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Research article

(Co-)fibration of generalized crossed modules

Hatice Gülsün Akay*

Department of Mathematics and Computer Science, Eskişehir Osmangazi University, 26040 Eskişehir, Turkey

* **Correspondence:** Email: hgulsun@ogu.edu.tr; Tel: +905063033107.

Abstract: Crossed modules are algebraic structures that generalize the concept of group extensions. They involve group-like objects (often groups or groupoids) with additional structure and mappings between them that satisfy certain properties. Generalized crossed modules further extend this concept to higher-dimensional settings or more general algebraic contexts. In this paper, we studied the fibration and co-fibration of generalized crossed modules.

Keywords: fibration; generalized crossed module; induced; (co)-limit; pullback
Mathematics Subject Classification: 18A35, 18D30, 18G45

1. Introduction

Fibred categories were introduced by Grothendieck in [10]. In [9], he established the category of fibrations over a fixed base category $\mathcal{B}$ and demonstrated that it is a reflective subcategory of the category of all categories over $\mathcal{B}$. Additionally, he discussed the preservation of limits by fibrations and applied some results to categories of sheaves. Furthermore, he discussed co-fibrations and dualized the results on concerning fibrations.

In [5], Brown and Sivera explored fibred and co-fibred categories, particularly focusing on certain colimit calculations of algebraic homotopical invariants for spaces. They emphasized the potential for such calculations based on various Higher Homotopy van Kampen Theorems, detailed in [3]. Among their work, they established that fibred categories preserve colimits, that is, if $\Psi\colon \mathcal{X} \to \mathcal{B}$ is a fibration and $A \in \mathcal{B}$, then the inclusion map $\mathcal{X}/A \to \mathcal{X}$ preserves colimits of connected diagrams. Also, they gave the relation between pushout and co-fibration. Moreover, they illustrated these results for homotopical calculations in groupoids, as well as for modules and crossed modules, in both cases over groupoids.

In [11], it was shown that the category of crossed modules over commutative algebras is both fibred and co-fibred. They established that, if $\varphi\colon \mathcal{P} \to Q$ is a ring morphism, then there exists a pair of adjoint functors (φ^*, φ_*), where $\varphi^*\colon \mathcal{XM}od/Q \to \mathcal{XM}od/\mathcal{P}$ and $\varphi_*\colon \mathcal{XM}od/\mathcal{P} \to \mathcal{XM}od/Q$, given by pullback and induced crossed modules, respectively. In [7], the fibration of the category of 2-crossed modules over groups was studied.

A crossed module of groups (P, C, ∂) is defined by a group morphism $\partial\colon P \to C$ together with a (left) action of C on P satisfying the following relations:

CM1) $\partial({}^{c}p) = c\partial(p)c^{-1}$,

CM2) ${}^{\partial(p_1)}p_2 = p_1 p_2 p_1^{-1}$,

for all $p, p_1, p_2 \in P$ and $c \in C$. A crossed module is called crossed-C module when it has the same fixed codomain C. Crossed modules of groups were given first in [13, 14]. The author defined this structure as models for (homotopy) 2-types.

Generalized crossed modules were introduced by Yavari and Salemkar in [15]. They defined the generalized crossed module on a group morphism $\partial\colon P \to C$ with arbitrary actions of C on C and P on P instead of the usual conjugation actions. Thus, they generalized the concept of crossed module. Furthermore, they studied the relations between epimorphisms and surjective morphisms.

The pullback crossed module of groups was given by Brown and Higgins in [4]. They constructed it over a crossed C-module and a fixed group morphism $\nu\colon G \to C$, which led to the definition of a crossed C-module in the sense of a pullback diagram. This construction yielded the definition of a functor $\nu^{\star}\colon \mathcal{XM}\mathrm{od}/G \to \mathcal{XM}\mathrm{od}/C$, which has a left adjoint to the induced functor. (Co)-limits of crossed modules were studied for various algebraic structures over time [1, 2, 6, 8, 12].

In this paper, we give the notions of fibration and (co-)fibration of generalized crossed modules in detail. We then construct the pullback and induced generalized crossed modules. Also, we get a functor

$$\nu^{\star}\colon GCM/C \to GCM/G,$$

which has a right adjoint functor, that is

$$\nu_{\star}\colon GCM/G \to GCM/C.$$

2. Generalized crossed modules

We recall the definition of a generalized crossed module from [15].

Definition 2.1. *A generalized crossed module (G, C, ∂) consists of a group morphism $\partial\colon G \to C$, together with the following properties,*

i) *an action of G on G, denoted by $g_1 \odot_G g_2$, for every $g_1, g_2 \in G$,*

ii) *an action of C on C, denoted by $c_1 \odot_C c_2$, for every $c_1, c_2 \in C$,*

iii) *an action of C on G, denoted by ${}^{c}g$, for every $c \in C$, $g \in G$,*

satisfying the conditions:

GCM1) $\partial({}^{c}g) = c \odot_C \partial(g)$,

GCM2) ${}^{\partial(g)}g' = g \odot_G g'$, *for all $g, g' \in G$ and $c \in C$. If ∂ only satisfies condition $GCM1$, we get a pre-generalized crossed module.*

Remark 2.2. *Throughout this paper, an action of G on G is denoted by $\cdot$ instead of $\odot_G$ for any group G.*

A morphism $(f, f')\colon (G, C, \partial) \to (G', C', \partial')$ of generalized crossed modules consists of group morphisms $f\colon G \to G'$ and $f'\colon C \to C'$ such that the diagram

$$\begin{array}{ccc} G & \xrightarrow{f} & G' \\ {\scriptstyle \partial}\downarrow & & \downarrow{\scriptstyle \partial'} \\ C & \xrightarrow[f']{} & C' \end{array}$$

is commutative, i.e., $f'\partial = \partial' f$ and

$$f({}^{c}g) = {}^{f'(c)} f(g)$$

for all $c \in C$ and $g \in G$. Thus, we get the category of generalized crossed modules, denoted by GCM.

Some examples of generalized crossed modules are given below:

Example 2.3. *If (G, C, ∂) is any crossed module, then it is also a generalized crossed module.*

Example 2.4. *Let $\partial\colon G \to C$ be a group morphism. If all actions are trivial, then ∂ becomes a generalized crossed module.*

Example 2.5. *Let C and G be two groups. If the action of G on G is trivial and the actions of C on C and C on G are arbitrary, then the trivial morphism $1\colon G \to C$ is a generalized crossed module.*

Example 2.6. *Every group gives a generalized crossed module. If D is a group, then (D, D, id_D) is a generalized crossed module by the arbitrary action of D on itself. Thus, we get the functor*

$$\lambda\colon GRP \to GCM,$$

which is the right adjoint of the functor

$$\lambda'\colon GCM \to GRP,$$

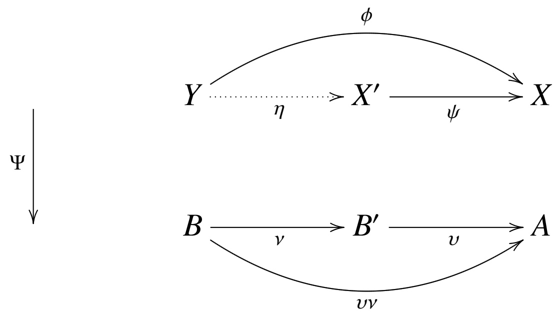
where (D, C, ∂) is a generalized crossed module and $\lambda'(D, C, \partial) = D$.

3. Fibred and co-fibred categories

Now, we give the definitions of fibration and co-fibrations of categories from [5].

Definition 3.1. *Let $\Psi\colon \mathcal{X} \to \mathcal{B}$ be a functor. A morphism $\psi\colon X' \to X$ in $\mathcal{X}$ over $\upsilon := \Psi(\psi)$ is called cartesian if and only if for all $\nu\colon B \to B'$ in $\mathcal{B}$ and $\phi\colon Y \to X$ with $\Psi(\phi) = \upsilon\nu$ there is a unique morphism $\eta\colon Y \to X'$ with $\Psi(\eta) = \nu$ and $\phi = \psi\eta$.*

This is given by the following diagram:



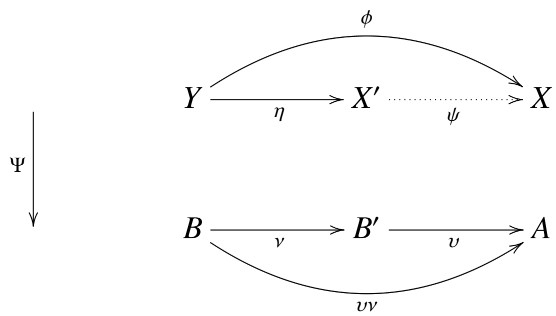
It is easy to show that ψ is an isomorphism if and only if ψ is a cartesian morphism over an isomorphism, and that cartesian morphisms are closed under composition.

A morphism $\beta\colon Y \to X'$ is called vertical, with respect to Ψ, if and only if $\Psi(\beta)$ is an identity morphism in $\mathcal{B}$. In particular, for $A \in \mathcal{B}$ we write $\mathcal{X}/A$ called fibre over A, for the subcategory of $\mathcal{X}$ consistings of those morphisms β with $\Psi(\beta) = id_A$.

Definition 3.2. *The functor $\Psi\colon \mathcal{X} \to \mathcal{B}$ is a fibration or category fibred over $\mathcal{B}$ if and only if $\upsilon\colon B' \to A$ in $\mathcal{B}$ and X in $\mathcal{X}/A$ there is a cartesian morphism $\psi\colon X' \to X$ over υ : Such a ψ called a cartesian lifting of X along υ.*

In other words, in a category fibred over $\mathcal{B}$, $\Psi\colon \mathcal{X} \to \mathcal{B}$, we can pullback objects of $\mathcal{X}$ along any arrow of $\mathcal{B}$.

Definition 3.3. *Let $\Psi\colon \mathcal{X} \to \mathcal{B}$ be a functor. A morphism $\eta\colon Y \to X'$ in $\mathcal{X}$ over $\nu := \Psi(\eta)$ is called cocartesian if and only if for all $\upsilon\colon B' \to A$ in $\mathcal{B}$ and $\phi\colon Y \to X$ with $\Psi(\phi) = \upsilon\nu$ there is a unique morphism $\psi\colon X' \to X$ with $\Psi(\psi) = \upsilon$ and $\phi = \psi\eta$. This is given by the following diagram:*



It is easy to show that ν is an isomorphism if and only if ν is a cocartesian morphism over an isomorphism, and that cocartesian morphisms are closed under composition.

Definition 3.4. *The functor $\Psi\colon \mathcal{X} \to \mathcal{B}$ is a co-fibration or category co-fibred over $\mathcal{B}$ if and only if $\nu\colon B \to B'$ in $\mathcal{B}$ and Y in $\mathcal{X}/B$ there is a cocartesian morphism $\eta\colon Y \to Y'$ over ν : Such a η called a cartesian lifting of Y along ν.*

Proposition 3.5. *Let $\Psi\colon \mathcal{X} \to \mathcal{B}$ be a fibration of categories. Then $\eta\colon Y \to X'$ in $\mathcal{X}$ over $\nu\colon B \to B'$ in $\mathcal{B}$ is cocartesian if and only if for all $\phi'\colon Y \to X_1$ over ν there is an unique morphism $\eta'\colon X' \to X_1$ in $\mathcal{X}/B'$ with $\phi' = \eta'\eta$, [5].*

3.1. Fibration of generalized crossed modules

In this section, we will show that the forgetful functor

$$\theta\colon GCM \to GRP,$$

which takes $(P, C, \iota) \in GCM$ in its base group C, is a fibration.

Theorem 3.6. *The forgetful functor $\theta\colon GCM \to GRP$ is fibred.*

Proof. To prove that θ is fibred, we will get the pullback generalized crossed module. Let (P, C, ι) be a generalized crossed module and let $\upsilon\colon G \to C$ be a group morphism. Define

$$\upsilon^\star(P) = \{(p, g) \in P \times G \mid \iota(p) = \upsilon(g)\},$$

and $\iota^\star\colon \upsilon^\star(P) \to G$ by $\iota^\star(p, g) = g$ for all $(p, g) \in \upsilon^\star(P)$. The actions of G on G and $\upsilon^\star(P)$ on $\upsilon^\star(P)$ are componentwise, the action of G on $\upsilon^\star(P)$ is defined by

$${}^{g}(p, g') = ({}^{\upsilon(g)}p, g \cdot g'),$$

for all $g \in G$ and $(p, g') \in \upsilon^\star(P)$. Then, $(\upsilon^\star(P), G, \iota^\star)$ is a generalized crossed G-module with the following equations:

GCM1)

$$\begin{aligned}\iota^\star({}^{g}(p, g')) &= \iota^\star({}^{\upsilon(g)}p, g \cdot g') \\ &= g \cdot g' \\ &= g \cdot \iota^\star(p, g').\end{aligned}$$

GCM2)

$$\begin{aligned}{}^{\iota^\star(p,g)}(p', g') &=^{g} (p', g') \\ &= ({}^{\upsilon(g)}p', g \cdot g') \\ &= ({}^{\iota(p)}p', g \cdot g') \\ &= (p \cdot p', g \cdot g') \\ &= (p, g) \cdot (p', g'),\end{aligned}$$

for all $(p, g), (p', g') \in \upsilon^\star(P)$. Moreover, $(\upsilon', \upsilon)\colon (\upsilon^\star(P), G, \iota^\star) \to (P, C, \iota)$ is a generalized crossed module morphism with $\upsilon'(p, g) = p$;

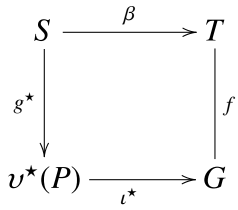
$$\begin{aligned}\upsilon'({}^{g'}(p, g)) &= \upsilon'({}^{\upsilon(g)}p, g \cdot g') \\ &=^{\upsilon(g)} p \\ &=^{\upsilon(g)} \upsilon'(p, g)\end{aligned}$$

for all $(p, g) \in \upsilon^\star(P)$ and $g' \in G$.

Suppose that $f\colon T \to G$ is any group morphism, S is a group and $(g, \upsilon f)\colon (S, T, \beta) \to (P, C, \iota)$ is a generalized crossed module morphism with $p(g, \upsilon f) = \upsilon f$. Then, there exists a unique generalized crossed module morphism $(g^\star, f)\colon (S, T, \beta) \to (\upsilon^\star(P), G, \iota^\star)$ such that

$$(\upsilon', \upsilon)(g^\star, f) = (g, \upsilon f), \quad p(g^\star, f) = f.$$

We define $g^{\star}(s) = (g(s), f\beta(s))$ for all $s \in S$. Considering the diagram below:



For all $t \in T$ and $s \in S$, we get

$$\begin{aligned} g^{\star}({}^{t}s) &= (g({}^{t}s), f\beta({}^{t}s)) \\ &= ({}^{\upsilon f(t)}g(s), f(t) \cdot f\beta(s)) \\ &=^{f(t)} (g(s), f\beta(s)) \\ &=^{f(t)} (g^{\star}(s)), \end{aligned}$$

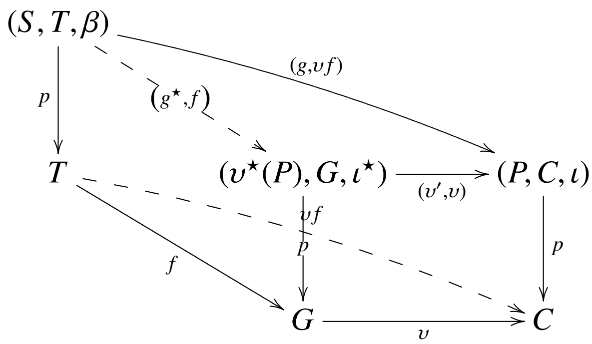
and

$$\begin{aligned} \iota^{\star} g^{\star}(s) &= \iota^{\star}(g(s), f\beta(s)) \\ &= f\beta(s). \end{aligned}$$

Then, $(g^{\star}, f)$ is a generalized crossed module morphism. Furthermore, for all $s \in S$ and $t \in T$, we get

$$\begin{aligned} (\upsilon', \upsilon)(g^{\star}, f)(s,t) &= (\upsilon', \upsilon)(g^{\star}(s), f(t)) \\ &= (\upsilon' g^{\star}(s), \upsilon f(t)) \\ &= (g(s), \upsilon f(t)). \end{aligned}$$

Then the diagram



commutes. Finally, let $(g^{\star\prime}, f') \colon (S, T, \beta) \to (\upsilon^{\star}(P), G, \iota^{\star})$ be a generalized crossed module morphism as the same property of $(g^{\star}, f)$. Clearly $f' = f$. Define $g^{\star\prime}(s) = (p, g)$ for all $s \in S$ and for some $p \in P, g \in G$. Then, for all $s \in S$,

$$\begin{aligned} g^{\star\prime}(s) &= (p, g) \\ &= (\upsilon'(p, g), \iota^{\star}(p, g)) \\ &= (\upsilon' g^{\star\prime}(s), \iota^{\star} g^{\star\prime}(s)) \\ &= (g(s), f'\beta(s)) \\ &= (g(s), f\beta(s)) \\ &= g^{\star}(s), \end{aligned}$$

so, we get

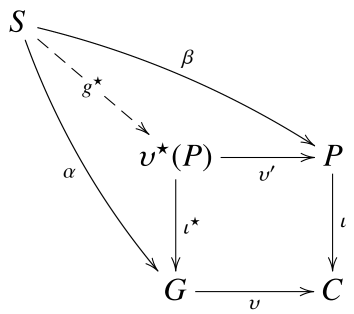
$$g^{\star\prime}(s) = (p, g) = (g(s), f\beta(s)) = g^{\star}(s).$$

That is, $(g^{\star}, f)$ is unique.

Consequently, we get a cartesian morphism $(\upsilon', \upsilon)\colon (\upsilon^{\star}(P), G, \iota^{\star}) \to (P, C, \iota)$, for group morphism $\upsilon\colon G \to C$ and generalized crossed module (P, C, ι).

□

Corollary 3.7. *In the category of generalized crossed module,* $(\upsilon^{\star}(P), G, \iota^{\star})$ *and the following diagram*

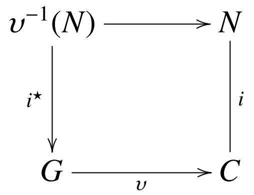


are called a pullback generalized crossed module and pullback diagram, respectively.

Example 3.8. *Let* $i\colon N \hookrightarrow C$ *be the inclusion map and* N *be a normal subgroup of* C. $(\upsilon^{\star}(N), G, i^{\star}) \cong (\upsilon^{-1}(N), G, i^{\star})$ *is the pullback generalized crossed module where,*

$$\begin{aligned} \upsilon^{\star}(N)) &= \{(n, g) \mid i(n) = \upsilon(g),\ n \in N,\ g \in G\} \\ &\cong \{g \in G \mid \upsilon(g) = n,\ n \in N\} \\ &\cong \upsilon^{-1}(N). \end{aligned}$$

See the pullback diagram below:

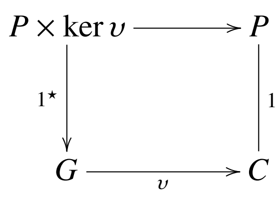


Particularly, if $N = 1$, *then* $\upsilon^{\star}(\{1\}) \cong \ker\upsilon$ *and so* $(\ker\upsilon, G, i^{\star})$ *is a pullback generalized crossed module. Consequently, kernels are particular examples of pullbacks. Moreover, if* υ *is surjective and* $N = C$, *then* $\upsilon^{\star}(C) \cong G$.

Example 3.9. *Let* $1\colon P \to C$ *be a generalized crossed module. Then,*

$$\begin{aligned} \upsilon^{\star}(P) &= \{(p, g) \in P \times G \mid \upsilon(g) = 1(p) = 1\} \\ &\cong P \times \ker\upsilon. \end{aligned}$$

See the pullback diagram below:



So, if υ *is injective, then* $\upsilon^{\star}(P) \cong P$. *Furthermore, if* $P = \{1\}$, *then* $\upsilon^{\star}(P) \cong \ker\upsilon$.

Corollary 3.10. *A pullback generalized crossed module $(\upsilon^\star(P), G, \iota^\star)$ for the group morphism $\upsilon\colon G \to C$ gives a functor*

$$\upsilon^\star\colon GCM/C \to GCM/G,$$

where objects and morphisms are defined as

$$\upsilon^\star(P, C, \iota) = (\upsilon^\star(P), G, \iota^\star)$$

and

$$\upsilon^\star(f, id_C) = (\upsilon^\star f, id_G)$$

such that $\upsilon^\star f(p, g) = (fp, g)$.

Proposition 3.11. *For each generalized crossed module morphism,$(\sigma, id_C)\colon (P, C, \iota) \to (P', C, \iota')$, there is a morphism*

$$(\upsilon^\star(\sigma), id_G)\colon (\upsilon^\star(P), G, \iota^\star) \to (\upsilon^\star(P'), G, \iota^{\star\prime}),$$

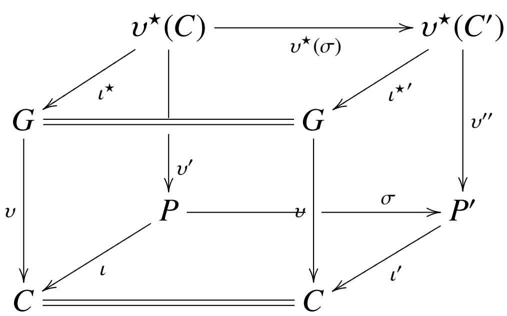
which is a unique morphism and satisfies the following equality,

$$(\sigma, id_C)(\upsilon', \upsilon) = (\upsilon'', \upsilon)(\upsilon^\star(\sigma), id_G).$$

Proof.

$$\begin{aligned} \upsilon''(\upsilon^\star\sigma(p, g)) &= \upsilon''(\sigma(p), g) \\ &= \sigma(p) \\ &= \sigma(\upsilon'(p, g)) \\ &= \sigma\upsilon'(p, g) \end{aligned}$$

for all $(p, g) \in \upsilon^\star(P)$. Thus, the diagram



is commutative. □

Proposition 3.12. *If $h_1\colon G \to C$ and $h_2\colon T \to G$ are two morphisms of groups, then $(h_1h_2)^\star$ and $h_2^\star h_1^\star$ are naturally isomorphic, i.e.,*

$$h_2^\star h_1^\star \simeq (h_1h_2)^\star.$$

Proof. Given any generalized crossed module (P, C, ι), we define $f\colon h_2^\star h_1^\star(P) \to (h_1h_2)^\star(P)$ as $f((p, h_2(t)), t) = (p, t)$ for all $((p, h_2(t)), t) \in h_2^\star h_1^\star(P)$. It is clear that f is well-defined and a group morphism. Also,

$$\begin{aligned} f(t' \cdot (p, h_2(t), t)) &= f((h_2)(t') \cdot (p, h_2(t)), t't) \\ &= f((((h_1(h_2)(t')) \cdot p, h_2(t')h_2(t)), t't) \\ &= f((((h_1h_2)(t') \cdot p, h_2(t't)), t't) \\ &= ((h_1h_2)(t'), t't) \\ &= t' \cdot (p, t) \\ &= t' \cdot f((p, h_2(t)), t) \end{aligned}$$

and

$$\begin{aligned} \overline{\iota^\star} f((p, h_2(t)), t) &= \overline{\iota^\star}(p, t) \\ &= t \\ &= id_T \overline{\overline{\iota^\star}}((p, h_2(t)), t) \end{aligned}$$

for all $((p, h_2(t)), t) \in h_2^\star h_1^\star(P)$. Thus, the diagram

$$\begin{array}{ccc} h_2^\star h_1^\star(P) & \xrightarrow{\overline{\overline{\iota^\star}}} & T \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle id_T} \\ (h_1h_2)^\star(P) & \xrightarrow[\overline{\iota^\star}]{} & T \end{array}$$

is commutative. Then, (f, id_T) is a generalized crossed module morphism. It is clear that (f, id_T) is an isomorphism.

Additionaly, for each generalized crossed module morphism $(\sigma, id_C)\colon (P, C, \iota) \to (S, C, \lambda)$ and $((p, h_2(t)), t) \in h_2^\star h_1^\star(P)$, we get

$$\begin{aligned} (((h_1h_2)^\star\sigma) f_P)((p, h_2(t)), t) &= ((h_1h_2)^\star\sigma)(p, t) \\ &= (\sigma(p), t) = f_S((\sigma(p), h_2(t), t)) \\ &= f_S(h_2^\star h_1^\star\sigma)((p, h_2(t)), t), \end{aligned}$$

so, the diagram

$$\begin{array}{ccc} (h_2^\star h_1^\star(P), T, \overline{\overline{\iota^\star}}) & \xrightarrow{(h_2^\star h_1^\star\sigma, id_T)} & (h_2^\star h_1^\star(S), T, \overline{\overline{\iota^\star}}) \\ {\scriptstyle (f_P, id_T)}\downarrow & & \downarrow{\scriptstyle (f_S, id_T)} \\ ((h_1h_2)^\star(P), T, \overline{\iota^\star}) & \xrightarrow[((h_1h_2)^\star\sigma, id_t)]{} & ((h_1h_2)^\star(S), T, \overline{\iota^\star}) \end{array}$$

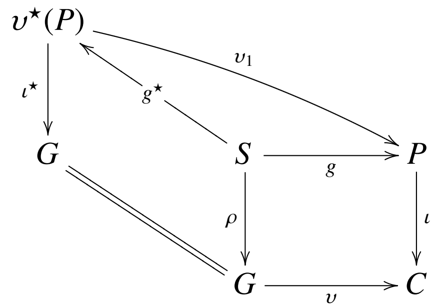
commutes. □

Proposition 3.13. *Let $\theta\colon GCM \to GRP$ be fibred, $\upsilon\colon G \to C$ be a group morphism and $\upsilon^\star\colon GCM/C \to GCM/G$ is chosen. Then, there is a bijection,*

$$GCM_\upsilon((S, G, \rho), (P, C, \iota)) \cong GCM/G((S, G, \rho), (\upsilon^\star(P), G, \iota^\star)),$$

which is natural in $(S, G, \rho) \in GCM/G$ and $(P, C, \iota) \in GCM/C$ where $GCM_\upsilon((S, G, \rho), (P, C, \iota))$ consists of those morphisms $f \in GCM_\upsilon((S, G, \rho), (P, C, \iota))$ with $\theta(f) = \upsilon$.

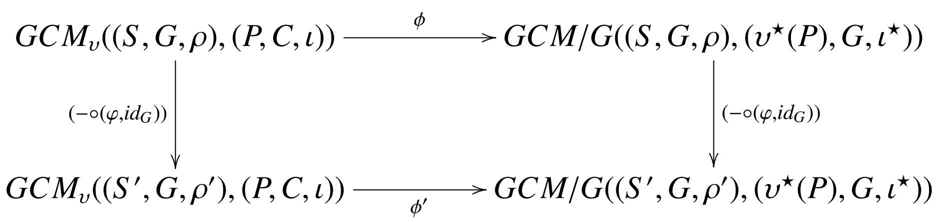
Proof. Define $\phi\colon GCM_\upsilon((S,G,\rho),(P,C,\iota)) \to GCM/G((S,G,\rho),(\upsilon^\star(P),G,\iota^\star))$ by $\phi(g,\upsilon) = (g^\star, id_G)$ such that $g^\star(s) = (g(s),\rho(s))$. Consider that $\phi(g, id_G) = \phi(h, id_G)$ for $(g,\upsilon),(h,\upsilon) \in GCM_\upsilon((S,G,\rho),(P,C,\iota))$. Then, we get $(g^\star, id_G) = (h^\star, id_G)$ and so $(g(s),\rho(s)) = (h(s),\rho(s))$, namely $g = h$. Thus, ϕ is one to one. Assume that $(g^\star, id_G) \in GCM/G((S,G,\rho),(\upsilon^\star(P),G,\iota^\star))$. Then, there is a morphism $(\upsilon_1 g^\star,\upsilon) \in GCM_\upsilon((S,G,\rho),(P,C,\iota))$ where $\upsilon_1(p,g) = p$, such that $\phi(\upsilon_1 g^\star,\upsilon) = (g^\star, id_G)$. Consider the following diagram:



It is clear that $\phi(\upsilon_1 g^\star,\upsilon) = ((\upsilon_1 g^\star)^\star, id_G)$ and then,

$$\begin{aligned} (\upsilon_1 g^\star)^\star(s) &= ((\upsilon_1 g^\star)(s),\rho(s)) \\ &= (\upsilon_1(g(s),\rho(s)),\rho(s)) \\ &= (g(s),\rho(s)) \\ &= g^\star(s). \end{aligned}$$

Thus, ϕ is a bijection. Furthermore, the following diagram



is commutative, since for $(\varphi, id_G)\colon (S',G,\rho') \to (S,G,\rho)$ and $s' \in S'$, then

$$\begin{aligned} (g^\star\varphi)(s') &= (g(\varphi(s')),\rho(\varphi(s'))) \\ &= ((g\varphi)(s'),(\rho\varphi)(s')) \\ &= (g\varphi^\star)(s') \end{aligned}$$

and

$$\begin{aligned} (-\circ(\varphi, id_G))(\phi(g,\upsilon)) &= (-\circ(\varphi, id_G))(g^\star, id_G) \\ &= (g^\star\varphi, id_G) \\ &= ((g\varphi)^\star, id_G) \\ &= \phi'(g\varphi,\upsilon) \\ &= (\phi'(-\circ(\varphi, id_G)))(g,\upsilon). \end{aligned}$$

Thus, ϕ is natural in (S,G,ρ). Moreover, for $(\sigma, id_C)\colon (P,C,\iota) \to (P',C,\iota')$, we get

$$\begin{aligned} (\sigma g)^\star(s) &= (\sigma(g(s)),\rho(s)) = ((\sigma)g(s),\rho(s)) \\ &= (\upsilon^\star\sigma)(g(s),\rho(s)) \\ &= (\upsilon^\star\sigma)g^\star(s) \end{aligned}$$

for $s \in S$ and then

$$\begin{aligned}((\upsilon^\star\sigma, id_C)\circ -)\phi(g,\upsilon) &= (\upsilon^\star\sigma, id_C)(g^\star, id_C)\\ &= ((\upsilon^\star\sigma)g^\star, id_C)\\ &= ((\sigma g)^\star, id_C)\\ &= \phi''(\sigma g, \upsilon)\\ &= (\phi''((\varphi, id_C)\circ -))(g,\upsilon).\end{aligned}$$

Thus, the diagram is

$$\begin{array}{ccc} GCM_\nu((S,G,\rho),(P,C,\iota)) & \xrightarrow{\phi} & GCM/G((S,G,\rho),(\upsilon^\star(P),G,\iota^\star)) \\ \Big\downarrow{\scriptstyle (\sigma, id_C)\circ -} & & \Big\downarrow{\scriptstyle (\upsilon^\star\sigma, id_G)\circ -} \\ GCM_\nu((S,G,\rho),(P',C,\iota')) & \xrightarrow[\phi'']{} & GCM/G((S,G,\rho),(\upsilon^\star(P'),G,\iota^{\star\prime})) \end{array}$$

commutative. So, ϕ is natural in (P,C,ι). □

3.2. Co-fibration of generalized crossed modules

Now we give the dual of Theorem 3.6.

Theorem 3.14. *The forgetful functor $\theta\colon GCM \to GRP$ is co-fibred.*

Let P be a group and C_P be a free group generated by $C \times P$ with the relation

$$(c,p)(c,p') = (c,pp')$$

for all $c \in C$ and $p, p' \in P$. Thus, C acts on C_P by

$$^{c'}(c,p) = (c'c, p)$$

for $c' \in C$ and $(c,p) \in C_P$.

Proposition 3.15. *Let (P,G,η) be a generalized crossed module and $\nu\colon G \to C$ be a group morphism. If C_P is a free group, then we get the following commutative diagram:*

$$\begin{array}{ccc} P & \longrightarrow & C_P \\ {\scriptstyle \eta}\Big\downarrow & \searrow^{\nu\eta} & \Big|{\scriptstyle \eta'} \\ G & \xrightarrow[\nu]{} & C \end{array}$$

Define $\eta'\colon C_P \to C$ by $\nu\eta$, that is $\eta'(c,p) = c\cdot\nu\eta(p)$ for all $(c,p) \in C_P$. Then, we get

$$\begin{aligned}\eta'(^{c'}(c,p)) &= \eta'(c'c,p)\\ &= (c'c)\cdot\nu\eta(p)\\ &= c'\cdot(c\cdot\nu\eta(p))\\ &= c'\cdot\nu'(c,p)\end{aligned}$$

for all $c' \in C$ and $(c,p) \in C_P$. Thus, η' is the free pre-generalized crossed module generated by P.

Proposition 3.16. *Let (P, G, η) be a generalized crossed module and let $\nu\colon G \to C$ be a group morphism. Then, the induced generalized crossed module $\nu_\star(P)$ is generated, as a group, by the set $C \times P$, with the following relations,*

$$i)\ (c, p)(c, p') = (c, pp')$$
$$ii)\ (c,{}^{g} p) = (c\nu(g), p)$$
$$iii)\ (c, p) \cdot (c' p') = (c \cdot \nu\eta(p) \cdot c', p')$$

for all $(c, p), (c', p') \in C_P$ and $g \in G$.

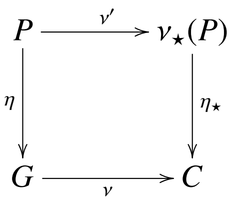
The action of C on $\nu_\star(P)$ defined by ${}^{c}(c', p) = (cc', p)$ for all $c' \in C$ and $(c, p) \in C_P$. Thus,

$$\eta_\star\colon \nu_\star(P) \to C$$

is given by $\eta_\star(c, p) = c \cdot \nu\eta(p)$, is a generalized crossed module, and the morphism

$$(\nu', \nu)\colon (P, G, \eta) \to (\nu_\star(P), C, \eta_\star)$$

is a generalized crossed module morphism. Define $\nu'(p) = (1, p)$ for all $p \in P$ and consider the diagram below:



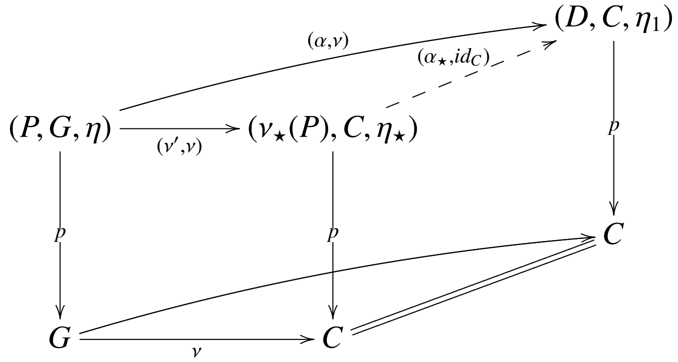
$$\begin{aligned} \nu'({}^{g}p) = (1,{}^{g} p) &= (1\nu(g), p) \\ &= (\nu(g), p) \\ &={}^{\nu(g)} (1, p) \\ &={}^{\nu(g)} \nu'(p) \end{aligned}$$

and

$$\eta_\star\nu'(p) = \eta_\star(1, p) = \nu\eta(p)$$

for all $g \in G$ and $p, p' \in P$.

Let (D, C, η_1) be any generalized crossed module and $(\alpha, \nu)\colon (P, G, \eta) \to (D, C, \eta_1)$ be any generalized crossed module morphism. Then, there is a unique generalized crossed module morphism $(\alpha_\star, id_C)\colon (\nu_\star(P), C, \eta_\star) \to (D, C, \eta_1)$ such that the diagram



commutes, i.e., $(\alpha_\star, id_C)(\nu', \nu) = (\alpha, \nu)$. Define $\alpha_\star(c, p) = c \cdot \alpha(p)$ for all $(c, p) \in \nu_\star(P)$. Then, $(\alpha_\star, id_C)$ is a generalized crossed module morphism, since

$$\begin{aligned} \alpha_\star({}^{c'}(c,p)) &= \alpha_\star((c'c, p)) \\ &= (c'c) \cdot \alpha(p) \\ &= c' \cdot (c \cdot \alpha(p)) \\ &= c' \cdot \alpha_\star(c,p) \\ &=^{id_C(c')} (\alpha_\star(c,p)) \end{aligned}$$

and

$$\begin{aligned} \alpha_\star \nu'(p) &= \alpha_\star(1, p) \\ &= 1 \cdot \alpha(p) \\ &= \alpha(p) \end{aligned}$$

for all $(c, p) \in \nu_\star(P)$ and $c' \in C$. Moreover,

$$\alpha_\star \nu'(p) = \alpha_\star(1, p) = \alpha(p)$$

for all $p \in P$.

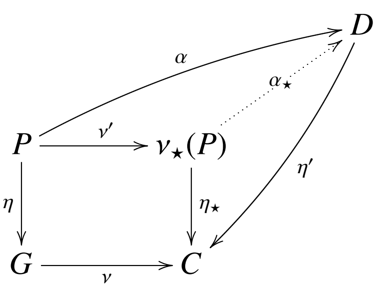
Let $(\alpha'_\star, id_C)\colon (\nu_\star(P), C, \eta_\star) \to (D, C, \eta_1)$ be any generalized crossed module morphism with $\rho(\alpha'_\star, id_C) = id_C$ and $(\nu', \nu)(\alpha'_\star, id_C) = (\alpha, \nu)$. For all $(c, p) \in \nu_\star(P)$,

$$\begin{aligned} \alpha_\star(c,p) &= c \cdot \alpha(p) \\ &= c \cdot \alpha'_\star \nu'(p) \\ &= c \cdot \alpha'_\star(1,p) \\ &=^{id_C(c)} \alpha'_\star(1,p) \\ &= \alpha'_\star({}^{c}(1,p)) \\ &= \alpha'_\star(c,p). \end{aligned}$$

Then, $(\alpha'_\star, id_C)$ is unique.

Thus, we get a cocartesian morphism $(\nu', \nu)\colon (P, G, \eta) \to (\nu_\star(P), C, \eta_\star)$, for group morphism $\nu\colon G \to C$ and generalized crossed module (P, G, η).

Corollary 3.17. *In the category of generalized crossed module, $(\nu_\star(P), C, \eta_\star)$ is called induced generalized crossed module with the following diagram:*



Corollary 3.18. *An induced generalized crossed module $(\nu_\star(P), C, \eta_\star)$ for the group morphism $\nu\colon G \to C$ gives a functor*

$$\nu_\star\colon GCM/G \to GCM/C,$$

which is the left adjoint functor of

$$\nu^\star\colon GCM/C \to GCM/G.$$

Proposition 3.19. *Let $\nu_1 : G \to C$ and $\nu_2 : S \to G$ be two group morphisms. Then, $(\nu_1\nu_2)_\star$ and $\nu_{2\star}\nu_{1\star}$ are naturally isomorphic.*

Proposition 3.20. *Let $\theta\colon GCM \to GRP$ be a co-fibred, $\upsilon\colon G \to C$ be a group morphism and a functor $\nu_\star\colon GCM/G \to GCM/C$ is chosen. Then, there is a bijection*

$$GCM_\upsilon((P, G, \eta), (D, C, \eta_1)) \cong GCM/C((\nu_\star(P), C, \eta_\star), (D, C, \eta_1)),$$

which is natural in $(P, G, \eta) \in GCM/G$, $(D, C, \eta_1) \in GCM/C$ where $GCM_\upsilon((P, G, \eta), (D, C, \eta_1))$ consists of those morphisms $f \in GCM_\upsilon((P, G, \eta), (D, C, \eta_1))$ with $\theta(f) = \upsilon$.

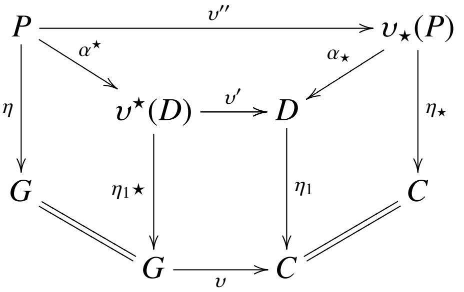
We deduce the following immediately from these discussions.

Corollary 3.21. *The category of generalized crossed module fibred and co-fibred over GRP, by the forgetful functor $\theta\colon GCM \to GRP$.*

Proof. For any group morphism $\upsilon\colon G \to C$, there is an adjoint functor pair $(\upsilon^\star, \upsilon_\star)$ as previously stated in Corollary 3.18. That is, a bijection

$$\Phi\colon GCM/C((\upsilon_\star(P), C, \eta_\star), (D, C, \eta_1)) \to GCM/G((P, G, \eta), (\upsilon^\star(D), G, \eta_1{}^\star))$$

exists and is natural in $(P, G, \eta) \in GCM/G$, $(D, C, \eta_1) \in GCM/C$. It is clear that $\Phi(\alpha_\star) = \alpha^\star$ and $\Phi^{-1}(\alpha^\star) = \alpha_\star$. See diagram below:



□

4. Conclusions

Yavari and Salemkar [15] defined the generalized crossed module on a group morphism $\partial\colon P \to C$ with arbitrary actions of C on C and P on P, instead of the usual conjugation actions. Thus, they generalized the concept of crossed modules. The category of generalized crossed modules provides a rich framework for studying various categorical structures and properties. In this paper, we extend well-known results from crossed modules to generalized crossed modules. It is shown that the forgetful functor $\theta\colon GCM \to GRP$ is both fibred and co-fibred. Moreover, a pair of adjoint functors $(\nu^\star, \nu_\star)$, where $\nu^\star\colon GCM/C \to GCM/G$ and $\nu_\star\colon GCM/G \to GCM/C$, is obtained for the group morphism $\nu\colon G \to C$.

Conflict of interest

The author declares that they have no conflict of interest to disclose.

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