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Research article

On the Rayleigh-Taylor instability for the two coupled fluids

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Abstract: In this paper, we considered the Rayleigh-Taylor (RT) instability for two incompressible, immisicible, invisid coupled fluids, which were Euler and magnetohydrodynamic with zero resistivity. Under the action of the uniform gravitational field, the two fluids interacted at a free interface. We utilized the flow map to denote the Lorentz force under the Lagrangian coordinates. We first showed the ill-posedness to the linear problem around the RT steady state solution. By virtue of such an ill-posed result, we showed that the nonlinear system is also ill-posed.

Keywords: two coupled fluids; RT instability; incompressible fluids
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1. Introduction

In this paper we are devoted to the following Euler and magnetohydrodynamics coupled system in Ω:

$$\begin{cases} \rho_+\partial_t u_+ + \rho_+ u_+\cdot\nabla u_+ + div(p_+ I) = -g\rho_+ e_3, \\ div u_+ = 0. \end{cases} \quad \text{in} \quad \Omega_+(t) \tag{1.1}$$

and

$$\begin{cases} \rho_-\partial_t u_- + \rho_- u_-\cdot\nabla u_- + div(p_- I - h_-\otimes h_-) = -g\rho_- e_3, \\ \partial_t h_- + u_-\cdot\nabla h_- - h_-\cdot\nabla u_- = 0, \\ div u_- = 0, div h_- = 0, \end{cases} \quad \text{in} \quad \Omega_-(t) \tag{1.2}$$

where $\Omega = \mathbb{R}^2\times(-1,1)\subset\mathbb{R}^3$ is divided into Ω_- and Ω_+ by a moving free surface $\Sigma(t)$. As shown in the above systems (1.1) and (1.2), the "upper fluid" is called Euler fluid, which is occupying Ω_+, and the "lower fluid", which is occupying Ω_-, is magnetohydrodynamics fluid. We use $(u_\pm, p_\pm, h_-)$ to describe the fluid velocity, pressure, and magnetic field. The subscript "±"refers to "upper/lower" fluid. I is the identity matrix, $\rho_\pm$ denotes the densities of the respective fluids, $g > 0$ is the gravitational constant, and $e_3 = (0, 0, 1)$.

The conditions on $\Sigma(t)$ are as follows:

$$\begin{cases} [u \cdot \nu]|_{\Sigma(t)} = 0, \\ h_- \cdot \nu|_{\Sigma(t)} = 0, \\ [(p + g\rho x_3)\nu]|_{\Sigma(t)} = (h_- \otimes h_-)\nu|_{\Sigma(t)}, \end{cases} \tag{1.3}$$

where ν is the normal vector of $\Sigma(t)$.

At the fixed boundary $x_3 = \pm 1$, we impose the conditions:

$$u_+(t, x_1, x_2, 1) \cdot e_3 = u_-(t, x_1, x_2, -1) \cdot e_3 = 0, \tag{1.4}$$

for any $t \geq 0$, $(x_1, x_2) \in \mathbb{R}^2$.

In order to overcome the mathematical difficulties brought about by the evolution of the free interface over time, the Lagrangian coordinates are introduced. Define the following reversible maps:

$$\varphi^0_\pm : \Omega_\pm \longrightarrow \Omega_\pm(0), \tag{1.5}$$

satisfying $\Sigma_0 = \varphi^0_\pm\{x_3 = 0\}$ and $\{x_3 = \pm 1\} = \varphi^0_\pm\{x_3 = \pm 1\}$. $\varphi^0_\pm$ are continuous across $\{x_3 = 0\}$. Define invertible flow maps $\varphi_\pm$ which solve

$$\begin{cases} \partial_t \varphi_\pm(t, x) = u_\pm(t, \varphi_\pm(t, x)), \\ \varphi_\pm(0, x) = \varphi^0_\pm(x). \end{cases} \tag{1.6}$$

In this paper, (t, y) with $y = \varphi(t, x)$ and $(t, x) \in \mathbb{R}^+ \times \Omega$ denote Eulerian coordinates and Lagrangian coordinates, respectively. Since the two-layer fluids may slip each other, the slip map must be introduced. Define $S_\pm : \mathbb{R}^2 \times \mathbb{R}^+ \to \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^2 \times (-1, 1)$ by

$$S_-(t, x_1, x_2) = \varphi_-^{-1}(t, \varphi_+(t, x_1, x_2, 0)), \tag{1.7}$$

Now, we define the corresponding unknown functions in the Lagrangian coordinate

$$\begin{cases} v_\pm(t, x) = u_\pm(t, \varphi_\pm(t, x)), \\ b_-(t, x) = h_-(t, \varphi_-(t, x)), \qquad (t, x) \in \mathbb{R}^+ \times \Omega. \\ q_\pm(t, x) = p_\pm(t, \varphi_\pm(t, x)), \end{cases} \tag{1.8}$$

Denote by $A_\pm := ((D\varphi_\pm)^{-1})^T$, where D is the derivative of the coordinates x and superscript T is the matrix transpose. Then, the evolution equations for $v_\pm, b_-, q_\pm, \varphi_\pm$ become

$$\begin{cases} \partial_t \varphi^i_+ = v^i_+, \\ \rho_+ \partial_t v^i_+ + A^{ik}_+ \partial_k q_+ = 0, \\ A^{jk}_+ \partial_k v^j_+ = 0, \\ \partial_t \varphi^i_- = v^i_-, \\ \rho_- \partial_t v^i_- + A^{ik}_- \cdot \partial_k q_- = b^j_- A^{ik}_- \partial_k b^i_-, \\ A^{jk}_- \partial_k v^j_- = 0, \\ \partial_t b^i_- = b^j_- A^{jk}_- \partial_k v^i_-, \\ A^{jk}_- \cdot \partial_k b^j_- = 0. \end{cases} \tag{1.9}$$

In the above system, we have used the Einstein summation convention. The corresponding conditions on $\Sigma(t)$ are

$$\begin{cases}(v_+(t,x_1,x_2,0)-v_-(t,S_-(t,x_1,x_2)))\cdot \nu(t,x_1,x_2,0)=0,\\ (q_+(t,x_1,x_2,0)-q_-(t,S_-(t,x_1,x_2)))\cdot \nu(t,x_1,x_2,0)\\ =g[\rho]\varphi_+^3(t,x_1,x_2)\nu(t,x_1,x_2,0)-(b_-\otimes b_-)(t,S_-(t,x_1,x_2))\nu(t,x_1,x_2,0),\end{cases} \tag{1.10}$$

where

$$\nu=\frac{\partial_1\varphi_+\times\partial_2\varphi_+}{|\partial_1\varphi_+\times\partial_2\varphi_+|}, \tag{1.11}$$

is the unit normal vector to the interface $\Sigma(t)=\varphi_+(t,\{x_3=0\})$, and φ_+^3 is the third component of φ_+. Finally, we require the impermeability conditions

$$v_-(t,x_1,x_2,-1)\cdot e_3=v_+(t,x_1,x_2,1)\cdot e_3=0. \tag{1.12}$$

In the Lagrangian coordinates, the magnetic field b_- can be expressed by virtue of φ_- as in [1,2]. Applying A_-^{il} to the seventh equation of (1.9), we achieve

$$\begin{aligned}A_-^{il}\partial_t b_-^i&=A_-^{il}b_-^jA_-^{jk}\partial_k v_-^i\\&=A_-^{il}b_-^jA_-^{jk}(\partial_t\partial_k\varphi_-^i)\\&=-b_-^i\partial_tA_-^{il}.\end{aligned}$$

Thus, we have $\partial_t(A_-^{il}b_-^i)=0$, which implies $A_-^{il}b_-^i=A_-^{il,0}b_-^{i,0}$ and

$$b_-^i=\partial_l\varphi_-^iA_-^{jl,0}b_-^{j,0}. \tag{1.13}$$

Now, we check the last equation of (1.9). Applying the geometric identities, we have

$$J=J^0\quad\text{and}\quad\partial_k(JA_-^{ik})=0,$$

where $J=|D\varphi|$. Utilizing $A_-^{ik}\partial_k$ to (1.13), one gets

$$\begin{aligned}A_-^{ik}\partial_k b_-^i&=\frac{J}{J^0}A_-^{ik}\partial_k(\partial_l\varphi_-^iA_-^{jl,0}b_-^{j,0})\\&=\frac{1}{J_0}\partial_k(JA_-^{ik}\partial_l\varphi_-^iA_-^{jl,0}b_-^{j,0})-\frac{1}{J_0}\partial_k(JA_-^{ik})\partial_l\varphi_-^iA_-^{jl,0}b_-^{i,0}\\&=\frac{1}{J_0}\partial_k(JA_-^{jk,0}b_-^{j,0})=\frac{1}{J_0}\partial_k(J_0A_-^{jk,0}b_-^{j,0})\\&=\frac{J_0}{J_0}\partial_k(A_-^{jk,0}b_-^{j,0})=\partial_k(A_-^{jk,0}b_-^{j,0})=A_-^{jk,0}\partial_kb_-^{j,0}.\end{aligned} \tag{1.14}$$

The compatibility conditions for the initial value are imposed as follows:

$$A_-^{jk,0}\partial_kb^{j,0}=0. \tag{1.15}$$

Combining (1.14), we have

$$A_-^{jk}\partial_kb_-^j=0,\quad\text{for all}\quad 0\le t\le T. \tag{1.16}$$

For simplicity, we assume that

$$A_-^{il,0}b_{i,0} = \bar{M}_l. \tag{1.17}$$

By virtue of (1.13) and (1.17), we can use the forcing term by the flow map φ_- to represent the Lorentz term in the fifth equation of (1.9). Thus, (1.9) becomes a two-fluids Navier-stokes system:

$$\begin{cases} \partial_t\varphi_\pm^i = v_\pm^i, \\ \rho\partial_t v_+^i + A_+^{ik}\partial_k q_+ = 0, \\ \rho_-\partial_t v_-^i + A_-^{ik}\partial_k q_- - \bar{M}_l\bar{M}_r\partial_{lr}^2\varphi_-^i = 0, \\ A_\pm^{jk}\partial_k v_\pm^j = 0, \end{cases} \tag{1.18}$$

where the magnetic field $\bar{M}$ can be considered as a vector parameter.

The conditions (1.10) can be expressed as

$$\begin{aligned} &[q_+(t,x_1,x_2,0) - q_-(t,S_-(t,x_1,x_2))]v_i(t,x_1,x_2,0) \\ =&g[\rho]\varphi_+^3(t,x_1,x_2,0)v_i(t,x_1,x_2,0) - \bar{M}_l\bar{M}_m(\partial\varphi_-^i\partial_m\varphi_-^j)(t,S_-(t,x_1,x_2))v_j(t,x_1,x_2,0). \end{aligned} \tag{1.19}$$

The boundary conditions are the same as (1.12).

We have known that $v_\pm = 0$, $\varphi_\pm = Id$, $q_\pm = const$ are steady -state solutions to the systems (1.18), (1.19), and (1.12). Then, $v = e_3$, $A = Id$, $S_- = Id_{\{x_3=0\}}$. The linearized equation system near the steady-state solution is

$$\begin{cases} \partial_t\varphi_\pm = v_\pm, \\ \rho_+\partial_t v_+ + \nabla q_+ = 0, \\ \rho_-\partial_t v_- + \nabla q_- - \bar{M}_l\bar{M}_m\partial_{lm}^2\varphi_- = 0, \\ divv_\pm = 0. \end{cases} \tag{1.20}$$

The corresponding jump and fixed boundary conditions are

$$[[v\cdot e_3]] = 0, [[q]]e_3 = g[\rho]\varphi^3 e_3 - \bar{M}_3\bar{M}_l\partial_l\varphi, \tag{1.21}$$

$$v_-(t,x_1,x_2,-1)\cdot e_3 - v_+(t,x_1,x_2,1)\cdot e_3 = 0, \tag{1.22}$$

where $[[\cdot]]$ denotes the interfacial jump quantity on the boundary $\{x_3 = 0\}$. Our aim is to study the Rayleigh-Taylor (RT) problem, so we suppose

$$\rho_+ > \rho_- \Leftrightarrow [\rho] > 0. \tag{1.23}$$

RT instability is a ubiquitous phenomenon in nature, widely existing in various research fields such as astrophysics, atmospheric and oceanic science, laser fusion, and magnetic confinement fusion [3–6]. Before further discussion, we first review some results with regard to the RT instability problems. The studies on the RT instability can be traced back to the pioneering work due to Rayleigh [7] and Taylor [8]. From then on, many interesting physical phenomena and numerical simulations come from both physical and numerical experiments. Li and Luo [9] studied the effect of a vertical magnetic field on the RT instability of 2d nonideal magnetic fluids by constructing numerical solutions. We refer to [10] and references therein for a general research of the physics about RT instability. However, there are only very few analytical results from the mathematical point of view. Recently, Guo and Tice [11, 12] studied the linear and nonlinear RT instability for Euler and Navier-Stokes fluids by the variational method or the modified variational method. In these papers, they discovered that the viscosity and surface tension have an impact on the RT instability. When considering the magnetic field, the RT instability appears by the Lorentz force. The theoretical discussion about the influence of magnetic fields was proposed by Kruskal and Schwarzchild in [13]. They found that the horizontal magnetic field can affect the development of RT instability but cannot suppress the growth of instability. Jiang et al. [1, 14–16] used the similar method as [11, 12] and employed the new techniques to discuss the RT instability for magnetohydrodynamics (MHD) fluids, as well as revealed the magnetic effect to the instability. In this paper we consider the mechanism for the effect of the magnetic field in the ideal fluid and magnetohydrodynamic coupled through the free interface.

2. Notations and main results

We first introduce some definitions that are applicable throughout the paper. Define the horizontal Fourier transform for a function $g \in L^2(\Omega)$ as follows:

$$\hat{g}(\xi_1, \xi_2, x_3) = \int_{\mathbb{R}^2} g(x_1, x_2, x_3) e^{-i(x_1\xi_1 + x_2\xi_2)} dx_1 dx_2. \tag{2.1}$$

Due to the Fubini and Parseval theorems, one has that

$$\int_{\Omega} \|g(x)\|^2 dx = \frac{1}{4\pi^2} \int_{\Omega} \|\hat{g}(\xi, x_3)\|^2 d\xi dx_3. \tag{2.2}$$

Define the piecewise Sobolev space $H^s(\Omega)$ for any $s \in \mathbb{R}$ as follows:

$$H^s(\Omega) = \{g | g_+ \in H^s(\Omega_+), g_- \in H^s(\Omega_-)\}$$

equipped with the following norm:

$$\|g\|^2_{H^s(\Omega)} = \|g\|^2_{H^s(\Omega_+)} + \|g\|^2_{H^s(\Omega_-)},$$

and

$$\begin{aligned} \|g\|^2_{H^k(\Omega_\pm)} &:= \sum_{j=0}^{k} \int_{\mathbb{R}^2 \times I_\pm} (1 + |\xi|^2)^{k-j} |\partial^j_{x_3} \hat{g}_\pm(\xi, x_3)|^2 d\xi dx_3 \\ &= \sum_{j=0}^{k} \int_{\mathbb{R}^2} (1 + |\xi|^2)^{k-j} \|\partial^j_{x_3} \hat{g}_\pm(\xi, x_3)\|^2_{L^2(I_\pm)} d\xi, \end{aligned} \tag{2.3}$$

for $I_- = (-1, 0)$ and $I_+ = (0, 1)$.

Next, we will give the main theorems. The first one is concerned with the linearized systems (1.20)–(1.22).

Theorem 2.1. *Give a constant vector $\bar{M} = (M, 0, 0)$, then for any k, the linear systems* (1.20)–(1.22) *are ill-posed in $H^k(\Omega)$. To be precise, for any fixed $k, j \in N$ with $j \geq k$, $T_0 > 0$, and $\alpha > 0$,* (1.20)–(1.22) *have the solutions $\{(\varphi_n, v_n, q_n)\}_{n=1}^{\infty}$ which satisfy*

$$\|\varphi_n(0)\|_{H^j} + \|v_n(0)\|_{H^j} + \|q_n(0)\|_{H^j} \leq \frac{1}{n}, \tag{2.4}$$

but

$$\|v_n(t)\|_{H^k} \geq \|\varphi_n(t)\|_{H^k} \geq \alpha, \quad \textit{for all} \quad t \geq T_0. \tag{2.5}$$

Remark 2.2. The ill-posedness in the above theorem implies that the solutions to the linear systems (1.20)–(1.22) established in Theorem 3.6 depend disconstinuously on the initial conditions.

With the linear instability in hand, there is the nonlinear instability as follows:

Theorem 2.3. *For any $k \geq 4$, the perturbed problem* (4.2)–(4.6) *does not have the property EE(k).*

Remark 2.4. We can extend the conclusions in Theorems 2.1 and 2.3 to the general horizontal magnetic field $\bar{M} = (M_1, M_2, 0)$. In practice, since the L^2- norm of the velocity remains unchanged under the horizontal rotation, one may rotate the coordinates so that $\bar{M} = (M, 0, 0)$ with $M = \sqrt{M_1^2 + M_2^2}$.

The paper is arranged as follows. In Section 1, we introduce the Lagrangian coordinates and linearize the nonlinear system. Some notations and main results are given in Section 2. In Section 3 we establish the growing mode solution to the linearized system and prove the uniqueness of the solution and discontinuous dependence on the initial value. In the last section, we investigate the ill-posedness of the nonlinear system.

3. Ill-posedness of linearized problems (1.20)

When discussing the posedness of linearized Eqs (1.20)–(1.22), studying normal mode solutions is a standard practice. To this end, for some $\lambda > 0$, suppose a normal mode ansatz as follows:

$$v_\pm(t,x) = e^{\lambda t} w_\pm(x), q_\pm(t,x) = e^{\lambda t}\tilde{q}_\pm(x), \varphi_\pm(t,x) = e^{\lambda t}\tilde{\varphi}_\pm(x). \tag{3.1}$$

Substituting the above ansatz into the systems (1.20)–(1.22) and eliminating the unknown $\tilde{\varphi}_\pm$ by using $(1.20)_1$ and $(1.20)_3$, we arrive at the following system:

$$\begin{cases} \lambda\rho_+ w_+ + \nabla\tilde{q}_+ = 0, \\ \lambda\rho_- w_- + \nabla\tilde{q}_- - \dfrac{1}{\lambda}\bar{M}_l\bar{M}_m\partial^2_{lm}w_- = 0, \\ divw_\pm = 0. \end{cases} \tag{3.2}$$

At the same time, the jump and boundary conditions become

$$[[w^3]] = 0, [[\tilde{q}]]e_3 = \frac{1}{\lambda}g[\rho]w^3e_3 - \frac{1}{\lambda}\bar{M}_3\bar{M}_l\partial_l w, \tag{3.3}$$

and

$$w^3_+(x_1,x_2,1) = w^3_-(x_1,x_2,-1) = 0. \tag{3.4}$$

Since the coefficients in (3.2) depend only on the x_3 variable, we can adopt the horizontal Fourier transformation to (3.2) to reduce them into ordinary differential equations (ODEs) in terms of x_3 with each spatial frequency as parameters. Define

$$\kappa_\pm, \psi_\pm, \theta_\pm, \pi_\pm : (-1,1) \to \mathbb{R},$$

so that

$$\kappa_\pm(x_3) = i\hat{w}^1_\pm(\xi_1, \xi_2, x_3),$$

$$\psi_\pm(x_3) = i\hat{w}^2_\pm(\xi_1, \xi_2, x_3),$$

$$\theta_\pm(x_3) = \hat{w}^3_\pm(\xi_1, \xi_2, x_3),$$

and

$$\pi_\pm(x_3) = \hat{\hat{q}}(\xi_1, \xi_2, x_3).$$

Then, we have

$$\mathcal{F}(divw_\pm) = \xi_1\phi_\pm + \xi_2\psi_\pm + \theta'_\pm, \tag{3.5}$$

where $\mathcal{F}$ means the Fourier transformation and $' = \frac{d}{dx_3}$.

Note that we only consider $\bar{M} = (M, 0, 0)$, and make the Fourier transform for (3.2), then we achieve the following system of ODEs:

$$\begin{cases} \lambda\rho_+\kappa_+ - \xi_1\pi_+ = 0, \\ \lambda\rho_+\psi_+ - \xi_2\pi_+ = 0, \\ \lambda\rho_+\theta_+ + \pi'_+ = 0, \\ \lambda^2\rho_-\kappa_- - \lambda\xi_1\pi_- + M^2\xi_1^2\kappa_- = 0, \\ \lambda^2\rho_-\psi_- - \lambda\xi_2\pi_- + M^2\xi_1^2\psi_- = 0, \\ \lambda^2\rho_-\theta_- + \lambda\pi'_- + M^2\xi_1^2\theta_- = 0, \\ \xi_1\kappa_\pm + \xi_2\psi_\pm + \theta'_\pm = 0, \end{cases} \tag{3.6}$$

subject to the jump conditions

$$[[\theta]] = 0, [[\lambda\pi]] = g[\rho]\theta(0), \tag{3.7}$$

and corresponding fixed boundary conditions

$$\theta_-(-1) = 0, \theta_+(1) = 0. \tag{3.8}$$

Eliminating $\pi_\pm$ from the Eq (3.6), one has

$$\begin{cases} \lambda^2\rho_+(|\xi|^2\theta_+ - \theta''_+) = 0, \\ \lambda^2\rho_-(|\xi|^2\theta_- - \theta''_-) = B^2\xi_1^2(|\xi|^2\theta_- - \theta''_-). \end{cases} \tag{3.9}$$

Equations (3.7) and (3.8) become

$$[[\theta]] = 0, \lambda^2[[\rho\theta']] - B^2\xi_1^2\theta'_- + g[\rho]|\xi|^2\theta = 0, \tag{3.10}$$

$$\theta_-(-1) = 0, \theta_+(1) = 0. \tag{3.11}$$

In what follows, we will devote ourselves to build a solution for (3.9)–(3.11) based on the variational method, which deduces a solution for the system (3.6)–(3.8). Then, we will derive an exponential growth solution of time for the system (1.20)–(1.22).

Multiply θ_+, θ_- to $(3.9)_1$ and $(3.9)_2$, add the resulting equations, and integrate over $(0,1)$ and $(-1,0)$, respectively. After integration by parts, we get

$$-\frac{1}{2}\lambda^2\int_{-1}^{1}\rho(|\xi|^2|\theta|^2+|\theta'|^2)dx_3=\frac{1}{2}[\int_{-1}^{0}B^2\xi_1^2(|\xi|^2|\theta_-|^2+|\theta_-'|^2)dx_3-g[\rho]|\xi|^2\theta^2(0)], \tag{3.12}$$

where we used boundary and jump conditions. We would like to find a growing mode solution to the system (3.2), which requires that there exists $\lambda > 0$. One can utilize the variational method to look for the smallest value μ as follows:

$$\begin{aligned}\mu &= \mu(|\xi|)\\ &= inf\{\frac{1}{2}[\int_{-1}^{0}B^2\xi_1^2(|\xi|^2|\theta_-|^2+|\theta_-'|^2)dx_3-g[\rho]|\xi|^2\theta^2(0)]\Big|\int_{-1}^{1}\rho(|\xi|^2|\theta|^2+|\theta'|^2)dx_3=2\}.\end{aligned} \tag{3.13}$$

Define

$$E(\theta)=\frac{1}{2}[\int_{-1}^{0}B^2\xi_1^2(|\xi|^2|\theta_-|^2+|\theta_-'|^2)dx_3-g[\rho]|\xi|\theta^2(0)], \tag{3.14}$$

and

$$J(\theta)=\frac{1}{2}\int_{-1}^{1}\rho(|\xi|^2|\theta|^2+|\theta'|^2)dx_3. \tag{3.15}$$

It is convenient to introduce the set $\mathcal{A}$

$$\mathcal{A}=\{\theta\in H_0^1(-1,1)|J(\theta)=1\}.$$

For any $|\xi|>0$, let

$$-\lambda^2=\inf_{\theta\in\mathcal{A}}E(\theta)<0,$$

which is equivalent to

$$-\lambda^2=\inf_{\theta\in H_0^1(-1,1)}\frac{E(\theta)}{J(\theta)}. \tag{3.16}$$

We want to find the minimizer of E on the set $\mathcal{A}$ and show the existence and negativity of the infimum.

Proposition 3.1. *E can obtain the infimum on $\mathcal{A}$ for any fixed $|\xi|\geq 0$. If θ is a minimizer and $-\lambda^2 := E(\theta)$, then (θ,λ^2) solves* (3.9) *with* (3.10) *and* (3.11). *Moreover, θ is smooth when limited to* $(-1,0)$ *or* $(0,1)$.

Proof. For any $\theta\in\mathcal{A}$, we estimate $E(\theta)$ as follows:

$$\begin{aligned}E(\theta) &\geq -\frac{1}{2}g[\rho]|\xi|^2|\theta(0)|^2\\ &= -\frac{1}{2}|\xi|g[\rho]|\xi|\int_{-1}^{0}\partial_{x_3}|\theta_-|^2dx_3\\ &\geq -|\xi|g[\rho]\frac{1}{2}\int_{-1}^{0}(|\xi|^2|\theta_-|^2+|\theta_-'|^2)dx_3\\ &\geq -\frac{g[\rho]}{\rho_-}|\xi|.\end{aligned} \tag{3.17}$$

Therefore, E has a lower bound on $\mathcal{A}$. Take $\theta_n \in \mathcal{A}$ as a minimizing sequence, then we get the boundedness of θ_n in $H_0^1(-1,1)$, which implies that there exists $\theta \in H_0^1(-1,1)$ to guarantee that θ_n is weakly convergent to θ in $H_0^1(-1,1)$ and strongly convergent in $L^2(-1,1)$. Thus, we have

$$E(\theta) \leq \liminf_{n\to\infty} E(\theta_n) = \inf_{\mathcal{A}} E. \tag{3.18}$$

Thus, E takes the infimum over $\mathcal{A}$ and θ is a minimizer.

For $s \in \mathbb{R}$ and any $\theta_0 \in H_0^1(-1,1)$, define $\theta(s) = \theta + s\theta_0$, then

$$E(\theta(s)) + \lambda^2 J(\theta(s)) \geq 0, \tag{3.19}$$

follows from (3.16). Let $L(s) = E(\theta(s)) + \lambda^2 J(\theta(s))$, then there is $L(s) \geq 0$ for any $s \in \mathbb{R}$ and $L(0) = 0$. This leads to $L'(0) = 0$. By virtue of (3.14) and (3.15), we derive

$$\begin{aligned} L'(0) \quad = \quad & \int_{-1}^{0} B^2\xi_1^2(|\xi|^2\theta_- \cdot (\theta_0)_- + \theta_-' \cdot (\theta_0)_-')dx_3 - g[\rho]|\xi|^2\theta(0)\theta_0(0) \\ & +\lambda^2 \int_{-1}^{1} \rho(|\xi|^2\theta \cdot \theta_0 + \theta' \cdot \theta_0')dx_3 = 0. \end{aligned} \tag{3.20}$$

By selecting θ_0 with compact support in either $(-1,0)$ or $(0,1)$, one can get that θ solves Eq (3.9) in a weak sense. By standard bootstrap arguments, we may demonstrate that $\theta_- \in H^k(-1,0)$(resp., $\theta_- \in H^k(0,1)$) for all $k \geq 0$ and, hence, it is smooth when limited to the respective interval. This means that $\theta_\pm$ are classical solutions to the Eq (3.9). The remainder is to show that (3.10) is established. For each $\theta_0 \in C_c^\infty(-1,1)$, we obtain

$$(\lambda^2[[\rho\theta']] - B^2\xi_1^2\theta_-' + g[\rho]|\xi|^2\theta)\theta_0(0) = 0. \tag{3.21}$$

Since $\theta_0(0)$ can be chosen arbitrary, we yield the second jump condition in (3.10). The conditions $[[\theta]] = 0$ and $\theta_-(-1) = \theta_+(1) = 0$ are satisfied trivially since $\theta \in H_0^1(-1,1) \hookrightarrow C_0^{0,\frac{1}{2}}(-1,1)$. □

Remark 3.2. (3.17) implies $-\lambda^2 = \inf_{\theta\in\mathcal{A}} E(\theta) \geq -\frac{g[\rho]}{\rho_-}|\xi|$ and, hence,

$$\lambda \leq \sqrt{\frac{g[\rho]}{\rho_-}|\xi|}. \tag{3.22}$$

Corollary 3.3. *For any* $|\xi| > 0$*, system* (3.6) *has a solution* $(\kappa_\pm, \psi_\pm, \theta_\pm, \pi_\pm)$ *with* $\lambda = \lambda(|\xi|) > 0$ *. Moreover, this solution satisfies* (3.7) *and* (3.8) *and is smooth when limited to* $(-1,0)$ *or* $(0,1)$.

Proof. By solving (3.6), we get

$$\begin{aligned} \pi_+ &= \frac{-\lambda\rho_+\theta_+'}{|\xi|^2}, \qquad \pi_- = \frac{-(\lambda^2\rho_- + M^2\xi_1^2)\theta_-'}{\lambda|\xi|^2}, \\ \kappa_\pm &= -\frac{\xi_1\theta_\pm'}{|\xi|^2}, \qquad \psi_\pm = \frac{\xi_2\theta_\pm'}{|\xi|^2}. \end{aligned} \tag{3.23}$$

From Proposition 3.1, it is obvious that $\pi_\pm = \pi_\pm(\xi, x_3), \theta_\pm = \theta_\pm(\xi, x_3)$, and $\psi_\pm = \psi_\pm(\xi, x_3)$ are smooth over the interval $(-1,0)$ or $(0,1)$. Furthermore, the jump and boundary conditions (3.7) and (3.8) are satisfied. □

Lemma 3.4. *Let R_1, ξ_1 satisfy*

$$\frac{e^{2R_1}-1}{e^{2R_1}+1} \geq \frac{1}{2}, \quad \text{and} \quad |\xi_1| < \frac{g[\rho]}{4M^2} < R_1, \tag{3.24}$$

then the eigenvalue $\lambda = \lambda(|\xi|)$ satisfies

$$\lambda \geq \sqrt{\frac{g[\rho]}{\rho_+ + \rho_-}|\xi|}. \tag{3.25}$$

Proof. Denote $\bar{\theta}$ by

$$\bar{\theta}(x_3) = \begin{cases} e^{|\xi|x_3} - e^{|\xi|(2-x_3)} & x_3 \in [0,1), \\ e^{-|\xi|x_3} - e^{|\xi|(2+x_3)} & x_3 \in (-1,0), \end{cases} \tag{3.26}$$

then

$$E(\bar{\theta}) = \frac{1}{2}|\xi|[M^2\xi_1^2(e^{4|\xi|}-1) - g[\rho]|\xi|(1-e^{2|\xi|})^2],$$

$$J(\bar{\theta}) = \frac{1}{2}(\rho_+ + \rho_-)(e^{4|\xi|}-1)|\xi|,$$

so

$$\begin{aligned} \frac{E(\bar{\theta})}{J(\bar{\theta})} &= |\xi|(\frac{M^2\xi_1^2}{(\rho_+ + \rho_-)|\xi|} - \frac{g[\rho](e^{2|\xi|}-1)}{(\rho_+ + \rho_-)(e^{2|\xi|}+1)}) \\ &\leq |\xi|\frac{1}{\rho_+ + \rho_-}(\frac{g[\rho]}{4} - \frac{g[\rho]}{2}) \\ &= -\frac{g[\rho]}{4(\rho_+ + \rho_-)}|\xi|. \end{aligned}$$

Since $-\lambda^2 = \inf_{\theta \in H_0^1(-1,1)} \frac{E(\theta)}{J(\theta)}$, the result follows. □

Define

$$\mathbb{D} := \{\xi = (\xi_1, \xi_2) | |\xi_1| < \frac{g[\rho]}{4M^2}, |\xi| > R_1\}. \tag{3.27}$$

Obviously, $\mathbb{D}$ is a symmetrical domain.

Lemma 3.5. *Let $\xi \in \mathbb{D}$, $\kappa_\pm, \psi_\pm, \theta_\pm$, and $\pi_\pm$ be the solutions to* (3.6) *constructed in Corollary 3.3 , then for each $k \geq 0$, the following inequalities are valid:*

$$\|\theta(\xi)\|_{H^k(-1,1)} \leq A_k \sum_{j=0}^{k} |\xi|^{j-\Delta(j)}, \tag{3.28}$$

$$\|\kappa(\xi)\|_{H^k(-1,1)} + \|\psi(\xi)\|_{H^k(-1,1)} + \|\pi(\xi)\|_{H^k(-1,1)} \leq B_k \sum_{j=0}^{k} |\xi|^j, \tag{3.29}$$

where

$$\Delta(j) = \begin{cases} 0, & \text{if} \quad j = 0, \\ 1, & \text{if} \quad j \neq 0. \end{cases}$$

Moreover,

$$\sqrt{\|\kappa\|^2_{L^2(-1,1)}+\|\psi\|^2_{L^2(-1,1)}+\|\theta\|^2_{L^2(-1,1)}}\geq D, \tag{3.30}$$

where $A_k, B_k, D>0$ are constants depending on ρ, M, R_1, and g.

Proof. $\theta(\xi)\in\mathcal{A}$ implies that there are constants $A_0, A_1>0$ so that

$$\|\theta\|_{L^2(-1,1)}\leq A_0, \|\theta\|_{H^1(-1,1)}\leq A_1.$$

By (3.9), we have

$$|\xi|^2\theta_\pm=\theta''_\pm. \tag{3.31}$$

Thus,

$$\|\theta''\|^2_{L^2(-1,1)}=|\xi|\||\xi|\theta\|^2_{L^2(-1,1)}\leq A_2|\xi|, \tag{3.32}$$

where we used $\theta\in\mathcal{A}$. Combining (3.31) and (3.32), we arrive at

$$\|\theta^{(k+1)}\|^2_{L^2(-1,1)}\leq A_{k+1}|\xi|^k,\ \text{for any}\quad k\geq 0,$$

which verifies (3.28). Employing (3.23) with $|\xi|\geq R_1$, we get

$$\|\theta^{(k)}\|_{L^2(-1,1)}+\|\psi^{(k)}\|_{L^2(-1,1)}\leq\frac{2}{|\xi|}\|\theta^{(k)}\|_{L^2(-1,1)}\leq B_k|\xi|^k, \tag{3.33}$$

for any $k\geq 0$. By virtue of the expression of π on (3.23), (3.22), and (3.25), with $|\xi|\geq R_1$, one has

$$\begin{aligned}
&\|\pi^{(k)}_-\|_{L^2(-1,0)}+\|\pi^{(k)}_+\|_{L^2(0,1)}\\
=\ &\frac{\lambda\rho_+}{|\xi|^2}\|\theta^{(k+1)}_+\|_{L^2(0,1)}+\frac{\lambda^2\rho_-+M^2\xi_1^2}{\lambda|\xi|^2}\|\theta^{(k+1)}_-\|_{L^2(-1,0)}\\
\leq\ &\frac{\sqrt{\frac{g[\rho]}{\rho_-}}\rho_+}{|\xi|^{\frac{3}{2}}}\|\theta^{(k+1)}_+\|_{L^2(0,1)}+\Big(\frac{\sqrt{\frac{g[\rho]}{\rho_-}}\rho_+}{|\xi|^{\frac{3}{2}}}+\frac{2M^2\sqrt{\frac{g[\rho]}{\rho_++\rho_-}}}{|\xi|^{\frac{1}{2}}}\Big)\|\theta^{(k+1)}_-\|_{L^2(-1,0)}\\
\leq\ &B_k|\xi|^k.
\end{aligned} \tag{3.34}$$

Combining (3.33) and (3.34), one can achieve (3.29).

Equation (3.30) follows from that for any fixed $|\xi|>0$, $\theta(|\xi|)\in\mathcal{A}$, and (3.23). □

In Corollary 3.3, we have achieved the solution to (1.20) for the fixed spatial frequency $\xi\in\mathbb{R}^2$. In rest of this section, we will establish the solution to (1.20) by using Fourier synthesis.

Theorem 3.6. *Let* $1\leq R_1\leq R_2<R_3<\infty$ *with* R_1 *satisfy* (3.24). *Suppose a real-valued and radial symmetric function $f\in C_0^\infty(\mathbb{R}^2)$ and $B(0,R_2)\subset supp(f)\subset B(0,R_3)$. For $\xi\in\mathbb{R}^2$, define*

$$\hat{w}(\xi,x_3)=-i\kappa(\xi,x_3)e_1-i\psi(\xi,x_3)e_2+\theta(\xi,x_3)e_3, \tag{3.35}$$

where κ,ψ,θ,π are the solutions constructed in Proposition 3.1 and Corollary 3.3 with $\lambda(\xi)>0$.

Denote

$$\varphi(t,x)=\frac{1}{4\pi^2}\int_{\mathbb{R}^2}f(\xi)\hat{w}(\xi,x_3)e^{\lambda(\xi)t}e^{ix'\xi}d\xi, \tag{3.36}$$

$$v(t,x)=\frac{1}{4\pi^2}\int_{\mathbb{R}^2}\lambda(\xi)\hat{w}(\xi,x_3)e^{\lambda(\xi)t}e^{ix'\xi}d\xi, \tag{3.37}$$

$$q(t,x)=\frac{1}{4\pi^2}\int_{\mathbb{R}^2}\lambda(\xi)f(\xi)\pi(\xi,x_3)e^{\lambda(\xi)t}e^{ix'\xi}d\xi, \tag{3.38}$$

where $x'\cdot\xi=x_1\xi_1+x_2\xi_2$, then (φ,v,q) is a real-valued solution to the linearized problem (1.20) *with the corresponding conditions. For any $k\in\mathbb{N}$, the following inequality is valid:*

$$\|\varphi(0)\|_{H^k}+\|v(0)\|_{H^k}+\|q(0)\|_{H^k}\le\tilde{C}_k(\int_{\mathbb{R}^2}(1+|\xi|^2)^{k+1}|f(\xi)|^2d\xi)^{\frac{1}{2}}<\infty, \tag{3.39}$$

in which the positive constant $\tilde{C}_k$ depends on $\rho,|M|,R_1$, and g. Moreover, $\varphi(t),v(t),q(t)\in H^k(\Omega_\pm)$ for every $t>0$ satisfies the following estimates:

$$\begin{cases} e^{t\sqrt{\bar{c}_2R_2}}\|\varphi(0)\|_{H^k}\le\|\varphi(t)\|_{H^k}\le e^{t\sqrt{\bar{c}_1R_3}}\|\varphi(0)\|_{H^k},\\ e^{t\sqrt{\bar{c}_2R_2}}\|v(0)\|_{H^k}\le\|v(t)\|_{H^k}\le e^{t\sqrt{\bar{c}_1R_3}}\|v(0)\|_{H^k},\\ e^{t\sqrt{\bar{c}_2R_2}}\|q(0)\|_{H^k}\le\|q(t)\|_{H^k}\le e^{t\sqrt{\bar{c}_1R_3}}\|q(0)\|_{H^k},\end{cases} \tag{3.40}$$

where $\bar{c}_1=\frac{g[\rho]}{\rho_-},\bar{c}_2=\frac{g[\rho]}{4(\rho_++\rho_-)}$.

Proof. Fix $\xi\in\mathbb{R}$, and

$$\begin{aligned}\varphi(t,x)&=f(\xi)\hat{w}(\xi,x_3)e^{\lambda(\xi)t}e^{ix'\cdot\xi},\\ v(t,x)&=\lambda(\xi)f(\xi)\hat{w}(\xi,x_3)e^{\lambda(\xi)t}e^{ix'\cdot\xi},\\ q(t,x)&=\lambda(\xi)f(\xi)\pi(\xi,x_3)e^{\lambda(\xi)t}e^{ix'\cdot\xi},\end{aligned}$$

are solutions to (1.20). Due to $B(0,R_2)\subset supp(f)\subset B(0,R_3)$, the following inequalities follow from Lemma 3.5:

$$\sup_{\xi\in supp(f)}\|\partial_{x_3}^k\hat{w}(\xi,\cdot)\|_{L^\infty}<\infty,$$

and

$$\sup_{\xi\in supp(f)}\|\partial_{x_3}^k\pi(\xi,\cdot)\|_{L^\infty}<\infty,$$

for every $k\in\mathbb{N}$.

Meanwhile, $\lambda(\xi)\le\sqrt{\frac{g[\rho]}{\rho_-}|\xi|}$. This boundedness indicates that the functions given by (3.36)–(3.38) are also a solution to (1.20).

For any $k\ge0$, by applying Lemma 3.5, and where f is compactly supported, we easily achieve the estimate (3.39). According to (3.22) and (3.25), one has

$$0<\sqrt{\bar{c}_2R_2}\le\sqrt{\frac{g[\rho]}{4(\rho_++\rho_-)}|\xi|}\le\lambda(|\xi|)\le\sqrt{\frac{g[\rho]}{\rho_-}|\xi|}\le\sqrt{\bar{c}_1R_3},$$

which derives the bounds (3.40). □

Now, we will study the ill-posedness for the linearized problem. Suppose that (φ, v, q) is the solution to (1.20). Further, assume that the solution is band-limited at radius $R > 0$, that is,

$$\bigcup_{x_3\in(-1,1)} \text{supp}(|\hat{\varphi}(\cdot, x_3)| + |\hat{v}(\cdot, x_3)| + |\hat{q}(\cdot, x_3)|) \subset B(0, R).$$

Since in the lower fluid the equation in (1.20) has Lorenz force, it is appropriate to use the second-derivative of velocity. Differentiating the second equation and the fifth Eq (1.20) in time and removing φ_- by using the fourth equation, one arrives at

$$\begin{cases} \rho_+\partial_{tt}v_+ + \nabla\partial_t q_+ = 0, \\ \rho_-\partial_{tt}v_- + \nabla\partial_t q_- - M^2\partial_{11}^2 v_- = 0, \\ divv_\pm = div(\partial_t v_\pm) = 0, \end{cases} \tag{3.41}$$

where we used $\bar{M} = (M, 0, 0)$. We impose the conditions:

$$[[v^3]] = [[\partial_t v^3]] = 0, \quad [[\partial_t q]]e_3 = g[\rho]v^3 e_3, \tag{3.42}$$

$$\partial_t v_-^3(t, x_1, x_2, -1) = \partial_t v_+^3(t, x_1, x_2, 1) = 0. \tag{3.43}$$

$\partial_t v(0)$ satisfies

$$\begin{cases} \rho_+\partial_t v_+(0) = -\nabla q_+(0), \\ \rho_-\partial_t v_-(0) = -\nabla q_-(0) + M^2\partial_{11}^2\varphi_-(0). \end{cases} \tag{3.44}$$

The first result is about the estimate of energy in terms of v for the evolution Eq (3.41).

Lemma 3.7. *For solutions to* (3.41)–(3.43)*, we have*

$$\frac{1}{2}\frac{d}{dt}\Big(\int_\Omega \rho|\partial_t v|^2 dx + \int_{\mathbb{R}^2\times(-1,0)} M^2|\partial_1 v_-|^2 dx - \int_{\mathbb{R}^2} g[\rho]|v_3(x', 0)|^2 dx'\Big) = 0. \tag{3.45}$$

Proof. Multiply $(3.41)_1$ and $(3.41)_2$ by $\partial_t v_\pm(t)$ and integrate over $\Omega_\pm$, respectively. After integration by parts and employing $(3.41)_3$, we arrive at

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_+} \rho_+|\partial_t v_+|^2 dx - \int_{\mathbb{R}^2} \partial_t q_+ \partial_t v_+^3\big|_{x_3=0} dx' = 0, \tag{3.46}$$

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_-} \rho_-|\partial_t v_-|^2 dx - \int_{\mathbb{R}^2} \partial_t q_- \partial_t v_-^3\big|_{x_3=0} dx' + \frac{1}{2}\frac{d}{dt}\int_{\Omega_-} M^2|\partial_1 v_-|^2 dx = 0. \tag{3.47}$$

Adding (3.46) and (3.47), using $(3.42)_2$, we yield (3.7). □

Lemma 3.8. *If v satisfies that $v \in H^1(\Omega)$ is band-limited at radius $R > 0$, $divv = 0$, and $v^3(t, x_1, x_2, \pm1) = 0$, then we arrive at*

$$\int_{\mathbb{R}^2} g[\rho]|v_3(x', 0)|^2 dx' \leq (R^2 + 1)g[\rho]\int_\Omega |v|^2 dx. \tag{3.48}$$

Proof. Utilizing the horizontal Fourier transform to $divv = 0$ and denoting

$$\kappa(x_3) = i\hat{v^1}(\xi_1, \xi_2, x_3), \psi(x_3) = i\hat{v^2}(\xi_1, \xi_2, x_3) \quad \text{and} \quad \theta(x_3) = \hat{v^3}(\xi_1, \xi_2, x_3), \tag{3.49}$$

we have

$$\xi_1\kappa + \xi_2\psi + \theta' = 0. \tag{3.50}$$

From (2.2), (3.49), and (3.50), one has

$$\begin{aligned}
\int_{\mathbb{R}^2} g[\rho]|v^3(x',0)|^2 dx' &= \frac{1}{4\pi^2}\int_{\mathbb{R}^2} g[\rho]|\theta(0)|^2 d\xi \\
&= \frac{g[\rho]}{4\pi^2}\int_{\mathbb{R}^2}\int_0^1 \partial_{x_3}|\theta(0)|^2 dx_3 d\xi \\
&\leq \frac{g[\rho]}{4\pi^2}\int_{\mathbb{R}^2}\int_{-1}^1 (|\theta|^2 + |\theta'|^2) dx_3 d\xi \\
&= \frac{g[\rho]}{4\pi^2}\int_{\mathbb{R}^2}\int_{-1}^1 (|\theta|^2 + |\xi_1\kappa|^2 + |\xi_2\psi|^2) dx_3 d\xi \\
&\leq \frac{g[\rho]}{4\pi^2}\int_{\mathbb{R}^2}\int_{-1}^1 (|\theta|^2 + R^2|\kappa|^2 + R^2|\psi|^2) dx_3 d\xi \\
&= g[\rho]\int_{\mathbb{R}^2}\int_{-1}^1 (|v^3|^2 + R^2|v^1|^2 + R^2|v^2|^2) dx \\
&\leq (R^2+1)g[\rho]\int_{\mathbb{R}} |v|^2 dx,
\end{aligned}$$

which gives (3.48). □

We may now derive growth estimates for $v(t)$ and $\partial_t v(t)$.

Proposition 3.9. *If v is a solution to* (3.41) *and is also band-limited at radius $R > 0$, the following estimate holds:*

$$\|v(t)\|^2_{L^2(\Omega)} + \|\partial_t v(t)\|^2_{L^2(\Omega)} \leq ce^{(\frac{(R^2+1)g[\rho]}{\rho_-}+1)t}(\|v(0)\|_{H^1(\Omega)} + \|\partial_t v(0)\|_{L^2(\Omega)}), \tag{3.51}$$

where c depends on ρ, M, g, R.

Proof. Integrate (3.7) with regard to time from 0 to t to achieve

$$\begin{aligned}
&\textstyle\int_\Omega \rho|\partial_t v|^2 dx + \int_{\mathbb{R}^2\times(-1,0)} M^2|\partial_1 v_-|^2 dx - \int_{\mathbb{R}^2} |v^3(t,x',0)|^2 dx' \\
&= \textstyle\int_\Omega \rho|\partial_t v(0)|^2 dx + \int_{\mathbb{R}^2\times(-1,0)} M^2|\partial_1 v_-(0)|^2 dx - \int_{\mathbb{R}^2} g[\rho]|v^3(0,x',0)dx'.
\end{aligned}$$

Thus, we have

$$\int_\Omega \rho|\partial_t v|^2 dx \leq A + \int_{\mathbb{R}^2} g[\rho]|v^3(t,x',0)|^2 dx', \tag{3.52}$$

where

$$A = \int_\Omega \rho|\partial_t v(0)|^2 dx + \int_{\mathbb{R}^2\times(-1,0)} M^2|\partial_1 v_-(0)|^2 dx \leq \rho_+\|\partial_t v(0)\|_{L^2(\Omega)} + M^2\|\partial_1 v(0)\|^2_{L^2}. \tag{3.53}$$

We apply (3.48) to (3.52) to get the inequality

$$\int_\Omega \rho|\partial_t v|^2 dx \le A + (R^2+1)g[\rho]\int_\Omega |v|^2 dx,$$

which implies

$$\|\partial_t v(t)\|^2 \le \frac{A}{\rho_-} + \frac{(R^2+1)g[\rho]}{\rho_-}\int_\Omega |v|^2 dx. \tag{3.54}$$

By virtue of the Cauchy-Schwartz inequality, one can show that

$$\begin{aligned}\partial_t\|v(t)\|^2 = 2\langle\partial_t v(t), v(t)\rangle &\le \|\partial_t v(t)\|_{L^2(\Omega)}\\ &\le \frac{A}{\rho_-} + (\frac{(R^2+1)g[\rho]}{\rho_-}+1)\|v(t)\|^2_{L^2(\Omega)},\end{aligned} \tag{3.55}$$

where $\langle\cdot,\cdot\rangle$ denotes the L^2 inner product. Applying the Gronwall inequality to (3.55), we derive

$$\|v(t)\|^2_{L^2(\Omega)} \le e^{(\frac{(R^2+1)g[\rho]}{\rho_-}+1)t}(\|v(0)\|^2_{L^2(\Omega)} + \frac{A}{(R^2+1)g[\rho]}). \tag{3.56}$$

Combining (3.55) and (3.56), we obtain

$$\|\partial_t v(t)\|^2_{L^2(\Omega)} + \|v(t)\|^2_{L^2(\Omega)} \le ce^{(\frac{(R^2+1)g[\rho]}{\rho_-}+1)t}(\|\partial_t v(0)\|^2_{L^2(\Omega)} + \|v(0)\|^2_{H^1(\Omega)}),$$

where c depends $\rho_\pm, g, R, M$. □

We have shown the existence of the solutions to the Eq (1.20). To investigate the ill-posedness, we turn to verify the uniqueness and discontinuous dependence on the initial conditions of the solutions. To do this, we first build a projection operator related to the horizontal spatial frequency. Let Φ be a function that is infinitely differentiable and a compact support in $\mathbb{R}^2$, satisfies $\Phi \in [0,1], supp(\Phi) \subset B(0,1)$, and $\Phi(x) = 1$ for $x \in B(0,\frac{1}{2})$, then define $\Phi_R(x) = \Phi(\frac{x}{R})$ for $R > 0$. For $f \in L^2(\Omega)$, the projection operator P_R is defined by

$$P_R f = \mathcal{F}^{-1}(\Phi_R \mathcal{F} f). \tag{3.57}$$

It is easy to show that P_R verifies the following properties [11]:

(1) $P_R f$ is band-limited at radius R;

(2) P_R is a bounded linear operator on $H^k(\Omega)$ for all $k \ge 0$;

(3)P_R commutes with partial differential and multiplication by functions depending only on x_3;

(4) $P_R f = 0$ for all $R > 0$ if, and only if, $f = 0$.

Theorem 3.10. *Solutions to* (1.20) *are unique.*

Proof. We only need to prove that when the initial data is zero, the solutions to (1.20) are also zero. Assume that $\eta_\pm, v_\pm, q_\pm$ solve (1.20) with zero initial conditions. For any fixed $R > 0$, define $\eta_R = P_R\eta, v_R = P_R v, q_R = P_R q$, then η_R, v_R, and q_R also solve (1.20). Moreover, v_R also solves (3.41) with zero initial value. By virtue of (3.51), for any $t \ge 0$, we derive

$$\|v_R(t)\|_{L^2(\Omega)} = \|\partial_t v_R(t)\|_{L^2(\Omega)} = 0. \tag{3.58}$$

Thus, there is $\varphi_R(t) = q_R(t) = v_R(t) = 0$ for all $t \ge 0$. Due to the arbitrariness of R , we have that $\varphi(t) = v(t) = q(t) = 0$ for all $t \ge 0$. □

Lastly, we will show that the solution to the problem (1.20) is discontinuously dependent on the initial data.

Now, let us complete the proof of Theorem 2.1.

Proof. Fix $j \geq k \geq 0, \alpha > 0, T_0 > 0$. Let positive constants $\tilde{C}_k, R_1, D$ come from Lemma 3.5 and Theorem 3.6. For every $n \in \mathbb{N}$, take $R(n)$ large enough such that $R(n) > R_1$, $\sqrt{\frac{g[\rho]}{4(\rho_+ + \rho_-)}R(n)} > 1$, and

$$\frac{exp(T_0\sqrt{\frac{g[\rho]}{4(\rho_+ + \rho_-)}R(n)})}{(1 + (R(n) + 1)^2)^{j-k+1}} \geq \alpha^2 n^2 \frac{\bar{C}_j^2}{D^2}. \tag{3.59}$$

Choose a family of real-valued, radial, and compact supported functions f_n as f in (3.36)–(3.38) so that $B(0, R(n)) \subset supp(f_n) \subset B(0, R(n) + 1)$ and

$$\int_{\mathbb{R}^2} (1 + |\xi|^2)^{j+1} |f_n(\xi)|^2 d\xi = \frac{1}{\bar{C}_j^2 n^2}. \tag{3.60}$$

Take $R_2 = R(n)$ and $R_3 = R(n) + 1$ in Theorem 3.6 to get $\varphi_n(t), v_n(t), q_n(t) \in H^j(\Omega)$ that solves (1.20) for all $t \geq 0$. By virtue of (3.39) and (3.60), we have that (2.4) holds for all n. Due to the definition (2.2), there is

$$\begin{aligned}
\|\varphi_n(T_0)\|^2_{H^k(\Omega_\pm)} &= \sum_{j=0}^{k} \int_{\mathbb{R}^2} (1 + |\xi|^2)^{k-j} |\partial^j_{x_3} \hat{\varphi}_n(T_0, \xi)|^2 d\xi \\
&= \sum_{j=0}^{k} \int_{\mathbb{R}^2} (1 + |\xi|^2)^{k-j} \|\partial^j_{x_3} \hat{\varphi}_n(T_0, \xi, \cdot)\|^2_{L^2(-1,1)} d\xi \\
&\geq \int_{\mathbb{R}^2} (1 + |\xi|^2)^{k} \|\hat{\varphi}_n(T_0, \xi, \cdot)\|^2_{L^2(-1,1)} d\xi \\
&= \int_{\mathbb{R}^2} (1 + |\xi|^2)^{k} \|\varphi_n(T_0, \xi, \cdot)\|^2_{L^2(-1,1)} d\xi \\
&= \int_{\mathbb{R}^2} (1 + |\xi|^2)^{k} |f_n(\xi)|^2 e^{2T_0\lambda(\xi)} \|\hat{w}(\xi, \cdot)\|^2_{L^2(-1,1)} d\xi \\
&\geq \frac{exp(T_0\sqrt{\frac{g[\rho]}{4(\rho_+ + \rho_-)}R(n)})}{(1 + (R(n) + 1)^2)^{j-k+1}} \int_{\mathbb{R}^2} (1 + |\xi|^2)^{j+1} |f_n(\xi)|^2 e^{2T_0\lambda(\xi)} \|\hat{w}(\xi, \cdot)\|^2_{L^2(-1,1)} d\xi \\
&\geq \alpha^2 n^2 \frac{\bar{C}_j^2}{D^2} \int_{\mathbb{R}^2} (1 + |\xi|^2)^{j+1} |f_n(\xi)|^2 D^2 d\xi = \alpha^2,
\end{aligned}$$

where we used (3.25), (3.40), and (3.30).

Since $\lambda(|\xi|) \geq \sqrt{\frac{g[\rho]}{4(\rho_+ + \rho -)}R(n)} \geq 1$ on the support of f_n, we also arrive at

$$\|v_n(t)\|^2_{H^k} \geq \|\varphi_n(t)\|^2_{H^k} \geq \|\varphi_n(T_0)\|^2_{H^k}, \quad \text{for} \quad t \geq T_0.$$

We finish the proof of Theorem 2.1. □

4. Ill-posedness for the nonlinear problem

We focus on showing the ill-posedness of the nonlinear system. Since $A = Id, S_- = Id_{\{x_3=0\}}$, $v_\pm = 0, \varphi_\pm = Id, q_\pm = const$ are the steady-state solutions to (1.18), one can rewrite (1.18) using the perturbation equations near the steady-state solutions. Let

$$\varphi_\pm = Id + \tilde{\varphi}_\pm, \varphi_\pm^{-1} = Id - \zeta_\pm, q_\pm = const + \sigma_\pm, A_\pm = I - G_\pm, \tag{4.1}$$

where $G_\pm^T = \sum_{n=1}^{\infty}(-1)^{n-1}(D\tilde{\varphi}_\pm)^n$.

Substituting (4.1) into (1.18) with $\bar{M} = (M, 0, 0)$, we yield the following equations about $\tilde{\varphi}_\pm, v_\pm, \sigma_\pm$

$$\begin{cases} \partial_t \tilde{\varphi}_\pm = v_\pm, \\ \rho_+ \partial_t v_+ + (I - G_+)\nabla\sigma_+ = 0, \\ \rho_- \partial_t v_- + (I - G_-)\nabla\sigma_- - M^2\partial_{11}\tilde{\varphi}_- = 0, \\ div v_\pm - tr(G_\pm \nabla v_\pm) = 0, \end{cases} \tag{4.2}$$

where $tr(\cdot)$ is the matrix trace. The following compatibility conditions are required:

$$\zeta_\pm = \tilde{\varphi}_\pm \circ (Id - \zeta_\pm).$$

We impose the corresponding jump conditions as follows:

$$(v_+(t, x_1, x_2, 0) - v_-(t, S_-(x_1, x_2))) \cdot \nu(t, x_1, x_2, 0) = 0, \tag{4.3}$$

$$\begin{aligned}(\sigma_+(t, x_1, x_2, 0) - \sigma_-(t, S_-(x_1, x_2)) \cdot \nu(t, x_1, x_2, 0) = g[\rho]\tilde{\varphi}_+^3(t, x_1, x_2, 0)\nu(t, x_1, x_2, 0) \\ -M^2(e_1 + \partial_1\tilde{\varphi}_-)(e_1 + \partial_1\tilde{\varphi}_-^j)(t, S_-(t, x_1, x_2))\nu_j(t, x_1, x_2, 0),\end{aligned} \tag{4.4}$$

where

$$S_- = (Id_{\mathbb{R}^2} - \zeta_-) \circ (Id_{\mathbb{R}^2} + \tilde{\varphi}_+) = Id_{\mathbb{R}^2} + \tilde{\varphi}_+ - \zeta_- \circ (Id_{\mathbb{R}^2} + \tilde{\varphi}_+), \tag{4.5}$$

$$v_-(t, x_1, x_2, -1) \cdot e_3 = v_+(t, x_1, x_2, 1) \cdot e_3 = 0. \tag{4.6}$$

We collect the equation, jump, and boundary Eqs (4.2)–(4.6) as "the perturbed problem". For $k \geq 0$, we use the following abbreviation:

$$\|(\tilde{\varphi}, v, \sigma)(t)\|_{H^k} = \|\tilde{\varphi}(t)\|_{H^k} + \|v(t)\|_{H^k} + \|\sigma(t)\|_{H^k}. \tag{4.7}$$

Before proving it, we give an importance definition.

Definition 4.1. (**Property** $EE(k)$) For any $\delta, t_0, C > 0$, and the initial data $\tilde{\varphi}_0, v_0, \sigma_0$ meeting

$$\|(\tilde{\varphi}_0, v_0, \sigma_0)\|_{H^k} < \delta, \tag{4.8}$$

there exists $(\tilde{\varphi}, v, \sigma) \in L^\infty((0, t_0); H^3(\Omega))$, which solves the perturbed problems (4.2)–(4.6) on $\Omega \times (0, t_0)$ and satisfies:

(1) $\varphi(t) = Id + \tilde{\varphi}(t)$ is reversible and $\varphi^{-1}(t) = Id - \zeta(t)$ for $0 \leq t < t_0$, and

(2)

$$\sup_{0\leq t<t_0} \|(\tilde{\varphi}, v, \sigma)(t)\|_{H^3} \leq Q(\|(\tilde{\varphi}_0, v_0, \sigma_0)\|_{H^k}), \tag{4.9}$$

where $Q : [0, \delta) \to \mathbb{R}^+$ and $Q(y) \leq Cy$ for $z \in [0, \delta)$. We say the perturbed problems (4.2)–(4.6) has property $EE(k)$.

Next, we will use the proof by contradiction to prove Theorem 2.3.

Proof. For some $k \geq 4$, we assume that the problems (4.2)–(4.6) has the property $EE(k)$ of the above definition. For $n \in \mathbb{N}$, let $T = \frac{t_0}{2}, k \geq 4$, and $\alpha = 1$ in Theorem 2.1. Then, $\bar{\varphi}, \bar{v}, \bar{\sigma}$ solves (1.20) with $\bar{M} = (M, 0, 0)$ and the initial data satisfying

$$\|(\bar{\varphi}, \bar{v}, \bar{\sigma})(0)\|_{H^k} < \frac{1}{n},$$

but

$$\|\bar{v}(t)\|_{H^4} \geq \|\bar{\varphi}(t)\|_{H^4} \geq 1, \quad \text{for} \quad t \geq T. \tag{4.10}$$

For any $\varepsilon > 0$, denote

$$\bar{\varphi}_0^\varepsilon = \varepsilon\bar{\varphi}(0), \bar{v}_0^\varepsilon = \varepsilon\bar{v}(0), \bar{\sigma}_0^\varepsilon = \varepsilon\bar{\sigma}(0),$$

then we have

$$\|(\bar{\varphi}_0^\varepsilon, \bar{v}_0^\varepsilon, \bar{\sigma}_0^\varepsilon)\|_{H^k} < \frac{\varepsilon}{n}.$$

Select n such that $n > C, \frac{\varepsilon}{n} < \delta$, where C, δ are the constants in the above property $EE(k)$.

Due to $EE(k)$, the perturbed problem exists a solution $(\tilde{\varphi}^\varepsilon, v^\varepsilon, \sigma^\varepsilon) \in L^\infty((0, t_0); H^4(\Omega))$ with $(\bar{\varphi}_0^\varepsilon, \bar{v}_0^\varepsilon, \bar{\sigma}_0^\varepsilon)$ as the initial data. In addition,

$$\begin{aligned} \sup_{0 \leq t < t_0} \|(\tilde{\varphi}^\varepsilon, v^\varepsilon, \sigma^\varepsilon, \partial_t\sigma^\varepsilon)(t)\|_{H^4} &\leq Q(\|(\bar{\varphi}_0^\varepsilon, \bar{v}_0^\varepsilon, \bar{\sigma}_0^\varepsilon\|_{H^k}) \\ &\leq C\varepsilon\|(\bar{\varphi}, \bar{v}, \bar{\sigma})(0)\|_{H^k} < \varepsilon. \end{aligned} \tag{4.11}$$

Defining

$$\bar{\varphi}^\varepsilon = \frac{\tilde{\varphi}^\varepsilon}{\varepsilon}, \bar{v}^\varepsilon = \frac{v^\varepsilon}{\varepsilon}, \bar{\sigma}^\varepsilon = \frac{\sigma^\varepsilon}{\varepsilon}, \tag{4.12}$$

and inputting them into (4.11), we derive

$$\sup_{0 \leq t < t_0} \|(\bar{\varphi}^\varepsilon, \bar{v}^\varepsilon, \bar{\sigma}^\varepsilon, \partial_t\bar{\sigma}^\varepsilon)\|_{H^4} \leq 1, \tag{4.13}$$

and

$$(\bar{\varphi}^\varepsilon, \bar{v}^\varepsilon, \bar{\sigma}^\varepsilon)(0) = (\bar{\varphi}, \bar{v}, \bar{\sigma})(0). \tag{4.14}$$

We next demonstrate that

$$\lim_{\varepsilon \to 0}(\bar{\varphi}^\varepsilon, \bar{v}^\varepsilon, \bar{\sigma}^\varepsilon) = (\bar{\varphi}, \bar{v}, \bar{\sigma}),$$

where $(\bar{\varphi}, \bar{v}, \bar{\sigma})$ solves the linearized system (1.20) with $\bar{M} = (M, 0, 0)$. Substitute (4.12) into (4.2), then we have

$$\begin{cases} \partial_t\bar{\varphi}_\pm^\varepsilon = \bar{v}_\pm^\varepsilon, \\ \rho_+\partial_t\bar{v}_+^\varepsilon + (I - \varepsilon\bar{G}_+^\varepsilon)\nabla\bar{\sigma}_+^\varepsilon = 0, \\ \rho_-\partial_t\bar{v}_-^\varepsilon + (I - \varepsilon\bar{G}_-^\varepsilon)\nabla\bar{\sigma}_-^\varepsilon - M^2\partial_{11}\bar{\varphi}_-^\varepsilon = 0, \\ div\bar{v}_\pm^\varepsilon - tr(\bar{G}_\pm^\varepsilon\nabla\bar{v}_\pm^\varepsilon) = 0, \end{cases} \tag{4.15}$$

where

$$\bar{G}_\pm^\varepsilon := \frac{I - (I + \varepsilon D\bar{\varphi}_\pm^T)^{-1}}{\varepsilon}, \tag{4.16}$$

then $\bar{G}^{\varepsilon}_{\pm}$ is well-defined. Thus,

$$\begin{aligned}\|\bar{G}^{\varepsilon}_{\pm}\|_{H^2} &= \|\sum_{n=1}^{\infty}(-\varepsilon)^{n-1}(D\bar{\varphi}^{\varepsilon}_{\pm})^n\|_{H^2} \leq \sum_{n=1}^{\infty}\varepsilon^{n-1}\|(D\bar{\varphi}^{\varepsilon}_{\pm})^n\|_{H^2}\\ &\leq \sum_{n=1}^{\infty}(\varepsilon K_1)^{n-1}\|D\bar{\varphi}^{\varepsilon}_{\pm})\|^n_{H^2} \leq \sum_{n=1}^{\infty}\frac{1}{2^{n-1}}\|\bar{\varphi}^{\varepsilon}_{\pm})\|^n_{H^4}\\ &< \sum_{n=1}^{\infty}\frac{1}{2^{n-1}} = 2, \end{aligned} \tag{4.17}$$

where the positive constant K_1 is the optimal constant in the inequality $\|FH\|_{H^2} \leq K_1\|F\|_{H^2}\|H\|_{H^2}$. Take ε small enough so that $\varepsilon < \frac{1}{2K_1}$, then $\bar{G}^{\varepsilon}_{\pm}$ is uniform boundness in $L^{\infty}(0, t_0; H^2(\Omega))$.

Now we will show some convergence results. From $(4.15)_1$, one gets

$$\sup_{0\leq t<t_0}\|\partial_t\bar{\varphi}^{\varepsilon}_{\pm}(t)\|_{H^4} = \sup_{0\leq t<t_0}\|\bar{v}^{\varepsilon}_{\pm}(t)\|_{H^4} \leq 1. \tag{4.18}$$

Expanding $(4.15)_2$, we have

$$\rho_+\partial_t\bar{v}^{\varepsilon}_+ + \nabla\bar{\sigma}^{\varepsilon}_+ - \varepsilon\bar{G}^{\varepsilon}_+\nabla\sigma^{\varepsilon}_+ = 0, \tag{4.19}$$

whence

$$\lim_{\varepsilon\to 0}\sup_{0\leq t<t_0}\|\rho_+\partial_t\bar{v}^{\varepsilon}_+ + \nabla\bar{\sigma}^{\varepsilon}_+\|_{H^3} = 0, \tag{4.20}$$

and

$$\sup_{0\leq t<t_0}\|\partial_t\bar{v}^{\varepsilon}_+\|_{H^3} \leq K_3 \quad \text{for some constant} \quad K_3 > 0. \tag{4.21}$$

By virtue of $(4.15)_3$, we achieve

$$\rho_-\partial_t\bar{v}^{\varepsilon}_- + \nabla\bar{\sigma}^{\varepsilon}_- - \varepsilon\bar{G}^{\varepsilon}_-\nabla\sigma^{\varepsilon}_- - M^2\partial_{11}\bar{\varphi}^{\varepsilon}_- = 0, \tag{4.22}$$

which implies

$$\lim_{\varepsilon\to 0}\sup_{0\leq t<t_0}\|\rho_-\partial_t\bar{v}^{\varepsilon}_- + \nabla\bar{\sigma}^{\varepsilon}_- - M^2\partial_{11}\bar{\varphi}^{\varepsilon}_-\|_{H^2} = 0, \tag{4.23}$$

Thus, we have

$$\sup_{0\leq t<t_0}\|\partial_t\bar{v}^{\varepsilon}_-\|_{H^2} \leq K_4 \quad \text{for some constant} \quad K_4 > 0, \tag{4.24}$$

$(4.15)_4$ implies

$$\lim_{\varepsilon\to 0}\sup_{0\leq t<t_0}\|div\bar{v}^{\varepsilon}_{\pm}\|_{H^3} = 0. \tag{4.25}$$

The convergence results about the jump conditions are as follows. Due to the invertibility of $Id + \varepsilon\bar{\varphi}^{\varepsilon}$, denote $\bar{\zeta}^{\varepsilon}$ by

$$(Id + \varepsilon\bar{\varphi}^{\varepsilon})^{-1} = Id - \varepsilon\bar{\zeta}^{\varepsilon},$$

which means

$$\bar{\zeta}^{\varepsilon} = \bar{\varphi}^{\varepsilon} \circ (Id - \varepsilon\bar{\zeta}^{\varepsilon}),$$

then $S^{\varepsilon}_- : \mathbb{R}^2 \times \mathbb{R}^+ \to \mathbb{R}^2 \times \{0\}$ can be expressed by

$$S^{\varepsilon}_- = Id_{\mathbb{R}^2} + \varepsilon\bar{\varphi}^{\varepsilon}_+ - \varepsilon\bar{\zeta}^{\varepsilon} \circ (Id_{\mathbb{R}^2} + \varepsilon\bar{\varphi}^{\varepsilon}_+). \tag{4.26}$$

Hence,

$$\begin{aligned}\sup_{0\le t\le t_0}\|S_{-}^{\varepsilon}(t)-Id_{\mathbb{R}^2}\|_{L^\infty} &\le 2\varepsilon\sup_{0\le t<t_0}\|\bar{\varphi}^{\varepsilon}(t)\|_{L^\infty}\\ &\le 2\varepsilon K_2\sup_{0\le t<t_0}\|\bar{\varphi}^{\varepsilon}\|_{H^4}<2\varepsilon a K_2,\end{aligned}\tag{4.27}$$

where the positive constant K_2 is the Sobolev embedding constant in the trace mapping $H^4(\Omega)\hookrightarrow L^\infty(\mathbb{R}^2\times\{0\})$. Define $\bar{S}_{-}^{\varepsilon}=\frac{S_{-}^{\varepsilon}-Id_{\mathbb{R}^2}}{\varepsilon}$, then $\bar{S}_{-}^{\varepsilon}$ is uniform boundness in $L^\infty((0,t_0);L^\infty(\mathbb{R}^2\times\{0\}))$ by (4.27). Denote the normal at the interface by $\nu^\varepsilon=\frac{N^\varepsilon}{|N^\varepsilon|}$ with

$$\begin{aligned}N^\varepsilon &= (e_1+\varepsilon\partial_{x_1}\bar{\varphi}_{+}^{\varepsilon})\times(e_2+\varepsilon\partial_{x_2}\bar{\varphi}_{+}^{\varepsilon})\\ &= e_3+\varepsilon(e_1\times\partial_{x_2}\bar{\varphi}_{+}^{\varepsilon}+\partial_{x_1}\bar{\varphi}_{+}^{\varepsilon}\times e_2+\varepsilon\partial_{x_1}\bar{\varphi}_{+}^{\varepsilon}\times\partial_{x_2}\bar{\varphi}_{+}^{\varepsilon})\\ &:= e_3+\varepsilon\bar{N}^\varepsilon.\end{aligned}\tag{4.28}$$

As $\varepsilon\to 0$, one gets $|N^\varepsilon|>0$. The jump condition (4.3) can be rewritten as follows:

$$(\bar{v}_{+}^{\varepsilon}-\bar{v}_{-}^{\varepsilon}\circ(Id_{\mathbb{R}^2}+\varepsilon\bar{S}_{-}^{\varepsilon a}))\cdot(e_3+\varepsilon\bar{N}^\varepsilon)=0.\tag{4.29}$$

It is obvious that $\sup_{0\le t<t_0}\|\bar{N}^\varepsilon(t)\|_{L^\infty}$ is uniformly bounded since

$$\begin{aligned}&\sup_{0\le t<t_0}\|\bar{v}_{-}^{\varepsilon}\circ(Id_{\mathbb{R}^2}+\varepsilon\bar{S}_{-}^{\varepsilon})-\bar{v}_{-}^{\varepsilon}\|_{L^\infty}\\ &\le\sup_{0\le t<t_0}\|D\bar{v}^{\varepsilon}(t)\|_{L^\infty}\sup_{0\le t<t_0}\|\varepsilon\bar{S}_{-}^{\varepsilon}(t)\|_{L^\infty}\to 0\quad\text{as}\quad\varepsilon\to 0.\end{aligned}$$

Therefore,

$$\sup_{0\le t<t_0}\|e_3\cdot(\bar{v}_{+}^{\varepsilon}(t)-\bar{v}_{-}^{\varepsilon}(t))\|_{L^\infty}\to 0\quad\text{as}\quad\varepsilon\to 0.\tag{4.30}$$

For the jump condition (4.4), we can rewrite it as

$$\begin{aligned}&[\bar{\sigma}_{+}^{\varepsilon}-\bar{\sigma}_{-}^{\varepsilon}\circ(Id_{\mathbb{R}^2}+\varepsilon\bar{S}_{-}^{\varepsilon})-g[\rho]\bar{\varphi}_{+}^{\varepsilon,3}](e_3+\varepsilon N^\varepsilon)\\ &=-M^2(\bar{N}^{1,\varepsilon}+\partial_1\bar{\varphi}_{-}^{3,\varepsilon})\circ(Id_{\mathbb{R}^2}+\varepsilon\bar{S}_{-}^{\varepsilon})e_1-\varepsilon M^2\bar{F}^\varepsilon,\end{aligned}$$

where

$$\begin{aligned}\bar{Q}^\varepsilon &= [(\partial_1\bar{\varphi}_{-}^{\varepsilon}\cdot\bar{N}^\varepsilon)e_1+(\bar{N}^{1,\varepsilon}+\partial_1\bar{\varphi}_{-}^{3,\varepsilon})\partial_1\bar{\varphi}_{-}^{\varepsilon}\\ &\quad+\varepsilon(\partial_1\bar{\varphi}_{-}\cdot\bar{N}^\varepsilon)\partial_1\bar{\varphi}_{-}^{\varepsilon}]\circ(Id_{\mathbb{R}^2}+\varepsilon\bar{S}_{-}^{\varepsilon}).\end{aligned}$$

Obviously, $\sup_{0\le t<t_0}\|\bar{Q}^\varepsilon(t)\|_{L^\infty}$ is uniformly bounded. Thus, we achieve

$$\sup_{0\le t<t_0}\|(\bar{\sigma}_{+}^{\nu}-\bar{\sigma}_{-}^{\varepsilon}-g[\rho]\bar{\varphi}_{+}^{3,\varepsilon})e_3+M^2\partial_1\bar{\varphi}_{-}^{3,\varepsilon})e_1\|_{L^\infty}\to 0,\quad\text{as}\quad\varepsilon\to 0.\tag{4.31}$$

Collecting (4.13), (4.18), (4.21), and (4.24), there exists $(\bar{\varphi}^0,\bar{v}^0,\bar{\sigma}^0,\partial_t\bar{\sigma}^0,\partial_t\bar{\varphi}_0)\in L^\infty(0,t_0;H^4(\Omega))$ so that

$$(\bar{\varphi}^\varepsilon,\bar{v}^\varepsilon,\bar{\sigma}^\varepsilon,\partial_t\bar{\sigma}^\varepsilon,\partial_t\bar{\varphi}^\varepsilon)\to(\bar{\varphi}^0,\bar{v}^0,\bar{\sigma}^0,\partial_t\bar{\sigma}^0,\partial_t\bar{\varphi}_0)\quad\text{weak-* in}\quad L^\infty(0,t_0;H^4(\Omega)),$$

and

$$\partial_t \bar{v}^\varepsilon \to \partial_t v^0 \quad \text{weakly-}* \text{ in} \quad L^\infty(0, t_0; H^2(\Omega)). \tag{4.32}$$

By virtue of the lower semi-continuity, we derive

$$\sup_{0\le t<t_0} \|(\bar{\varphi}^0, \bar{v}^0, \bar{\sigma}^0)(t)\|_{H^4} \le 1. \tag{4.33}$$

Combining (4.13), (4.18), (4.21), and (4.24), one has

$$\limsup_{\varepsilon\to 0} \sup_{0\le t<t_0} \|(\partial_t\bar{\varphi}^\varepsilon, \partial_t\bar{v}^\varepsilon, \partial_t\bar{\sigma}^\varepsilon)\|_{H^2} < \infty.$$

By virtue of a conclusion in [17], $(\bar{\varphi}^\varepsilon, \bar{v}^\varepsilon, \bar{\sigma}^\varepsilon)$ is strongly pre-compact in $L^\infty(0, t_0; H^{\frac{11}{4}}(\Omega))$. So, we have

$$(\bar{\varphi}^\varepsilon, \bar{v}^\varepsilon, \bar{\sigma}^\varepsilon) \xrightarrow{strongly} (\bar{\varphi}^0, \bar{v}^0, \bar{\sigma}^0) \quad \text{in} \quad L^\infty(0, t_0; H^{\frac{11}{4}}(\Omega)). \tag{4.34}$$

Following from the above strong convergence, the convergence results (4.20) and (4.23), and the equation $\partial_t\bar{\varphi}^\varepsilon = \bar{v}^\varepsilon$, we arrive at

$$\begin{aligned} &\partial_t\bar{\varphi}^\varepsilon \xrightarrow{strongly} \partial_t\bar{\varphi}^0 \quad \text{in} \quad L^\infty(0, t_0; H^{\frac{11}{4}}(\Omega)), \\ &\partial_t\bar{v}^\varepsilon \xrightarrow{strongly} \partial_t\bar{v}^0 \quad \text{in} \quad L^\infty(0, t_0; L^2(\Omega)), \end{aligned}$$

and

$$\begin{cases} \partial_t\bar{\varphi}^0_\pm = \bar{v}^0_\pm, \\ \rho_+\partial_t\bar{v}^0_+ + \nabla\bar{\sigma}^0_+ = 0, \\ \rho_-\partial_t\bar{v}^0_- + \nabla\bar{\sigma}^0_- - M^2\partial^2_{11}\bar{\varphi}^0_- = 0, \\ div\bar{v}^0_\pm = 0. \end{cases} \tag{4.35}$$

Taking the limit for (4.14), there is

$$(\bar{\varphi}^0, \bar{v}^0, \bar{\sigma}^0)(0) = (\bar{\varphi}, \bar{v}, \bar{\sigma})(0). \tag{4.36}$$

Combining (4.30) and (4.31), we infer

$$\begin{aligned} &\bar{v}^0_+ \cdot e_3 = 0 \quad \text{on} \quad \{x_3 = 1\}, \quad \bar{v}^0_- \cdot e_3 = 0 \quad \text{on} \quad \{x_3 = -1\}, \\ &(\bar{v}^0_+ - \bar{v}^0_-) \cdot e_3 = 0 \quad \text{on} \quad \{x_3 = 0\}, \end{aligned} \tag{4.37}$$

and

$$(\bar{\sigma}^0_+ - \bar{\sigma}^0_- - g[\rho]\bar{\varphi}^{3,0}_+)e_3 + M^2\partial_1\bar{\varphi}^{3,0}_- e_1 = 0 \quad \text{on} \quad \{x_3 = 0\}. \tag{4.38}$$

(4.35)–(4.38) imply that $(\bar{\varphi}^0, \bar{v}^0, \bar{\sigma}^0)$ solves (1.20)–(1.23) with $\bar{M} = (M, 0, 0)$ and meets the initial conditions (4.24). Thereby,

$$(\bar{\varphi}^0, \bar{v}^0, \bar{\sigma}^0) = (\bar{\varphi}, \bar{v}, \bar{\sigma}) \quad \text{on} \quad [0, t_0) \times \Omega,$$

follows from Theorem 3.10. Furthermore, collect the inequality (4.33) and (4.10) to yield

$$2 \le \sup_{\frac{t_0}{2}\le t<t_0} \|(\bar{\varphi}^0, \bar{v}^0, \bar{\sigma}^0)(t)\|_{H^4} \le \sup_{0\le t<t_0} \|(\bar{\varphi}^0, \bar{v}^0, \bar{\sigma}^0)(t)\|_{H^4} \le 1,$$

which is a contradiction. Therefore, for any $k \ge 4$, the property $EE(k)$ is not valid for the perturbed problem. Theorem 2.3 has been proven. □

5. Conclusions

We investigated the RT instability problem of the two-phase flow coupled with ideal fluid and magnetohydrodynamic. We obtained the RT instability of linear problems by establishing a growth mode solution to the linearization problem near the steady-state solution. By virtue of the instability of linearization problems, we ultimately obtained the RT instability of nonlinear problems.

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Conflict of interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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