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Research article

Solution-tube and existence results for fourth-order differential equations system

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Abstract: In the present paper, we examine the existence of solutions to fourth-order differential equation systems when the L^1-Carathéodory function is on the right-hand side. A concept of *solution-tube* for these issues is presented. The concepts of upper and lower solutions for fourth-order differential equations are extended to systems owing to this idea.

Keywords: fourth-order systems of differential equations; iterative methods; solution-tubes; existence theorems; Leray-Schauder degree
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1. Introduction

In the present research, we prove existence results for a fourth-order differential equation system that takes the form:

$$\begin{cases} \varpi^{(4)}(t) = f(t, \varpi(t), \varpi'(t), \varpi''(t), \varpi'''(t)), \quad a.e.\ t \in \mathcal{J} = [0, 1], \\ \varpi(0) = \varpi_0, \quad \varpi'(0) = \varpi_1 \quad \text{and} \quad \varpi'' \in (BC), \end{cases} \tag{1.1}$$

where $f : [0, 1] \times \mathbb{R}^{4n} \to \mathbb{R}^n$ represents an L^1-Carathéodory function, $\varpi_0, \varpi_1 \in \mathbb{R}^n$ and (BC) can be the boundary conditions that are given by one of the following:

(SL) Strum-Liouville boundary conditions on $\mathcal{J}$

$$\begin{aligned} A_0\varpi(0) - \beta_0\varpi'(0) &= r_0, \\ A_1\varpi(1) + \beta_1\varpi'(1) &= r_1. \end{aligned} \tag{1.2}$$

(P) Periodic boundary conditions on $\mathcal{J}$

$$\begin{aligned} \varpi(0) &= \varpi(1), \\ \varpi'(0) &= \varpi'(1), \end{aligned} \tag{1.3}$$

where $(A_i)_{i\in\{0,1\}} \in \mathcal{M}_{n\times n}(R)$, such that

$$\forall i \in \{0,1\}, \exists \kappa_i \geq 0 : \langle \varpi, A_i\varpi \rangle \geq \kappa_i \|\varpi\|^2, \forall \varpi \in \mathbb{R}^n$$

$$\forall i \in \{0,1\}, r_i \in \mathbb{R} : \beta_i \in \{0,1\}, \ \kappa_i + \beta_i > 0.$$

We refer to [1–3] for further findings that were achieved in the specific instance of a boundary value issue for only one differential equation of the fourth-order ($n = 1$), for more details, please see [4–6]. Existence results for higher-order differential equations can be found in [7, 8], and the general case of N^{th} order systems is discussed in [9–11].

The concept of the solution-tube of problem (1.1) is presented in this work; see [12–14]. This idea is inspired by [15] and [16], where solution-tubes for second and third order differential equations systems are defined, respectively, as follows:

$$\begin{cases} \varpi''(t) = f(t, \varpi(t), \varpi'(t)), & a.e.\ t \in \mathcal{J}, \\ \varpi \in (BC), \end{cases} \tag{1.4}$$

and

$$\begin{cases} \varpi'''(t) = f(t, \varpi(t), \varpi'(t), \varpi''(t)), & a.e.\ t \in \mathcal{J}, \\ \varpi(0) = \varpi_0,\ \varpi' \in (BC). \end{cases} \tag{1.5}$$

We prove that the system (1.1) has solutions. For this system, we employ the concept of a solution tube, which extends to systems the ideas of lower and upper solutions to the fourth-order differential equations presented in [17–19].

The structure of this paper is given as follows: This article will utilize the notations, definitions, and findings found in Section 2. In Section 3, we provide the idea of a solution-tube to get existence results for fourth-order differential equation systems. We then go on to demonstrate the practicality of our results through two examples.

2. Preliminaries

In this section, we recall some notations, definitions, and results that we will use in this article. The scalar product and the Euclidian norm in $\mathbb{R}^n$ are denoted by $\langle , \rangle$ and $\|\cdot\|$, respectively. Also, let $C^k(\mathcal{J}, \mathbb{R}^n)$ be the Banach space of the k-times continuously differentiable functions ϖ associated with the norm

$$\|\varpi\|_k = \max\left\{\|\varpi\|_0, \|\varpi'\|_0, ..., \left\|\varpi^{(k)}\right\|_0\right\},$$

where

$$\|\varpi\|_0 = \max\{\varpi(t) : t \in \mathcal{J}\}.$$

The space of integral functions is denoted by $L^1(\mathcal{J}, \mathbb{R}^n)$, with the usual norm $\|\cdot\|_{L^1}$. The Sobolev space of functions in $C^{k-1}(\mathcal{J}, \mathbb{R}^n)$, where $k \geq 1$ and the $(k-1)^{th}$ derivative is denoted by $W^{k,1}(\mathcal{J}, \mathbb{R}^n)$.

For $\varpi_0, \varpi_1 \in \mathbb{R}^n$, we have the following:

$$C_{\varpi_0}(\mathcal{J}, \mathbb{R}^n) := \{\varpi \in C(\mathcal{J}, \mathbb{R}^n) : \varpi(0) = \varpi_0\},$$

$$C^1_{\varpi_0,\varpi_1}(\mathcal{J}, \mathbb{R}^n) := \{\varpi \in C^1(\mathcal{J}, \mathbb{R}^n) : \varpi(0) = \varpi_0,\ \varpi'(0) = \varpi_1\},$$

$$C^k_B(\mathcal{J}, \mathbb{R}^n) = \left\{\varpi \in C^k(\mathcal{J}, \mathbb{R}^n) : \varpi \in (BC)\right\},$$

$$W^{k,1}_B(\mathcal{J}, \mathbb{R}^n)) = \left\{\varpi \in W^{k,1}(\mathcal{J}, \mathbb{R}^n)) : \varpi \in (BC)\right\},$$

$$C^{k+1}_{\varpi_0,B}(\mathcal{J}, \mathbb{R}^n) = \left\{\varpi \in C^{k+1}(\mathcal{J}, \mathbb{R}^n) : \varpi(0) = \varpi_0,\ \varpi^{(k)} \in (BC)\right\},$$

$$W^{k+1,1}_{\varpi_0,B}(\mathcal{J}, \mathbb{R}^n)) = \left\{\varpi \in W^{k+1,1}(\mathcal{J}, \mathbb{R}^n)) : \varpi(0) = \varpi_0,\ \varpi^{(k)} \in (BC)\right\},$$

$$C^{k+2}_{\varpi_0,\varpi_1,B}(\mathcal{J}, \mathbb{R}^n) = \left\{\varpi \in C^{k+2}(\mathcal{J}, \mathbb{R}^n) : \varpi(0) = \varpi_0,\ \varpi'(0) = \varpi_1,\ x^{(k)} \in (BC)\right\},$$

$$W^{k+2,1}_{\varpi_0,\varpi_1,B}(\mathcal{J}, \mathbb{R}^n)) = \left\{\varpi \in W^{k+2,1}(\mathcal{J}, \mathbb{R}^n)) : \varpi(0) = \varpi_0,\ \varpi'(0) = \varpi_1,\ \varpi^{(k)} \in (BC)\right\}.$$

Definition 2.1. *A function $f : \mathcal{J} \times \mathbb{R}^{4n} \to \mathbb{R}^n$ is called an L^1-Carathéodory function if*

(i) For every $(\varpi, y, q, p) \in \mathbb{R}^{4n}$, the function $t \mapsto f(t, \varpi, y, q, p)$ is measurable;
(ii) The function $(\varpi, y, q, p) \mapsto f(t, \varpi, y, q, p)$ is continuous for a.e. $t \in \mathcal{J}$;
(iii) For every $r > 0$, there exists a function $h_r \in L^1(\mathcal{J}, [0, \infty))$ such that $\|f(t, \varpi, y, q, p)\| \leq h_r(t)$ for a.e. $t \in \mathcal{J}$ and for all $(\varpi, y, q, p) \in \mathbb{D}$, where

$$\mathbb{D} = \{(\varpi, y, q, p) \in \mathbb{R}^{4n} : \|\varpi\| \leq r,\ \|y\| \leq r,\ \|q\| \leq r,\ \|p\| \leq r\}.$$

Definition 2.2. *A function $F : C^3(\mathcal{J}, \mathbb{R}^n) \times \mathcal{J} \to L^1(\mathcal{J}, \mathbb{R}^n)$ is integrally bounded, if for every bounded subset $B \subset C^3(\mathcal{J}, \mathbb{R}^n)$, there exists an integral function $h_B \in L^1(\mathcal{J}, [0, \infty))$ so that $\|F(\varpi, \alpha)(t)\| \leq h_B(t)$, for $\forall t \in \mathcal{J}$, $(\varpi, \alpha) \in B \times \mathcal{J}$.*

The operator $N_F : C^3(\mathcal{J}, \mathbb{R}^n) \times \mathcal{J} \to C_0(\mathcal{J}, \mathbb{R}^n)$ will be associated with F and defined by

$$N_F(\varpi)(t) = \int_0^t F(\varpi, \alpha)(s)ds.$$

We now state the following results:

Theorem 2.1. *[20] Let $F : C^3(\mathcal{J}, \mathbb{R}^n) \times \mathcal{J} \to L^1(\mathcal{J}, \mathbb{R}^n)$ be continuous and integrally bounded, then N_F is continuous and completely continuous.*

Lemma 2.1. *[21] Let E be a Banach space. Let $v : \mathcal{J} \to E$ be an absolutely continuous function, then for*

$$\{t \in \mathcal{J} : v(t) = 0 \text{ and } v'(t) \neq 0\},$$

the measure is zero.

Lemma 2.2. *[22] For $w \in W^{2,1}(\mathcal{J};\mathbb{R})$ and $\varepsilon \geq 0$,* ***assume that one of the next properties*** *is satisfied:*

(i) $w''(t) - \varepsilon w(t) \geq 0$; *for almost every* $t \in \mathcal{J}, \kappa_0 w(0) - \nu_0 w'(0) \leq 0, \kappa_1 w(1) + \nu_1 w'(1) \leq 0$; *where* $\kappa_i, \nu_i \geq 0, \max\{\kappa_i, \nu_i\} > 0;\ i = 0, 1;$ *and* $\max\{\kappa_0, \kappa_1, \varepsilon\} > 0$,

(ii) $w''(t) - \varepsilon w(t) \geq 0$; *for almost every* $t \in \mathcal{J},\ \varepsilon > 0,\ w(0) = w(1),\ \ w'(1) - w'(0) \leq 0$,

(iii) $w''(t) - \varepsilon w(t) \geq 0$; *for almost every* $t \in [0, t_1] \cup [t_2, 1], \varepsilon > 0, w(0) = w(1),\ \ w'(1) - w'(0) \leq 0$, $w(t) \leq 0, t \in [t_1, t_2]$.

Then $w(t) \leq 0$, $\forall t \in [0, 1]$.

Lemma 2.3. *[22] Let $f \in C(\mathcal{J} \times \mathbb{R}^{2n}, \mathbb{R}^n)$ be a L^1-Carathéodory function (see definition in [22]). Consider the following problem:*

$$\begin{cases} \varpi''(t) = f(t, \varpi(t), \varpi'(t)), & a.e.\ t \in \mathcal{J}, \\ \varpi \in (BC). \end{cases} \tag{2.1}$$

Let $\varepsilon > 0$, and (z, N) a solution-tube of (2.1) given in Definition 2.3 of [22]. If $\varpi \in W_B^{2,1}(\mathcal{J}, \mathbb{R}^n)$ satisfies

$$\begin{aligned} \Pi(t) &= \frac{\langle \varpi(t) - z(t), \varpi''(t) - z''(t)\rangle + \|\varpi'(t) - z'(t)\|^2}{\|\varpi(t) - z(t)\|} - \frac{\langle \varpi(t) - z(t), \varpi'(t) - z'(t)\rangle^2}{\|\varpi(t) - z(t)\|^3} - \varepsilon \|\varpi(t) - z(t)\| \\ &\geq N''(t) - \varepsilon N(t), \end{aligned}$$

a.e. on

$$\{t \in \mathcal{J} : \|\varpi(t) - z(t)\| > N(t)\}.$$

Then

$$\|\varpi(t) - z(t)\| \leq N(t) \ \ for\ every\ t \in \mathcal{J}.$$

Now, we recall some properties of the Leray Schauder degree. The interested reader can see [23,24].

Theorem 2.2. *Let E be a Banach space and $U \subset E$ is an open bounded set. We define $K_{\partial U}(\overline{U}, E) = \{f : \overline{U} \to E$, where f is compact and $f(\varpi) \neq \varpi$, for every $\varpi \in \partial U\}$, the Leary-Schauder degree on U of $(Id - f)$ is an integer $deg(Id - f, U, 0)$ satisfying the following properties:*

(i) (*Existence*) *If $deg(Id - f, U, 0) \neq 0$, then $\exists \varpi \in U$, s.t.,*

$$\varpi - f(\varpi) = 0.$$

(ii) (*Normalization*) *If $0 \in U$, then $deg(Id, U, 0) = 1$.*

(iii) (*Homotopy invariance*) *If $h : \overline{U} \times \mathcal{J} \to E$ is a compact such that $\varpi - h(\varpi, \alpha) \neq 0$ for each $(\varpi, \alpha) \in \partial U \times \mathcal{J}$, then*

$$deg(Id - h(., \alpha), U, 0) = deg(Id - h(., 0), U, 0),\ \ for\ every\ \alpha \in \mathcal{J}.$$

(iv) (*Excision*) *If $V \subset U$ is open and $\varpi - f(\varpi) \neq 0$ for all $\varpi \in \overline{U} \setminus V$, then*

$$deg(Id - f, U, 0) = deg(Id - f, V, 0).$$

(v) (*Additivity*) *If $U_1, U_2 \subset U$ are disjoint and open, such that $\overline{U} = \overline{U_1 \cup U_2}$ and $\varpi - f(\varpi) \neq 0$ for all $\varpi \in \partial U_1 \cup \partial U_2$, then*

$$deg(Id - f, U, 0) = deg(Id - f, U_1, 0) + deg(Id - f, U_2, 0).$$

3. Main results

In this section, we define the solution-tube to the problem (1.1). This definition is important for our discussion about the existence results. A solution to this problem is a function $\varpi \in W^{4,1}(\mathcal{J},\mathbb{R}^n)$ satisfying (1.1). Now, we define the tube solution of problem (1.1), where the functions $z \in W^{4,1}(\mathcal{J},\mathbb{R}^n)$ and $N \in W^{4,1}(\mathcal{J},[0,\infty)$ are chosen before studying the existence of this problem.

Definition 3.1. *Let $(z,N) \in W^{4,1}(\mathcal{J},\mathbb{R}^n) \times W^{4,1}(\mathcal{J},[0,\infty))$. The couple (z,N) is solution-tube of (1.1), if*

(i) $N''(t) \geq 0, \ \forall\, t \in \mathcal{J}$.

(ii) For almost every $t \in \mathcal{J}$ and for all $(\varpi, y, q, p) \in \mathbb{F}$,

$$\langle q - z''(t), f(,t,\varpi,y,q) - z'''(t)\rangle + \|p - z'''(t)\|^2 \geq N''(t)N^4(t) + (N'''(t))^2\,,$$

where

$$\mathbb{F} = \Big\{(\varpi,y,q,p) \in \mathbb{R}^{4n} \ : \ \begin{array}{l} \|\varpi - z\,(t)\| \leq N\,(t)\,, \\ \|y - z'\,(t)\| \leq N'\,(t)\,, \\ \|q - z''\,(t)\| = N''\,(t)\,, \\ \langle q - z''(t), p - z'''(t)\rangle = N''(t)N'''(t)\Big\}. \end{array}$$

(iii) $z^{(4)}(t) = f(t,\varpi,y,z''(t),z'''(t))$, *a.e.* $t \in [0,\ 1]$ *such that* $N''(t) = 0$ *and* $(\varpi,y) \in \mathbb{R}^{2n}$*, such that* $\|\varpi - z\,(t)\| \leq N\,(t)$ *and* $\|y - z'\,(t)\| \leq N'\,(t)$.

(iv) With (1.2), we have

$$\|r_0 - (A_0 z''(0) - \beta_0 z'''(0))\| \leq \kappa_0 N''(0) - \beta_0 N'''(0),$$

$$\|r_1 - (A_1 z''(1) + \beta_1 z'''(1))\| \leq \kappa_1 N''(1) + \beta_1 N'''(1).$$

If (BC) is given by (1.3), then

$$z''(0) = z''(1),\ N'''(0) = N''(1),$$
$$\|z'''(1) - z'''(0)\| \leq N'''(1) - N'''(0).$$

(v) $\|\varpi_0 - z(0)\| \leq N(0),\ \|\varpi_1 - z'(0)\| \leq N'(0)$.

The next notation will be used

$$\mathrm{T}(z,N) = \Big\{\varpi \in C^2(\mathcal{J},\ \mathbb{R}^n) : \|\varpi''(t) - z''(t)\| \leq N''(t),\ \|\varpi'(t) - z'(t)\| \leq N'\,(t)$$
$$\text{and } \|\varpi'''(t) - z'''(t)\| \leq N''(t) \ for\ all\ t \in \mathcal{J}\}.$$

The next hypotheses will be used:

$(\mathcal{F}1)$ $f : \mathcal{J} \times \mathbb{R}^{4n} \to \mathbb{R}^n$ is a L^1-Carathéodory function.

$(\mathcal{H}1)$ There exists $(z,N) \in W^{4,1}(\mathcal{J},\mathbb{R}^n) \times W^{4,1}(\mathcal{J},[0,\infty))$ a solution-tube of the main system (1.1).

The next family of problems should be considered to prove the general existence theorem that will be presented:

$$\begin{cases}\varpi^{(4)}(t)-\varepsilon\varpi''(t)=f_\alpha^\varepsilon(t,\varpi(t),\varpi'(t),\varpi''(t),\varpi'''(t)), \quad a.e.\ t\in\mathcal{J},\\ \varpi(0)=\varpi_0,\ \varpi'(0)=\varpi_1\ and\ \varpi''\in(BC),\end{cases} \tag{3.1}$$

where $\varepsilon,\alpha\in\mathcal{J}$ and $f_\alpha^\varepsilon:\mathcal{J}\times\mathbb{R}^{4n}\to\mathbb{R}^n$ is defined by

$$f_\alpha^\varepsilon(t,\varpi,y,q,p)=\begin{cases}\alpha\left(\frac{N''(t)}{\|q-z''(t)\|}f_1(t,\varpi,y,\tilde{q},\check{p})-\varepsilon\tilde{q}\right)-\varepsilon(1-\alpha)z''(t)\\ \quad+\left(1-\frac{\alpha N''(t)}{\|q-z''(t)\|}\right)\left(z^{(4)}(t)+\frac{N^{(4)}(t)}{\|q-z''(t)\|}(q-z''(t))\right),\ if\ \|q-z''(t)\|>N''(t),\\ \alpha\left(f_1(t,\varpi,y,q,p)-\varepsilon q\right)-\varepsilon(1-\alpha)z''(t)\\ \quad+(1-\alpha)\left(z^{(4)}(t)+\frac{N^{(4)}(t)}{N''(t)}(q-z''(t))\right), \qquad otherwise,\end{cases}$$

where (z,N) is the solution-tube of (1.1),

$$f_1(t,\varpi,y,q,p)=\begin{cases}f(t,\bar{\varpi},\hat{y},q,p), & if\ \|\varpi-z(t)\|>N(t)\ and\ \|y-z'(t)\|>N'(t),\\ f(t,\varpi,y,q,p), & otherwise,\end{cases}$$

$$\bar{\varpi}(t)=\frac{N(t)}{\|\varpi-z(t)\|}(\varpi-z(t))+z(t), \tag{3.2}$$

$$\hat{y}(t)=\frac{N'(t)}{\|y-z'(t)\|}(y-z'(t))+z'(t), \tag{3.3}$$

$$\tilde{q}(t)=\frac{N''(t)}{\|q-z''(t)\|}(q-z''(t))+z''(t), \tag{3.4}$$

$$\check{p}(t)=p+\left(N'''(t)-\frac{\langle q-z''(t),p-z'''(t)\rangle}{\|q-z''(t)\|}\right)\left(\frac{q-z''(t)}{\|q-z''(t)\|}\right), \tag{3.5}$$

and where we mean

$$\frac{N^{(4)}(t)}{N''(t)}(q-z''(t))=0 \text{ on } \{t\in\mathcal{J}:\|q(t)-z''(t)\|=N''(t)=0\}.$$

We associate with f_α^ε the operator $F^\varepsilon:C^3(\mathcal{J},\mathbb{R}^n)\times\mathcal{J}\to L^1(\mathcal{J},\mathbb{R}^n)$ defined by

$$F^\varepsilon(\varpi,\alpha)(t)=f_\alpha^\varepsilon(t,\varpi(t),\varpi'(t),\varpi''(t),\varpi'''(t)).$$

Similarly to the Lemma 3.3 and Propositions 3.4 in [20] and results in [25], we need the following auxiliary results:

Lemma 3.1. *Assume ($\mathcal{H}1$). If a function $\varpi\in W^{4,1}_{\varpi_0,\varpi_1,B}(\mathcal{J},\mathbb{R}^n)$ satisfies*

$$\frac{\left\langle\varpi''(t)-z''^{(4)}(t)\right\rangle+\|\varpi'''(t)-z'''(t)\|^2}{\|\varpi''(t)-z''(t)\|}-\frac{\langle\varpi''(t)-z''(t),\varpi'''(t)-z'''(t)\rangle^2}{\|\varpi''(t)-z''(t)\|^3}-\varepsilon\|\varpi''(t)-z''(t)\|$$
$$\geq N^{(4)}(t)-\varepsilon N''(t),$$

for a.e. $t\in\{t\in\mathcal{J}:\|\varpi''(t)-z''(t)\|>N''(t)\}$, then $\varpi\in T(z,N)$.

Proof. By assumption

$$\varpi' \in W^{3,1}_{\varpi_1,B}(\mathcal{J},\mathbb{R}^n),\ \varpi'' \in W^{2,1}_{B}(\mathcal{J},\mathbb{R}^n),$$

and thus, from applying Lemma 2.3 to ϖ'', we obtain

$$\|\varpi''(t) - z''(t)\| \le N''(t), \forall t \in \mathcal{J}.$$

On

$$\{t \in \mathcal{J} : \|\varpi'(t) - z'(t)\| > N'(t),\ \|\varpi'(t) - z'(t)\|' \le \|\varpi''(t) - z''(t)\| \le N''(t).\}$$

The function

$$t \to \|\varpi'(t) - z'(t)\| - N'(t),$$

is nonincreasing on $\mathcal{J}$. Since

$$\left\|\varpi'_0 - z'(0)\right\| \le N'(0),$$

we get

$$\|\varpi'(t) - z'(t)\| \le N'(t), \forall t \in \mathcal{J},$$

hence

$$\|\varpi(t) - z(t)\|' \le \|\varpi'(t) - z'(t)\| \le N'(t).$$

The function

$$t \to \|\varpi(t) - z(t)\| - N(t),$$

is nonincreasing on $\mathcal{J}$ and since

$$\|\varpi(0) - z(0)\| \le N(0),$$

we obtain

$$\|\varpi(t) - z(t)\| \le N(t), \forall t \in \mathcal{J}.$$

□

Proposition 3.1. *Assume* ($\mathcal{F}1$) *and* ($\mathcal{H}1$) *hold. Then the operator* F^ε *that was defined earlier is continuous and integrally bounded.*

Proof. First, we will prove that F^ε is integrally bounded. If $\varpi \in \mathcal{B}$, where $\mathcal{B}$ is a bounded set of $C^3(\mathcal{J},\mathbb{R}^n)$, $\exists K > 0$ that satisfies $\left\|\varpi^{(i)}(t)\right\| \le K$, $\forall t \in \mathcal{J}$, where $i = 0,1,2,3$. Then $f^\varepsilon_\alpha(t,.,.,.,.)$ is bounded in E, it can be observed that

$$\begin{aligned}\|F^\varepsilon(\varpi,\alpha)(t)\| &= \left\|f^\varepsilon_\alpha(t,\varpi(t),\varpi'(t),\varpi''(t),\varpi'''(t))\right\| \\ &\le \max\{\|f(t,\varpi,y,q,p)\|,(\varpi,y,q,p)\in E\} + |N''(t)| + \|z''(t)\| + \|z^{(4)}(t)\| + |N^{(4)}(t)|,\end{aligned}$$

for all $\alpha \in \mathcal{J}$ and almost every $t \in \mathcal{J}$, where

$$\begin{aligned}E = \Big\{(u,y,q,p) \in \mathbb{R}^{4n} : \|u\| \le \|z\|_0 + \|N\|_0, \|y\| \le \|z'\|_0 + \|N'\|_0, \\ \|q\| \le \|z''\|_0 + \|N''\|_0,\ \ \|p\| \le 2\|\varpi'''\|_0 + \|z'''\|_0 + \|N'''\|_0\}.\end{aligned}$$

As f is L^1-Carathéodory, $z \in W^{4,1}(\mathcal{J},\mathbb{R}^n)$ and $N \in W^{4,1}(\mathcal{J},[0,\infty))$, it is easy to see that F^ε is integrally bounded.

In order prove the continuity, we should firstly prove that if $\left(\varpi_p, \alpha_p\right) \to (\varpi, \alpha)$ in $C^3(\mathcal{J}, \mathbb{R}^n) \times \mathcal{J}$, then

$$f^{\varepsilon}_{\alpha_p}(t, \varpi_p(t), \varpi'_p(t), \varpi''_p(t), \varpi'''_p(t)) \to f^{\varepsilon}_{\alpha}(t, \varpi(t), \varpi'(t), \varpi''(t), \varpi'''(t)) \ a.e.\ t \in \mathcal{J}. \tag{3.6}$$

Using the fact that f is L^1-Carathéodory, and from the definition of f^{ε}_{α}, it can be concluded that (3.6) is true a.e. on $\{t \in \mathcal{J} : \|\varpi''(t) - z''(t)\| \neq N''(t)\}$. Then, by Lemma 2.1 and Proposition 3.5 in [22], we easily show that $\breve{\varpi}'''_n(t) \to \varpi'''(t)$ on

$$\{t \in \mathcal{J} : \|\varpi''(t) - z''(t)\| = N''(t) > 0\},$$

where $\breve{\varpi}'''_n(t)$, is defined as (3.5). Then, (3.6) is satisfied on

$$\{t \in \mathcal{J} : \|\varpi''(t) - z''(t)\| = N''(t) > 0\}.$$

For

$$A = \{t \in \mathcal{J} : \|\varpi''(t) - z''(t)\| = N''(t) = 0\},$$

where $\varpi''(t) = z''(t)$, and by Lemma 2.1, it is not hard to see that $\varpi'''(t) = z'''(t)$, $N'''(t) = 0$ and $N^{(4)}(t) = 0$, $\forall t \in A$, which means,

$$\begin{aligned} f^{\varepsilon}_{\alpha}(t, \varpi(t), \varpi'(t), \varpi''(t), \varpi'''(t)) &= \alpha\Big(f_1(t, \varpi(t), \varpi'(t), \varpi''(t), \varpi'''(t)) - \varepsilon\varpi''(t)\Big) + (1-\alpha)\left(z^{(4)}(t) - \varepsilon z''(t)\right) \\ &= \alpha f_1(t, \varpi(t), \varpi'(t), z''(t), z'''(t)) - \varepsilon z''(t) + (1-\alpha)\, z^{(4)}(t), \end{aligned}$$

a.e. on A. By the solution tube hypothesis (Definition 3.1 condition (iii)), we have

$$\begin{aligned} f^{\varepsilon}_{\alpha}(t, \varpi(t), \varpi'(t), \varpi''(t), \varpi'''(t)) &= \alpha z^{(4)}(t) + (1-\alpha)\, z^{(4)}(t) - \varepsilon z''(t) \\ &= z^{(4)}(t) - \varepsilon z''(t), \end{aligned}$$

a.e. on A. Consequently, (3.6) must be true a.e. on $\mathcal{J}$. Using the Lebesgue-dominated convergence theorem, and since F^{ε} is integrally bounded, the proof can be concluded. □

Now, we can obtain our general existence result. We follow the method of proof given in [20].

Theorem 3.1. *Assume* $(\mathcal{F}1)$, $(\mathcal{H}1)$, *and the following conditions are satisfied:*

$(\mathcal{H}_k)$ *For every solution* ϖ *of the related system* (3.1), $\exists K > 0$, *so that*

$$\|\varpi'''(t)\| < K, \ \forall t \in \mathcal{J}.$$

Then, problem (1.1) *has a solution* $\varpi \in W^{4,1}(\mathcal{J}, \mathbb{R}^n) \cap T(z, N)$.

Proof. We first show that if $(\varpi, N) \in W^{4,1}_{\varpi_0, \varpi_1, B}(\mathcal{J}, \mathbb{R}^n) \times W^{4,1}(\mathcal{J}, [0, \infty))$ is a solution of (3.1), then

$$\|\varpi''(t) - z''(t)\| \leq N''(t), \forall t \in \mathcal{J}.$$

For the set

$$\{t \in \mathcal{J} : \|\varpi''(t) - z''(t)\| > N''(t)\}.$$

By the definition of $\widetilde{\varpi''}$ and $\breve{\varpi}'''(t)$ (as (3.4) and (3.5)), we have

$$\|\widetilde{\varpi''}(t) - z''(t)\| = N''(t), \tag{3.7}$$

$$< \widetilde{\varpi''}(t) - z''\,(t)\,, \breve{\varpi}'''(t) - z'''(t) >= N''(t)N'''(t).$$

Also

$$\|\breve{\varpi}'''(t) - z'''(t)\|^2 = \|\varpi'''(t) - z'''(t)\|^2 + (N'''(t))^2 - \frac{\langle \varpi''(t) - z''\,(t)\,, \varpi'''(t) - z'''(t)\rangle^2}{\|\varpi''(t) - z''(t)\|^2}.$$

Then, by $(\mathcal{H}1)$, we obtain

$$\begin{aligned}
&\frac{\langle \varpi''(t) - z''(t) - z^{(4)}(t)\rangle + \|\varpi'''(t) - z'''(t)\|^2}{\|\varpi''(t) - z''(t)\|} - \frac{\langle \varpi''(t) - z''(t), \varpi'''(t) - z'''(t)\rangle^2}{\|\varpi''(t) - z''(t)\|^3} - \varepsilon\,\|\varpi''(t) - z''(t)\| \\
=&\frac{\langle \varpi''(t) - z''(t), f_\alpha^\varepsilon(t, \varpi(t), \varpi'(t), \varpi''(t), \varpi'''(t)) + \varepsilon\varpi''(t)\rangle}{\|\varpi''(t) - z''(t)\|} + \frac{1}{\|\varpi''(t) - z''(t)\|}(\|\varpi'''(t) - z'''(t)\|^2 \\
&-\frac{\langle \varpi''(t) - z''(t), \varpi'''(t) - z'''(t)\rangle^2}{\|\varpi''(t) - z''(t)\|^2}) - \varepsilon\,\|\varpi''(t) - z''(t)\| \\
=&\frac{\langle \varpi''(t) - z''(t), \frac{\alpha N''(t)}{\|\varpi''(t) - z''(t)\|}(f_1(t, \varpi(t), \varpi'(t), \widetilde{\varpi''}(t), \breve{\varpi}'''(t)) - z^{(4)}(t))\rangle}{\|\varpi''(t) - z''(t)\|} \\
&+\frac{\langle \varpi''(t) - z''(t), (1 - \frac{\alpha N''(t)}{\|\varpi''(t) - z''(t)\|})\frac{N^{(4)}(t)(\varpi''(t) - z''(t))}{\|\varpi''(t) - z''(t)\|}\rangle}{\|\varpi''(t) - z''(t)\|} - \varepsilon\frac{\langle \varpi''(t) - z''(t), \alpha(\widetilde{\varpi''}(t) - z''(t)) - (\varpi''(t) - z''(t))\rangle}{\|\varpi''(t) - z''(t)\|} \\
&+\frac{\|\breve{\varpi}'''(t) - z'''(t)\|^2 - (N'''(t))^2}{\|\varpi''(t) - z''(t)\|} - \varepsilon\,\|\varpi''(t) - z''(t)\| \\
=&\frac{\alpha}{\|\varpi''(t) - z''(t)\|}\left\langle \widetilde{\varpi''}(t) - z''(t), f_1(t, \varpi(t), \varpi'(t), \widetilde{\varpi''}(t), \breve{\varpi}'''(t)) - z^{(4)}(t)\right\rangle + N^{(4)}(t)\left(1 - \frac{\alpha N''(t)}{\|\varpi''(t) - z''(t)\|}\right) \\
&-\varepsilon\,\|\varpi''(t) - z''(t)\| + \varepsilon\,\|\varpi''(t) - z''(t)\| - \varepsilon\frac{\langle \varpi''(t) - z''(t), \alpha(\widetilde{\varpi''}(t) - z''(t))\rangle}{\|\varpi''(t) - z''(t)\|} + \frac{\|\breve{\varpi}'''(t) - z'''(t)\|^2 - (N'''(t))^2}{\|\varpi''(t) - z''(t)\|} \\
\geq&\frac{\alpha}{\|\varpi''(t) - z''(t)\|}\left(N''(t) + (N'''(t))^2 - \|\breve{\varpi}'''(t) - z'''(t)\|^2\right) + N^{(4)}(t) - \frac{\alpha N''(t)}{\|\varpi''(t) - z''(t)\|} \\
&-\alpha\;\varepsilon N''(t) + \frac{\|\breve{\varpi}'''(t) - z''(t)\|^2 - (N'''(t))^2}{\|\varpi''(t) - z''(t)\|} \\
=&N^{(4)}(t) - \alpha\;\varepsilon N''(t) + \frac{(1-\alpha)\left(\|\breve{\varpi}'''(t) - z'''(t)\|^2 - (N'''(t))^2\right)}{\|\varpi''(t) - z''(t)\|} \\
\geq&N^{(4)}(t) - \varepsilon N''(t),
\end{aligned}$$

on

$$\{t \in \mathcal{J} : \|\varpi''\,(t) - z''\,(t)\| > N''\,(t)\}.$$

Using Lemma 3.1, it can be observed that any solutions to system (3.1) are in $T(z, N)$ and then, in U, where

$$U = \left\{\varpi \in C^3(\mathcal{J}, \mathbb{R}^n) : \|u^{(i)}\|_0 \leq \|z^{(i)}\|_0 + \|N^{(i)}\|_0 + 1,\ i = 1, 0, 2;\ \|\varpi'''\|_0 \leq K\right\}.$$

Fix $\varepsilon \in \mathcal{J}$ such that the operator $L_\varepsilon : C_B^1(\mathcal{J}, \mathbb{R}^n) \to C_0(\mathcal{J}, \mathbb{R}^n)$ given by

$$L_\varepsilon\,(\varpi)\,(t) = \varpi'(t) - \varpi'(0) - \varepsilon\int_0^t \varpi(s)ds$$

is invertible.

Consider the linear operator $D : C^3_{\varpi_0,\varpi_1,B}(\mathcal{J},\mathbb{R}^n) \to C^1_B(\mathcal{J},\mathbb{R}^n)$ defined by

$$D(\varpi) = \varpi''.$$

It can be easily confirmed that D is invertible.

A solution to (1.1) is a fixed point of the operator

$$K = D^{-1} o L_\varepsilon^{-1} o N_{F^\varepsilon} : C^3(\mathcal{J},\mathbb{R}^n) \times \mathcal{J} \to C^3_{\varpi_0,\varpi_1,B}(\mathcal{J},\mathbb{R}^n) \subset C^3(\mathcal{J},\mathbb{R}^n).$$

Using Proposition 3.1 and Theorem 2.1, and since the operators D and L_ε are continuous, it can be concluded that K is completely continuous and fixed point free on ∂U. Let

$$K_0 : C^3(\mathcal{J},\mathbb{R}^n) \times \mathcal{J} \to C^3(\mathcal{J},\ \mathbb{R}^n)$$

by $K_0(\varpi,\alpha) = \alpha K(\varpi, 0)$. Because $F^\varepsilon(.,0)$ is integrally bounded, there exists an open bounded set $\mathcal{K} \subset C^3(\mathcal{J},\mathbb{R}^n)$, where

$$U \subset \mathcal{K} \text{ and } K_0\left(C^3(\mathcal{J},\mathbb{R}^n) \times \mathcal{J}\right) \subset \mathcal{K},$$

it can be implied from the homotopic and the excision properties of the Leray-Schauder theorem that

$$\begin{aligned} 1 = deg\,(Id,\mathcal{K},0) &= deg\,(Id - K_0\,(.,1)\,,\mathcal{K},0) = deg\,(Id - K\,(.,0)\,,\mathcal{K},0) \\ &= deg\,(Id - K\,(.,0)\,,U,0) = deg\,(Id - K\,(.,1)\,,U,0)\,. \end{aligned}$$

As a result, there exists a solution $\varpi \in T(z,N)$ for $\alpha = 1$ to (3.1), which also can solve (1.1) by definition of f_1^ε. The proof is complete. □

Now, following from our general existence theorem (Theorem 3.1), other existence results will be presented. We will consider the following assumptions:

($\mathcal{H}$2) There exist a function $\gamma \in L^1(\mathcal{J},[0,\infty))$ and a Borel measurable function $\Psi \in C([0,\infty),[1,\infty))$ s.t.

(i) $\|f(t,\varpi,y,q,p)\| \leq \gamma\,(t)\,\Psi\,(\|p\|)\,, \forall t \in \mathcal{J}$ and $\forall\,(\varpi,y,q,p) \in \mathbb{R}^{4n}$, where $\|\varpi - z\,(t)\| \leq N\,(t)$, $\|y - z'\,(t)\| \leq N'\,(t)$ and $\|q - z''\,(t)\| \leq N''\,(t)$,

(ii) $\forall c \geq 0$, we have

$$\int_c^\infty \frac{d\tau}{\Psi\,(\tau)} = \infty.$$

($\mathcal{H}$3) There exist, a function $\gamma \in L^1(\mathcal{J},[0,\infty))$ and a Borel measurable function $\Psi \in C([0,\infty],]0,\infty))$ s.t.

(i) $\|\langle p, f(t,\varpi,y,q,p)\rangle\| \leq \Psi\,(\|p\|)\,(\gamma\,(t) + \|p\|)\,, \forall t \in \mathcal{J}$ and $\forall\,(\varpi,y,q,p) \in \mathbb{R}^{4n}$, where $\|\varpi - z\,(t)\| \leq N\,(t)$, $\|y - z'\,(t)\| \leq N'\,(t)$ and $\|q - z''\,(t)\| \leq N''\,(t)$,

(ii) $\forall c \geq 0$, we have

$$\int_c^\infty \frac{\tau d\tau}{\Psi\,(\tau) + \tau} = \infty.$$

($\mathcal{H}4$) $\exists r, b > 0$, $c \geq 0$ and a function $h \in L^1(\mathcal{J}, \mathbb{R})$ s.t. $\forall t \in \mathcal{J}$, $\forall (\varpi, y, q, p) \in \mathrm{R}^{4n}$, where

$$\|\varpi - z(t)\| \leq N(t), \|y - z'(t)\| \leq N'(t), \|q - z''(t)\| \leq N''(t),$$

and $\|p\| \geq r$, then

$$(b + c\|q\|)\,\sigma(t, \varpi, y, q, p) \geq \|p\| - h(t),$$

where

$$\sigma(t, \varpi, y, q, p) = \frac{\langle q, f(t, \varpi, y, q, p)\rangle + \|p\|^2}{\|p\|} - \frac{\langle p, f(t, \varpi, y, q, p)\rangle \langle q, p\rangle}{\|p\|^3}.$$

($\mathcal{H}5$) $\exists a \geq 0$ and $l \in L^1(\mathcal{J}, \mathbb{R})$ s.t.

$$\|f(t, \varpi, y, q, p)\| \leq a\left(\langle q, f(t, \varpi, y, q, p)\rangle + \|p\|^2\right) + l(t),$$

$\forall t \in \mathcal{J}$ and $\forall (\varpi, y, q, p) \in \mathbb{R}^{4n}$, where

$$\|\varpi - z(t)\| \leq N(t), \|y - z'(t)\| \leq N'(t),$$

and

$$\|q - z''(t)\| \leq N''(t).$$

Theorem 3.2. *Assume* ($\mathcal{F}1$), ($\mathcal{H}1$)*, and* ($\mathcal{H}2$) *are satisfied. If* (BC) *is given by* (1.2) *with* $\max\{\beta_0, \beta_1\}$ >0*, then system* (1.1) *has at least one solution* $\varpi \in T(z, N) \cap W^{4,1}(\mathcal{J}, \mathbb{R}^n)$.

Proof. Theorem 3.1 will guarantee the existence of a solution if we can obtain a priori bound on the third derivative of any solution ϖ to (3.1). It is known that $\varpi \in T(z, N)$ from the Theorem 3.1 proof. Therefore, since (BC) is given by (1.2) with $\max\{\beta_0, \beta_1\} > 0$, $\exists k > 0$, s.t.

$$min\{\|\varpi'''(0)\|, \|\varpi'''(1)\|\} \leq k.$$

Now, let $R > k$ such that

$$\int_k^R \frac{ds}{\Psi(s)} > L = \|\gamma\|_{L^1} + \varepsilon\|N''\|_0 + \|z^{(4)}\|_{L^1} + \left\|N^{(4)}\right\|_{L^1}.$$

Suppose there exists $t_1 \in [0, 1]$ s.t. $\|\varpi'''(t_1)\| \geq R$. Then, there exists $t_0 \neq t_1 \in [0, 1]$ such that $\|\varpi'''(t_0)\| = k$ and $\|\varpi'''(t)\| \geq k$, $\forall t \in [t_0, t_1]$. Let us assume that $t_0 < t_1$. Thus, by ($\mathcal{H}2$), almost everywhere on $[t_0, t_1]$, we have

$$\begin{aligned}
\|\varpi'''(t)\|' &= \frac{\langle \varpi'''(t)\rangle}{\|\varpi'''(t)\|} \leq \left\|\varpi^{(4)}(t)\right\| \\
&\leq \left\|f(t, \varpi(t), \varpi'(t), \varpi''(t), \varpi'''(t))\right\| + \varepsilon\|\varpi''(t) - z''(t)\| + \left\|z^{(4)}(t)\right\| + \left|N^{(4)}(t)\right| \\
&\leq \|\gamma(t)\|\,\Psi(\|\varpi'''(t)\|) + \varepsilon\|N''(t)\|_0 + \|z^{(4)}(t)\| + \left\|N^{(4)}\right\|_{L^1}.
\end{aligned}$$

So,

$$\int_{t_0}^{t_1} \frac{\|\varpi'''(t)\|' t}{\Psi(\|\varpi'''(t)\|)} dt \leq L.$$

Then, we have

$$\int_{t_0}^{t_1} \frac{\|\varpi'''(t)\|' t}{\Psi(\|\varpi'''(t)\|)} dt = \int_{\|\varpi'''(t_0)\|}^{\|\varpi'''(t_1)\|} \frac{ds}{\Psi(s)} \geq \int_k^R \frac{ds}{\Psi(s)} > L,$$

which contradict the assumptions. So, for any solution ϖ of (3.1), $\exists R > 0$ s.t. $\|\varpi'''(t)\| < R$, $\forall t \in \mathcal{J}$. □

If $(\mathcal{H}2)$ is replaced by $(\mathcal{H}3)$, extra assumptions are needed.

Theorem 3.3. *Assume $(\mathcal{F}1)$, $(\mathcal{H}1)$, $(\mathcal{H}3)$, and $(\mathcal{H}4)$ or $(\mathcal{H}5)$ are satisfied. Then, there exists a solution $\varpi \in T(z, N) \cap W^{4,1}(\mathcal{J}, \mathbb{R}^n)$ to* (1.1).

For this end, we need the next three Lemmas.

Lemma 3.2. *[20] Let $r, k \geq 0$, $N \in L^1([0, 1], \mathbb{R})$ and $\Psi \in C([0, \infty[,]0, \infty[)$ be a Borel measurable function s.t.*

$$\int_r^\infty \frac{\tau d\tau}{\Psi(\tau)} > \|N\|_{L^1([0,\ 1],\mathbb{R})} + k.$$

Then $\exists K > 0$, s.t. $\|\varpi'\|_0 < K$, $\forall \varpi \in W^{2,1}([0, 1], \mathbb{R}^n)$ satisfy:

(i) $min_{t\in[0,\ 1]} \|\varpi'(t)\| \leq r$;

(ii) $\|\varpi'\|_{L^1([t_0,\ t_1],R)} \leq k$ for every interval $[t_0, t_1] \subset \{t \in [0, 1] : \|\varpi'(t)\| \geq r\}$;

(iii) $|\langle \varpi'(t), \varpi''(t)\rangle| \leq \Psi(\|\varpi'(t)\|)(N(t) + \|\varpi'(t)\|)$ a.e. on

$$\{t \in [0, 1] : \|\varpi'(t)\| \geq r\}.$$

Lemma 3.3. *[20] Let $r, \nu > 0$, $\gamma \geq 0$ and $N \in L^1([0, 1], \mathbb{R})$. Then there exists a nondecreasing function $\omega \in C[0, \infty[, [0, \infty[)$ s.t.*

$$\|\varpi'\|_{L^1([t_0,\ t_1],\mathbb{R})} \leq \omega(\|\varpi\|_0),$$

and

$$\min_{t\in[0,\ 1]} \|\varpi'(t)\| \leq \max\{r, ; \omega(\|\varpi\|_0)\}.$$

$\forall u \in W^{2,1}([0, 1], \mathbb{R}^n)$ and

$$\{t \in [t_0, t_1] : \|\varpi'(t)\| \geq r\},$$

the following inequality

$$(\nu + \gamma\|\varpi(t)\|)\,\sigma_0(t, \varpi) + \frac{\gamma\langle \varpi(t), \varpi'(t)\rangle^2}{\|\varpi(t)\|\|\varpi'(t)\|} \geq \|\varpi'(t)\| - N(t)$$

is satisfied, where

$$\sigma_0(t, \varpi) = \frac{\langle \varpi(t), \varpi''(t)\rangle + \|\varpi'^2}{\|\varpi'(t)\|} - \frac{\langle \varpi'(t), \varpi''(t)\rangle\langle \varpi(t), \varpi'(t)\rangle}{\|\varpi'(t)\|^3}.$$

Lemma 3.4. *[20] Let $K > 0$, and $N \in L^1([0, 1], \mathbb{R})$. Then there exists an increasing function $\omega \in C([0, \infty[,]0, \infty[)$ s.t. $\|\varpi'\|_{L^1([0,\ 1],\mathbb{R})} \leq \omega(\|\varpi\|_0)$ for all $\varpi \in W^{2,1}([0, 1], \mathbb{R}^n)$ that satisfies*

$$\|\varpi''(t)\| \leq k(\langle \varpi(t), \varpi''(t)\rangle + \|\varpi'(t)\|^2) + N(t),$$

for almost every $t \in [0, 1]$.

Proof of Theorem 3.3. Similarly to the previous proof, we need Theorem 3.1 to prove that the third derivative of all solutions ϖ to (3.1) is bounded. Let ϖ be a solution to (3.1), where $\varpi \in \mathrm{T}(z, N)$ from Theorem 3.1 proof. We obtain from $(\mathcal{H}3)$,

$$\begin{aligned}
&|\langle \varpi'''(t), \varpi^{(4)}(t)\rangle| \\
&\leq |\langle \varpi'''(t), f(t, \varpi(t), \varpi'(t), \varpi''(t), \varpi'''(t))\rangle| + (\varepsilon \|\varpi''(t) - z''(t)\| + \|z^{(4)}(t)\| + |N^{(4)}(t)|) \|\varpi'''(t)\| \\
&\leq (\gamma(t) + \|\varpi'''(t)\|) \Psi(\|\varpi'''(t)\|) + (\varepsilon |N''(t)| + \|z^{(4)}(t)\| + |N^{(4)}(t)|) \|\varpi'''(t)\| \\
&\leq (\Psi(\|\varpi'''(t)\|) + \|\varpi'''(t)\|) + (\gamma(t) + \|\varpi'''(t)\| + \varepsilon |N''(t)| + \|z^{(4)}(t)\| + |N^{(4)}(t)|),
\end{aligned}$$

for almost every $t \in [0, 1]$. Thus, condition (iii) of Lemma 3.2 is verified, where

$$\psi(\tau) = \Psi(\tau) + \tau \ \textit{and} \ \ N(\tau) = \gamma(\tau) + \varepsilon |N''(\tau)| + \|z^{(4)}(\tau)\| + |N^{(4)}(t)|.$$

Therefore, it is enough to prove that conditions (i) and (ii) are verified. $(\mathcal{H}4)$ guarantees that a.e. on

$$\{t \in [0, 1] : \|\varpi'''(t)\| \geq r\},$$

we have

$$\begin{aligned}
\sigma_0(t, \varpi'') &= \frac{\langle \varpi''(t)\rangle + \|\varpi'''(t)\|^2}{\|\varpi'''(t)\|} - \frac{\langle \varpi'''(t)\rangle \langle \varpi''(t), \varpi'''(t)\rangle}{\|\varpi'''(t)\|^3} \\
&= \alpha\, \sigma(t, \varpi(t), \varpi'(t), \varpi''(t), \varpi'''(t)) + (1-\alpha) \|\varpi'''(t)\| \\
&\quad + \frac{(1-\alpha) \langle \varpi''(t) + (\varepsilon + \frac{N^{(4)}(t)}{N''(t)}) (\varpi''(t) - z''(t))\rangle}{\|\varpi'''(t)\|} \\
&\quad - \frac{(1-\alpha) \langle \varpi'''(t) + (\varepsilon + \frac{N^{(4)}(t)}{N''(t)}) (\varpi''(t) - z''(t))\rangle \langle \varpi''(t), \varpi'''(t)\rangle}{\|\varpi'''(t)\|^3} \\
&\geq \alpha\, \sigma(t, \varpi(t), \varpi'(t), \varpi''(t), \varpi'''(t)) + (1-\alpha) \|\varpi'''(t)\| \\
&\quad - \frac{2 (\|z''(t)\| + |N''(t)|) \left(\left\|z^{(4)}(t)\right\| + \varepsilon |N''(t)| + \left|N^{(4)}(t)\right|\right)}{r}.
\end{aligned}$$

Thus, we have

$$(b + c \|\varpi''(t)\|) \sigma_0(t, \varpi'') + c \frac{\langle \varpi''(t), \varpi'''(t)\rangle^2}{\|\varpi''(t)\| \|\varpi'''(t)\|} \geq \alpha \|\varpi'''(t)\| + b(1-\alpha) \|\varpi'''(t)\| - h(t) - \delta_0, (t),$$

where

$$\delta_0(t) = \frac{2}{r} (b + c \|z''(t)\| + c |N''(t)|) (\|z''(t)\| + |N''(t)|) (\|z^{(4)}(t)\| + \varepsilon |N''(t)| + |N^{(4)}(t)|).$$

If we take

$$z = \min_{\alpha \in [0, 1]} \{\alpha + b(1-\alpha)\}, \ \nu = \tfrac{b}{z} \text{ and } \theta = \tfrac{c}{\eta},$$

we can apply Lemma 3.3 to $\varpi''([0, 1], \mathbb{R}^n)$. Thus, conditions of Lemma 3.2 are verified. Moreover, if $(\mathcal{H}5)$ holds, we have

$$\|\varpi^{(4)}(t)\| \leq \alpha \|f(t, \varpi(t), \varpi'(t), \varpi''(t), \varpi'''(t))\| + \varepsilon \|\varpi''(t) - z''(t)\| + \|z^{(4)}(t)\| + |N^{(4)}(t)|$$

$$
\begin{aligned}
&\leq \alpha a\left(\langle \varpi''(t),\ f(t,\varpi(t),\varpi'(t),\varpi''(t),\varpi'''(t))\rangle + \|\varpi'''(t)\|^2\right) + l(t)\\
&\quad + \varepsilon\,|N''(t)| + \|z^{(4)}(t)\| + |N^{(4)}(t)|\\
&\leq a\left(\left\langle \varpi''(t),\ \varpi^{(4)}(t)\right\rangle + \|\varpi'''(t)\|^2\right) + \varepsilon\,|N''(t)| + \left\|z^{(4)}(t)\right\| + \left|N^{(4)}(t)\right|\\
&\quad - a\,(1-\alpha)\left\langle \varpi''(t),\ z^{(4)}(t) + \left(\frac{N^{(4)}(t)}{N''(t)} + \varepsilon\right)(\varpi''(t) - z''(t))\right\rangle\\
&\leq a\left(\left\langle \varpi''(t),\ \varpi^{(4)}(t)\right\rangle + \|\varpi'''(t)\|^2\right) + \varepsilon\,|N''(t)| + \left\|z^{(4)}(t)\right\| + \left|N^{(4)}(t)\right|\\
&\quad + a\,(\|z''(t)\| + |N''(t)|)\left(\|z^{(4)}(t)\| + |N^{(4)}(t)| + \varepsilon N''(t)\right).
\end{aligned}
$$

Therefore, if Lemma 3.4 is applied to $\varpi''([0,\ 1],\mathbb{R}^n)$, all conditions of Lemma 3.2 are satisfied. As a result, for all solutions ϖ of (3.1), $\|\varpi'''\|_0 < K$ for some constant $K > 0$. □

From the previous results, we obtain the following consequence:

Corollary 3.1. *Assume* $(\mathcal{F}1)$, $(\mathcal{H}1)$, $(\mathcal{H}2)$, *and* $(\mathcal{H}4)$ *or* $(\mathcal{H}5)$ *are satisfied. Then, we have a solution* $\varpi \in T(z,N) \cap W^{4,1}([0,\ 1],\mathbb{R}^n)$ *to the system* (1.1).

Remark 3.1. *Definition 3.1 is associated to the definitions of lower and upper solutions to the fourth-order differential equation. These definitions are used in [17], and introduce them for problems (1.1) and (1.2).*

Definition 3.2. *Let* $n = 1$ *and* $\varpi_0 = \varpi_1 = 0$.
A function $\kappa \in C^4(]0,1[) \cap C^3(\mathcal{J})$ *is called a lower solution to (1.1) and (1.2), if*

(i) $\kappa^{(4)}(t) \geq f(t,\kappa(t),\kappa'(t),\kappa''(t),\kappa'''(t))$ *for every* $t \in \mathcal{J}$;
(ii) $\kappa(0) = \kappa'(0) = 0$;
(iii) $A_0\kappa''(0) - \beta_0\kappa'''(0) \leq r_0$ *and* $A_1\kappa''(1) + \beta_1\kappa'''(1) \leq r_1$.

On the other hand, an upper solution to (1.1) and (1.2) is a function $\nu \in C^4(]0,1[)\cap C^3(\mathcal{J})$ *that satisfies (i)–(iii) with reversed inequalities.*

Similarly to Remark 3.2 in [20], we consider the following assumptions:

(A) There exist lower and upper solutions, κ *and* ν*, respectively, to* (1.1) *and* (1.2)*, where* $\kappa \leq \nu$.
(B) There exists a solution-tube (z,N) *to (1.1) and (1.2).*
(C) There exist lower and upper solutions, $\kappa \leq \nu$*, to (1.1) and (1.2) s.t.*

  (i) $\kappa''(t) \leq \nu''(t))$ *for all* $t \in \mathcal{J}$;
  (ii) $f(t,\nu(t),\nu'(t),q,p) \leq f(t,\varpi,y,q,p) \leq f(t,\kappa(t),\kappa'(t),q,p)$; $\forall t \in \mathcal{J}$ *and* $(\varpi,\ y,\ q,p) \in \mathbb{R}^{4n}$ *such that* $\kappa(t) \leq \varpi \leq \nu(t)$ *and* $\kappa'(t) \leq \varpi'(t) \leq \nu'(t)$.

It can be easily checked that

- *If* (B) *holds with* z *and* N *of class* C^4*, and* $z(0) = N(0) = 0$*, then* (A) *holds.*

Indeed, $\kappa = z - N$ *and* $\nu = z + N$ *are respectively lower and upper solutions of* (1.1) *and* (1.2)*. However,* (A) *does not imply* (B).
Noting that (B) is more general than (C), see [17]; i.e.,

- *If* (C) *is verified, then* (B) *is verified.*

Taking $z = (\kappa + \nu)/2$ *and* $N = (\nu - \kappa)/2$. *But,* (B) *does not imply* (C) *(ii) and* $\kappa(0) = \nu(0) = 0$.

Next, we present two examples to illustrate the applicability of Theorem 3.3.

Example 3.1. *Consider the following system:*

$$\begin{cases} \varpi^{(4)}(t) = \varpi'''(t) + \|\varpi'''(t)\| \left(\|\varpi''(t)\|^2 \varpi'(t) - \langle \varpi'(t),\ \varpi''(t)\rangle \varpi''(t)\right) - \xi, \ a.e.\ t \in \mathcal{J}, \\ \varpi(0) = 0,\ \varpi'(0) = 0,\ A_0\varpi''(0) = 0,\ A_1\varpi''(1) + \beta_{t=1}\varpi'''(1) = 0, \end{cases} \tag{3.8}$$

here $\xi \in \mathbb{R}^n, \|\xi\| = 1$, *and* A_i *and* β_i *are given before for* $i = 0, 1$. *Show that when* $z \equiv 0,\ N(t) = \frac{t^3}{6}$, (z, N) *is a solution-tube of* (3.8). *We have* $(\mathcal{H}3)$ *and* $(\mathcal{H}4)$ *are verified for*

$$\Psi(\tau) = 3\tau + 1, \gamma(t) = 0,\ b = 1,\ c = 0,\ r > 0, h(\tau) = \frac{2\tau}{r} + \tau^5.$$

Owing to the Theorem 3.3*, the problem* (3.8) *has at least one solution* ϖ *s.t.*

$$\|\varpi(t)\| \leq \frac{t^3}{6},\ \left\|\varpi(t)'\right\| \leq \frac{t^2}{2}\ \text{and}\ \left\|\varpi(t)''\right\| \leq t\ \text{for all}\ t \in \mathcal{J}.$$

Example 3.2. *Consider the following system:*

$$\begin{cases} \varpi^{(4)}(t) = \varpi''(t)\left(\|\varpi'''(t)\|^2 + 1\right) + \varphi(t),\ a.e.\ t \in \mathcal{J}, \\ \varpi(0) = 0,\ \varpi'(0) = 0,\ \varpi''(0) = \varpi''(1),\ \varpi'''(0) = \varpi'''(1), \end{cases} \tag{3.9}$$

where $\varphi \in L^\infty(\mathcal{J}, \mathbb{R}^n)$ *with* $\|\varphi\|_{L^\infty} \leq 1$. *Show that for* $z \equiv 0,\ N(t) = \frac{t^2}{2}$, (z, N) *is a solution-tube of* (3.9). *We have* $(\mathcal{H}3)$ *and* $(\mathcal{H}5)$ *are verified when*

$$\Psi(\tau) = \tau^2 + 2,\ \gamma(t) = 0,\ a = 1,\ l(t) = 3.$$

By Theorem 3.3*, the problem* (3.9) *has at least one solution* ϖ *s.t.*

$$\|\varpi(t)\| \leq \frac{t^2}{2},\ \|\varpi'(t)\| \leq t,\ \|\varpi''(t)\| \leq 1, \forall t \in \mathcal{J}.$$

4. Conclusions

Our paper discusses the existence of solutions for fourth-order differential equation systems, focusing particularly on cases involving L1-Carathéodory functions on the right-hand side of the equations. We first, introduced the concept of a solution-tube, which is an innovative approach that extends the concepts of upper and lower solutions applicable to fourth-order equations into the domain of systems. It outlines the mathematical framework necessary to demonstrate that solutions exist for these types of differential equation systems under specified boundary conditions (such as Sturm-Liouville and periodic conditions). The paper stands on prior results regarding higher-order differential equations, providing a fresh perspective and methodology that can be used to explore further developments in the field. In addition to presenting the theoretical underpinnings, we also illustrated the practicality of our results with examples, contributing to the mathematical discourse on differential equations and our solutions, which ultimately serves as a scholarly contribution to understanding the dynamics of fourth-order systems and the existence of their solutions; please see [26, 27].

Author contributions

Bouharket Bendouma: Conceptualization, formal analysis, Writing-original draft preparation; Fatima Zohra Ladrani and Keltoum Bouhali: investigation, Methodology; Ahmed Hammoudi and Loay Alkhalifa: Writing-review and editing. All authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

The authors declare that there is no conflict of interest.

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