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Research article

A symplectic fission scheme for the association scheme of rectangular matrices and its automorphisms

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Abstract: In this paper, a symplectic fission scheme for the association scheme of $m \times n$ rectangular matrices over the finite field $\mathbb{F}_q$, denoted by $\mathrm{SMat}(m \times n, q)$, is constructed, where q is a power of a prime number. We discuss its association classes and inner automorphism group. In particular, we determine the intersection numbers and automorphism group of $\mathrm{SMat}(m \times n, q)$ for $m = 1$ and $m = 2$.

Keywords: association scheme; fission scheme; intersection number; automorphism
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1. Introduction

The concept of the association scheme together with the partially balanced incomplete block designs was defined in its own right by Bose and Shimamoto in 1952 [2]. It was introduced to describe the balance relations among the treatments of partially balanced incomplete block designs. Association schemes have close connections with coding theory, graph theory, and finite group theory and, in particular, provide a framework for studying codes and designs. By the 1980s, association scheme theory had become an important branch of algebraic combinatorics, and the research work on association scheme theory had grown tremendously; see [1].

The study of association schemes in China was started by Chang and Hsu in the late 1950s. In the mid 1960s, Wan constructed a family of association schemes on Hermitian matrices and computed the parameters of the lower-dimensional ones and started a new direction of construction of association schemes on matrices. The association scheme theory developed later indicates the association schemes of maximal totally isotropic subspaces and of Hermitian matrices are what is known as primitive P-polynomial and Q-polynomial association schemes. In the late 1970s, Wang continued the study of association schemes of matrices. He derived formulas for the parameters of association schemes of Hermitian matrices and construct association schemes using rectangular matrices and alternate matrices. Later, Wan et al., studied the association schemes of symmetric matrices in odd characteristic. In the 1990s, Wang, with his students, studied the association schemes of symmetric matrices and quadratic forms in even characteristic. Besides the parameters of these association schemes, they discussed the subschemes, quotient schemes, and duality and automorphisms [3, 4, 6, 7]. So, the study of association schemes of matrices reaches a more complete stage. The results on association schemes of matrices are collected in [5].

Let $\mathbb{F}_q$ be the finite field with q elements, and $\mathcal{M}_{mn}(\mathbb{F}_q)$ be the set of $m\times n$ matrices over $\mathbb{F}_q$, where q is a power of a prime number and $m \le n$. For brevity, we write $\mathcal{M}_{mn}(\mathbb{F}_q)$ by $\mathcal{M}_{mn}$. Let $GL_n(\mathbb{F}_q)$ be the general linear group of degree n over $\mathbb{F}_q$ and $G_0 = GL_m(\mathbb{F}_q)\times GL_n(\mathbb{F}_q)$(a direct product). The group G_0 acts on $\mathcal{M}_{mn}$:

$$\begin{array}{rcl} G_0 \times \mathcal{M}_{mn} & \longrightarrow & \mathcal{M}_{mn} \\ ((P, Q), X) & \longmapsto & PXQ. \end{array}$$

Let T_0 be the group of right translation of $\mathcal{M}_{mn}$, and G be the group generated by G_0 and T_0. Then G acts transitively on $\mathcal{M}_{mn}$, which determines an association scheme $(\mathcal{M}_{mn}, \{R_i\}_{0\le i\le m})$, where

$$R_i = \{(X, Y) \in \mathcal{M}_{mn} \times \mathcal{M}_{mn} | \mathrm{rank}(X - Y) = i\}.$$

It is called the association scheme of rectangular matrices and denoted by $\mathrm{Mat}(m \times n, q)$.

Lemma 1.1. *[5]* (i) *When $m = 1$,* $\mathrm{Mat}(m\times n, q)$ *is a trivial association scheme, and its automorphism group is $Sym(q^n)$.*

(ii) *When $1 < m \le n$, each automorphism of the association scheme* $\mathrm{Mat}(m \times n, q)$ *must have the following form:*

$$X \mapsto PX^{\sigma}Q + A, \forall X \in \mathcal{M}_{mn},$$

where $P \in GL_m(\mathbb{F}_q)$, $Q \in GL_n(\mathbb{F}_q)$, $A \in \mathcal{M}_{mn}$, and σ is an automorphism of $\mathbb{F}_q$.

In addition, if $m = n$, the following mapping is also an automorphism

$$X \mapsto P({}^tX)^{\sigma}Q + A, \forall X \in \mathcal{M}_{mn},$$

where ${}^tX$ is the transpose of X.

Next, let $n = 2\nu$. We replace the group G_0 with $\overline{G_0} = GL_m(\mathbb{F}_q) \times Sp_{2\nu}(\mathbb{F}_q)$, where $Sp_{2\nu}(\mathbb{F}_q)$ is the symplectic group of degree 2ν over $\mathbb{F}_q$. Then $\overline{G}$, generated by $\overline{G_0}$ and T_0, acts transitively on $\mathcal{M}_{mn}$, which determines a fission scheme of $\mathrm{Mat}(m \times n, q)$. We call it the symplectic fission scheme of $\mathrm{Mat}(m \times n, q)$, denoted by $\mathrm{SMat}(m \times n, q)$. In this paper, we discuss the association classes and inner automorphism group of $\mathrm{SMat}(m\times n, q)$. In particular, we determine the intersection numbers and automorphism group for $m = 1$ and $m = 2$.

2. Preliminaries

2.1. Definition of association schemes

Definition 2.1. *Let X be a nonempty set of cardinality n and $R_0, R_1, \cdots, R_d$ be subsets of $X \times X$ that satisfy the following conditions:*

(i) $R_0 = \{(x, x)|x \in X\}$;
(ii) $X \times X = R_0 \cup R_1 \cup \cdots \cup R_d, R_i \cap R_j = \emptyset (i \neq j)$;
(iii) for each $i \in \{0, 1, \cdots, d\}$, there exists some $i' \in \{0, 1, \cdots, d\}$ such that $R_i^t = R_{i'}$, where $R_i^t = \{(x, y)|(y, x) \in R_i\}$;
(iv) for any $i, j, k \in \{0, 1, \cdots, d\}$, the number

$$p_{ij}^k =| \{z \in X \mid (x, z) \in R_i, (z, y) \in R_j\} |$$

is constant whenever $(x, y) \in R_k$.

Such a configuration $\mathfrak{X} = (X, \{R_i\}_{0 \leq i \leq d})$ is called an association scheme of class d on X. R_0 is called the trivial or diagonal relation, while the others are called nontrivial relations. Note that d is the number of nontrivial relations. The numbers p_{ij}^k are called the intersection numbers of $\mathfrak{X}$. The association scheme $\mathfrak{X}$ is said to be commutative if
(v) $p_{ij}^k = p_{ji}^k$ for all $i, j, k \in \{0, 1, ..., d\}$.
Further, $\mathfrak{X}$ is said to be symmetric (or Bose-Mesner type) if
(vi) $i' = i$ for all $i \in \{0, 1, ..., d\}$.

Example 2.1. *[5] Let G be a finite group acting transitively on a finite set* Ω. *This induces an action on* $\Omega \times \Omega$*: for* $(x, y) \in \Omega \times \Omega$ *and* $\sigma \in G$, $(x, y)^\sigma = (x^\sigma, y^\sigma)$. *Then G no longer acts transitively on* $\Omega \times \Omega$ *if* $|\Omega| = n > 1$. *Let* $R_0, R_1, \cdots, R_d$ *be the orbits of G on* $\Omega \times \Omega$*, where* $R_0 = \{(x, x)|x \in \Omega\}$. *Then* $\mathfrak{X} = (\Omega, \{R_i\}_{0 \leq i \leq d})$ *is an association scheme (not necessarily commutative).*

Let $\mathfrak{X} = (X, \{R_i\}_{0 \leq i \leq d})$ be an association scheme of class d on X and $k_i = p_{ii'}^0$. The number k_i is the number of $y \in X$ such that $(x, y) \in R_i$ for any fixed $x \in X$. It is called the valency of R_i. Clearly,

$$k_0 = 1, |X| = k_0 + k_1 + \cdots + k_d.$$

Let δ be the Kronecker delta: $\delta_{ij} = 0$ for $i \neq j$, and $\delta_{ii} = 1$. Then the following holds:

$$p_{0j}^k = \delta_{jk},\ p_{i0}^k = \delta_{ik},\ p_{ij}^0 = k_i \delta_{ij'},\ \sum_{j=0}^{d} p_{ij}^k = k_i,\ k_\gamma p_{\alpha\beta}^\gamma = k_\beta p_{\alpha'\gamma}^\beta = k_\alpha p_{\gamma\beta'}^\alpha, \tag{2.1}$$

where $\alpha, \beta, \gamma, \alpha', \beta' \in \{0, 1, \cdots, d\}$, $R_{\alpha'} = \{(x, y)|(y, x) \in R_\alpha\}$, and $R_{\beta'} = \{(x, y)|(y, x) \in R_\beta\}$.

Let $\mathfrak{X} = (X, \{R_i\}_{0 \leq i \leq d})$ and $\mathfrak{X}' = (X, \{S_j\}_{0 \leq j \leq d'})$ be two association schemes on X. If each relation S_j is a union of some R_i, then $\mathfrak{X}'$ is said to be a fusion scheme of $\mathfrak{X}$, and $\mathfrak{X}$ is said to be a fission scheme of $\mathfrak{X}'$. Furthermore, let $\mathfrak{Y} = (Y, \{T_i\}_{0 \leq k \leq d})$ is an association scheme satisfying $|X| = |Y|$. If a bijection $f : X \to Y$ induces a permutation $\sigma(f)$ on $\{0, 1, \cdots, d\}$ by $(f(x), f(z)) \in T_{i^{\sigma(f)}}$ for $(x, z) \in R_i$, f is called an isomorphism between $\mathfrak{X}$ and $\mathfrak{Y}$. In this case, $\mathfrak{X}$ and $\mathfrak{Y}$ are said to be isomorphic. An isomorphism f from an association scheme $\mathfrak{X}$ to itself is called an automorphism. The set of all automorphisms of $\mathfrak{X}$ becomes a group, called the automorphism group of $\mathfrak{X}$ and denoted by $\mathrm{Aut}\mathfrak{X}$. An automorphism f of $\mathfrak{X}$ is called an inner automorphism if it induces the identity permutation on $0, 1, \cdots, d$, i.e., $i^{\sigma(f)} = i (i = 0, 1, \cdots, d)$. Clearly, the set of inner automorphisms of $\mathfrak{X}$ becomes a normal subgroup of $\mathrm{Aut}\mathfrak{X}$, denoted by $\mathrm{Inn}\mathfrak{X}$. The quotient group $\mathrm{Aut}\mathfrak{X}/\mathrm{Inn}\mathfrak{X}$ is called the outer automorphism group of $\mathfrak{X}$.

2.2. Symplectic geometry over the finite field

Let

$$K = \begin{pmatrix} & I^{(\nu)} \\ -I^{(\nu)} & \end{pmatrix},$$

where $I^{(\nu)}$ is the $\nu \times \nu$ identity matrix. The set of all $2\nu \times 2\nu$ matrices T over $\mathbb{F}_q$ satisfying $TK^tT = K$ forms a group with respect to the matrix multiplication, called the symplectic group of degree 2ν over $\mathbb{F}_q$, and is denoted by $Sp_{2\nu}(\mathbb{F}_q)$. A $2\nu \times 2\nu$ matrix T is called a generalized symplectic matrix of degree 2ν over $\mathbb{F}_q$ if $TK^tT = kK$ for some $k \in \mathbb{F}_q^*$. The set of generalized symplectic matrices of degree 2ν over $\mathbb{F}_q$ forms a group with respect to the matrix multiplication, which is called the generalized symplectic group of degree 2ν over $\mathbb{F}_q$ and denoted by $GSp_{2\nu}(\mathbb{F}_q)$.

Let $\mathbb{F}_q^{(2\nu)}$ be the 2ν-dimensional row vector space over $\mathbb{F}_q$. There is a natural action of $Sp_{2\nu}(\mathbb{F}_q)$ on $\mathbb{F}_q^{(2\nu)}$ by the vector matrix multiplication as follows:

$$\begin{array}{rcl} \mathbb{F}_q^{(2\nu)} \times Sp_{2\nu}(\mathbb{F}_q) & \longrightarrow & \mathbb{F}_q^{(2\nu)} \\ (\alpha, T) & \longmapsto & \alpha T. \end{array}$$

The space $\mathbb{F}_q^{(2\nu)}$ together with this action is called the 2ν-dimensional symplectic space over $\mathbb{F}_q$. Suppose that P is an m-dimensional vector subspace of $\mathbb{F}_q^{(2\nu)}$. We use the same letter P to denote a matrix representation of P, i.e., P is an $m \times 2\nu$ matrix whose rows form a basis of P. It is clear that a matrix representation of a subspace is not unique. Two $m \times 2\nu$ matrices P_1 and P_2 of rank m represent the same subspace if and only if there is an $m \times m$ nonsingular matrix Q such that $P_1 = QP_2$. A subspace P is said to be of type (m, s) if the dimension of P is m and $\text{rank}(PK^tP) = 2s$.

Lemma 2.1. *[5] Subspaces of type (m, s) exist in $\mathbb{F}_q^{(2\nu)}$ if and only if $2s \le m \le \nu + s$.*

Lemma 2.2. *[5] Let P_1 and P_2 be two m-dimensional subspaces of $\mathbb{F}_q^{(2\nu)}$. Then there is a $T \in Sp_{2\nu}(\mathbb{F}_q)$ such that $P_1 = AP_2T$, where $A \in GL_m(\mathbb{F}_q)$, if and only if P_1 and P_2 are of the same type. In other words, $Sp_{2\nu}(\mathbb{F}_q)$ acts transitively on each set of subspaces of the same type.*

Corallary 2.1. *Let P be a subspace of type (m, s) in $\mathbb{F}_q^{(2\nu)}$, where $2s \le m \le \nu + s$. Then there are $A \in GL_m(\mathbb{F}_q)$ and $T \in Sp_{2\nu}(\mathbb{F}_q)$ such that*

$$APT = \begin{array}{c} \begin{array}{cccccc} s & m-2s & \nu+s-m & s & m-2s & \nu+s-m \end{array} \\ \begin{pmatrix} I^{(s)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I^{(s)} & 0 & 0 \\ 0 & I^{(m-2s)} & 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{l} s \\ s \\ m-2s \end{array} \end{array}.$$

Lemma 2.3. *[5] Let $2s \le m \le \nu + s$. Then the number of subspaces of type (m, s) in $\mathbb{F}_q^{(2\nu)}$ is given by*

$$N(m, s; 2\nu) = q^{2s(\nu+s-m)} \frac{\prod_{i=\nu+s-m+1}^{\nu}(q^{2i} - 1)}{\prod_{i=1}^{s}(q^{2i} - 1)\prod_{i=1}^{m-2s}(q^{i} - 1)}.$$

In this paper, we define $\Pi_{i\in\phi} f(i) = 1$, where ϕ is empty set and $f(i)$ is a function about i. For example, $\Pi_{i=2}^{1}(q^i - 1) = 1$.

3. Association classes of SMat$(m \times n, q)$

Let $n = 2\nu$ and $\overline{G_0} = GL_m(\mathbb{F}_q) \times Sp_{2\nu}(\mathbb{F}_q)$. By the introduction, $\overline{G}$, generated by $\overline{G_0}$ and T_0, acts transitively on $\mathcal{M}_{mn}$, which determines the symplectic fission scheme of Mat$(m \times n, q)$, denoted by SMat$(m \times n, q)$. Let $R_0, R_1, \cdots, R_d$ be the orbits of $\overline{G}$ on $\mathcal{M}_{mn} \times \mathcal{M}_{mn}$, where $R_0 = \{(X, X) \mid X \in \mathcal{M}_{mn}\}$. Then SMat$(m \times n, q) = (\mathcal{M}_{mn}, \{R_i\}_{0 \le i \le d})$.

Definition 3.1. *A matrix in $\mathcal{M}_{mn}$ is said to be of type (t, s), if the subspace generated by its row vectors is of type (t, s) in $\mathbb{F}_q^{(2\nu)}$. Two matrices P and Q in $\mathcal{M}_{mn}$ are said to be S-equivalent, denoted by $P \sim Q$, if there exist $A \in GL_m(\mathbb{F}_q)$ and $T \in Sp_{2\nu}(\mathbb{F}_q)$ such that $P = AQT$.*

Obviously, the S-equivalence between matrices is an equivalent relationship, and the equivalent classes are the orbits of $\overline{G_0}$ acting on $\mathcal{M}_{mn}$.

Theorem 3.1. *Let $P \in \mathcal{M}_{mn}$ be of type (t, s), then $2s \le t \le min\{m, \nu + s\}$, and*

$$P \sim M(t,s) = \begin{array}{c} \begin{matrix} s & t-2s & \nu+s-t & s & t-2s & \nu+s-t \end{matrix} \\ \begin{pmatrix} I^{(s)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I^{(s)} & 0 & 0 \\ 0 & I^{(t-2s)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} s \\ s \\ t-2s \\ m-t \end{matrix} \end{array}.$$

Proof. Obviously, $0 \le t \le m$. By Lemma 2.2, we have $2s \le t \le \min\{m, \nu + s\}$.

Since $\dim(P) = t$, there is $A_1 \in GL_m(\mathbb{F}_q)$ such that

$$A_1 P = \begin{pmatrix} P_1 \\ 0 \end{pmatrix},$$

where P_1 is the matrix representation of a subspace of type (t, s) in $\mathbb{F}_q^{(2\nu)}$. Then, by Corollary 2.1, there are $A_2 \in GL_t(\mathbb{F}_q)$ and $T \in Sp_{2\nu}(\mathbb{F}_q)$ such that

$$A_2 P_1 T = \begin{array}{c} \begin{matrix} s & t-2s & \nu+s-t & s & t-2s & \nu+s-t \end{matrix} \\ \begin{pmatrix} I^{(s)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I^{(s)} & 0 & 0 \\ 0 & I^{(m-2s)} & 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} s \\ s \\ t-2s \end{matrix} \end{array}.$$

Let

$$A_3 = \begin{array}{c} \begin{matrix} t & m-t \end{matrix} \\ \begin{pmatrix} A_2 & 0 \\ 0 & I^{(m-t)} \end{pmatrix} \begin{matrix} t \\ m-t \end{matrix} \end{array},$$

then $A_3 A_1 P T = M(t, s)$. The theorem holds. □

By the above theorem and Lemma 2.2, we obtain the necessary and sufficient conditions for two matrices to be S-equivalent immediately.

Theorem 3.2. *Let $P,\ Q \in \mathcal{M}_{mn}$, then P and Q are S-equivalent if and only if they are of the same type.*

Theorem 3.3. *Let $n = 2\nu$, then* $\mathrm{SMat}(m \times n, q) = (\mathcal{M}_{mn}, \{R_{(t,s)}\})$*, where*

$$(X,\ Y) \in R_{(t,s)} \ \textit{if and only if} \ Y - X \sim M(t, s),$$

$X,\ Y \in \mathcal{M}_{mn}$ and $2s \leq t \leq min\{m, \nu + s\}$.

The class number d of $\mathrm{SMat}(m \times n, q)$ *satisfies*

$$d + 1 = \begin{cases} (m+1)(m+3)/4, & \textit{if } 0 \leq m \leq \nu \textit{ and } m \textit{ is odd} ; \\ (m+2)^2/4, & \textit{if } 0 \leq m \leq \nu \textit{ and } m \textit{ is even} ; \\ (4m\nu - 2\nu^2 - m^2 + 2m + 2\nu + 3)/4, & \textit{if } \nu < m \leq 2\nu \textit{ and } m \textit{ is odd} ; \\ (4m\nu - 2\nu^2 - m^2 + 2m + 2\nu + 4)/4, & \textit{if } \nu < m \leq 2\nu \textit{ and } m \textit{ is even} . \end{cases}$$

The valency of $R_{(t,s)}$ is given by

$$k_{(t,s)} = q^{t(t-1)/2} \Pi_{i=m-t+1}^{m} (q^i - 1) N(t, s; 2\nu),$$

where $N(t, s; 2\nu)$ is defined in Lemma 2.3.

Proof. We discuss the orbits of $\overline{G}$ on $\mathcal{M}_{mn} \times \mathcal{M}_{mn}$ first. Let $P,\ Q \in \mathcal{M}_{mn}$ and $\tau_1 : X \mapsto X - P$ for each $X \in \mathcal{M}_{mn}$. Then $\tau_1 \in \overline{G}$ and under this transformation, (P, Q) could be carried into $(0, Q - P)$. Suppose $Q - P$ is of type (t, s), then $2s \leq t \leq \min\{m, \nu + s\}$ and there is $\tau_2 \in \overline{G_0}$ such that $(\tau_2(0), \tau_2(Q - P) = (0, M(t, s))$ by Theorem 3.1. By Theorem 3.2, different $(0, M(t, s))$ represent different orbits. Thus $\mathrm{SMat}(m \times n, q) = (\mathcal{M}_{mn}, \{R_{(t,s)}\})$.

In addition, let $\mathrm{SMat}(m \times n, q)$ is an association scheme of class d, then $d + 1$ is the number of S-equivalent classes, which is the number of (t, s). Clearly, $0 \leq t \leq m$. From $2s \leq t \leq \min\{m, \nu + s\}$, we deduce $t - \nu \leq s \leq [t/2]$. If $t \leq \nu$, then s can take $[t/2] + 1$ values. If $t > \nu$, then s can take $[t/2] - (t - \nu) + 1$ values. This means that

(i) If $0 \leq m \leq \nu$, then

$$d + 1 = \Sigma_{t=0}^{m}([t/2] + 1) = m + 1 + \Sigma_{t=0}^{m}[t/2].$$

(ii) If $\nu < m \leq 2\nu$, then the number of $\mathrm{SMat}(m \times n, q)$ is

$$d + 1 = \Sigma_{t=0}^{\nu}([[t/2] + 1) + \Sigma_{t=\nu+1}^{m}([t/2] - (t - \nu) + 1) = m + 1 - \Sigma_{s=1}^{m-\nu} s + \Sigma_{t=0}^{m}[t/2].$$

The results in the theorem can be obtained through simple calculations.

Finally, let's calculate the valency of $R_{(t,s)}$. By [5], the number of matrices of rank t in $\mathcal{M}_{mn}$ is

$$n_t = q^{t(t-1)/2} \frac{\Pi_{i=m-t+1}^{m}(q^i - 1)\Pi_{i=n-t+1}^{n}(q^i - 1)}{\Pi_{i=1}^{t}(q^i - 1))}.$$

For different t-dimensional subspaces, there are the same number of representation matrices of rank t in $\mathcal{M}_{mn}$. The number of t-dimensional subspaces in $\mathbb{F}_q^{(n)}$ is

$$N(t, n) = \frac{\prod_{i=n-t+1}^{n}(q^i - 1)}{\prod_{i=1}^{t}(q^i - 1)}.$$

Thus, there are $n_t/N(t, n)$ matrices in $\mathcal{M}_{mn}$ that represent the same t-dimensional subspace. By Lemma 2.3, there are $n_t N(t, s; 2\nu)/N(t, n)$ matrices in $\mathcal{M}_{mn}$ that represent the same subspace of type (t, s). The theorem holds. □

Theorem 3.4. *Let $m = 1$, then* $\mathrm{SMat}(m \times n, q) = (\mathcal{M}_{mn}, \{R_{(t,s)}\})$ *be a trivial association scheme, where* $(t, s) = (0, 0), (1, 0)$. *The valencies are* $k_{(0,0)} = 1, k_{(1,0)} = q^{2\nu} - 1$. *For the intersection numbers,*

(i) when $i = (0, 0)$, $p^k_{ij} = \delta_{jk}$.

(ii) when $i = (1, 0)$, p^k_{ij} could be obtained by the following table, whose rows are indexed by the value of j and columns indexed by the value of k.

	$(0, 0)$	$(1, 0)$
$(0, 0)$	0	1
$(1, 0)$	$q^{2\nu} - 1$	$q^{2\nu} - 2$

Proof. The values of (t, s), $k_{(0,0)}$ and $k_{(1,0)}$ could be obtained by Theorem 3.3 immediately. For the intersection numbers,

(i) when $i = (0, 0)$, $p^k_{ij} = \delta_{jk}$ by (2.1).

(ii) when $i = (1, 0)$, $p^{(0,0)}_{(1,0)(0,0)} = 0$, $p^{(1,0)}_{(1,0)(0,0)} = 1$, $p^{(0,0)}_{(1,0)(1,0)} = k_{(1,0)}$ and $p^{(1,0)}_{(1,0)(0,0)} + p^{(1,0)}_{(1,0)(1,0)} = k_{(1,0)}$ by Eq (2.1). Thus, $p^{(1,0)}_{(1,0)(1,0)} = k_{(1,0)} - 1$. □

Example 3.1. *Let $m = 1, n = 2\nu = 2$ and $q = 2$, then* $\mathrm{SMat}(m \times n, q) = (\mathcal{M}_{mn}, \{R_{(t,s)}\})$ *be a trivial association scheme, where $(t, s) = (0, 0), (1, 0)$. The valencies are $k_{(0,0)} = 1, k_{(1,0)} = 3$. For the intersection numbers,*

(i) $p^{(0,0)}_{(0,0)(0,0)} = p^{(1,0)}_{(0,0)(1,0)} = 1$ and $p^{(1,0)}_{(0,0)(0,0)} = p^{(0,0)}_{(0,0)(1,0)} = 0$.

(ii) $p^{(0,0)}_{(1,0)(0,0)} = 0, = p^{(1,0)}_{(1,0)(0,0)} = 1$, $p^{(0,0)}_{(1,0)(1,0)} = 3$, and $p^{(1,0)}_{(1,0)(1,0)} = 2$.

Example 3.2. *Let $m = 1, n = 2\nu = 4$ and $q = 3$, then* $\mathrm{SMat}(m \times n, q) = (\mathcal{M}_{mn}, \{R_{(t,s)}\})$ *be a trivial association scheme, where $(t, s) = (0, 0), (1, 0)$. The valencies are $k_{(0,0)} = 1, k_{(1,0)} = 80$. For the intersection numbers,*

(i) $p^{(0,0)}_{(0,0)(0,0)} = p^{(1,0)}_{(0,0)(1,0)} = 1$ and $p^{(1,0)}_{(0,0)(0,0)} = p^{(0,0)}_{(0,0)(1,0)} = 0$.

(ii) $p^{(0,0)}_{(1,0)(0,0)} = 0, = p^{(1,0)}_{(1,0)(0,0)} = 1$, $p^{(0,0)}_{(1,0)(1,0)} = 80$, and $p^{(1,0)}_{(1,0)(1,0)} = 79$.

Theorem 3.5. *Let $m = 2$, then the association classes of* $\mathrm{SMat}(m \times n, q)$ *are $R_{(0,0)}, R_{(1,0)}, R_{(2,0)}$(vanishes when $n = 2$), and $R_{(2,1)}$. Their valencies are $k_{(0,0)} = 1, k_{(1,0)} = (q + 1)(q^{2\nu} - 1)$, $k_{(2,0)} = q(q^{2\nu} - 1)(q^{2(\nu-1)} - 1)$(vanishes when $n = 2$), $k_{(2,1)} = q^{2\nu-1}(q - 1)(q^{2\nu} - 1)$. For the intersection numbers,*

(i) when $i = (0, 0)$, $p^k_{ij} = \delta_{jk}$.

(ii) when $i \neq (0, 0)$, the intersection numbers p^k_{ij} could be obtained by Tables 1–3. The rows are indexed by the value of j and columns indexed by the value of k. When $n = 2$, there are no matrices of type $(2, 0)$, thus related intersection numbers p^k_{ij} disappear.

Table 1. $m = 2, i = (1, 0)$.

	$(0, 0)$	$(1, 0)$	$(2, 0)$	$(2, 1)$
$(0, 0)$	0	1	0	0
$(1, 0)$	$q^{2\nu+1} + q^{2\nu} - q - 1$	$q^{2\nu} + q^2 - q - 2$	$q^2 + q$	$q^2 + q$
$(2, 0)$	0	$q^{2\nu} - q^2$	$q^{2\nu} + q^{2\nu-1} - q^2 - 2q - 1$	$q^{2\nu} + q^{2\nu-1} - q^2 - q$
$(2, 1)$	0	$q^{2\nu+1} - q^{2\nu}$	$q^{2\nu+1} - q^{2\nu-1}$	$q^{2\nu+1} - q^{2\nu-1} - q - 1$

Table 2. $m = 2, i = (2,0)$.

	$(0,0)$	$(1,0)$	$(2,0)$	$(2,1)$
$(0,0)$	0	0	1	0
$(1,0)$	0	$q^{2v} - q^2$	$q^{2v} + q^{2v-1}$ $-q^2 - 2q - 1$	$q^{2v} + q^{2v-1} - q^2 - q$
$(2,0)$	$q^{4v-1} - q^{2v+1}$ $-q^{2v-1} + q$	$q^{4v-2} - 2q^{2v}$ $-q^{2v-1} + q^2 + q$	$q^{4v-2} - q^{2v}$ $-3q^{2v-1} + q^2 + 3q$	$q^{4v-2} - 2q^{2v}$ $-q^{2v-1} + q^2 + q$
$(2,1)$	0	$q^{4v-1} - q^{4v-2}$ $-q^{2v+1} + q^{2v}$	$q^{4v-1} - q^{4v-2}$ $-q^{2v+1} + q^{2v-1}$	$q^{4v-1} - q^{4v-2} - q^{2v+1}$ $+q^{2v} - q^{2v-1} + q$

Table 3. $m = 2, i = (2,1)$.

	$(0,0)$	$(1,0)$	$(2,0)$	$(2,1)$
$(0,0)$	0	0	0	1
$(1,0)$	0	$q^{2v+1} - q^{2v}$	$q^{2v+1} - q^{2v-1}$	$q^{2v+1} - q^{2v-1} - q - 1$
$(2,0)$	0	$q^{4v-1} - q^{4v-2}$ $-q^{2v+1} + q^{2v}$	$q^{4v-1} - q^{4v-2}$ $-q^{2v+1} + q^{2v-1}$	$q^{4v-1} - q^{4v-2} - q^{2v+1}$ $+q^{2v} - q^{2v-1} + q$
$(2,1)$	$q^{4v} - q^{4v-1}$ $-q^{2v} + q^{2v-1}$	$q^{4v} - 2q^{4v-1}$ $+q^{4v-2} - q^{2v} + q^{2v-1}$	$q^{4v} - 2q^{4v-1}$ $+q^{4v-2} - q^{2v} + q^{2v-1}$	$q^{4v} - 2q^{4v-1}$ $+q^{4v-2} - 2q^{2v} + 3q^{2v-1}$

Proof. The values of (t, s), $k_{(0,0)}$ and $k_{(1,0)}$ could be obtained by Theorem 3.3 immediately. For the intersection numbers,

(i) when $i = (0,0)$, $p_{ij}^{k} = \delta_{jk}$ by (2.1).

(ii) when $i \neq (0,0)$, we only take the 1st table as an example to give the proof in the case of $i = (1,0)$; others' are similar.

Let

$$X = \begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_{2v} \\ y_1 & y_2 & y_3 & \cdots & y_{2v} \end{pmatrix}.$$

(1) By (2.1), $p_{i0}^{k} = \delta_{ik},\ p_{ij}^{0} = k_i \delta_{ij'}$. Thus,

$$p_{(1,0)(0,0)}^{(0,0)} = p_{(1,0)(0,0)}^{(2,0)} = p_{(1,0)(0,0)}^{(2,1)} = p_{(1,0)(2,0)}^{(0,0)} = p_{(1,0)(2,1)}^{(0,0)} = 0,$$

and

$$p_{(1,0)(0,0)}^{(1,0)} = 1,\ p_{(1,0)(1,0)}^{(0,0)} = k_{(1,0)} = q^{2v+1} + q^{2v} - q - 1.$$

(2) We will compute the value of $p_{(1,0)(1,0)}^{(1,0)}$.

By definition, $p_{(1,0)(1,0)}^{(1,0)} = |\{X \in \mathcal{M}_{2n} | X \sim M(1,0), X - M(1,0) \sim M(1,0)\}|$.

If $(x_1, x_2, \cdots, x_{2v}) = 0$, then $(y_2, \cdots, y_{2v}) = 0$ and $y_1 \neq 0$. In this case, X has $q - 1$ choices.

If $(x_1, x_2, \cdots, x_{2v}) \neq 0$, then by $X \sim M(1,0)$, we have $(y_1, y_2, \cdots, y_{2v}) = k(x_1, x_2, \cdots, x_{2v})$, where $k \in \mathbb{F}_q$. At this time, if $(x_2, \cdots, x_{2v}) = 0$, X has $q^2 - q - 1$ choices. If $(x_2, \cdots, x_{2v}) \neq 0$, then $k = 0$, and X has $q(q^{2v-1} - 1)$ choices.

Above all, $p^{(1,0)}_{(1,0)(1,0)} = q - 1 + q^2 - q - 1 + q(q^{2\nu-1} - 1) = q^{2\nu} + q^2 - q - 2$.

(3) When $n > 2$, we will compute the value of $p^{(2,0)}_{(1,0)(1,0)}$.

By definition, $p^{(2,0)}_{(1,0)(1,0)} = |\{X \in \mathcal{M}_{2n} | X \sim M(1,0), X - M(2,0) \sim M(1,0)\}|$.

If $(x_1, x_2, \cdots, x_{2\nu}) = 0$, then $y_2 = 1$ and $(y_3, \cdots, y_{2\nu}) = 0$. In this case, X has q choices.

If $(x_1, x_2, \cdots, x_{2\nu}) \neq 0$, then by $X \sim M(1,0)$, we have $(y_1, y_2, \cdots, y_{2\nu}) = k(x_1, x_2, \cdots, x_{2\nu})$, where $k \in \mathbb{F}_q$. At this time, $(x_3, \cdots, x_{2\nu}) = 0$, then X has q^2 choices.

Above all, $p^{(2,0)}_{(1,0)(1,0)} = q^2 + q$.

(4) When $n > 2$, we will compute the value of $p^{(2,0)}_{(1,0)(2,0)}$.

By definition, $p^{(2,0)}_{(1,0)(2,0)} = |\{X \in \mathcal{M}_{2n} | X \sim M(1,0), X - M(2,0) \sim M(2,0)\}|$.

If $(x_1, x_2, \cdots, x_{2\nu}) = 0$, then $(y_1, y_2, \cdots, y_{2\nu}) \neq 0$, $(y_2 - 1, y_3, \cdots, y_{2\nu}) \neq 0$ and $y_{\nu+1} = 0$. In this case, X has $q^{2\nu-1} - q - 1$ choices.

If $(x_1, x_2, \cdots, x_{2\nu}) \neq 0$, then by $X \sim M(1,0)$, we have $(y_1, y_2, \cdots, y_{2\nu}) = k(x_1, x_2, \cdots, x_{2\nu})$, where $k \in \mathbb{F}_q$. At this time, if $(x_3, \cdots, x_{2\nu}) = 0$, then X has $q^3 - q^2 - q$ choices. If $(x_3, \cdots, x_{2\nu}) = 0$, then X has $q^{2\nu} - q^3$ choices.

Above all, $p^{(2,0)}_{(1,0)(2,0)} = q^{2\nu} + q^{2\nu-1} - q^2 - 2q - 1$.

(5) Other values of p^k_{ij} could be obtained by $\sum_{j=0}^{d} p^k_{ij} = k_i,\ k_\gamma p^\gamma_{\alpha\beta} = k_\beta p^\beta_{\alpha'\gamma} = k_\alpha p^\alpha_{\gamma\beta'}$ in (2.1).

□

Example 3.3. *Let $m = 2, n = 2\nu = 4$ and $q = 2$, then the association classes of* SMat$(m \times n, q)$ *are* $R_{(0,0)}, R_{(1,0)}, R_{(2,0)}$, *and* $R_{(2,1)}$. *Their valencies are* $k_{(0,0)} = 1, k_{(1,0)} = 45$, $k_{(2,0)} = 90$, $k_{(2,1)} = 120$. *For the intersection numbers,*

(i) when $i = (0,0)$, $p^{(0,0)}_{(0,0)(0,0)} = p^{(1,0)}_{(0,0)(1,0)} = p^{(2,0)}_{(0,0)(2,0)} = p^{(2,1)}_{(0,0)(2,1)} = 1$, and $p^k_{ij} = 0$ for other cases.

(ii) when $i \neq (0,0)$, the intersection numbers p^k_{ij} are as follows (Tables 4–6).

Table 4. $m = 2, i = (1,0)$.

	(0, 0)	(1, 0)	(2, 0)	(2, 1)
(0, 0)	0	1	0	0
(1, 0)	45	16	6	6
(2, 0)	0	12	15	18
(2, 1)	0	16	24	21

Table 5. $m = 2, i = (2,0)$.

	(0, 0)	(1, 0)	(2, 0)	(2, 1)
(0, 0)	0	0	1	0
(1, 0)	0	12	15	18
(2, 0)	90	30	34	30
(2, 1)	0	48	40	42

Table 6. $m = 2, i = (2,1)$.

	(0,0)	(1,0)	(2,0)	(2,1)
(0,0)	0	0	0	1
(1,0)	0	16	24	21
(2,0)	0	48	40	42
(2,1)	120	56	56	56

4. Automorphism group of $\mathrm{SMat}(m \times n, q)$

In this section, we give the inner automorphism group of $\mathrm{SMat}(m \times n, q)$ as follows.

Theorem 4.1. *Let $n = 2\nu$ and q is a power of a prime number.*

(i) *When $m = 1$, the automorphism group of* $\mathrm{SMat}(m \times n, q)$ *is $Sym(q^n)$.*

(ii) *When $1 < m \le n$, each inner automorphism of the association scheme* $\mathrm{SMat}(m\times n, q)$ *must have the following form:*

$$\tau_{P,Q,A,\sigma}: \quad X \mapsto PX^{\sigma}Q + A, \forall X \in \mathcal{M}_{mn},$$

where $P \in GL_m(\mathbb{F}_q)$, $Q \in GSp_{2\nu}(\mathbb{F}_q)$, $A \in \mathcal{M}_{mn}$, and σ is an automorphism of $\mathbb{F}_q$.

In addition, if $m = n$, the following mapping is also an inner automorphism

$$X \mapsto P({}^tX)^{\sigma}Q + A, \forall X \in \mathcal{M}_{nn}.$$

Proof. (i) When $m = 1$, $\mathrm{SMat}(m \times n, q)$ is a trivial association scheme, thus its automorphism group is $Sym(q^n)$.

(ii) When $1 < m \le n$, let $P \in GL_m(\mathbb{F}_q)$, $Q \in GSp_{2\nu}(\mathbb{F}_q)$, $A \in \mathcal{M}_{mn}$, and σ be an automorphism of $\mathbb{F}_q$. For $(X, Y) \in R_{(s,t)}$,

$$(\tau_{P,Q,A,\sigma}(X), \tau_{P,Q,A,\sigma}(Y)) = (PX^{\sigma}Q + A, PY^{\sigma}Q + A) \in R_{(s,t)}.$$

Thus, every $\tau_{P,Q,A,\sigma}$ is an inner automorphism of $\mathrm{SMat}(m \times n, q)$.

Conversely, let τ be an inner automorphism of $\mathrm{SMat}(m\times n, q)$, i.e., τ induces the identity permutation on (t, s). It is easy to verify that τ is also an automorphism of $\mathrm{Mat}(m \times n, q)$. Thus, we can assume $\tau = \tau_{P,Q,A,\sigma}$, where $P \in GL_m(\mathbb{F}_q)$, $Q \in GL_n(\mathbb{F}_q)$, $A \in \mathcal{M}_{mn}$, and σ is an automorphism of $\mathbb{F}_q$. In the following, we will prove $Q \in GSp_{2\nu}(\mathbb{F}_q)$, that is, $QK^tQ = kK$, where $k \in \mathbb{F}_q^*$. Denote the i-th unit row vector by e_i.

(1) Let $1 \le i \le n$ and

$$M = \begin{pmatrix} e_i \\ 0 \end{pmatrix} \in \mathcal{M}_{mn}.$$

Since $(0, M) \in R_{(1,0)}$, $(\tau(0), \tau(M)) = (A, PM^{\sigma}Q + A) \in R_{(1,0)}$, i.e., $PM^{\sigma}Q \sim M(1,0)$. Thus,

$$0 = PM^{\sigma}QK^t(PM^{\sigma}Q) = MQK^t(MQ) = \begin{pmatrix} \alpha_iK^t\alpha_i & 0 \\ 0 & 0 \end{pmatrix},$$

where α_i be the i-th row vector of Q. This means that $(QK^tQ)_{ii} = 0$ when $1 \le i \le n$.

(2) Let $1 \leq i \neq j \leq \nu$ and

$$M = \begin{pmatrix} e_i \\ e_j \\ 0 \end{pmatrix} \in \mathcal{M}_{mn}.$$

Since $(0, M) \in R_{(2,0)}$, $(\tau(0), \tau(M)) = (A, PM^\sigma Q + A) \in R_{(2,0)}$, i.e., $PM^\sigma Q \sim M(2, 0)$. Thus,

$$0 = PM^\sigma QK^t(PM^\sigma Q) = MQK^t(MQ) = \begin{pmatrix} 0 & \alpha_i K^t \alpha_j & 0 \\ \alpha_j K^t \alpha_i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This means that $(QK^tQ)_{ij} = 0$ when $1 \leq i \neq j \leq \nu$. Similarly, it can be proven that $(QK^tQ)_{ij} = 0$ when $\nu \leq i \neq j \leq 2\nu$ or $1 \leq i \leq \nu, \nu < j \leq 2\nu (j - i \neq \nu)$.

(3) Let $1 \leq i \leq \nu$, $j = i + \nu$, and M be shown as the case (2). Since $(0, M) \in R_{(2,1)}$, $(\tau(0), \tau(M)) = (A, PM^\sigma Q + A) \in R_{(2,1)}$, i.e., $PM^\sigma Q \sim M(2, 1)$. Thus, the rank of

$$MQK^t(MQ) = \begin{pmatrix} 0 & \alpha_i K^t \alpha_j & 0 \\ \alpha_j K^t \alpha_i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is 2. Let $\alpha_i K^t \alpha_j = k_i$, then $k_i \neq 0$, and

$$QK^tQ = \begin{pmatrix} & J \\ -J & \end{pmatrix},$$

where $J = diag\{k_1, k_2, \cdots, k_\nu\}$.

(4) Let $1 \leq i < j \leq \nu$, and

$$M = \begin{pmatrix} e_i + e_j \\ e_{\nu+i} - e_{\nu+j} \\ 0 \end{pmatrix} \in \mathcal{M}_{mn},$$

then $(0, M) \in R_{(2,0)}$. Through similar discussions, we have

$$0 = MQK^t(MQ) = \begin{pmatrix} 0 & k_i - k_j & 0 \\ k_j - k_i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus $k_i = k_j$. If we assume $k_i = k_j = k$, then $QK^tQ = kK$. Thus $Q \in GSp_{2\nu}(\mathbb{F}_q)$. The theorem now follows from Lemma 1.1.

It should be pointed out the cases (2) and (4) don't appear when $n = 2$. □

When $m = 2$, by Theorem 3.5, all valencies $k_{(s,t)}$ of $\mathrm{SMat}(m \times n, q)$ are distinct. Thus, each automorphism of $\mathrm{SMat}(m \times n, q)$ must be an inner automorphism. We have the following theorem.

Theorem 4.2. *When $m = 2$, each automorphism of the association scheme* $\mathrm{SMat}(m \times n, q)$ *must have the following form:*

$$\tau_{P,Q,A,\sigma} : \quad X \mapsto PX^\sigma Q + A, \forall X \in \mathcal{M}_{mn},$$

where $P \in GL_m(\mathbb{F}_q)$, $Q \in GSp_{2\nu}(\mathbb{F}_q)$, $A \in \mathcal{M}_{mn}$, and σ is an automorphism of the finite field $\mathbb{F}_q$.

In addition, if $n = 2$, the following mapping is also an automorphism

$$X \mapsto P({}^tX)^\sigma Q + A, \forall X \in \mathcal{M}_{nn}.$$

5. Conclusions

In this paper, we construct a symplectic fission scheme for the association scheme of $m \times n$ rectangular matrices over the finite field $\mathbb{F}_q$, denoted by $\mathrm{SMat}(m \times n, q)$. Its association classes and inner automorphism group are discussed. In particular, we determine the intersection numbers and automorphism group of $\mathrm{SMat}(m \times n, q)$ for $m = 1$ and $m = 2$.

Author contributions

Yang Zhang, Shuxia Liu, and Liwei Zeng: Conceptualization, Methodology, Validation, Writing-original draft, Writing-review and editing. All authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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