geq 0$$

and the Lipschitz condition

$$|f(t, x) - f(t, y)| + |g(t, x) - g(t, y)| \leq L(|x - y|), \forall x, y \in \Re, t \geq 0.$$

Without loss of generality, suppose L. Moreover, the solution is sample continuous.

Theorem 2.3. *(Liu [12]) Let X_t^α be the α-path of the regular uncertain differential equation*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t.$$

If the linear growth, Lipschitz, and regular conditions hold, then X_t^α is a strictly increasing function with respect to α at each time $t > 0$.

Theorem 2.4. *(Liu [12]) Let X_t^α be the α-path of the regular uncertain differential equation*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t.$$

Then X_t^α is a continuous function with respect to α at each time $t > 0$.

Theorem 2.5. *(Yao-Chen formula [2]) Let X_t and X_t^α be the solution and α-path of the regular uncertain differential equation*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t,$$

respectively. Then

$$\mathcal{M}\{X_t \le X_t^\alpha, \forall t\} = \alpha,$$

$$\mathcal{M}\{X_t > X_t^\alpha, \forall t\} = 1 - \alpha.$$

Definition 2.5. *(Liu [14]; Chen and Ralescu [15]; Ye [16]) Let C_t be a Liu process, and let Z_t be an uncertain process. If there exist two sample-continuous uncertain processes μ_t and δ_t such that*

$$Z_t = Z_0 + \int_0^t \mu_s \mathrm{d}s + \int_0^t \delta_s \mathrm{d}C_s,$$

for any $t \ge 0$, then Z_t is called a general Liu process with drift μ_t and diffusion δ_t. Furthermore, Z_t has an uncertain differential

$$\mathrm{d}Z_t = \mu_t \mathrm{d}t + \delta_t \mathrm{d}C_t,$$

and a first-order derivative

$$\dot{Z}_t = \mu_t + \delta_t \dot{C}_t,$$

where $\dot{C}_t$ is the formal derivative $\mathrm{d}C_t/\mathrm{d}t$.

Definition 2.6. *(Zhang [11]) Let C_t be a Liu process, and let Z_t be an uncertain process. If there exist sample-continuous uncertain processes μ_t and δ_t such that*

$$Z_t = Z_0 + \dot{Z}_0 t + \int_0^t \int_0^s \mu_r \mathrm{d}r\mathrm{d}s + \int_0^t \int_0^s \delta_r \mathrm{d}C_r \mathrm{d}s$$

for any $t \ge 0$, then Z_t is called a second-order Liu process and has a second-order derivative

$$\ddot{Z}_t = \mu_t + \delta_t \dot{C}_t,$$

where $\ddot{Z}_t$ is the formal second-order derivative $\mathrm{d}^2 Z_t/\mathrm{d}t^2$.

Theorem 2.6. *(Zhang [11]) Let $f(t, s)$ be an uncertain field, and let X_t and Y_s be general Liu processes. For any partition of the closed region $[0, a] \times [0, b]$ with*

$$0 = t_1 < t_2 < \cdots < t_{n+1} = a, 0 = s_1 < s_2 < \cdots < s_{m+1} = b,$$

the mesh is written as

$$\Delta = \max_{\substack{1\le i\le n \\ 1\le j\le m}} \sqrt{(t_{i+1} - t_i)^2 + (s_{i+1} - s_i)^2}.$$

Then the double Liu integral of $f(t, s)$ with respect to X_t and Y_s is defined as

$$\iint_{[0,a]\times[0,b]} f(t, s)\mathrm{d}X_t \mathrm{d}Y_s = \lim_{\Delta\to 0} \sum_{i=1}^{n} \sum_{j=1}^{m} f(t_i, s_i)(X_{t_{i+1}} - X_{t_i})(Y_{s_{j+1}} - Y_{s_j}),$$

provided that there is an uncertain variable to which the above sum converges almost surely as $\Delta \to 0$. In this case, the uncertain field $f(t, s)$ is said to be integrable with respect to X_t and Y_s.

Theorem 2.7. *Zhang [11] (Fubini's Theorem) Let X_t and Y_s be general Liu processes. Suppose $f(t,s)$ is an integrable uncertain field with respect to X_t and Y_s. Then*

(1) $\int_0^a f(t,s)\mathrm{d}X_t$ and $\int_0^b f(t,s)\mathrm{d}Y_s$ exist almost surely;
(2) $\int_0^a f(t,s)\mathrm{d}X_t$ and $\int_0^b f(t,s)\mathrm{d}Y_s$ are integrable with respect to Y_s and X_t, respectively;
(3)

$$\iint_{[0,a]\times[0,b]} f(t,s)\mathrm{d}Y_s\mathrm{d}X_t = \int_0^b \int_0^a f(t,s)\mathrm{d}X_t\mathrm{d}Y_s, a.s.$$

Definition 2.7. *(Liu [12]) Suppose f and g are continuous functions. An uncertain differential equation*

$$\mathrm{d}X_t = f(t,X_t)\mathrm{d}t + g(t,X_t)\mathrm{d}C_t$$

is said to be regular if

$$g(t,X_t) > 0, \forall t > 0.$$

Definition 2.8. *(Zhang [11]) An uncertain process X_t is called a solution of the higher-order uncertain integral equation if*

$$\begin{aligned} X_t = &\sum_{i=0}^{n-1} \frac{t^i}{i!} X_0^i + \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} f(t, X_t, X_t^{(1)}, X_t^{(2)}, \ldots, X_t^{(n-1)})\mathrm{d}t_n\mathrm{d}t_{n-1}\ldots\mathrm{d}t_2\mathrm{d}t_1 \\ &+ \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} g(t, X_t, X_t^{(1)}, X_t^{(2)}, \ldots, X_t^{(n-1)})\mathrm{d}C_{t_n}\mathrm{d}t_{n-1}\ldots\mathrm{d}t_2\mathrm{d}t_1, \end{aligned}$$

where C_t is a Liu process, and f and g are continuous functions. Equivalently, the above equation can be simply written as the differential form

$$X_t^{(n)} = f(t, X_t, X_t^{(1)}, X_t^{(2)}, \ldots, X_t^{(n-1)}) + g(t, X_t, X_t^{(1)}, X_t^{(2)}, \ldots, X_t^{(n-1)})\dot{C}_t,$$

that is called a higher-order uncertain diferential equation.

Theorem 2.8. *(Liu [17]) A function $\Phi^{-1} : (0,1) \to \Re$ is the inverse uncertainty distribution of an uncertain variable ξ if and only if it is continuous and*

$$\mathcal{M}\{\xi \le \Phi^{-1}(\alpha)\} = \alpha.$$

3. The inverse uncertainty distribution to some higher-order UDEs

Initially, for the significant research conclusions obtained in this paper, such as Theorem 3.6, a schematic diagram is employed for a more intuitive representation of the research process, as shown in Figure 3.

Step 1 Defined the α-path(Definition 3.1) and Defined the regularity(Definition 3.2);
Step 2 Prove the solution is sample-continuous(Theorem 3.1) if f and g satisfy conditions of linear growth, Lipschitz and regularity;
Step 3 Prove X_t^α is a continuous function with respect to α(Theorem 3.2);
Step 4 Prove X_t^α is a continuous and strictly increasing function with respect to α(Theorem 3.3);
Step 5 Prove

$$\mathcal{M}\{X_t \le X_t^\alpha, \forall t\} = \alpha,$$

$$\mathcal{M}\{X_t > X_t^\alpha, \forall t\} = 1 - \alpha.$$

If an additional condition (H) is imposed(Theorem 3.4);
Step 6 Prove at any time $t > 0$

$$\mathcal{M}\{X_t \le X_t^\alpha\} = \alpha,$$

$$\mathcal{M}\{X_t > X_t^\alpha\} = 1 - \alpha;$$

Step 7 $\Psi_t^{-1}(\alpha) = X_t^\alpha$;(Theorem 3.6)

Figure 3. The proof process of the $\Psi_t^{-1}(\alpha)$ for second-order UDEs.

Next, we introduce Lemma 3.1, which serves as a foundation for the proof of Theorems 3.3, 3.4, 3.7, and 3.8.

Lemma 3.1. ***Lemma 3.1.*** *Let X_t satisfies the following higher-order UDE:*

$$\begin{cases} X_t^{(n)} &= f(t, X_t, X_t^{(1)}, X_t^{(2)}, \dots, X_t^{(n-1)}) + g(t, X_t, X_t^{(1)}, X_t^{(2)}, \dots, X_t^{(n-1)}\dot{C}_t, \\ X_0(\gamma) &= X_0, X_0^{(k)}(\gamma) = X_0^k, k = 1, 2, \dots, n-1. \end{cases}$$

Then, X_t also satisfies the following integral equation:

$$\begin{aligned} X_t = &\sum_{k=0}^{n-1} \frac{t^k}{k!} X_0^k + \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} f(s, X_s^{(1)}, X_s^{(2)}, \dots, X_s^{(n-1)}) \mathrm{d}s \\ &+ \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} g(s, X_s, X_s^{(1)}, X_s^{(2)}, \dots, X_s^{(n-1)}) \mathrm{d}C_s. \end{aligned}$$

For simplicity and clarity, we denote $X_t(\gamma)$ as X_t. Henceforth, we shall use X_t to represent the solution to the UDE and X_t^α to denote its corresponding α-path.

Proof. Introduce two variables, Z_t and W_t, which respectively satisfy the following two equations:

$$\begin{cases} Z_t^{(n)} &= 0, \\ Z_t\Big|_{t=0} &= Z_0, Z_t^{(k)}\Big|_{t=0} = Z_0^k, k = 1, 2, \dots, n-1. \end{cases}$$

$$\begin{cases} W_t^{(n)} &= f(t, W_t, W_t^{(1)}, W_t^{(2)}, \dots, W_t^{(n-1)}) + g(t, W_t, W_t^{(1)}, W_t^{(2)}, \dots, W_t^{(n-1)})\dot{C}_t, \\ W_t^{(k)}\Big|_{t=0} &= 0. \end{cases}$$

It can be easily derived that $X_t = Z_t + W_t$. For Z_t, we integrate both sides directly:

$$
\begin{aligned}
Z_t &= \int_0^t Z_s^{(1)} \mathrm{d}s \\
&= -\int_0^t Z_s^{(1)} \mathrm{d}(t-s) \\
&= -Z_s^{(1)}(t-s)|_0^t - \frac{1}{2}\int_0^t Z_s^{(2)} \mathrm{d}(t-s)^2 \\
&= Z_0^1 t + \frac{1}{2} Z_0^2 t^2 - \frac{1}{3\times 2}\int_0^t Z_s^{(3)}(s) \mathrm{d}(t-s)^3 \\
&\cdots \\
&= Z_0^1 t + \frac{1}{2} Z_0^2 t^2 + \frac{1}{3\times 2} Z_0^3 t^3 + \cdots + \frac{1}{(n-1)!} Z_0^{n-1} t^{(n-1)} \\
&= \sum_{k=0}^{n-1} \frac{t^k}{k!} Z_0^k.
\end{aligned}
\tag{3.1}
$$

For W_t, we use intregration by parts repeatedly as follows:

$$
\begin{aligned}
W_t &= \int_0^t W_s^{(1)} \mathrm{d}s \\
&= -\int_0^t W_s^{(1)} \mathrm{d}(t-s) \\
&= -W_s^{(1)}(t-s)|_0^t + \int_0^t (t-s) W_s^{(2)} \mathrm{d}s \\
&= -\frac{1}{2} W_0^2 t^2 + \frac{1}{2}\int_0^t (t-s)^2 W_s^{(3)} \mathrm{d}s \\
&\cdots \\
&= \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1} f(s, W_s, W_s^{(1)}, W_s^{(2)}, \ldots, W_s^{(n-1)}) \mathrm{d}s \\
&+ \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1} g(s, W_s, W_s^{(1)}, W_s^{(2)}, \ldots, W_s^{(n-1)}) \mathrm{d}C_s.
\end{aligned}
\tag{3.2}
$$

Thus, adding Eqs (3.1) and (3.2), we derive the integral equation satisfied by X_t:

$$
\begin{aligned}
X_t &= Z_t + W_t \\
&= \sum_{k=0}^{n-1} \frac{t^k}{k!} X_0^k + \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1} X_s^{(n)}(s) \mathrm{d}s \\
&= \sum_{k=0}^{n-1} \frac{t^k}{k!} X_0^k + \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1} f(s, X_s, X_s^{(1)}, X_s^{(2)}, \ldots, X_s^{(n-1)}) \mathrm{d}s \\
&+ \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1} g(s, X_s, X_s^{(1)}, X_s^{(2)}, \ldots, X_s^{(n-1)}) \mathrm{d}C_s.
\end{aligned}
$$

□

Definition 3.1. *Let α be a number between 0 and 1. An uncertain differential equation*

$$\begin{cases} \frac{\mathrm{d}^n X_t}{\mathrm{d}t^n} = f(t, X_t, \ldots, \frac{\mathrm{d}^{n-1}X_t}{\mathrm{d}t^{n-1}}) + g(t, X_t, \ldots, \frac{\mathrm{d}^{n-1}X_t}{\mathrm{d}t^{n-1}})\frac{\mathrm{d}C_t}{\mathrm{d}t}, \\ X_t\Big|_{t=0} = X_0, \frac{\mathrm{d}X_t}{\mathrm{d}t}\Big|_{t=0} = X_1, \ldots, \frac{\mathrm{d}^n X_t}{\mathrm{d}t^n}\Big|_{t=0} = X_n \end{cases}$$

is said to have an α-path X_t^α if it solves the corresponding ordinary differential equation

$$\begin{cases} \frac{\mathrm{d}^n X_t^\alpha}{\mathrm{d}t^n} = f(t, X_t^\alpha, \ldots, \frac{\mathrm{d}^{n-1}X_t^\alpha}{\mathrm{d}t^{n-1}}) + |g(t, X_t^\alpha, \ldots, \frac{\mathrm{d}^{n-1}X_t^\alpha}{\mathrm{d}t^{n-1}})|\Phi^{-1}(\alpha), \\ X_t^\alpha\Big|_{t=0} = X_0, \frac{\mathrm{d}X_t^\alpha}{\mathrm{d}t}\Big|_{t=0} = X_1, \ldots, \frac{\mathrm{d}^n X_t^\alpha}{\mathrm{d}t^n}\Big|_{t=0} = X_n, \end{cases}$$

where $\Phi^{-1}(\alpha)$ is the inverse standard normal uncertainty distribution, i.e.,

$$\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}.$$

Example 1. *In the uncertain spring vibration equation [18] denoted as*

$$\begin{cases} \frac{\mathrm{d}^2 X_t}{\mathrm{d}t^2} + 20\frac{\mathrm{d}X_t}{\mathrm{d}t} + 64X_t = \frac{\mathrm{d}C_t}{\mathrm{d}t}, \\ X_t\Big|_{t=0} = 0, \frac{\mathrm{d}X_t}{\mathrm{d}t}\Big|_{t=0} = 0. \end{cases}$$

The solution is given by

$$X_t = \frac{1}{12}\int_0^t (\exp(-4t+4s) - \exp(-16t+16s))\mathrm{d}C_s,$$

it has an α-path

$$X_t^\alpha = (\frac{3}{16} - \frac{1}{4}\exp(-4\alpha) + \frac{1}{16}\exp(-16\alpha))\frac{\sqrt{3}}{12\pi}\ln\frac{t}{1-t},$$

and the inverse uncertainty distribution is expressed as

$$\Psi_t^{-1}(\alpha) = (\frac{3}{16} - \frac{1}{4}\exp(-4t) + \frac{1}{16}\exp(-16t))\frac{\sqrt{3}}{12\pi}\ln\frac{\alpha}{1-\alpha}.$$

Note 1.

(1) There are generally two approaches to obtaining the inverse uncertain distribution $\Psi_t^{-1}(\alpha)$ of the solution to a UDE. One approach involves directly solving the equation to obtain $\Psi_t^{-1}(\alpha)$ [18], and the other involves solving its α-path, which entails solving a family of deterministic ordinary differential equations to determine $\Psi_t^{-1}(\alpha)$.

(2) The first method is straightforward, allowing for the direct computation of the solution X_t, and consequently, the inverse uncertainty distribution. Example 1 is linear and thus allows for the computation of X_t. However, for the majority of equations, analytical solutions cannot be derived, and currently, there are no numerical methods available for directly solving uncertain systems. Therefore, this method has significant limitations. The second method does not suffer from such limitations, as it involves solving a family of ordinary differential equations for the α-path, which has a variety of solution methods. Consequently, the second method is more meaningful in practical applications and represents the direction of research in this paper.

(3) Regarding X_t^α and $\Psi_t^{-1}(\alpha)$, if we consider them as bivariate functions, they represent the same function. If we view them as univariate functions with a single parameter, X_t^α is a function with α as the parameter and t as the independent variable. $\Psi_t^{-1}(\alpha)$ is a function with t as the parameter and α as the independent variable. Therefore, knowing X_t^α allows us to determine $\Psi_t^{-1}(\alpha)$ by simply interchanging the roles of the parameter and the independent variable.

Definition 3.2. *Suppose f and g are continuous functions. An uncertain differential equation*

$$\frac{\mathrm{d}^n X_t}{\mathrm{d}t^n} = f(t, X_t, \ldots, \frac{\mathrm{d}^{n-1}X_t}{\mathrm{d}t^{n-1}}) + g(t, X_t, \ldots, \frac{\mathrm{d}^{n-1}X_t}{\mathrm{d}t^{n-1}})\frac{\mathrm{d}C_t}{\mathrm{d}t}$$

is said to be regular if

$$g(t, X_t, \ldots, \frac{\mathrm{d}^{n-1}X_t}{\mathrm{d}t^{n-1}}) > 0, \forall t > 0.$$

Theorem 3.1. *The uncertain differential equation is*

$$\begin{cases} X_t^{(n)} = f(t, X_t, X_t^{(1)}, X_t^{(2)}, \ldots, X_t^{(n-1)}) + g(t, X_t, X_t^{(1)}, X_t^{(2)}, \ldots, X_t^{(n-1)})\dot{C}_t, \\ X_t\Big|_{t=0} = y_0, X_t^{(1)}\Big|_{t=0} = y_1, \ldots, X_t^{(n-1)}\Big|_{t=0} = y_{n-1} \end{cases}$$

where

$$\boldsymbol{X} = (X_t, X_t^{(1)}, X_t^{(2)}, \ldots, X_t^{(n-1)}).$$

It has a unique solution if the functions $f(t, \boldsymbol{X})$ and $g(t, \boldsymbol{X})$ satisfy the linear growth condition

$$|f(t, \boldsymbol{X})| + |g(t, \boldsymbol{X})| \leq L(1 + \|\boldsymbol{X}\|), \forall \boldsymbol{X} \in \Re^n, t \geq 0, \|\boldsymbol{X}\| = \sum_{i=1}^{n} |X_i|,$$

and the Lipschitz condition

$$|f(t, \boldsymbol{X}) - f(t, \boldsymbol{Z})| + |g(t, \boldsymbol{X}) - g(t, \boldsymbol{Z})| \leq L(|\boldsymbol{X} - \boldsymbol{Z}|), \forall \boldsymbol{X}, \boldsymbol{Z} \in \Re^n, t \geq 0.$$

Without loss of generality, suppose $L \geq 2$. Moreover, the solution is sample continuous.

Proof. At first, Theorem 14.1 in [12] says that there exists an event Λ with $\mathcal{M}\{\Lambda\} = 1$, such that $C_t(\gamma)$ is Lipschitz continuous with respect to t for each $\gamma \in \Lambda$. Next, we prove the existence and continuity of the solution by a successive approximation method. Define

$$X_t^{(0)}(\gamma) = y_0, \ldots, Y_{t_{n-1}}^{(0)}(\gamma) = y_{n-1},$$

and

$$\begin{cases} \mathrm{d}X_t = Y_{t,1}\mathrm{d}t, \\ \mathrm{d}Y_{t,1} = Y_{t,2}\mathrm{d}t, \\ \ldots \\ \mathrm{d}Y_{t,n-1} = f(t, X_t, Y_{t,1}, \ldots, Y_{t,n-1})\mathrm{d}t + g(t, X_t, Y_{t,2}, \ldots, Y_{t,n-1})\mathrm{d}C_t, \\ X_t\Big|_{t=0} = y_0, Y_{t,1}\Big|_{t=0} = y_1, \ldots, Y_{t,n-1}\Big|_{t=0} = y_{n-1}. \end{cases}$$

By integrating,

$$
\begin{cases}
X_t^{(n)}(\gamma) = y_0 + \int_0^t Y_{t,1}^{(n-1)}(\gamma)\mathrm{d}s,\\
Y_{t,1}^{(n)}(\gamma) = y_1 + \int_0^t Y_{t,2}^{(n-1)}(\gamma)\mathrm{d}s,\\
\cdots\\
Y_{t,n-1}^{(n)}(\gamma) = y_{n-1} + \int_0^t f(s, X_s^{(n-1)}(\gamma), \ldots, Y_{s,n-1}^{(n-1)}(\gamma))\mathrm{d}s + \int_0^t g(s, X_s^{(n-1)}(\gamma), \ldots, Y_{s,n-1}^{(n-1)}(\gamma))\mathrm{d}C_s(\gamma),\\
X_t\Big|_{t=0} = y_0, Y_{t,1}\Big|_{t=0} = y_1, \ldots, Y_{t,n-1}\Big|_{t=0} = y_{n-1},
\end{cases}
$$

for each integer n. It follows from the linear growth condition that

$$
\begin{aligned}
&|X_t^{(1)}(\gamma) - X_t^{(0)}(\gamma)| + |Y_{t,1}^{(1)}(\gamma) - Y_{s,1}^{(0)}(\gamma)| + \cdots + |Y_{t,n-1}^{(1)}(\gamma) - Y_{t,n-1}^{(0)}(\gamma)|\\
=&|\int_0^t Y_{s,1}^0(\gamma)\mathrm{d}s| + |\int_0^t Y_{s,2}^0(\gamma)\mathrm{d}s| + \cdots + |\int_0^t f(s, X_s^0(\gamma), \ldots, Y_{s,n-1}^0(\gamma))\mathrm{d}s|\\
&+ |\int_0^t g(s, X_s^0(\gamma), \ldots, Y_{s,n-1}^0(\gamma))\mathrm{d}C_s(\gamma)|\\
\leq&\int_0^t |y_1|\mathrm{d}s + \int_0^t |y_2|\mathrm{d}s + \cdots + \int_0^t |f(s, X_s^0(\gamma), \ldots, Y_{s,n-1}^0(\gamma))|\mathrm{d}s\\
&+ K_\gamma \int_0^t |g(s, X_s^0(\gamma), \ldots, Y_{s,n-1}^0(\gamma))|\mathrm{d}s\\
\leq&\int_0^t (|y_1| + \cdots + |y_{n-1}|)\mathrm{d}s + \int_0^t L(1 + |y_0| + |y_1| + \cdots + |y_{n-1}|)\mathrm{d}s\\
&+ K_\gamma \int_0^t L(1 + |y_0| + |y_1| + \cdots + |y_{n-1}|)\mathrm{d}s\\
\leq&\int_0^t (1 + |y_0| + |y_1| + \cdots + |y_{n-1}|)\mathrm{d}s + \int_0^t L(1 + |y_0| + |y_1| + \cdots + |y_{n-1}|)\mathrm{d}s\\
&+ K_\gamma \int_0^t L(1 + |y_0| + |y_1| + \cdots + |y_{n-1}|)\mathrm{d}s\\
=&\int_0^t (1 + L + K_\gamma \cdot L)(1 + |y_0| + |y_1| + \cdots + |y_{n-1}|)\mathrm{d}s\\
\leq&\int_0^t (2 + K_\gamma)L(1 + |y_0| + |y_1| + \cdots + |y_{n-1}|)\mathrm{d}s\\
=&(1 + |y_0| + |y_1| + \cdots + |y_{n-1}|)L(2 + K_\gamma)t
\end{aligned}
$$

for any $t \geq 0$, where K_γ is the Lipschitz constant to the sample path $C_t(\gamma)$. Assume

$$
\begin{aligned}
&|X_t^{(k)}(\gamma) - X_t^{(k-1)}(\gamma)| + |Y_{t,1}^{(k)}(\gamma) - Y_{t,1}^{(k-1)}(\gamma)| + \cdots + |Y_{t,n-1}^{(k)}(\gamma) - Y_{t,n-1}^{(k-1)}(\gamma)|\\
\leq&(1 + |y_0| + |y_1| + \cdots + |y_{n-1}|)\frac{L^k(2 + K_\gamma)^k}{k!}t^k.
\end{aligned}
$$

It follows from the Lipschitz condition that

$$\begin{aligned}
&|X_t^{(k+1)}(\gamma)-X_t^{(k)}(\gamma)|+|Y_{t,1}^{(k+1)}(\gamma)-Y_{t,1}^{(k)}(\gamma)|+\cdots+|Y_{t_{n-1}}^{(k+1)}(\gamma)-Y_{t_{n-1}}^{(k)}(\gamma)|\\
\le&\int_0^t|Y_{s,1}^{(k)}(\gamma)-Y_{s,1}^{(k-1)}(\gamma)|\mathrm{d}s+\cdots+\int_0^t|Y_{s,n-1}^{(k)}(\gamma)-Y_{s,n-1}^{(k-1)}(\gamma)|\mathrm{d}s\\
&+\int_0^t|f(s,X_s^{(k)}(\gamma),\ldots,Y_{s,n-1}^{(k)}(\gamma))-f(s,X_s^{(k-1)}(\gamma),\ldots,Y_{s,n-1}^{(k-1)}(\gamma))|\mathrm{d}s\\
&+K_\gamma\int_0^t|g(s,X_s^{(k)}(\gamma),\ldots,Y_{s,n-1}^{(k)}(\gamma))-g(s,X_s^{(k-1)}(\gamma),\ldots,Y_{s,n-1}^{(k-1)}(\gamma))|\mathrm{d}s\\
\le&\int_0^t(|X_s^{(k)}(\gamma)-X_s^{(k-1)}(\gamma)|+|Y_{s,1}^{(k)}(\gamma)-Y_{s,1}^{(k-1)}(\gamma)|+\cdots+|Y_{s,n-1}^{(k)}(\gamma)-Y_{s,n-1}^{(k-1)}(\gamma)|)\mathrm{d}s\\
&+L\int_0^t(|X_s^{(k)}(\gamma)-X_s^{(k-1)}(\gamma)|+|Y_{s,1}^{(k)}(\gamma)-Y_{s,1}^{(k-1)}(\gamma)|+\cdots+|Y_{s,n-1}^{(k)}(\gamma)-Y_{s,n-1}^{(k-1)}(\gamma)|)\mathrm{d}s\\
&+K_\gamma L\int_0^t(|X_s^{(k)}(\gamma)-X_s^{(k-1)}(\gamma)|+|Y_{s,1}^{(k)}(\gamma)-Y_{s,1}^{(k-1)}(\gamma)|+\cdots+|Y_{s,n-1}^{(k)}(\gamma)-Y_{s,n-1}^{(k-1)}(\gamma)|)\mathrm{d}s\\
\le&(1+|y_0|+|y_1|+\cdots+|y_{n-1}|)\frac{L^{k+1}(2+K_\gamma)^{k+1}}{(k+1)!}t^{k+1}.
\end{aligned}$$

The above induction shows that

$$|Y_{t,n-1}^{(k+1)}(\gamma)-Y_{t,n-1}^{(k)}(\gamma)|\le(1+|y_0|+|y_1|+\cdots+|y_{n-1}|)\frac{L^{n+1}(2+K_\gamma)^{n+1}}{(n+1)!}t^{n+1}$$

for each integer n. This means that, for each $\gamma\in\Lambda$, the sequence $X_t^{(n)}(\gamma)$ converges uniformly on any given time interval as $n\to\infty$. Furthermore, since $X_t^{(0)}$ is constant, obviously it is continuous. Suppose $X_t^{(k)}$ is continuous. Then, $X_t^{(k)}$ is also continuous with respect to t. So, by induction, $X_t^{(n)}$ is continuous for all n. Provided $X_t^{(n)}$ converges uniformly, that is $\lim\limits_{n\to\infty}X_t^{(n)}(\gamma)=X_t(\gamma)$, the limit function X_t is also continuous.

The solution to the first equation below is $X_t(\gamma)$.

$$\begin{cases}
X_t(\gamma)=y_0+\int_0^t Y_{t,1}(\gamma)\mathrm{d}s,\\
Y_{t,1}(\gamma)=y_1+\int_0^t Y_{t,2}(\gamma)\mathrm{d}s,\\
\cdots\\
Y_{t,n-1}(\gamma)=y_{n-1}+\int_0^t f(s,X_s(\gamma),\ldots,Y_{s,n-1}(\gamma))\mathrm{d}s+\int_0^t g(s,X_s(\gamma),\ldots,Y_{s,n-1}(\gamma))\mathrm{d}C_s(\gamma),\\
X_t\Big|_{t=0}=y_0,Y_{t,1}\Big|_{t=0}=y_1,\ldots,Y_{t,n-1}\Big|_{t=0}=y_{n-1}.
\end{cases}$$

We then prove that the solution is unique. Assume that both $\mathbf{X}_t$ and $\mathbf{X}_t^*$ are solutions of the UDE. For each $\gamma\in\Lambda$, it follows from the Lipschitz condition that

$$\mathbf{X}_t(\gamma)-\mathbf{X}_t^*(\gamma)\le L(2+K_\gamma)\int_0^t(\mathbf{X}_s(\gamma)-\mathbf{X}_s^*(\gamma))\mathrm{d}s.$$

By using the Gronwall inequality, we obtain

$$\mathbf{X}_t(\gamma)-\mathbf{X}_t^*(\gamma)=0\cdot\exp(\mathrm{L}(2+\mathrm{K}_\gamma)\mathrm{t})=0.$$

Hence $\mathbf{X}_t=\mathbf{X}_t^*$. The uniqueness is verified. The theorem is proved. □

Note 2. The conditions on f and g in the aforementioned theorem are stringent, leading to an ideal conclusion: the global existence of solutions. However, in practical problems, such a perfect conclusion is not necessary. Therefore, we can relax the condition, for instance, by discussing the properties of solutions within a bounded domain. In this case, the linear growth condition for f and g is not required. Instead, it is sufficient that f and g have an upper bound M on the domain G. This also allows us to derive th einequality

$$\begin{aligned}&|X_t^{(1)}(\gamma) - X_t^{(0)}(\gamma)| + |Y_{t_1}^{(1)}(\gamma) - Y_{s_1}^{(0)}(\gamma)| + \cdots + |Y_{t_{n-1}}^{(1)}(\gamma) - Y_{t_{n-1}}^{(0)}(\gamma)| \\ &\leq (1 + |y_0| + |y_1| + \cdots + |y_{n-1}|)L(2 + K_\gamma)t.\end{aligned}$$

From this inequality, we can infer the existence, continuity, and uniqueness of the solution on the domain. These are more practical conditions for applications, and the examples provided later in the paper all meet this criterion.

Theorem 3.2. *Let X_t^α be the α-path of the uncertain differential equation*

$$\begin{cases} X_t^{(n)} = f(t, X_t, X_t^{(1)}, X_t^{(2)}, \ldots, X_t^{(n-1)}) + g(t, X_t, X_t^{(1)}, X_t^{(2)}, \ldots, X_t^{(n-1)})\dot{C}_t, \\ X_t\Big|_{t=0} = y_0, X_t^{(1)}\Big|_{t=0} = y_1, \ldots, X_t^{(n-1)}\Big|_{t=0} = y_{n-1}. \end{cases}$$

If f and g satisfy the Lipschitz and regular conditions hold, then X_t^α is a continuous function with respect to α at each time $t > 0$.

Proof. Since f and g satisfy the Lipschitz conditions, it follows from Theorem 3.1 that the UDEs

$$\begin{cases} \mathrm{d}X_t^\alpha = Y_{t,1}^\alpha \mathrm{d}t, \\ \mathrm{d}Y_{t,1}^\alpha = Y_{t,2}^\alpha \mathrm{d}t, \\ \cdots \\ \mathrm{d}Y_{t,n-1}^\alpha = f(t, X_t^\alpha, Y_{t,1}^\alpha, \ldots, Y_{t,n-1}^\alpha)\mathrm{d}t + g(t, X_t^\alpha, Y_{t,2}^\alpha, \ldots, Y_{t,n-1}^\alpha)\Phi^{-1}(\alpha)\mathrm{d}t, \\ X_t\Big|_{t=0} = y_0, Y_{t_1}\Big|_{t=0} = y_1, \ldots, Y_{t,n-1}\Big|_{t=0} = y_{n-1}. \end{cases}$$

have a unique solution X_t^α, where $\Phi^{-1}(\alpha)$ is the inverse standard normal uncertainty distribution. For any numbers α and β between 0 and 1, it follows from the Lipschitz condition that

$$\begin{aligned}&|X_t^\alpha - X_t^\beta| + |Y_{t,1}^\alpha - Y_{t,1}^\beta| + \cdots + |Y_{t,n-1}^\alpha - Y_{t,n-1}^\beta| \\ &\leq \int_0^t |Y_{s,1}^\alpha - Y_{s,1}^\beta| \mathrm{d}s + \cdots + \int_0^t |Y_{s,n-1}^\alpha - Y_{s,n-1}^\beta| \mathrm{d}s \\ &\quad + \int_0^t |f(s, X_s^\alpha, \ldots, Y_{s,n-1}^\alpha) - f(s, X_s^\beta, \ldots, Y_{s,n-1}^\beta)| \mathrm{d}s \\ &\quad + \int_0^t |g(s, X_s^\alpha, \ldots, Y_{s,n-1}^\alpha) - g(s, X_s^\beta, \ldots, Y_{s,n-1}^\beta)||\Phi^{-1}(\alpha)| \mathrm{d}s \\ &\quad + \int_0^t |g(s, X_s^\beta, \ldots, Y_{s,n-1}^\beta)||\Phi^{-1}(\alpha) - \Phi^{-1}(\beta)|\end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^t (|X_s^\alpha - X_s^\beta| + |Y_{s,1}^\alpha - Y_{s,1}^\beta| + \cdots + |Y_{s,n-1}^\alpha - Y_{s,n-1}^\beta|)\mathrm{d}s \\
&\quad + L\int_0^t (|X_s^\alpha - X_s^\beta| + |Y_{s,1}^\alpha - Y_{s,1}^\beta| + \cdots + |Y_{s,n-1}^\alpha - Y_{s,n-1}^\beta|)\mathrm{d}s \\
&\quad + L|\Phi^{-1}(\alpha)|\int_0^t (|X_s^\alpha - X_s^\beta| + |Y_{s,1}^\alpha - Y_{s,1}^\beta| + \cdots + |Y_{s,n-1}^\alpha - Y_{s,n-1}^\beta|)\mathrm{d}s \\
&\quad + |\Phi^{-1}(\alpha) - \Phi^{-1}(\beta)|\int_0^t |g(s, X_s^\beta, \ldots, Y_{s,n-1}^\beta)|\mathrm{d}s \\
&\leq (2+\Phi^{-1}(\alpha))L\int_0^t (|X_s^\alpha - X_s^\beta| + |Y_{s,1}^\alpha - Y_{s,1}^\beta| + \cdots + |Y_{s,n-1}^\alpha - Y_{s,n-1}^\beta|)\mathrm{d}s \\
&\quad + |\Phi^{-1}(\alpha) - \Phi^{-1}(\beta)|\int_0^t |g(s, X_s^\beta, \ldots, Y_{s,n-1}^\beta)|\mathrm{d}s.
\end{aligned}
$$

By using the Gronwall inequality, we obtain

$$
\begin{aligned}
&|X_t^\alpha - X_t^\beta| + |Y_{t,1}^\alpha - Y_{t,1}^\beta| + \cdots + |Y_{t,n-1}^\alpha - Y_{t,n-1}^\beta| \\
&\leq |\Phi^{-1}(\alpha) - \Phi^{-1}(\beta)|\int_0^t |g(s, X_s^\beta, \ldots, Y_{s,n-1}^\beta)|\mathrm{d}s \cdot L(2+\Phi^{-1}(\alpha))t.
\end{aligned}
$$

Thus

$$
\lim_{\alpha\to\beta} |X_t^\alpha - X_t^\beta| = 0
$$

for each time $t > 0$. Hence X_t^α is continuous with respect to α.

Note 3. The proof of Theorem 3.2 is pivotal as it establishes a necessary condition for X_t^α to serve as an inverse uncertainty distribution. This condition will be instrumental in the subsequent proofs of Theorems 3.6 and 3.10.

Theorem 3.3. *Let X_t^α be the α-path of the regular uncertain differential equation*

$$
\frac{\mathrm{d}^2 X_t}{\mathrm{d}t^2} = f(t, X_t, \frac{\mathrm{d}X_t}{\mathrm{d}t}) + g(t, X_t, \frac{\mathrm{d}X_t}{\mathrm{d}t})\frac{\mathrm{d}C_t}{\mathrm{d}t}.
$$

If the conditions

$$
f(x, y_1, z) \leq f(x, y_2, z), g(x, y_1, z) \leq g(x, y_2, z), \forall y_1 < y_2 \tag{H}
$$

are met and the linear growth, Lipschitz, and regular conditions hold, then X_t^α is a continuous and strictly increasing function with respect to α at each time $t > 0$.

Proof. Since f and g satisfy the linear growth and Lipschitz conditions, the α-path X_t^α is continuous with respect to t. Let Φ^{-1} be the inverse standard normal uncertainty distribution, and let α and β be numbers with $0 < \alpha < \beta < 1$. Write the second-order UDEs in the form of a system of equations:

$$
\begin{cases}
\mathrm{d}X_t^\alpha = Y_t^\alpha \mathrm{d}t, \\
\mathrm{d}Y_t^\alpha = f(t, X_t^\alpha, Y_t^\alpha) + g(t, X_t^\alpha, Y_t^\alpha)\Phi^{-1}(\alpha)\mathrm{d}t, \\
X_0^\alpha = X_0, Y_0^\alpha = Y_0.
\end{cases} \tag{3.3}
$$

$$\begin{cases} \mathrm{d}X_t^\beta = Y_t^\beta \mathrm{d}t, \\ \mathrm{d}Y_t^\beta = f(t, X_t^\beta, Y_t^\beta) + g(t, X_t^\beta, Y_t^\beta)\Phi^{-1}(\beta)\mathrm{d}t, \\ X_0^\beta = X_0, Y_0^\beta = Y_0. \end{cases} \tag{3.4}$$

Define μ and ν as:

$$\mu(T,t) = (T-t)[f(t, X_t^\alpha, Y_t^\alpha) + g(t, X_t^\alpha, Y_t^\alpha)\Phi^{-1}(\alpha)],$$
$$\nu(T,t) = (T-t)[f(t, X_t^\beta, Y_t^\beta) + g(t, X_t^\beta, Y_t^\beta)\Phi^{-1}(\beta)].$$

We have

$$\mu(T,0) = T[f(0, X_0, Y_0) + g(0, X_0, Y_0)\Phi^{-1}(\alpha)],$$
$$\nu(T,0) = T[f(0, X_0, Y_0) + g(0, X_0, Y_0)\Phi^{-1}(\beta)].$$

Since $g(0, X_0, Y_0) > 0$ (regular condition), we have

$$\mu(T,0) < \nu(T,0).$$

Due to the continuous nature of f and g, and the continuity of X_t^α and Y_t^α, we can infer that the composite functions μ and ν are also continuous with respect to t.

By the continuity of μ and ν, there exists a small number $r > 0$ such that

$$\mu(T,t) < \nu(T,t), \forall t \in [0, r].$$

Thus, by Lemma 3.1, we have:

$$\begin{aligned} X_T^\alpha &= X_0 + Y_0 T + \int_0^T (T-t) f(t, X_t^\alpha, Y_t^\alpha)\mathrm{d}t + \int_0^T (T-t) g(t, X_t^\alpha, Y_t^\alpha)\Phi^{-1}(\alpha)\mathrm{d}t \\ &< X_0 + Y_0 T + \int_0^T (T-t) f(t, X_t^\beta, Y_t^\beta)\mathrm{d}t + \int_0^T (T-t) g(t, X_t^\beta, Y_t^\beta)\Phi^{-1}(\beta)\mathrm{d}t \\ &= X_T^\beta \end{aligned}$$

for any time $T \in (0, r]$.

If for any $t > r$, $X_t^\alpha < X_t^\beta$, and the theorem holds. We will prove that by contradiction.

Suppose there exists a time $b > r$ at which X_t^α and X_t^β first meet, i.e.,

$$X_b^\alpha = X_b^\beta, X_t^\alpha < X_t^\beta, \forall t \in (0, b).$$

The next phase of our proof will be to compare Y_t^α and Y_t^β in a similar way. Write

$$\overline{\mu}(t) = f(t, X_t^\alpha, Y_t^\alpha) + g(t, X_t^\alpha, Y_t^\alpha)\Phi^{-1}(\alpha),$$
$$\overline{\nu}(t) = f(t, X_t^\beta, Y_t^\beta) + g(t, X_t^\beta, Y_t^\beta)\Phi^{-1}(\beta).$$

Since $g(0, X_0, Y_0) > 0$, we have

$$\overline{\mu}(0) < \overline{\nu}(0).$$

By the continuity of $\overline{\mu}$ and $\overline{\nu}$, there exists a small number $\overline{r} > 0$ such that

$$\overline{\mu}(t) < \overline{\nu}(t), \forall t \in [0, \overline{r}].$$

Thus,

$$\begin{aligned} Y_t^\alpha &= Y_0 + \int_0^t f(s, X_s^\alpha, Y_s^\alpha)\mathrm{d}s + \int_0^t g(s, X_s^\alpha, Y_s^\alpha)\Phi^{-1}(\alpha)\mathrm{d}s \\ &< Y_0 + \int_0^t f(s, X_s^\beta, Y_s^\beta)\mathrm{d}s + \int_0^t g(s, X_s^\beta, Y_s^\beta)\Phi^{-1}(\beta)\mathrm{d}s \\ &= Y_t^\beta, \end{aligned}$$

for any time $t \in (0, \overline{r}]$.

We will prove $Y_t^\alpha < Y_t^\beta$ for $\overline{r} < t < b$ by contradiction.

Suppose there exists a time $\overline{r} < \overline{b} < b$ at which Y_t^α and Y_t^β first meet, i.e.,

$$Y_{\overline{b}}^\alpha = Y_{\overline{b}}^\beta, Y_t^\alpha < Y_t^\beta, \forall t \in (0, \overline{b}).$$

Due to conditions (H), we have

$$f(\overline{b}, X_{\overline{b}}^\alpha, Y_{\overline{b}}^\alpha) < f(\overline{b}, X_{\overline{b}}^\beta, Y_{\overline{b}}^\beta), 0 < g(\overline{b}, X_{\overline{b}}^\alpha, Y_{\overline{b}}^\alpha) < g(\overline{b}, X_{\overline{b}}^\beta, Y_{\overline{b}}^\beta).$$

Then

$$\overline{\mu}(\overline{b}) < \overline{\nu}(\overline{b}).$$

By the continuity of $\overline{\mu}$ and $\overline{\nu}$, there exists a time $\overline{a} \in (0, \overline{b})$ such that

$$\overline{\mu}(t) < \overline{\nu}(t), t \in [\overline{a}, \overline{b}].$$

Thus,

$$\begin{aligned} Y_{\overline{b}}^\alpha &= Y_{\overline{a}}^\alpha + \int_{\overline{a}}^{\overline{b}} f(s, X_s^\alpha, Y_s^\alpha)\mathrm{d}s + \int_{\overline{a}}^{\overline{b}} g(s, X_s^\alpha, Y_s^\alpha)\Phi^{-1}(\alpha)\mathrm{d}s \\ &< Y_{\overline{a}}^\beta + \int_{\overline{a}}^{\overline{b}} f(s, X_s^\beta, Y_s^\beta)\mathrm{d}s + \int_{\overline{a}}^{\overline{b}} g(s, X_s^\beta, Y_s^\beta)\Phi^{-1}(\beta)\mathrm{d}s \\ &= Y_{\overline{b}}^\beta, \end{aligned}$$

which is in contradiction with the assumption $Y_{\overline{b}}^\alpha = Y_{\overline{b}}^\beta$. Therefore,

$$Y_t^\alpha < Y_t^\beta, \forall 0 < t < b.$$

Integrate Eqs (3.3) and(3.4)

$$X_b^\alpha = X_0 + \int_0^b Y_t^\alpha \mathrm{d}t,$$

$$X_b^\beta = X_0 + \int_0^b Y_t^\alpha \mathrm{d}t,$$

and we have

$$X_b^\alpha < X_b^\beta,$$

which is in contradiction with the assumption $X_b^\alpha = X_b^\beta$. The theorem is proved. □

Note 4. Condition (H) is indispensable in the proof of the theorem. Should condition (H) not hold, it would be impossible to establish the relationship between $X^{\alpha}_{\overline{b}}$ and $X^{\beta}_{\overline{b}}$ when proving $\overline{\mu}(\overline{b}) < \overline{\nu}(\overline{b})$ and thus the proof would be untenable.

By Theorem 3.3, we have established that X_t is a continuous and strictly increasing function with respect to α. We wonder whether this collection of α-paths can determine the inverse uncertainty distribution of the solutions to second-order UDEs. The following Theorem 3.4 provides a definitive answer.

Theorem 3.4. *Let X_t and X_t^{α} be the solution and α-path of a regular uncertain differential equation*

$$\frac{\mathrm{d}^2 X_t}{\mathrm{d}t^2} = f(t, X_t, \frac{\mathrm{d}X_t}{\mathrm{d}t}) + g(t, X_t, \frac{\mathrm{d}X_t}{\mathrm{d}t})\frac{\mathrm{d}C_t}{\mathrm{d}t}.$$

If the conditions (H) are met and the linear growth, Lipschitz, and regular conditions hold, then we have

$$\mathcal{M}\{X_t \le X_t^{\alpha}, \forall t\} = \alpha,$$

$$\mathcal{M}\{X_t > X_t^{\alpha}, \forall t\} = 1 - \alpha.$$

Proof. Theorem 14.3 in [12] constructs an event Λ_1 with $\mathcal{M}\{\Lambda_1\} = \alpha$, and shows that for each $\gamma \in \Lambda_1$, there exists a small number $\delta > 0$ such that

$$\frac{C_s(\gamma) - C_t(\gamma)}{s - t} < \Phi^{-1}(\alpha - \delta) \tag{3.5}$$

for any times s and t with $s > t$, where Φ^{-1} is the inverse standard normal uncertainty distribution. Since f and g satisfy the linear growth and Lipschitz conditions, X_t^{α} and $X_t(\gamma)$ are continuous with respect to t. Write the second-order UDE in the form of a system of equations:

$$\begin{cases} \mathrm{d}X_t = Y_t \mathrm{d}t, \\ \mathrm{d}Y_t = f(t, X_t, Y_t)\mathrm{d}t + g(t, X_t, Y_t)\mathrm{d}C_t, \\ X_t\Big|_{t=0} = X_0, Y_t\Big|_{t=0} = Y_0. \end{cases} \tag{3.6}$$

Define λ, μ, and ν as:

$$\lambda(T, t, s) = (T - t)[f(t, X_t(\gamma), Y_t(\gamma)) + g(t, X_t(\gamma), Y_t(\gamma))\frac{C_s(\gamma) - C_t(\gamma)}{s - t}],$$

$$\mu(T, t) = (T - t)[f(t, X_t(\gamma), Y_t(\gamma)) + g(t, X_t(\gamma), Y_t(\gamma))\Phi^{-1}(\alpha - \delta)],$$

$$\nu(T, t) = (T - t)[f(t, X_t^{\alpha}, Y_t^{\alpha}) + g(t, X_t^{\alpha}, Y_t^{\alpha})\Phi^{-1}(\alpha)].$$

Due to the continuous nature of f and g, and the continuity of $X_t(\gamma)$ and $Y_t(\gamma)$, we can infer that the composite functions λ, μ, and ν are also continuous with respect to t.

Since $g(0, X_0, Y_0) > 0$ (regular condition), we have

$$\mu(T, 0) < \nu(T, 0).$$

By the continuity of μ and ν, there exists a small number $r > 0$ such that

$$\mu(T, t) < \nu(T, t), \forall t \in [0, r].$$

By inequality (3.5), for any time $t \in [0, r]$ and any time $s \in (t, \infty)$, we have

$$\begin{aligned}\lambda(T, t, s) &= (T - t)[f(t, X_t(\gamma), Y_t(\gamma)) + g(t, X_t(\gamma), Y_t(\gamma))\frac{C_s(\gamma) - C_t(\gamma)}{s - t}] \\ &< (T - t)[f(t, X_t(\gamma), Y_t(\gamma)) + g(t, X_t(\gamma), Y_t(\gamma))\Phi^{-1}(\alpha - \delta)] \\ &= \mu(T, t) < \nu(T, t).\end{aligned}$$

By Lemma 3.1, it follows that

$$\begin{aligned}X_T(\gamma) &= X_0 + Y_0 T + \int_0^T (T - t)f(t, X_t(\gamma), Y_t(\gamma))\mathrm{d}t + \int_0^T (T - t)g(t, X_t(\gamma), Y_t(\gamma))\mathrm{d}C_t(\gamma) \\ &= X_0 + Y_0 T + \lim_{\Delta\to 0} \sum_{i=1}^{k} \lambda(T, t_{i+1}, t_i)(t_{i+1} - t_i) \\ &\leq X_0 + Y_0 T + \lim_{\Delta\to 0} \sum_{i=1}^{k} \mu(T, t_i)(t_{i+1} - t_i) \\ &= X_0 + Y_0 T + \int_0^T \mu(T, t)\mathrm{d}t \\ &< X_0 + Y_0 T + \int_0^T \nu(T, t)\mathrm{d}t \\ &= X_0 + Y_0 T + \int_0^T (T - t)f(t, X_t^\alpha, Y_t^\alpha)\mathrm{d}t + \int_0^T (T - t)g(t, X_t^\alpha, Y_t^\alpha)\Phi^{-1}(\alpha)\mathrm{d}t \\ &= X_T^\alpha\end{aligned}$$

for any time $T \in (0, r]$.

We will then prove $X_t(\gamma) < X_t^\alpha$ for all $t > r$ by contradiction.

Suppose there exists a time $b > r$ at which $X_t(\gamma)$ and X_t^α first meet, i.e.,

$$X_b(\gamma) = X_b^\alpha, X_t(\gamma) < X_t^\alpha, \forall t \in (0, b).$$

The next phase of our proof will be to compare Y_t^α and $Y_t(\gamma)$ in a similar way. Write

$$\overline{\lambda}(t, s) = f(t, X_t(\gamma), Y_t(\gamma)) + g(t, X_t(\gamma), Y_t(\gamma))\frac{C_s(\gamma) - C_t(\gamma)}{s - t},$$

$$\overline{\mu}(t) = f(t, X_t(\gamma), Y_t(\gamma)) + g(t, X_t(\gamma), Y_t(\gamma))\Phi^{-1}(\alpha - \delta),$$

$$\overline{\nu}(t) = f(t, X_t^\alpha, Y_t^\alpha) + g(t, X_t^\alpha, Y_t^\alpha)\Phi^{-1}(\alpha).$$

Since $g(0, X_0, Y_0) > 0$, we have

$$\overline{\mu}(0) < \overline{\nu}(0).$$

By the continuity of $\overline{\mu}$ and $\overline{\nu}$, there exists a small number $\overline{r} > 0$ such that

$$\overline{\mu}(t) < \overline{\nu}(t), \forall t \in [0, \overline{r}].$$

By inequality (3.5), for any time $t \in [0, \overline{r}]$ and any time $s \in (t, \infty)$, we have

$$\begin{aligned}\overline{\lambda}(t,s) &= f(t, X_t(\gamma), Y_t(\gamma)) + g(t, X_t(\gamma), Y_t(\gamma))\frac{C_s(\gamma) - C_t(\gamma)}{s-t}\\ &< f(t, X_t(\gamma), Y_t(\gamma)) + g(t, X_t(\gamma), Y_t(\gamma))\Phi^{-1}(\alpha - \delta)\\ &= \overline{\mu}(t) < \overline{\nu}(t).\end{aligned}$$

Thus,

$$\begin{aligned}Y_t(\gamma) &= Y_0 + \int_0^t f(s, X_s(\gamma), Y_s(\gamma))\mathrm{d}s + \int_0^t g(s, X_s(\gamma), Y_s(\gamma))\mathrm{d}C_s(\gamma)\\ &= Y_0 + \lim_{\Delta\to 0}\sum_{i=1}^{k}\overline{\lambda}(t_{i+1}, t_i)(t_{i+1} - t_i)\\ &\le Y_0 + \lim_{\Delta\to 0}\sum_{i=1}^{k}\overline{\mu}(t_i)(t_{i+1} - t_i)\\ &= Y_0 + \int_0^t \overline{\mu}(t)\mathrm{d}t\\ &< Y_0 + \int_0^t \overline{\nu}(t)\mathrm{d}t\\ &= Y_0 + \int_0^t f(s, X_s^\alpha, Y_s^\alpha)\mathrm{d}s + \int_0^t g(s, X_s^\alpha, Y_s^\alpha)\Phi^{-1}(\alpha)\mathrm{d}s\\ &= Y_t^\alpha\end{aligned}$$

for any time $t \in (0, \overline{r}]$.

We will prove $Y_t(\gamma) < Y_t^\alpha$ for all $\overline{r} < t < b$ by contradiction.

Suppose there exists a time $\overline{r} < \overline{b} < b$ at which $Y_t(\gamma)$ and Y_t^α first meet, i.e.,

$$Y_{\overline{b}}(\gamma) = Y_{\overline{b}}^\alpha, Y_t(\gamma) < Y_t^\alpha, \forall t \in (0, \overline{b}).$$

Due to conditions (H), we have

$$f(\overline{b}, X_{\overline{b}}(\gamma), Y_{\overline{b}}(\gamma)) < f(\overline{b}, X_{\overline{b}}^\alpha, Y_{\overline{b}}^\alpha), 0 < g(\overline{b}, X_{\overline{b}}(\gamma), Y_{\overline{b}}(\gamma)) < g(\overline{b}, X_{\overline{b}}^\alpha, Y_{\overline{b}}^\alpha),$$

and then

$$\overline{\mu}(\overline{b}) < \overline{\nu}(\overline{b}).$$

By the continuity of $\overline{\mu}$ and $\overline{\nu}$, there exists a time $\overline{a} \in (0, \overline{b})$ such that

$$\overline{\mu}(t) < \overline{\nu}(t), t \in [\overline{a}, \overline{b}].$$

By inequality (3.5), for any time $t \in [\overline{a}, \overline{b}]$ and any time $s \in (t, \infty)$, we have

$$\begin{aligned}\overline{\lambda}(t,s) &= f(t, X_t(\gamma), Y_t(\gamma)) + g(t, X_t(\gamma), Y_t(\gamma))\frac{C_s(\gamma) - C_t(\gamma)}{s-t}\\ &< f(t, X_t(\gamma), Y_t(\gamma)) + g(t, X_t(\gamma), Y_t(\gamma))\Phi^{-1}(\alpha - \delta)\\ &= \overline{\mu}(t) < \overline{\nu}(t).\end{aligned}$$

Thus,

$$\begin{aligned}
Y_{\overline{b}}(\gamma) &= Y_{\overline{a}}(\gamma) + \int_{\overline{a}}^{\overline{b}} f(s, X_s(\gamma), Y_s(\gamma))\mathrm{d}s + \int_{\overline{a}}^{\overline{b}} g(s, X_s(\gamma), Y_s(\gamma))\mathrm{d}C_s(\gamma) \\
&= Y_{\overline{a}}(\gamma) + \lim_{\Delta\to 0}\sum_{i=1}^{k} \overline{\lambda}(t_{i+1}, t_i)(t_{i+1} - t_i) \\
&\le Y_{\overline{a}}(\gamma) + \lim_{\Delta\to 0}\sum_{i=1}^{k} \overline{\mu}(t_i)(t_{i+1} - t_i) \\
&= Y_{\overline{a}}(\gamma) + \int_{\overline{a}}^{\overline{b}} \overline{\mu}(t)\mathrm{d}s \\
&< Y_{\overline{a}}^{\alpha} + \int_{\overline{a}}^{\overline{b}} \overline{\nu}(t)\mathrm{d}s \\
&= Y_{\overline{a}}^{\alpha} + \int_{\overline{a}}^{\overline{b}} f(s, X_s^{\alpha}, Y_s^{\alpha})\mathrm{d}s + \int_{\overline{a}}^{\overline{b}} g(s, X_s^{\alpha}, Y_s^{\alpha})\Phi^{-1}(\alpha)\mathrm{d}s \\
&= Y_{\overline{b}}^{\alpha},
\end{aligned}$$

which is in contradiction with $Y_{\overline{b}}(\gamma) = Y_{\overline{b}}^{\alpha}$. Therefore,

$$Y_t(\gamma) < Y_t^{\alpha}, \forall 0 < t < b.$$

Integrate Eq (3.6),

$$X_b(\gamma) = X_0 + \int_0^b Y_t(\gamma)\mathrm{d}t,$$

$$X_b^{\alpha} = X_0 + \int_0^b Y_t^{\alpha}\mathrm{d}t,$$

and we have

$$X_b(\gamma) < X_b^{\alpha},$$

which is in contradiction with $X_b(\gamma) = X_b^{\alpha}$. Therefore,

$$X_t(\gamma) < X_t^{\alpha}, \forall t > 0.$$

Since $\mathcal{M}\{\Lambda_1\} = \alpha$, we have

$$\mathcal{M}\{X_t \le X_t^{\alpha}, \forall t\} \ge \alpha. \tag{3.7}$$

Theorem 14.3 in [12] constructs an event Λ_2 with $\mathcal{M}\{\Lambda_2\} = 1 - \alpha$, and shows that for each $\gamma \in \Lambda_2$, there exists a small number $\delta > 0$ such that

$$\frac{C_s(\gamma) - C_t(\gamma)}{s - t} > \Phi^{-1}(\alpha + \delta) \tag{3.8}$$

for any times s and t with $s > t$. Write

$$\lambda(T, t, s) = (T - t)[f(t, X_t(\gamma), Y_t(\gamma)) + g(t, X_t(\gamma), Y_t(\gamma))\frac{C_s(\gamma) - C_t(\gamma)}{s - t}],$$

$$\mu(T,t) = (T-t)[f(t,X_t(\gamma),Y_t(\gamma)) + g(t,X_t(\gamma),Y_t(\gamma))\Phi^{-1}(\alpha+\delta)],$$
$$\nu(T,t) = (T-t)[f(t,X_t^\alpha,Y_t^\alpha) + g(t,X_t^\alpha,Y_t^\alpha)\Phi^{-1}(\alpha)].$$

Since $g(0,X_0,Y_0) > 0$ (regular condition), we have

$$\mu(T,0) > \nu(T,0).$$

By the continuity of μ and ν, there exists a small number $r > 0$ such that

$$\mu(T,t) > \nu(T,t), \forall t \in [0,r].$$

By inequality (3.8), for any time $t \in [0,r]$ and any time $s \in (t,\infty)$, we have

$$\begin{aligned}\lambda(T,t,s) &= (T-t)[f(t,X_t(\gamma),Y_t(\gamma)) + g(t,X_t(\gamma),Y_t(\gamma))\frac{C_s(\gamma)-C_t(\gamma)}{s-t}]\\ &> (T-t)[f(t,X_t(\gamma),Y_t(\gamma)) + g(t,X_t(\gamma),Y_t(\gamma))\Phi^{-1}(\alpha+\delta)]\\ &= \mu(T,t) > \nu(T,t).\end{aligned}$$

By Lemma 3.1, it follows that

$$\begin{aligned}X_T(\gamma) &= X_0 + Y_0T + \int_0^T (T-t)f(t,X_t(\gamma),Y_t(\gamma))\mathrm{d}t + \int_0^T (T-t)g(t,X_t(\gamma),Y_t(\gamma))\mathrm{d}C_t(\gamma)\\ &= X_0 + Y_0T + \lim_{\Delta\to 0}\sum_{i=1}^{k}\lambda(T,t_{i+1},t_i)(t_{i+1}-t_i)\\ &\ge X_0 + Y_0T + \lim_{\Delta\to 0}\sum_{i=1}^{k}\mu(T,t_i)(t_{i+1}-t_i)\\ &= X_0 + Y_0T + \int_0^T \mu(T,t)\mathrm{d}t\\ &> X_0 + Y_0T + \int_0^T \nu(T,t)\mathrm{d}t\\ &= X_0 + Y_0T + \int_0^T (T-t)f(t,X_t^\alpha,Y_t^\alpha)\mathrm{d}t + \int_0^T (T-t)g(t,X_t^\alpha,Y_t^\alpha)\Phi^{-1}(\alpha)\mathrm{d}t\\ &= X_T^\alpha\end{aligned}$$

for any time $T \in (0,r]$.

We will then prove $X_t(\gamma) < X_t^\alpha$ for all $t > r$ by contradiction.

Suppose there exists a time $b > r$ at which $X_t(\gamma)$ and X_t^α first meet, i.e.,

$$X_b(\gamma) = X_b^\alpha, X_t(\gamma) > X_t^\alpha, \forall t \in (0,b).$$

The next phase of our proof will be to compare Y_t^α and $Y_t(\gamma)$ in a similar way. Write

$$\overline{\lambda}(t,s) = f(t,X_t(\gamma),Y_t(\gamma)) + g(t,X_t(\gamma),Y_t(\gamma))\frac{C_s(\gamma)-C_t(\gamma)}{s-t},$$

$$\overline{\mu}(t) = f(t, X_t(\gamma), Y_t(\gamma)) + g(t, X_t(\gamma), Y_t(\gamma))\Phi^{-1}(\alpha + \delta),$$

$$\overline{\nu}(t) = f(t, X_t^\alpha, Y_t^\alpha) + g(t, X_t^\alpha, Y_t^\alpha)\Phi^{-1}(\alpha).$$

Since $g(0, X_0, Y_0) > 0$, we have

$$\overline{\mu}(0) > \overline{\nu}(0).$$

By the continuity of $\overline{\mu}$ and $\overline{\nu}$, there exists a small number $\overline{r} > 0$ such that

$$\overline{\mu}(t) > \overline{\nu}(t), \forall t \in [0, \overline{r}].$$

By inequality (3.8), for any time $t \in [0, \overline{r}]$ and any time $s \in (t, \infty)$, we have

$$\begin{aligned}
\overline{\lambda}(t, s) &= f(t, X_t(\gamma), Y_t(\gamma)) + g(t, X_t(\gamma), Y_t(\gamma))\frac{C_s(\gamma) - C_t(\gamma)}{s - t} \\
&> f(t, X_t(\gamma), Y_t(\gamma)) + g(t, X_t(\gamma), Y_t(\gamma))\Phi^{-1}(\alpha + \delta) \\
&= \overline{\mu}(t) > \overline{\nu}(t).
\end{aligned}$$

Thus,

$$\begin{aligned}
Y_t(\gamma) &= Y_0 + \int_0^t f(s, X_s(\gamma), Y_s(\gamma))\mathrm{d}s + \int_0^t g(s, X_s(\gamma), Y_s(\gamma))\mathrm{d}C_s(\gamma) \\
&= Y_0 + \lim_{\Delta \to 0} \sum_{i=1}^{k} \overline{\lambda}(t_{i+1}, t_i)(t_{i+1} - t_i) \\
&\geq Y_0 + \lim_{\Delta \to 0} \sum_{i=1}^{k} \overline{\mu}(t_i)(t_{i+1} - t_i) \\
&= Y_0 + \int_0^t \overline{\mu}(t)\mathrm{d}t \\
&> Y_0 + \int_0^t \overline{\nu}(t)\mathrm{d}t \\
&= Y_0 + \int_0^t f(s, X_s^\alpha, Y_s^\alpha)\mathrm{d}s + \int_0^t g(s, X_s^\alpha, Y_s^\alpha)\Phi^{-1}(\alpha)\mathrm{d}s \\
&= Y_t^\alpha
\end{aligned}$$

for any time $t \in (0, \overline{r}]$.

We will prove $Y_t(\gamma) > Y_t^\alpha$ for all $\overline{r} < t < b$ by contradiction.

Suppose there exists a time $\overline{r} < \overline{b} < b$ at which $Y_t(\gamma)$ and Y_t^α first meet, i.e.,

$$Y_{\overline{b}}(\gamma) = Y_{\overline{b}}^\alpha, Y_t(\gamma) > Y_t^\alpha, \forall t \in (0, b).$$

Due to conditions (H), we have

$$f(\overline{b}, X_{\overline{b}}(\gamma), Y_{\overline{b}}(\gamma)) > f(\overline{b}, X_{\overline{b}}^\alpha, Y_{\overline{b}}^\alpha), g(\overline{b}, X_{\overline{b}}(\gamma), Y_{\overline{b}}(\gamma)) > g(\overline{b}, X_{\overline{b}}^\alpha, Y_{\overline{b}}^\alpha) > 0,$$

and then

$$\overline{\mu}(\overline{b}) > \overline{\nu}(\overline{b}).$$

By the continuity of $\overline{\mu}$ and $\overline{\nu}$, there exists a time $\overline{a} \in (0, \overline{b})$ such that

$$\overline{\mu}(t) > \overline{\nu}(t), t \in [\overline{a}, \overline{b}].$$

By inequality (3.8), for any time $t \in [\overline{a}, \overline{b}]$ and any time $s \in (t, \infty)$, we have

$$\begin{aligned}
\overline{\lambda}(t, s) &= f(t, X_t(\gamma), Y_t(\gamma)) + g(t, X_t(\gamma), Y_t(\gamma))\frac{C_s(\gamma) - C_t(\gamma)}{s - t} \\
&> f(t, X_t(\gamma), Y_t(\gamma)) + g(t, X_t(\gamma), Y_t(\gamma))\Phi^{-1}(\alpha + \delta) \\
&= \overline{\mu}(t) > \overline{\nu}(t).
\end{aligned}$$

Thus,

$$\begin{aligned}
Y_{\overline{b}}(\gamma) &= Y_{\overline{a}}(\gamma) + \int_{\overline{a}}^{\overline{b}} f(s, X_s(\gamma), Y_s(\gamma))\mathrm{d}s + \int_{\overline{a}}^{\overline{b}} g(s, X_s(\gamma), Y_s(\gamma))\mathrm{d}C_s(\gamma) \\
&= Y_{\overline{a}}(\gamma) + \lim_{\Delta \to 0} \sum_{i=1}^{k} \overline{\lambda}(t_{i+1}, t_i)(t_{i+1} - t_i) \\
&\geq Y_{\overline{a}}(\gamma) + \lim_{\Delta \to 0} \sum_{i=1}^{k} \overline{\mu}(t_i)(t_{i+1} - t_i) \\
&= Y_{\overline{a}}(\gamma) + \int_{\overline{a}}^{\overline{b}} \overline{\mu}(t)\mathrm{d}s \\
&> Y_{\overline{a}}^{\alpha} + \int_{\overline{a}}^{\overline{b}} \overline{\nu}(t)\mathrm{d}s \\
&= Y_{\overline{a}}^{\alpha} + \int_{\overline{a}}^{\overline{b}} f(s, X_s^{\alpha}, Y_s^{\alpha})\mathrm{d}s + \int_{\overline{a}}^{\overline{b}} g(s, X_s^{\alpha}, Y_s^{\alpha})\Phi^{-1}(\alpha)\mathrm{d}s \\
&= Y_{\overline{b}}^{\alpha},
\end{aligned}$$

which is in contradiction with $Y_{\overline{b}}(\gamma) = Y_{\overline{b}}^{\alpha}$. Therefore,

$$Y_t(\gamma) > Y_t^{\alpha}, \forall 0 < t < b.$$

Integrate Eq (3.6),

$$X_b(\gamma) = X_0 + \int_0^b Y_t(\gamma)\mathrm{d}t,$$

$$X_b^{\alpha} = X_0 + \int_0^b Y_t^{\alpha}\mathrm{d}t,$$

and we have

$$X_b(\gamma) > X_b^{\alpha},$$

which is in contradiction with $X_b(\gamma) = X_b^{\alpha}$. Therefore,

$$X_t(\gamma) > X_t^{\alpha}, \forall t > 0.$$

Since $\mathcal{M}\{\Lambda_2\} = 1 - \alpha$, we have

$$\mathcal{M}\{X_t > X_t^{\alpha}, \forall t\} \geq 1 - \alpha. \tag{3.9}$$

It follows from (3.7), (3.9) and

$$\mathcal{M}\{X_t \le X_t^\alpha, \forall t\} + \mathcal{M}\{X_t > X_t^\alpha, \forall t\} \le 1,$$

that

$$\mathcal{M}\{X_t \le X_t^\alpha, \forall t\} = \alpha,$$

$$\mathcal{M}\{X_t > X_t^\alpha, \forall t\} = 1 - \alpha.$$

The proof is now complete. □

Theorem 3.5. *Let X_t and X_t^α be the solution and α-path of a regular uncertain differential equation*

$$\frac{\mathrm{d}^2 X_t}{\mathrm{d}t^2} = f(t, X_t, \frac{\mathrm{d}X_t}{\mathrm{d}t}) + g(t, X_t, \frac{\mathrm{d}X_t}{\mathrm{d}t})\frac{\mathrm{d}C_t}{\mathrm{d}t}.$$

If the conditions (H) are met, the linear growth, Lipschitz and regular conditions hold, then at any time $t > 0$, we have

$$\mathcal{M}\{X_t \le X_t^\alpha\} = \alpha,$$

$$\mathcal{M}\{X_t > X_t^\alpha\} = 1 - \alpha.$$

Proof. Note that $\{X_t \le X_t^\alpha\} \supset \{X_s \le X_s^\alpha, \forall s > 0\}$ holds for any time $t > 0$. By using the Theorems 3.3 and 3.4, we obtain

$$\mathcal{M}\{X_t \le X_t^\alpha\} \ge M\{X_s \le X_s^\alpha, \forall s > 0\} = \alpha.$$

Similarly, we also have

$$\mathcal{M}\{X_t > X_t^\alpha\} \ge M\{X_s > X_s^\alpha, \forall s > 0\} = 1 - \alpha.$$

Since $\{X_t \le X_t^\alpha\}$ and $\{X_t > X_t^\alpha\}$ are opposite events for each $t > 0$, the duality axiom makes

$$\mathcal{M}\{X_t \le X_t^\alpha\} + \mathcal{M}\{X_t > X_t^\alpha\} = 1.$$

The theorem is proved. □

Theorem 3.6. *Let X_t and X_t^α be the solution and α-path of the uncertain differential equation*

$$\frac{\mathrm{d}^2 X_t}{\mathrm{d}t^2} = f(t, X_t, \frac{\mathrm{d}X_t}{\mathrm{d}t}) + g(t, X_t, \frac{\mathrm{d}X_t}{\mathrm{d}t})\frac{\mathrm{d}C_t}{\mathrm{d}t},$$

respectively. If the conditions (H) are met, the linear growth, Lipschitz and regular conditions hold, then X_t has an inverse uncertainty distribution

$$\Psi_t^{-1}(\alpha) = X_t^\alpha.$$

Proof. Theorem 3.2 says that X_t^α is a continuous function with respect to α at each time $t > 0$. Theorem 3.4 says that

$$\mathcal{M}\{X_t \le X_t^\alpha\} = \alpha$$

for any $\alpha \in (0, 1)$. It follows from Theorem 2.8 that the inverse uncertainty distribution of X_t is $\Psi_t^{-1}(\alpha) = X_t^\alpha$. □

Determining the inverse uncertainty distribution of solutions to second-order UDEs is more complex than that for first-order UDEs. In our analysis, we employ Condition H, which is fundamentally derived from the comparison theorem. Readers can refer to publications on the comparison theorem for higher-order differential equations [19], partial order relations, and quasi-monotone functions [20].

We now turn to the discussion of higher-order UDEs. In Definition 2.7, we defined higher-order UDEs and presented them in the form of a system of Eq (3.10). The first component of the solution to system (3.10) is the solution to the equation as defined in Definition 2.7.

$$\begin{cases} \mathrm{d}X_t = Y_{t,1}\mathrm{d}t, \\ \mathrm{d}Y_{t,1} = Y_{t,2}\mathrm{d}t, \\ \cdots \\ \mathrm{d}Y_{t,n-1} = f(t, X_t, Y_{t,1}, \dots, Y_{t,n-1})\mathrm{d}t + g(t, X_t, Y_{t,1}, \dots, Y_{t,n-1})\mathrm{d}C_t, \\ X_t\Big|_{t=0} = X_0, Y_{t_1}\Big|_{t=0} = Y_1, \dots, Y_{t_{n-1}}\Big|_{t=0} = Y_{n-1}. \end{cases} \tag{3.10}$$

However, since f is an $(n+1)$-ary function, applying the condition of inverse monotonicity becomes more complex and less straightforward to prove. Yet, if our UDE is formulated as in Eq (3.11), we will proceed to provide a proof for the theorem below.

Theorem 3.7. *Let X_t^α be the α-path of a regular uncertain differential equation*

$$\frac{\mathrm{d}^n X_t}{\mathrm{d}t^n} = f(t, X_t) + g(t, X_t)\frac{\mathrm{d}C_t}{\mathrm{d}t}. \tag{3.11}$$

If the linear growth, Lipschitz and regular conditions hold, then X_t^α is a continuous and strictly increasing function with respect to α at each time $t > 0$.

Proof. Since f and g satisfy the linear growth and Lipschitz conditions, the α-path X_t^α is continuous with respect to t. Let Φ^{-1} be the inverse standard normal uncertainty distribution, and let α and β be numbers with $0 < \alpha < \beta < 1$. Write

$$\begin{cases} \mathrm{d}X_t^\alpha = Y_{t,1}^\alpha\mathrm{d}t, \\ \mathrm{d}Y_{t,1}^\alpha = Y_{t,2}^\alpha\mathrm{d}t, \\ \cdots \\ \mathrm{d}Y_{t,n-1}^\alpha = f(t, X_t^\alpha)\mathrm{d}t + g(t, X_t^\alpha)\Phi^{-1}(\alpha)\mathrm{d}t, \\ X_t^\alpha\Big|_{t=0} = X_0, Y_{t,1}^\alpha\Big|_{t=0} = Y_1, \dots, Y_{t,n-1}^\alpha\Big|_{t=0} = Y_{n-1}. \end{cases} \tag{3.12}$$

$$\begin{cases} \mathrm{d}X_t^\beta = Y_{t,1}^\beta\mathrm{d}t, \\ \mathrm{d}Y_{t,1}^\beta = Y_{t,2}^\beta\mathrm{d}t, \\ \cdots \\ \mathrm{d}Y_{t,n-1}^\beta = f(t, X_t^\beta)\mathrm{d}t + g(t, X_t^\beta)\Phi^{-1}(\beta)\mathrm{d}t, \\ X_t^\beta\Big|_{t=0} = X_0, Y_{t,1}^\beta\Big|_{t=0} = Y_1, \dots, Y_{t,n-1}^\beta\Big|_{t=0} = Y_{n-1}. \end{cases} \tag{3.13}$$

$$\mu(T, t) = (T - t)^{n-1}[f(t, X_t^\alpha) + g(t, X_t^\alpha)\Phi^{-1}(\alpha)],$$

$$\nu(T,t) = (T-t)^{n-1}[f(t,X_t^\beta) + g(t,X_t^\beta)\Phi^{-1}(\beta)],$$

$$\mu_1(t) = f(t,X_t^\alpha) + g(t,X_t^\alpha)\Phi^{-1}(\alpha),$$

$$\nu_1(t) = f(t,X_t^\beta) + g(t,X_t^\beta)\Phi^{-1}(\beta).$$

Due to the continuous nature of f and g, and the continuity of X_t^α and Y_t^α, we can infer that the composite functions μ, ν and μ_1, ν_1 are also continuous with respect to t.

Since $g(0, X_0) > 0$ (regular condition), we have

$$\mu(T,0) < \nu(T,0).$$

By the continuity of μ and ν, there exists a small number $r > 0$ such that

$$\mu(T,t) < \nu(T,t), \forall t \in [0,r].$$

By Lemma 3.1, it follows that

$$\begin{aligned} X_T^\alpha &= X_0 + \sum_{k=1}^{n-1} \frac{T^k}{k!} Y_k + \frac{1}{(n-1)!}[\int_0^T (T-t)^{n-1} f(t,X_t^\alpha)\mathrm{d}t + \int_0^T (T-t)^{n-1} g(t,X_t^\alpha)\Phi^{-1}(\alpha)\mathrm{d}t] \\ &< X_0 + \sum_{k=1}^{n-1} \frac{T^k}{k!} Y_k + \frac{1}{(n-1)!}[\int_0^T (T-t)^{n-1} f(t,X_t^\beta)\mathrm{d}t + \int_0^T (T-t)^{n-1} g(t,X_t^\beta)\Phi^{-1}(\beta)\mathrm{d}t] \\ &= X_T^\beta \end{aligned}$$

for any time $T \in (0, r]$.

We will prove $X_t^\alpha < X_t^\beta$ for any $t > r$ by contradiction.

Suppose there exists a time $b > r$ at which X_t^α and X_t^β first meet, i.e.,

$$X_b^\alpha = X_b^\beta, X_t^\alpha < X_t^\beta, \forall t \in (0,b).$$

Since $f(b, X_b^\beta) = f(b, X_b^\alpha), g(b, X_b^\beta) = g(b, X_b^\alpha) > 0$, and we have

$$\mu_1(b) < \nu_1(b).$$

By the continuity of μ_1 and ν_1, there exists a time $a \in (0, b)$ such that

$$\mu_1(t) < \nu_1(t), t \in [a,b].$$

Then, we have

$$\frac{(b-t)^{n-1}}{(n-1)!}\mu_1(t) < \frac{(b-t)^{n-1}}{(n-1)!}\nu_1(t), t \in [a,b).$$

Choose a_1 and b_1 such that $a < a_1 < b_1 < b$. Thus, we have

$$\frac{(b-t)^{n-1}}{(n-1)!}\mu_1(t) < \frac{(b-t)^{n-1}}{(n-1)!}\nu_1(t), t \in [a_1,b_1].$$

Note that any continuous function defined on a closed interval can always reach its minimum. Without loss of generality, suppose that $\nu_1(t)$-$\mu_1(t)$ reaches its minimum on $[a_1, b_1]$ at $t^* \in [a_1, b_1]$. Then, we have the following inequality:

$$\begin{aligned}\int_a^b \frac{(b-t)^{n-1}}{(n-1)!}(\nu_1(t)-\mu_1(t))\mathrm{d}t \geqslant & \int_{a_1}^{b_1} \frac{(b-t)^{n-1}}{(n-1)!}(\nu_1(t)-\mu_1(t))\mathrm{d}t \\ \geqslant & \int_{a_1}^{b_1} \frac{(b-b_1)^{n-1}}{(n-1)!}(\nu_1(t^*)-\mu_1(t^*))\mathrm{d}t \\ = & \frac{(b-b_1)^{n-1}}{(n-1)!}(\nu_1(t^*)-\mu_1(t^*))(b_1-a_1) \\ > & 0.\end{aligned}$$

Thus,

$$\begin{aligned}X_b^\alpha &= X_0 + \sum_{k=1}^{n-1} \frac{(b-a)^k}{k!} Y_k + \frac{1}{(n-1)!}[\int_a^b (b-t)^{n-1} f(t, X_t^\alpha)\mathrm{d}t \\ &+ \int_a^b (b-t)^{n-1} g(t, X_t^\alpha)\Phi^{-1}(\alpha)\mathrm{d}t] \\ &< X_0 + \sum_{k=1}^{n-1} \frac{(b-a)^k}{k!} Y_k + \frac{1}{(n-1)!}[\int_a^b (b-t)^{n-1} f(t, X_t^\beta)\mathrm{d}t \\ &+ \int_a^b (b-t)^{n-1} g(t, X_t^\beta)\Phi^{-1}(\beta)\mathrm{d}t] \\ &= X_b^\beta,\end{aligned}$$

which is in contradiction with $X_b(\gamma) = X_b^\alpha$. The theorem is proved. □

By Theorem 3.7, we have established that X_t is a continuous and strictly increasing function with respect to α. We wonder whether this collection of α-paths can determine the inverse uncertainty distribution of the solutions to higher-order UDEs. The following Theorem 3.8 provides a definitive answer.

Theorem 3.8. *Let X_t and X_t^α be the solution and α-path of the regular uncertain differential equation*

$$\frac{\mathrm{d}^n X_t}{\mathrm{d}t^n} = f(t, X_t) + g(t, X_t)\frac{\mathrm{d}C_t}{\mathrm{d}t}.$$

If the linear growth, Lipschitz and regular conditions hold, then

$$\mathcal{M}\{X_t \le X_t^\alpha, \forall t\} = \alpha,$$

$$\mathcal{M}\{X_t > X_t^\alpha, \forall t\} = 1 - \alpha.$$

Proof. Theorem 14.3 in [12] constructs an event Λ_1 with $\mathcal{M}\{\Lambda_1\} = \alpha$, and shows that for each $\gamma \in \Lambda_1$, there exists a small number $\delta > 0$ such that

$$\frac{C_s(\gamma) - C_t(\gamma)}{s-t} < \Phi^{-1}(\alpha - \delta), \tag{3.14}$$

for any times s and t with $s > t$, where Φ^{-1} is the inverse standard normal uncertainty distribution. Since f and g satisfy the linear growth and Lipschitz conditions, it follows from Theorem 3.1 that X_t^α and $X_t(\gamma)$ exist and are continuous with respect to t. Write the higher-order UDE in the form of a system of equations:

$$\begin{cases} dX_t = Y_{t,1}dt, \\ dY_{t,1} = Y_{t,2}dt, \\ \cdots \\ dY_{t,n-1} = f(t, X_t)dt + g(t, X_t)dC_t, \\ X_t\Big|_{t=0} = X_0, Y_{t_1}\Big|_{t=0} = Y_1, \ldots, Y_{t_{n-1}}\Big|_{t=0} = Y_{n-1}. \end{cases} \tag{3.15}$$

$$\lambda(T, t, s) = (T - t)^{n-1}[f(t, X_t(\gamma)) + g(t, X_t(\gamma))\frac{C_s(\gamma) - C_t(\gamma)}{s - t}],$$

$$\mu(T, t) = (T - t)^{n-1}[f(t, X_t(\gamma)) + g(t, X_t(\gamma))\Phi^{-1}(\alpha - \delta)],$$

$$\nu(T, t) = (T - t)^{n-1}[f(t, X_t^\alpha) + g(t, X_t^\alpha)\Phi^{-1}(\alpha)],$$

$$\lambda_1(t, s) = f(t, X_t(\gamma)) + g(t, X_t(\gamma))\frac{C_s(\gamma) - C_t(\gamma)}{s - t},$$

$$\mu_1(t) = f(t, X_t(\gamma)) + g(t, X_t(\gamma))\Phi^{-1}(\alpha - \delta),$$

$$\nu_1(t) = f(t, X_t^\alpha) + g(t, X_t^\alpha)\Phi^{-1}(\alpha).$$

Due to the continuous nature of f and g, and the continuity of $X_t(\gamma)$, we can infer that the composite functions λ, μ, ν and λ_1, μ_1, ν_1 are also continuous with respect to t.

Since $g(0, X_0) > 0$ (regular condition), we have

$$\mu(T, 0) < \nu(T, 0).$$

By the continuity of μ and ν, there exists a small number $r > 0$ such that

$$\mu(T, t) < \nu(T, t), \forall t \in [0, r].$$

By inequality (3.14), for any time $t \in [0, r]$ and any time $s \in (t, \infty)$, we have

$$\begin{aligned} \lambda(T, t, s) &= (T - t)^{n-1}[f(t, X_t(\gamma)) + g(t, X_t(\gamma))\frac{C_s(\gamma) - C_t(\gamma)}{s - t}] \\ &< (T - t)^{n-1}[f(t, X_t(\gamma)) + g(t, X_t(\gamma))\Phi^{-1}(\alpha - \delta) \\ &= \mu(T, t) < \nu(T, t). \end{aligned}$$

By Lemma 3.1, it follows that

$$\begin{aligned}
X_T(\gamma) &= X_0 + \sum_{k=1}^{n-1} \frac{T^k}{k!} Y_k + \frac{1}{(n-1)!}[\int_0^T (T-t)^{n-1} f(t, X_t(\gamma))\mathrm{d}t \\
&+ \int_0^T (T-t)^{n-1} g(t, X_t(\gamma))\mathrm{d}C_t(\gamma)] \\
&= X_0 + \sum_{k=1}^{n-1} \frac{T^k}{k!} Y_k + \frac{1}{(n-1)!} \lim_{\Delta\to 0} \sum_{i=1}^{k} \lambda(T, t_{i+1}, t_i)(t_{i+1} - t_i) \\
&\le X_0 + \sum_{k=1}^{n-1} \frac{T^k}{k!} Y_k + \frac{1}{(n-1)!} \lim_{\Delta\to 0} \sum_{i=1}^{k} \mu(T, t_i)(t_{i+1} - t_i) \\
&= X_0 + \sum_{k=1}^{n-1} \frac{T^k}{k!} Y_k + \frac{1}{(n-1)!} \int_0^T \mu(T, t)\mathrm{d}t \\
&< X_0 + \sum_{k=1}^{n-1} \frac{T^k}{k!} Y_k + \frac{1}{(n-1)!} \int_0^T \nu(T, t)\mathrm{d}t \\
&= X_0 + \sum_{k=1}^{n-1} \frac{T^k}{k!} Y_k + \frac{1}{(n-1)!} \int_0^T (T-t)^{n-1} f(t, X_t^{\alpha})\mathrm{d}t \\
&+ \frac{1}{(n-1)!} \int_0^T (T-t)^{n-1} g(t, X_t^{\alpha})\Phi^{-1}(\alpha)\mathrm{d}t \\
&= X_T^{\alpha}
\end{aligned}$$

for any time $T \in (0, r]$.

We will prove $X_t(\gamma) < X_t^{\alpha}$ for all $t > r$ by contradiction. Suppose there exists a time $b > r$ at which $X_t(\gamma)$ and X_t^{α} first meet, i.e.,

$$X_b(\gamma) = X_b^{\alpha}, X_t(\gamma) < X_t^{\alpha}, \forall t \in (0, b).$$

Since $f(b, X_b^{\alpha}) = f(b, X_b(\gamma)), g(b, X_b^{\alpha}) = g(b, X_b(\gamma)) > 0$, and we have

$$\mu_1(b) < \nu_1(b).$$

By the continuity of μ_1 and ν_1, there exists a time $a \in (0, b)$ such that

$$\mu_1(t) < \nu_1(t), t \in [a, b].$$

By inequality (3.14), for any time $t \in [a, b)$ and any time $s \in (t, \infty)$, we have

$$\begin{aligned}
(b-t)^{n-1}\lambda_1(t, s) &= (b-t)^{n-1}[f(t, X_t(\gamma)) + g(t, X_t(\gamma))\frac{C_s(\gamma) - C_t(\gamma)}{s-t}] \\
&< (b-t)^{n-1}[f(t, X_t(\gamma)) + g(t, X_t(\gamma))\Phi^{-1}(\alpha - \delta) \\
&= (b-t)^{n-1}\mu_1(t) < (b-t)^{n-1}\nu_1(t).
\end{aligned}$$

By a similar process as in Theorem 3.7, which indicates that the integral of a non-negative function with a positive interval is also positive, we have the following inequality:

$$\begin{aligned}
X_b(\gamma) &= X_0 + \sum_{k=1}^{n-1} \frac{(b-a)^k}{k!} Y_k + \frac{1}{(n-1)!} \int_a^b (b-t)^{n-1} f(t, X_t(\gamma)) \mathrm{d}t \\
&+ \frac{1}{(n-1)!} \int_a^b (b-t)^{n-1} g(t, X_t(\gamma)) \mathrm{d}C_t(\gamma) \\
&= X_0 + \sum_{k=1}^{n-1} \frac{(b-a)^k}{k!} Y_k + \frac{1}{(n-1)!} \lim_{\Delta\to 0} \sum_{i=1}^{k} (b-t_i)^{n-1} \lambda_1(t_{i+1}, t_i)(t_{i+1}-t_i) \\
&\le X_0 + \sum_{k=1}^{n-1} \frac{(b-a)^k}{k!} Y_k + \frac{1}{(n-1)!} \lim_{\Delta\to 0} \sum_{i=1}^{k} (b-t_i)^{n-1} \mu_1(t_i)(t_{i+1}-t_i) \\
&= X_0 + \sum_{k=1}^{n-1} \frac{(b-a)^k}{k!} Y_k + \frac{1}{(n-1)!} \int_a^b (b-t)^{n-1} \mu_1(t) \mathrm{d}t \\
&< X_0 + \sum_{k=1}^{n-1} \frac{(b-a)^k}{k!} Y_k + \frac{1}{(n-1)!} \int_a^b (b-t)^{n-1} \nu_1(t) \mathrm{d}t \\
&= X_0 + \sum_{k=1}^{n-1} \frac{(b-a)^k}{k!} Y_k + \frac{1}{(n-1)!} \int_a^b (b-t)^{n-1} f(t, X_t^\alpha) \mathrm{d}t \\
&+ \frac{1}{(n-1)!} \int_a^b (b-t)^{n-1} g(t, X_t^\alpha) \Phi^{-1}(\alpha) \mathrm{d}t \\
&= X_b^\alpha
\end{aligned}$$

which is in contradiction with $X_b(\gamma) = X_b^\alpha$. Therefore,

$$X_t(\gamma) < X_t^\alpha, \forall t > 0.$$

Since $\mathcal{M}\{\Lambda_1\} = \alpha$, we have

$$\mathcal{M}\{X_t \le X_t^\alpha, \forall t\} \ge \alpha. \tag{3.16}$$

Theorem 14.3 in [1] constructs an event Λ_2 with $\mathcal{M}\{\Lambda_2\} = 1 - \alpha$, and shows that for each $\gamma \in \Lambda_2$, there exists a small number $\delta > 0$ such that

$$\frac{C_s(\gamma) - C_t(\gamma)}{s - t} > \Phi^{-1}(\alpha + \delta) \tag{3.17}$$

for any times s and t with $s > t$. Write

$$\lambda(T, t, s) = (T-t)^{n-1}[f(t, X_t(\gamma)) + g(t, X_t(\gamma)) \frac{C_s(\gamma) - C_t(\gamma)}{s-t}],$$

$$\mu(T, t) = (T-t)^{n-1}[f(t, X_t(\gamma)) + g(t, X_t(\gamma)) \Phi^{-1}(\alpha + \delta)],$$

$$\nu(T, t) = (T-t)^{n-1}[f(t, X_t^\alpha) + g(t, X_t^\alpha) \Phi^{-1}(\alpha)],$$

$$\lambda_1(t, s) = f(t, X_t(\gamma)) + g(t, X_t(\gamma)) \frac{C_s(\gamma) - C_t(\gamma)}{s-t},$$

$$\mu_1(t) = f(t, X_t(\gamma)) + g(t, X_t(\gamma))\Phi^{-1}(\alpha + \delta),$$

$$\nu_1(t) = f(t, X_t^\alpha) + g(t, X_t^\alpha)\Phi^{-1}(\alpha).$$

Since $g(0, X_0) > 0$, we have

$$\mu(T, 0) > \nu(T, 0).$$

By the continuity of μ and ν, there exists a small number $r > 0$ such that

$$\mu(T, t) > \nu(T, t), \forall t \in [0, r].$$

By inequality (3.17), for any time $t \in [0, r]$ and any time $s \in (t, \infty)$, we have

$$\begin{aligned}\lambda(T, t, s) &= (T-t)^{n-1}[f(t, X_t(\gamma)) + g(t, X_t(\gamma))\frac{C_s(\gamma) - C_t(\gamma)}{s-t}]\\ &> (T-t)^{n-1}[f(t, X_t(\gamma)) + g(t, X_t(\gamma))\Phi^{-1}(\alpha + \delta)\\ &= \mu(T, t) > \nu(T, t).\end{aligned}$$

Thus,

$$\begin{aligned}X_T(\gamma) &= X_0 + \sum_{k=1}^{n-1}\frac{T^k}{k!}Y_k + \frac{1}{(n-1)!}\int_0^T (T-t)^{n-1}f(t, X_t(\gamma))\mathrm{d}t\\ &+ \frac{1}{(n-1)!}\int_0^T (T-t)^{n-1}g(t, X_t(\gamma))\mathrm{d}C_t(\gamma)\\ &= X_0 + \sum_{k=1}^{n-1}\frac{T^k}{k!}Y_k + \frac{1}{(n-1)!}\lim_{\Delta\to 0}\sum_{i=1}^{k}(T-t_i)^{n-1}\lambda(T, t_{i+1}, t_i)(t_{i+1}-t_i)\\ &\geq X_0 + \sum_{k=1}^{n-1}\frac{T^k}{k!}Y_k + \frac{1}{(n-1)!}\lim_{\Delta\to 0}\sum_{i=1}^{k}(T-t_i)^{n-1}\mu(T, t_i)(t_{i+1}-t_i)\\ &= X_0 + \sum_{k=1}^{n-1}\frac{T^k}{k!}Y_k + \frac{1}{(n-1)!}\int_0^T \mu(T, t)\mathrm{d}t\\ &> X_0 + \sum_{k=1}^{n-1}\frac{T^k}{k!}Y_k + \frac{1}{(n-1)!}\int_0^T \nu(T, t)\mathrm{d}t\\ &= X_0 + \sum_{k=1}^{n-1}\frac{T^k}{k!}Y_k + \frac{1}{(n-1)!}\int_0^T (T-t)^{n-1}f(t, X_t^\alpha)\mathrm{d}t\\ &+ \frac{1}{(n-1)!}\int_0^T (T-t)^{n-1}g(t, X_t^\alpha)\Phi^{-1}(\alpha)\mathrm{d}t\\ &= X_T^\alpha\end{aligned}$$

for any time $T \in (0, r]$.

We will prove $X_t(\gamma) > X_t^\alpha$ for all $t > r$ by contradiction.

Suppose there exists a time $b > r$ at which $X_t(\gamma)$ and X_t^α first meet, i.e.,

$$X_b(\gamma) = X_b^\alpha, X_t(\gamma) > X_t^\alpha, \forall t \in (0, b).$$

Since $f(b, X_b^\alpha) = f(b, X_b(\gamma)), g(b, X_b^\alpha) = g(b, X_b(\gamma)) > 0$, we have

$$\mu_1(b) > \nu_1(b).$$

By the continuity of μ_1 and ν_1, there exists a time $a \in (0, b)$ such that

$$\mu_1(t) > \nu_1(t), t \in [a, b].$$

By inequality (3.17), for any time $t \in [a, b]$ and any time $s \in (t, \infty)$, we have

$$\begin{aligned}(b-t)^{n-1}\lambda_1(t,s) &= (b-t)^{n-1}[f(t, X_t(\gamma)) + g(t, X_t(\gamma))\frac{C_s(\gamma) - C_t(\gamma)}{s-t}] \\ &> (b-t)^{n-1}[f(t, X_t(\gamma)) + g(t, X_t(\gamma))\Phi^{-1}(\alpha+\delta) \\ &= (b-t)^{n-1}\mu_1(t) > (b-t)^{n-1}\nu_1(t).\end{aligned}$$

Thus,

$$\begin{aligned}X_b(\gamma) &= X_0 + \sum_{k=1}^{n-1}\frac{(b-a)^k}{k!}Y_k + \frac{1}{(n-1)!}\int_a^b (b-t)^{n-1} f(t, X_t(\gamma))\mathrm{d}t \\ &+ \frac{1}{(n-1)!}\int_a^b (b-t)^{n-1} g(t, X_t(\gamma))\mathrm{d}C_t(\gamma) \\ &= X_0 + \sum_{k=1}^{n-1}\frac{(b-a)^k}{k!}Y_k + \frac{1}{(n-1)!}\lim_{\Delta\to 0}\sum_{i=1}^{k}(b-t)^{n-1}\lambda_1(t_{i+1}, t_i)(t_{i+1}-t_i) \\ &\geq X_0 + \sum_{k=1}^{n-1}\frac{(b-a)^k}{k!}Y_k + \frac{1}{(n-1)!}\lim_{\Delta\to 0}\sum_{i=1}^{k}(b-t)^{n-1}\mu_1(t_i)(t_{i+1}-t_i) \\ &= X_0 + \sum_{k=1}^{n-1}\frac{(b-a)^k}{k!}Y_k + \frac{1}{(n-1)!}\int_a^b (b-t)^{n-1}\mu_1(t)\mathrm{d}t \\ &> X_0 + \sum_{k=1}^{n-1}\frac{(b-a)^k}{k!}Y_k + \frac{1}{(n-1)!}\int_a^b (b-t)^{n-1}\nu_1(t)\mathrm{d}t \\ &= X_0 + \sum_{k=1}^{n-1}\frac{(b-a)^k}{k!}Y_k + \frac{1}{(n-1)!}\int_a^b (b-t)^{n-1} f(t, X_t^\alpha)\mathrm{d}t \\ &+ \frac{1}{(n-1)!}\int_a^b (b-t)^{n-1} g(t, X_t^\alpha)\Phi^{-1}(\alpha)\mathrm{d}t \\ &= X_b^\alpha,\end{aligned}$$

which is in contradiction with $X_b(\gamma) = X_b^\alpha$. Therefore,

$$X_t(\gamma) > X_t^\alpha, \forall t > 0.$$

Since $\mathcal{M}\{\Lambda_2\} = 1 - \alpha$, we have

$$\mathcal{M}\{X_t > X_t^\alpha, \forall t\} \geq 1 - \alpha. \tag{3.18}$$

It follows from (3.16), (3.18) and

$$\mathcal{M}\{X_t \leq X_t^\alpha, \forall t\} + \mathcal{M}\{X_t > X_t^\alpha, \forall t\} \leq 1$$

that

$$\mathcal{M}\{X_t \le X_t^\alpha, \forall t\} = \alpha,$$

$$\mathcal{M}\{X_t > X_t^\alpha, \forall t\} = 1 - \alpha.$$

hold. □

Theorem 3.9. *Let X_t and X_t^α be the solution and α-path of the regular uncertain differential equation*

$$\frac{\mathrm{d}^n X_t}{\mathrm{d}t^n} = f(t, X_t) + g(t, X_t)\frac{\mathrm{d}C_t}{\mathrm{d}t}.$$

If the linear growth, Lipschitz and regular conditions hold, then at any time $t > 0$, we have

$$\mathcal{M}\{X_t \le X_t^\alpha\} = \alpha,$$

$$\mathcal{M}\{X_t > X_t^\alpha\} = 1 - \alpha.$$

Proof. The proof follows a similar process to Theorem 3.5. □

Theorem 3.10. *Let X_t and X_t^α be the solution and α-path of the uncertain differential equation*

$$\frac{\mathrm{d}^n X_t}{\mathrm{d}t^n} = f(t, X_t) + g(t, X_t)\frac{\mathrm{d}C_t}{\mathrm{d}t},$$

respectively. If the linear growth, Lipschitz and regular conditions hold, then X_t has an inverse uncertainty distribution

$$\Psi_t^{-1}(\alpha) = X_t^\alpha.$$

Proof. The proof follows a similar process to Theorem 3.6. □

4. Numerical examples

Moving forward, we will provide examples to illustrate the significance of Condition (H).

Example 2. *For the uncertain Duffing system, the equation is given by*

$$m\frac{\mathrm{d}^2 X_t}{\mathrm{d}t^2} + \delta\frac{\mathrm{d}X_t}{\mathrm{d}t} + \alpha X_t + \beta X_t^3 - \zeta\cos\omega(t) - \gamma\frac{\mathrm{d}C_t}{\mathrm{d}t} = 0.$$

The initial values are $X_{t_0}(\gamma) = X_{t_0}^\alpha = 0.1$. For convenience, the parameters involved are illustrated in Table 1.

The uncertain external excitation represents the influence of climate on the object within the system. Figure 4 is a schematic diagram of the Duffing system, where the spring is a hardening spring.

We have

$$f(t, X_t, \frac{\mathrm{d}X_t}{\mathrm{d}t}) = \cos(t) - \frac{\mathrm{d}X_t}{\mathrm{d}t} - X_t - X_t^3, g(t, X_t, \frac{\mathrm{d}X_t}{\mathrm{d}t}) = 1.$$

It is straightforward to deduce that

$$\frac{\partial f(t, X_t, \frac{\mathrm{d}X_t}{\mathrm{d}t})}{\partial X_t} = -1 - 3X_t^2 < 0, \frac{\partial g(t, X_t, \frac{\mathrm{d}X_t}{\mathrm{d}t})}{\partial X_t} = 0.$$

Table 1. Parameter defintions and their corresponding values.

Parameters	Interpretation	Value
m	Mass	1kg
δ	Damping coefficient	1Ns/m
E	Elastic modulus	1000Pa
A	Cross-sectional area	0.01m^2
L	Length	0.1m
α	Linear stiffness coefficient	1N/m
β	Nonlinear stiffness coefficient	1N/m^3
ζ	Excitation coefficient	1N
ω	Excitation frequency	1rads/s
γ	Uncertain external excitation coefficient	1Ns/m

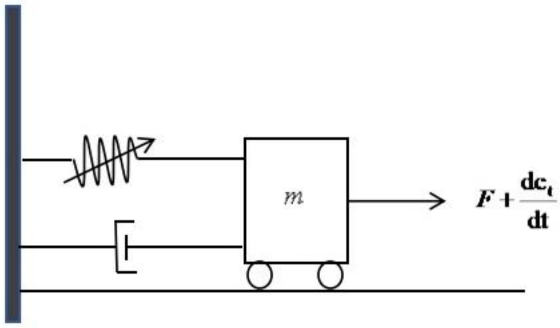

Figure 4. System schematic diagram.

Upon observing Figure 5, it can be seen that the α-path for $\alpha = 0.1$ and $\alpha = 0.2$ intersect at time t_1. The set of solutions X_t^α derived from the family of ordinary differential equations cannot substitute for the inverse uncertainty distribution of the solution to the UDE.

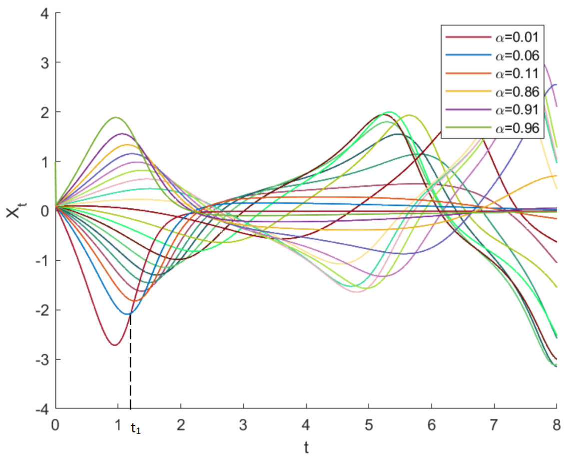

Figure 5. α-path of the Uncertain Duffing Equation of a hardening spring.

For $\alpha < 0$, the spring is a softening spring, where $\beta < 0$, the uncertain Duffing system is for

$$\frac{\mathrm{d}^2X_t}{\mathrm{d}t^2}+\frac{\mathrm{d}X_t}{\mathrm{d}t}-X_t-X_t^3-\cos(t)-\frac{\mathrm{d}C_t}{\mathrm{d}t}=0.$$

The initial values are $X_t(\gamma)\Big|_{t=0}=X_t^\alpha\Big|_{t=0}=0.1,\frac{\mathrm{d}X_t(\gamma)}{\mathrm{d}t}\Big|_{t=0}=\frac{\mathrm{d}X_t^\alpha}{\mathrm{d}t}\Big|_{t=0}=0.1$. *And we have*

$$f(t,X_t,\frac{\mathrm{d}X_t}{\mathrm{d}t})=\cos(t)-\frac{\mathrm{d}X_t}{\mathrm{d}t}+X_t+X_t^3, g(t,X_t,\frac{\mathrm{d}X_t}{\mathrm{d}t})=1.$$

Verify condition (H),

$$\frac{\partial f(t,X_t,\frac{\mathrm{d}X_t}{\mathrm{d}t})}{\partial X_t}=1+3X_t^2>0, \frac{\partial g(t,X_t,\frac{\mathrm{d}X_t}{\mathrm{d}t})}{\partial X_t}=0.$$

We define the region $G\in[0,100]\times[0,10]\times[0,10]$. *It is straightforward to determine that the maximum value of the function* f *is denoted as* $M=1100$, *which satisfies the conditions provided in the remarks. Furthermore, as depicted in Figure 6, there are no intersecting* α*-path. Hence, the set of solutions* X_t^α *derived from the family of ordinary differential equations corresponding to the* α*-path can be used to represent the inverse uncertainty distribution of the UDE.*

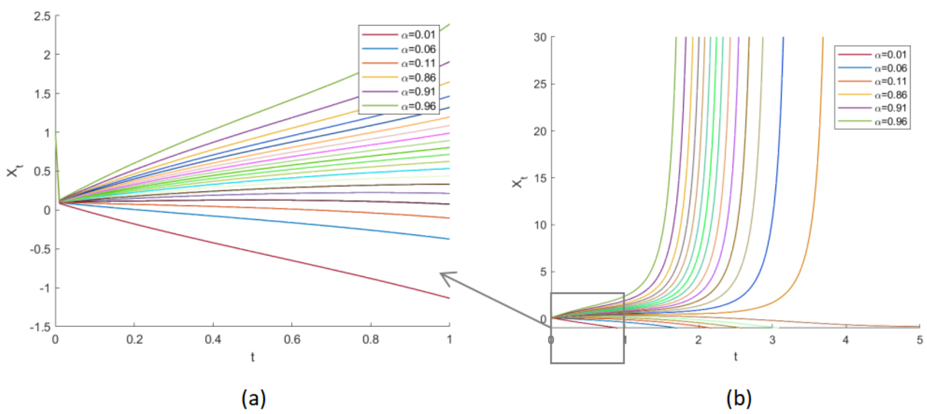

Figure 6. α-path of the Uncertain Duffing Equation of a softening spring.

Example 3. *The function g represents a commonly utilized activation function within neural networks. Its second-order uncertain differential equation is given by*

$$\frac{\mathrm{d}^2X_t}{\mathrm{d}t^2}-\frac{\mathrm{d}X_t}{\mathrm{d}t}-X_t-X_t^3-\cos(t)-\frac{1}{1+\exp(-X_t)}\frac{\mathrm{d}C_t}{\mathrm{d}t}=0.$$

The initial values are $X_t(\gamma)\Big|_{t=0}=X_t^\alpha\Big|_{t=0}=0.1,\frac{\mathrm{d}X_t(\gamma)}{\mathrm{d}t}\Big|_{t=0}=\frac{\mathrm{d}X_t^\alpha}{\mathrm{d}t}\Big|_{t=0}=0.1$, *where*

$$f(t,X_t,\frac{\mathrm{d}X_t}{\mathrm{d}t})=\cos(t)-\frac{\mathrm{d}X_t}{\mathrm{d}t}+X_t+X_t^3, g(t,X_t,\frac{\mathrm{d}X_t}{\mathrm{d}t})=\frac{1}{1+\exp(-X_t)},$$

$$\frac{\partial f(t,X_t,\frac{\mathrm{d}X_t}{\mathrm{d}t})}{\partial X_t}=1+3X_t^2>0, \frac{\partial g(t,X_t,\frac{\mathrm{d}X_t}{\mathrm{d}t})}{\partial X_t}=\frac{\exp(-X_t)}{(1+\exp(-X_t))^{-2}}>0.$$

We define the region $G\in[0,100]\times[0,10]\times[0,10]$. *It is easy to establish that the maximum value of the functions* f *and* g *is* $M=1100$, *which meets the conditions specified in the notes. Additionally, as* *shown in Figure 7, there are no intersections among the α-paths, whether in the short term (a) or the long term (b). So the set of solutions* X_t^α *derived from the α-path for a family of ordinary differential equations can be used to represent the inverse uncertainty distribution of the solution to the UDE.*

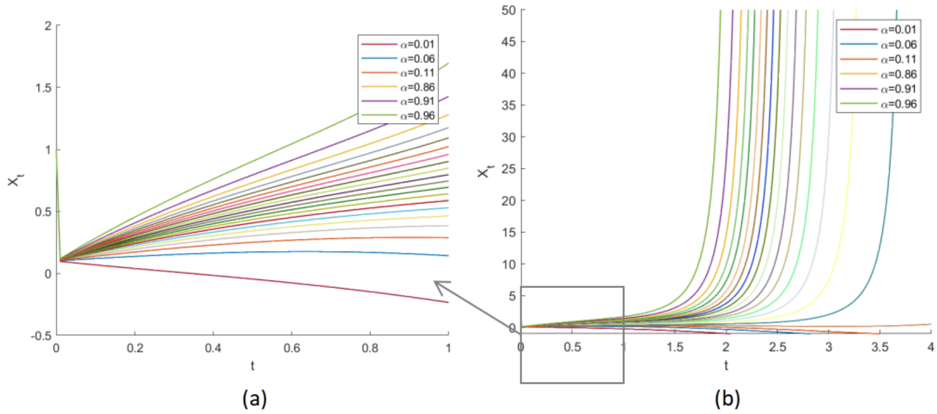

Figure 7. α-path of the UDE in Example 2.

Example 4.

$$\frac{d^5X_t}{dt^5} - \sin(t) - X_t - \frac{dC_t}{dt} = 0.$$

The initial values are

$$X_t(\gamma)\Big|_{t=0} = X_t^\alpha\Big|_{t=0} = 0.1, X_t^{(k)}(\gamma)\Big|_{t=0} = X_t^{\alpha(k)}\Big|_{t=0} = 0.1, k = 1, 2, \ldots, 5.$$

$$f(t, X_t) = \mathrm{s}(t) + X_t, g(t, X_t) = 1.$$

We define the region $G \in [0, 100] \times [0, 10]$*. It is easy to establish that the maximum value of the function* f *is* $M = 11$*, which meets the conditions specified in the notes. The α-path diagram is depicted in Figure 8, from which it is observable that the individual α-path do not intersect at any point in time. Thus, the set of solutions* X_t^α *derived from the α-path for a family of ordinary differential equations can be used to represent the inverse uncertainty distribution of the UDE.*

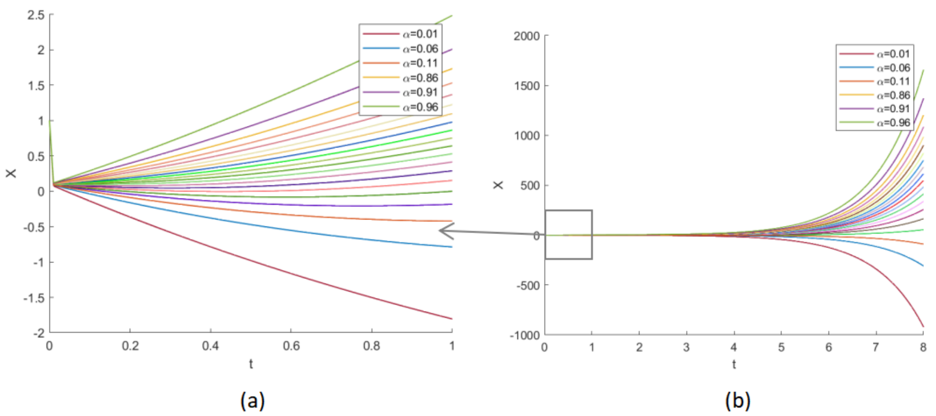

Figure 8. α-path of the UDE in Example 3.

5. Conclusions

The research presented in this paper is of significant importance and has broad applicability in practical applications. If the correct inverse uncertainty distribution of the solutions to UDEs cannot be obtained, it would be impossible to delve into the study of the behavior of UDEs and their solutions. Consequently, the construction of such UDE models would be meaningless.

This paper presents for the first time the inverse uncertainty distribution for second-order and a class of higher-order UDEs. (1) For second-order UDEs, if the conditions (H) are met and the linear growth, Lipschitz and regular conditions hold, we can determine the inverse uncertainty distribution of the second-order UDEs. (2) For a class of higher-order UDEs, if the linear growth, Lipschitz and regular conditions hold, we can similarly ascertain the inverse uncertainty distribution of the second-order UDEs. Based on the theorems presented in this paper, it is possible to conduct uncertain dynamical analyses of these types of UDEs, which constitutes a foundational work in the field of uncertainty theory. It is undeniable that these conditions have certain limitations. Scholars can continue to explore the statistical properties and other applications of the solutions to UDEs to potentially relax these constraints.

Based on our research, there are three areas for potential future exploration: (1) It merits investigation whether the sufficient conditions for determining the inverse unncertainty distribution of solutions to second-order or higher-order UDEs can be expanded and whether there exist necessary and sufficient conditions. (2) While uncertainty theory and this paper discuss the transition from ordinary differential equations to UDEs, our research group is currently working on more complex equations, such as uncertain fractional-order differential equations and uncertain functional differential equations with constant delays. We have conceived a potential sufficient condition for the α-paths of systems of UDEs to determine the inverse uncertainty distribution of their solutions. Additionally, uncertain partial differential equations also warrant investigation. (3) There is a multitude of differential equations that satisfy our conditions, with numerous examples provided in the final version under submission. On this basis, more in-depth issues can be explored, such as uncertain dynamics, uncertain bifurcation, and uncertain chaos.

Author contributions

Zeman Wang: Conceptualization, Data curation, Formal analysis, Investigation, Methodology, Software, Validation, Writing–original draft, Writing–review & editing; Qiubao Wang: Conceptualization, Funding acquisition, Project administration, Supervision, Validation, Writing–review & editing; Zhong Liu: Data curation, Formal analysis; Zikun Han: Data curation, Investigation; Xiuying Guo: Investigation, Supervision. All authors have read and approved the final version of the manuscript for publication.

Acknowledgments

This work was supported by the Innovation and Entrepreneurship Curriculum Construction in Hebei Province: Ordinary Differential Equations; Key Project of Natural Science Foundation of Hebei Province (Basic Discipline Research) (A2023210064).

Conflict of interest

All authors declare no conflict of interest that could affect the publication of this paper.

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