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Research article

Inverse source problems for multi-parameter space-time fractional differential equations with bi-fractional Laplacian operators

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Abstract: Two inverse source problems for a space-time fractional differential equation involving bi-fractional Laplacian operators in the spatial variable and Caputo time-fractional derivatives of different orders between 1 and 2 are studied. In the first inverse source problem, the space-dependent term along with the diffusion concentration is recovered, while in the second inverse source problem, the time-dependent term along with the diffusion concentration is identified. Both inverse source problems are ill-posed in the sense of Hadamard. The existence and uniqueness of solutions for both inverse source problems are investigated. Finally, several examples are presented to illustrate the obtained results for the inverse source problems.

Keywords: inverse source problem; fractional derivative; bi-fractional Laplacian operator; Ill-posedness; Mittag-Leffler type functions
Mathematics Subject Classification: 35R30, 35K10, 35A09, 35A01, 35A02

1. Introduction

Fractional calculus broadens the scope of traditional calculus by extending the concepts of differentiation and integration to non-integer, arbitrary orders. This mathematical framework is particularly valuable for describing systems that exhibit memory and hereditary properties, transcending the limitations of classical differential and integral calculus. Fractional derivatives, which form the cornerstone of fractional calculus, offer a nuanced approach to capturing the dynamics of processes where the past influences the present, a feature not adequately addressed by integer-order derivatives [1–3]. These derivatives are instrumental in formulating fractional differential equations (FDEs), which effectively model complex phenomena across various domains, including engineering, physics, biology, and finance [4, 5]. The adoption of FDEs allows for a more accurate representation of behaviors such as viscoelastic material response, anomalous diffusion, and memory effects, underscoring the relevance of fractional calculus in complex system analysis [6–11].

This paper addresses two inverse source problems (ISPs) defined for the following space-time PDE:

$$ {}^{C}D^{\xi_0}_{0_+,t}u(x,t) + \sum_{j=1}^{m-1} a_j {}^{C}D^{\xi_j}_{0_+,t}u(x,t) = -(-\Delta)^{\eta_1/2}u(x,t) + -(-\Delta)^{\eta_2/2}u(x,t) + F(x,t), \quad (x,t) \in \Omega_T, \quad (1.1) $$

subject to boundary conditions

$$ u(-1,t) = 0 = u(1,t), \quad t \in (0,T), \quad (1.2) $$

with non-homogeneous initial conditions

$$ u(x,0) = \rho(x), \quad x \in (-1,1), \quad (1.3) $$
$$ u_t(x,0) = \nu(x), \quad x \in (-1,1), \quad (1.4) $$

where ${}^{C}D^{\xi_j}_{0_+,t}$ stands for the Caputo fractional derivatives in time variable of order ξ_j, $1 < \xi_{m-1} < ... < \xi_1 < \xi_0 < 2$, and is defined by

$$ {}^{C}D^{\xi_j}_{0_+,t}h(t) := J^{2-\xi_j}_{0_+,t}\frac{d^2}{dt^2}h(t) = \frac{1}{\Gamma(2-\xi_j)}\int_0^t \frac{\frac{d^2}{d\tau^2}h(\tau)}{(t-\tau)^{\xi_j-1}}\, d\tau, \quad t > 0, $$

where $\Omega_T := \Omega \times (0,T)$, $\Omega \in (-1,1)$, and a_j, $j = 1,2,...,m-1$, $m \in \mathbb{N}$ are positive real constants. The Riemann-Liouville fractional integral of order $\xi_j > 0$ in time is considered here and is defined as:

$$ J^{\xi_j}_{0_+,t}h(t) := \frac{1}{\Gamma(\xi_j)}\int_0^t (t-\tau)^{\xi_j-1}h(\tau)\, d\tau \qquad \xi_j > 0. $$

Here, $(-\Delta)^{\eta_1/2}$ and $(-\Delta)^{\eta_2/2}$ denote the fractional Laplacian operators of orders $1 < \eta_1 \leq \eta_2 < 2$ in the spatial domain. These operators are defined through the spectral decomposition of the Laplacian. Consider $\{\bar{\lambda}_n, \psi_n\}$ as the eigenvalues and eigenfunctions, respectively, associated with the Helmholtz equation for the Laplacian operator in domain Ω, subject to Dirichlet boundary conditions on $\partial\Omega$:

$$ \begin{cases} -\Delta X_n = \bar{\lambda}_n \psi_n, & \text{in} \quad \Omega, \\ \quad \psi_n = 0, & \text{on} \quad \partial\Omega. \end{cases} \quad (1.5) $$

A straightforward calculation shows that $\bar{\lambda}_n = \left(\frac{n\pi}{2}\right)^2$. Consequently, it follows that

$$ \psi_n(x) = \sin\Big(\frac{n\pi}{2}(x+1)\Big), \quad \forall n \geq 0. $$

Define the operator

$$ Q^{\eta_1,\eta_2}_{\Omega} := -(-\Delta)^{\eta_1/2} - (-\Delta)^{\eta_2/2}, \ \ 1 < \eta_1 \leq \eta_2 < 2, $$

on Ω for

$$ h \in Dom(Q^{\eta_1,\eta_2}_{\Omega}) = \left\{ h = \sum_{n=1}^{\infty} c_n\psi_n \in L^2(\Omega) : \sum_{n=1}^{\infty} c_n^2\lambda_n^2 < \infty \right\} := \dot{\mathcal{H}}^{\eta_1,\eta_2}, $$

and

$$Q_D^{\eta_1,\eta_2}h(x) = -\sum_{n=1}^{\infty} c_n\lambda_n\psi_n(x) \quad \text{with} \quad \lambda_n = \bar{\lambda}_n^{\eta_1/2} + \bar{\lambda}_n^{\eta_2/2} \quad \forall\, n = 1, 2, ... \tag{1.6}$$

Note that the set $\psi_n(x)n = 1^\infty$ constitutes an orthonormal basis for $L^2(\Omega)$. It is evident that $\dot{\mathcal{H}}^{\eta 1,\eta_2}$, a Hilbert space, is a subset of $L^2(\Omega)$. This Hilbert space is equipped with the inner product $\langle\cdot,\cdot\rangle$, which denotes the conventional inner product in $L^2(\Omega)$:

$$\langle u, v\rangle_{\dot{\mathcal{H}}^{\eta_1,\eta_2}} = \langle Q_D^{\eta_1,\eta_2}u, Q_D^{\eta_1,\eta_2}v\rangle,$$

and induced norms

$$\|v\|_{\dot{\mathcal{H}}^{\eta_1,\eta_2}} = \|Q_D^{\eta_1,\eta_2}v\|_{L^2(\Omega)} = \left(\sum_{n=1}^{\infty}\lambda_n^2\langle v, \psi_n(x)\rangle^2\right)^{1/2}.$$

For instance, $\dot{\mathcal{H}}^{0,0} = L^2(\Omega)$, $\dot{\mathcal{H}}^{1,1} = \mathcal{H}^1 0(D)$, and $\dot{\mathcal{H}}^{2,2} = \mathcal{H}^2(D)\cap\mathcal{H}^1 0(D)$, all of which have equivalent norms. The dual space of $\dot{\mathcal{H}}^{\eta_1,\eta_2}$ for $\eta_1, \eta_2 > 0$ is denoted as $\dot{\mathcal{H}}^{-\eta_1,-\eta_2}$, which corresponds to the dual space $(\dot{\mathcal{H}}^{\eta_1,\eta_2})^*$. The notation $\langle h, \psi\rangle$ represents the action of h on a bounded linear functional ψ within $\dot{\mathcal{H}}^{\eta_1,\eta_2}$. It is found that $\dot{\mathcal{H}}^{-\eta_1,-\eta_2}$ is also a Hilbert space, characterized by the norm

$$\|\psi\|_{\dot{\mathcal{H}}^{-\eta_1,-\eta_2}} = \left(\sum_{n=1}^{\infty}\lambda_n^{-2}|\langle h, \psi_n(x)\rangle_*|^2\right)^{1/2}.$$

Additionally, if $h \in L^2(\Omega)$ and $\psi \in \dot{\mathcal{H}}^{\eta_1,\eta_2}$, then $\langle h, \psi\rangle_* = \langle h, \psi\rangle$, as illustrated in ([12], Chap. V). Now, we are going to discuss two ISPs for the given system (1.1)–(1.4).

1.1. Inverse Source Problem-I (ISP-I)

In this ISP-I, we focus on a source term defined as $F(x,t) = f(x)$. To thoroughly analyze the space-dependent source term $f(x)$ alongside $u(x,t)$, we require additional information commonly referred to as an over-specified condition, presented as

$$u(x, T) = \Phi(x), \quad x \in \Omega. \tag{1.7}$$

A classical solution to the ISP-I, namely a pair of functions $\{u(x,t), f(x)\}$, satisfies the conditions that $T^{\xi_0+\xi_j-1}f(x) \in C(\bar{\Omega})$, $\bar{\Omega}_T := \bar{\Omega}\times[0,T]$, $\bar{\Omega} \in [-1,1]$, $t^{\xi_0+\xi_j-1}u(x,t) \in C(\bar{\Omega}_T)$, $Q_\Omega^{\eta_1,\eta_2}u(\cdot,t) \in C(\bar{\Omega})$, $t^{2\xi_0+\xi_j-1}{}^C D_{0+,t}^{\xi_0}u(x,\cdot) \in C([0,T])$, and $t^{2\xi_0+\xi_j-1}{}^C D_{0+,t}^{\xi_j}u(x,\cdot) \in C([0,T])$, $\quad j = 1, 2, ..., m-1 \quad m \in \mathbb{N}$. Our investigation will cover the existence and uniqueness results for the solution of the ISP-I under specific assumptions about the given data.

1.2. Inverse Source Problem-II (ISP-II)

In this ISP-II, we examine a source term defined as $F(x,t) = q(t)f(x,t)$. To fully reconstruct the pair of functions $u(x,t), q(t)$, additional information, commonly referred to as an over-specified condition, is required and is provided by

$$\int_{-1}^{1} u(x,t)dx = E(t), \quad t \in [0,T]. \tag{1.8}$$

We define a classical solution for the ISP-II as the set $\{u(x,t), q(t)\}$, where

$$q(t) \in C[0,T], t^{\xi_0+\xi_j-1}u(x,t) \in C(\bar{\Omega}_T), Q_{\Omega}^{\eta_1,\eta_2}u(\cdot,t) \in C(\bar{\Omega}), t^{2\xi_0+\xi_j-1}{}^{C}D_{0_+,t}^{\xi_0}u(x,\cdot) \in C([0,T])$$

and

$$t^{2\xi_0+\xi_j-1}{}^{C}D_{0_+,t}^{\xi_j}u(x,\cdot) \in C([0,T]), \quad j = 1,2,...,m-1, \quad m \in \mathbb{N}.$$

We aim to demonstrate that, under specific conditions applied to the given data, a unique classical solution for the ISP-II exists.

The ISPs involving FDEs are a significant area of research in applied mathematics and physics. These problems aim to determine unknown parameters or inputs in a system governed by fractional derivatives, which generalize classical integer-order derivatives to non-integer orders. FDEs are particularly useful in modeling processes with memory effects and anomalous diffusion. Solving ISPs in this context involves techniques to reconstruct hidden information from observable data, often leading to applications in fields such as engineering, finance, and biology. Huntul et al. [13, 14] considered the IP of recovering the time-dependent source term for time fractional pseudoparabolic equation. The two ISPs for the time FDEs are studied in [15, 16]. Direct and ISPs involving the estimation of specific parameters using numerical techniques for a multi-term time FDE are examined in [17]. Li et al. [18] investigated the well-posedness and long-term asymptotic behavior of initial-boundary value problems for multi-term time FDEs. Lin et al. [19] studied the three dimensional meshfree analysis for time-Caputo and space-Laplacian fractional diffusion equation. The governing equation under consideration involves a linear combination of Caputo derivatives in time with decreasing orders in the interval $(0,1)$ and includes positive constant coefficients. The discussion focuses on ISPs involving the determination of a time-dependent source term for higher-order multi-term FDEs that incorporate the Caputo-Fabrizio derivative [20]. The direct and ISPs for integro-differential equations involving generalized fractional derivatives, along with appropriate over-specified conditions, are discussed in [21–23]. The ISP for a class of multi-term time FDEs with non-local boundary conditions is examined in [24]. Ilyas et al. [25] focused on examining two ISPs related to a multi-term time-fractional evolution equation that includes an involution term, bridging the characteristics of both the heat and wave equations. Suhaib et al. [26] examined an ISP to identify a time-dependent source term in a multi-term FDE, incorporating a non-local dynamic boundary condition and an integral-type overdetermination condition. The ISP of determining both diffusion and source terms simultaneously in a multi-term FDE is studied in [27]. Ilyas et al. [28] considered an ISP for a diffusion equation involving a fractional Laplacian operator in space and Hilfer fractional derivatives in time with Dirichlet zero boundary conditions.

The rest of the paper is organized as follows: in this section, we provide the definition of the multinomial Prabhakar and Mittag-Leffler functions and describe several of their properties. In Section 3, we formulate the solution of ISP-I, investigate the existence and uniqueness results for ISP-I, and present the ill-posedness of ISP-I. In Section refISPII, we present the solution of ISP-II, discuss the existence and uniqueness results for ISP-II, and also discuss the ill-posedness of ISP-II. We also provide some examples related to ISP-I and ISP-II. In the last section, we present the conclusion.

2. Prabhakar and Mittag-Leffler functions

In this section, we will define the multinomial Prabhakar and Mittag-Leffler functions and discuss some of their important properties.

Definition 1. *[29] For $\gamma > 0,\ \eta_i > 0,\ z_i \in \mathbb{C}, i = 1, 2, ..., m,\ m \in \mathbb{N}$, the multinomial Prabhakar function is defined as*

$$E^{\delta}_{(\eta_1,\eta_2,...,\eta_m),\gamma}(z_1, z_2, ..., z_m) := \sum_{k=0}^{\infty} \sum_{\substack{l_1+l_2+...+l_m=k\\ l_1\geq 0,...,l_m\geq 0}} \frac{(\delta)_k}{l_1!...l_m!} \frac{\prod_{i=1}^{m} z_i^{l_i}}{\Gamma\Big(\gamma + \sum_{i=1}^{m} \eta_i l_i\Big)},$$

where $(\delta)_k$ denotes the Pochhammer symbol

$$(\delta)_k = \delta(\delta + 1)...(\delta + k - 1), \quad k \in \mathbb{N}, \quad \delta_0 = 1.$$

Theorem 1. *[30] Let $1 \geq \eta_1 > \eta_2 > ... > \eta_m > 0, 0 < \eta_1\delta \leq \gamma \leq 1$, and $z_i > 0,\ i = 1, 2, ..., m$. Then*

$$E^{\delta}_{(\eta_1,\eta_2,...,\eta_m),\gamma}(z_1, z_2, ..., z_m) \in \mathcal{CMF},$$

where $\mathcal{CMF}$ represents the complete monotone function.

In the special case $\delta = 1$, the Pochhammer symbol yields $(1)k = k!$ and Definition 1 is the multinomial Mittag-Leffler function which is defined as follows:

Definition 2. *[31] For $\gamma > 0,\ \eta_i > 0,\ z_i \in \mathbb{C}, i = 1, 2, ..., m,\ m \in \mathbb{N}$, the multinomial Mittag-Leffler function is defined as:*

$$E_{(\eta_1,\eta_2,...,\eta_m),\gamma}(z_1, z_2, ..., z_m) := \sum_{k=0}^{\infty} \sum_{\substack{l_1+l_2+...+l_m=k\\ l_1\geq 0,...,l_m\geq 0}} (k; l_1, ..., l_m) \frac{\prod_{i=1}^{m} z_i^{l_i}}{\Gamma\Big(\gamma + \sum_{i=1}^{m} \eta_i l_i\Big)},$$

where $(k; l_1, ..., l_m) = \dfrac{k!}{l_1! \times ... \times l_m!}$.

Moreover, note that

$$E_{(\xi_1,\xi_2,...,\xi_m),\gamma}(z_1, z_2, ..., z_m) = E_{(\xi_m,...,\xi_2,\xi_1),\gamma}(z_m, ..., z_2, z_1). \tag{2.1}$$

Remark 1. *For $z_1 \neq 0$ and $z_2 = Z_3 = ... = z_m = 0$, $m \in \mathbb{N}$, the multinomial Mittag-Leffler function reduces to*

$$E_{(\eta_1,\eta_2,...,\eta_m),\gamma}(z_1, 0, ..., 0) = \sum_{k=0}^{\infty} \frac{z_1^k}{\Gamma(\gamma + \eta_1 k)} := E_{\eta_1,\gamma}(z_1). \tag{2.2}$$

Lemma 1. *[18] Let $0 < \gamma < 2$ and $0 < \eta_m < ... < \eta_1 < 1$ be given. Assume that $\eta_1\pi/2 < \mu < \eta_1\pi$, $\mu \le |arg(m_2\tau^{\eta_1})| \le \pi$, and there exist $K > 0$ such that $-K \le -m_1\tau^{\eta_1-\eta_2} < 0$ and $-K \le -m_2\tau^{\eta_1} < 0$. Then there exists a constant $C_0 > 0$ depending only on $\mu, K, \eta_i,\ i = 1, 2, ..., m$, and γ such that*

$$|E_{(\eta_1-\eta_m,...,\eta_1-\eta_2,\eta_1),\gamma}(z_m, ..., z_2, z_1)| \le \frac{C_0}{1 + |z_m|}.$$

For convenience, we use the following notation:

$$\mathcal{E}_{(\eta_1,\eta_2,...,\eta_m),\gamma}(\tau; \mu_1, \mu_2, ..., \mu_m) := \tau^{\gamma-1}E_{(\eta_1,\eta_2,...,\eta_m),\gamma}(-\mu_1\tau^{\eta_1}, -\mu_2\tau^{\eta_2}, ..., -\mu_m\tau^{\eta_m}), \tag{2.3}$$

where $\mu_i > 0,\ i = 1, 2, ..., m,\ \ m \in \mathbb{N}$.

Lemma 2. *[32] For $\eta_i, \gamma, \tau, \mu_i > 0,\ i = 1, 2, ..., m,\ m \in \mathbb{N}$ the Laplace transform of the multinomial Mittag-Leffler function is given by*

$$\pounds\{\mathcal{E}_{(\eta_1,\eta_2,...,\eta_m),\gamma}(\tau; \mu_1, ..., \mu_m)\} = \frac{s^{-\gamma}}{1 + \sum_{i=1}^{m} \mu_i s^{-\eta_i}}, \qquad if \qquad \Big|\sum_{i=1}^{m} \mu_i s^{-\eta_i}\Big| < 1.$$

Lemma 3. *[25] For $\eta_i,\beta, \gamma, \tau, \mu_i > 0,\ i = 1, 2, ..., m$, the Mittag-Leffler-type functions have the following properties:*

- ${}^{c}D^{\gamma}_{0_+,\tau}\left(E_{(\eta_1,\eta_2,...,\eta_m),1}(\tau; \mu_1, \mu_2, ..., \mu_m)\right) = \tau^{-\gamma}E_{(\eta_1,\eta_2,...,\eta_m),1-\gamma}(\tau; \mu_1, \mu_2, ..., \mu_m)$,
- ${}^{c}D^{\gamma}_{0_+,\tau}\left(t^{\gamma-\beta}E_{(\eta_1,\eta_2,...,\eta_m),\gamma-\beta+1}(\tau; \mu_1, \mu_2, ..., \mu_m)\right) = \tau^{-\beta}E_{(\eta_1,\eta_2,...,\eta_m),1-\beta}(\tau; \mu_1, \mu_2, ..., \mu_m)$,
- ${}^{RL}D^{\gamma}_{0_+,\tau}\left(\tau^{\gamma-1}E_{(\eta_1,\eta_2,...,\eta_m),\gamma}(\tau; \mu_1, \mu_2, ..., \mu_m)\right) = \tau^{-\beta}E_{(\eta_1,\eta_2,...,\eta_m),1-\beta}(\tau; \mu_1, \mu_2, ..., \mu_m)$,
- $J^{1-\gamma}_{0_+,\tau}\left(\tau^{\gamma-1}E_{(\eta_1,\eta_2,...,\eta_m),\gamma}(\tau; \mu_1, \mu_2, ..., \mu_m)\right) = E_{(\eta_1,\eta_2,...,\eta_m),1}(\tau; \mu_1, \mu_2, ..., \mu_m)$.

Lemma 4. *[32] For $g \in C^1([a, b])$ and $\eta_i, \mu_i > 0$, for $i = 1, 2, ..., m$, we have*

$$|g(\tau) * \mathcal{E}_{(\eta_1,\eta_2,...,\eta_m),\eta_1}(\tau; \mu_1, \mu_2, ..., \mu_m)| \le \frac{C}{\mu_1} \parallel g \parallel_{C^1([0,T])},$$

where $*$ represents the Laplace convolution and $\|g\|_{C^1[0,T]} = \sup_{t\in[0,T]} |g(t)| + \sup_{t\in[0,T]} |g'(t)|$.

Lemma 5. *[32] For $\eta_i, \gamma, \beta, \tau, \mu_i > 0,\ i = 1, 2, ..., m,\ \ m \in \mathbb{N}$, we have the following relation:*

$$\tau^{\gamma} * \mathcal{E}_{(\eta_1,\eta_2,...,\eta_m),\beta}(\tau; \mu_1, \mu_2, ..., \mu_m) = \Gamma(\gamma + 1)\mathcal{E}_{(\eta_1,\eta_2,...,\eta_m),\beta+\gamma+1}(\tau; \mu_1, \mu_2, ..., \mu_m).$$

Proposition 1. *[32] The following identities hold for Mittag-Leffler functions:*

- $\mathcal{E}_{\eta_1,3}(\tau; \mu_1) = \frac{\tau^2}{\Gamma(3)} - \mu_1\mathcal{E}_{\eta_1,3+\eta_1}(\tau; \mu_1)$,
- $\mathcal{E}_{(\eta_1,\eta_1-\eta_2),3-\eta_2}(\tau; \mu_1, \mu_2) + \mu_2\mathcal{E}_{(\eta_1,\eta_1-\eta_2),3+\eta_1-2\eta_2}(\tau; \mu_1, \mu_2) = \frac{\tau^{2-\eta_2}}{\Gamma(3-\eta_2)} - \mu_1\mathcal{E}_{(\eta_1,\eta_1-\eta_2),3+\eta_1-\eta_2}(\tau; \mu_1, \mu_2)$,

3. Inverse Source Problem-I

3.1. Formal solution of the ISP-I

The solution of the ISP-I (1.1)–(1.4) and (1.7) can be written by using Fourier's method:

$$u(t,x)=\sum_{n=1}^{\infty}T_n(t)\psi_n(x), \quad f(x)=\sum_{n=1}^{\infty}f_n\psi_n(x),$$

where $T_n(t)$ and f_n are the unknowns and satisfy the following fractional differential equation:

$$^{C}D^{\xi_1}_{0_+,t}T_n(t)+\sum_{j=1}^{m-1}a_j\,^{C}D^{\xi_j}_{0_+,t}T_n(t)=-\lambda_nT_n(t)+\langle f(x),\psi_n(x)\rangle, \tag{3.1}$$

where $\lambda_n = (\bar{\lambda}_n)^{\eta_1/2}+(\bar{\lambda}_n)^{\eta_2/2}$. Applying the Laplace transform and incorporating the initial conditions (1.3) and (1.4), we obtain

$$\begin{aligned}\mathcal{L}\{T_n(t);s\}=&\frac{s^{(\xi_0-1)}\Big\langle\rho(x),\psi_n(x)\Big\rangle}{s^{\xi_0}+\sum_{j=1}^{m-1}a_js^{\xi_j}-\lambda_n}+\frac{\sum_{j=1}^{m-1}a_js^{(\xi_j-1)}\Big\langle\rho(x),\psi_n(x)\Big\rangle}{s^{\xi_0}+\sum_{j=1}^{m-1}a_js^{\xi_j}-\lambda_n}+\frac{s^{(\xi_0-1)}\Big\langle\nu(x),\psi_n(x)\Big\rangle}{s^{\xi_0}+\sum_{j=1}^{m-1}\mu_js^{\xi_j}-\lambda_n}\\&+\frac{\sum_{j=1}^{m-1}a_js^{(\xi_j-1)}\Big\langle\nu(x),\psi_n(x)\Big\rangle}{s^{\xi_0}+\sum_{j=1}^{m-1}\mu_js^{\xi_j}-\lambda_n}+\frac{f_n}{s\left(s^{\xi_0}+\sum_{j=1}^{m-1}\mu_js^{\xi_j}-\lambda_n\right)}.\end{aligned}$$

Due to Lemma 2, we obtain

$$\begin{aligned}T_n(t)=&\,\mathcal{E}_{\boldsymbol{\xi},1}(..r..)\langle\rho(x),\psi_n(x)\rangle+\sum_{j=1}^{m-1}a_j\mathcal{E}_{\boldsymbol{\xi},\xi_0-\xi_j+1}(..r..)\langle\rho(x),\psi_n(x)\rangle+\mathcal{E}_{\boldsymbol{\xi},2}(..r..)\langle\nu(x),\psi_n(x)\rangle\\&+\sum_{j=1}^{m-1}a_j\mathcal{E}_{\boldsymbol{\xi},\xi_0-\xi_j+2}(..r..)\langle\nu(x),\psi_n(x)\rangle+\mathcal{E}_{\boldsymbol{\xi},\xi_0+1}(..r..)\langle f(x),\psi_n(x)\rangle,\end{aligned} \tag{3.2}$$

where $\boldsymbol{\xi}$ and $(..r..)$ are defined as

$$\boldsymbol{\xi}=(\xi_0,\xi_0-\xi_2,...,\xi_0-\xi_{m-1}) \qquad \text{and} \qquad (..r..)=(t;\lambda_n,a_1,a_2,...,a_{m-1}),$$

where $f_n=\langle f(x),\psi_n(x)\rangle$. To calculate the space-dependent source term f_n, we will employ the over-specified condition (1.7), which leads to the following expression:

$$f_n=\frac{1}{\mathcal{E}_{\boldsymbol{\xi},\xi_0+1}(..r..)|_{t=T}}\Bigg\{\langle\Phi(x),\psi_n(x)\rangle-\Bigg(\mathcal{E}_{\boldsymbol{\xi},1}(..r..)|_{t=T}\langle\rho(x),\psi_n(x)\rangle$$

$$+\sum_{j=1}^{m-1} a_j\mathcal{E}_{\xi,\xi_0-\xi_j+1}(..r..)|_{t=T}\langle\rho(x),\psi_n(x)\rangle+\mathcal{E}_{\xi,2}(..r..)|_{t=T}\langle\nu(x),\psi_n(x)\rangle$$
$$+\sum_{j=1}^{m-1} a_j\mathcal{E}_{\xi,\xi_0-\xi_j+2}(..r..)|_{t=T}\langle\nu(x),\psi_n(x)\rangle\Big)\Big\}. \tag{3.3}$$

Consequently, the solution to the ISP-I, specifically $\{u(x,t), f(x)\}$, is provided by

$$u(x,t)=\sum_{n=1}^{\infty}\Big(\mathcal{E}_{\xi,1}(..r..)\langle\rho(x),\psi_n(x)\rangle+\sum_{j=1}^{m-1} a_j\mathcal{E}_{\xi,\xi_0-\xi_j+1}(..r..)\langle\rho(x),\psi_n(x)\rangle+\mathcal{E}_{\xi,2}(..r..)\langle\nu(x),\psi_n(x)\rangle$$
$$+\sum_{j=1}^{m-1} a_j\mathcal{E}_{\xi,\xi_0-\xi_j+2}(..r..)\langle\nu(x),\psi_n(x)\rangle+\mathcal{E}_{\xi,\xi_0+1}(..r..)\langle f(x),\psi_n(x)\rangle\Big)\psi_n(x), \tag{3.4}$$

and

$$f(x)=\sum_{n=1}^{\infty}\frac{1}{\mathcal{E}_{\xi,\xi_0+1}(..r..)|_{t=T}}\Big\{\langle\Phi(x),\psi_n(x)\rangle-\Big(\mathcal{E}_{\xi,1}(..r..)|_{t=T}\langle\rho(x),\psi_n(x)\rangle$$
$$+\sum_{j=1}^{m-1} a_j\mathcal{E}_{\xi,\xi_0-\xi_j+1}(..r..)|_{t=T}\langle\rho(x),\psi_n(x)\rangle+\mathcal{E}_{\xi,2}(..r..)|_{t=T}\langle\nu(x),\psi_n(x)\rangle$$
$$+\sum_{j=1}^{m-1} a_j\mathcal{E}_{\xi,\xi_0-\xi_j+2}(..r..)|_{t=T}\langle\nu(x),\psi_n(x)\rangle\Big)\Big\}\psi_n(x). \tag{3.5}$$

Lemma 6. *For $g(.,t)\in C^2([-1,1)$ satisfying $g(-1,t)=0=g(1,t)$, we have*

$$|g_n(t)|\leq\frac{D_1}{|\lambda_n|^2}\|g^{(ii)}(x,t)\|,$$

where

$$g_n(t)=\big\langle g(x,t),X_n(x)\big\rangle. \tag{3.6}$$

Proof. From the expression of $g_n(t)$ given by (3.6) and integration by parts, we obtain

$$g_n=\frac{1}{|\lambda_n|^2}\Big\langle g^{(ii)}(x,t),X_n(x)\Big\rangle.$$

Using the Cauchy-Schwarz inequality, we have

$$|g_n|\leq\frac{1}{|\lambda_n|^2}\|g^{(ii)}(x,t)\|\|X_n(x)\|,$$

which implies

$$|g_n|\leq\frac{D_1}{|\lambda_n|^2}\|g^{(ii)}(x,t)\|,$$

where $\|X_n(x)\|\leq D_1$. □

3.2. Existence of the solution of the ISP-I

In this subsection, we will describe the classical solution of the ISP-I under the given data. Before proceeding further, we present the following lemma, which will be used to determine the continuity of the series obtained by taking the Caputo fractional derivative of the series solution.

Lemma 7. *(Lemma 15.2 [33], page 278) Let the fractional derivative* ${}^{C}D^{\eta}_{0_+,x}g_n(x)$ *exist for all* $n \in \mathbb{N}$ *and for every* $\epsilon > 0$*, the series* $\sum_{n=1}^{\infty} g_n(x)$ *and* $\sum_{n=1}^{\infty} {}^{C}D^{\eta}_{0_+,x}g_n(x)$ *are uniformly convergent on the subinterval* $[\epsilon, b]$*. Then*

$${}^{C}D^{\eta}_{0_+,x}\Big(\sum_{n=1}^{\infty} g_n(x)\Big) = \sum_{n=1}^{\infty} {}^{C}D^{\eta}_{0_+,x}g_n(x), \quad \eta > 0,\ 0 < x < 1.$$

Theorem 2. *Let* $\rho(x), \nu(x)$*, and* $\Phi(x)$ *satisfy the following conditions:*

(1) $\rho \in C^2(\Omega)$ *such that* $\rho(-1) = 0 = \rho(1)$.
(2) $\nu \in C^2(\Omega)$ *such that* $\nu(-1) = 0 = \nu(1)$.
(3) $\Phi \in C^2(\Omega)$ *such that* $\Phi(-1) = 0 = \Phi(1)$.

Then, there exists a classical solution of the ISP-I.

Proof. To establish the existence of a solution for the ISP-I, it is necessary to demonstrate the uniform convergence of the infinite series related to the functions $f(x)$, $u(x,t)$, ${}^{C}D^{\xi_0}_{0_+,t}u(x,t)$, and ${}^{C}D^{\xi_j}_{0_+,t}u(x,t)$, $j = 1,2,...,m-1$, $m \in \mathbb{N}$. Initially, we demonstrate that $T^{\xi_0+\xi_j-1}|f(x)|$ denotes a continuous function. By utilizing Lemma 1 and Eq (3.5), we derive

$$|f(x)| \leq \sum_{n=1}^{\infty} \frac{|\lambda_n|}{C_0}\Big\{|\Phi_n| - \frac{C_0}{|\lambda_n|}\Big(|\rho_n|T^{-\xi_0} + \sum_{j=1}^{m-1} a_j|\rho_n|T^{-\xi_j} + |\nu_n|T^{1-\xi_0} + \sum_{j=1}^{m-1} a_j|\nu_n|T^{1-\xi_j}\Big)\Big\}.$$

By Lemma 6, we obtain

$$\begin{aligned} T^{\xi_0+\xi_j-1}|f(x)| \leq \sum_{n=1}^{\infty} \frac{D_0}{|\lambda_n|}\Big\{\frac{|\lambda_n|}{C_0}\|\Phi''\|T^{\xi_0+\xi_i-1} - \Big(\|\rho''\|T^{\xi_j-1} + \sum_{j=1}^{m-1} a_j\|\rho''\|T^{\xi_0-1} + \|\nu''\|T^{\xi_j} \\ + \sum_{j=1}^{m-1} a_j\|\nu''\|T^{\xi_0}\Big)\Big\}. \end{aligned} \tag{3.7}$$

Since, $\lambda_n = (\frac{n\pi}{2})^{\eta_1} + (\frac{n\pi}{2})^{\eta_2}$ and $1 < \eta_1 \leq \eta_2 < 2$, by (3.7), we can conclude that the series $T^{\xi_0+\xi_j-1}|f(x)|$ converges uniformly for $x \in \Omega_T$. Consequently, by the Weierstrass M-test, the series $T^{\xi_0+\xi_j-1}|f(x)|$ represents a continuous function. Subsequently, we demonstrate that $t^{\xi_0+\xi_j-1}|u(x,t)|$ denotes a continuous function. Utilizing Lemma 1 and Eq (3.4), we derive the ensuing inequality:

$$|u(x,t)| \leq \sum_{n=1}^{\infty} \frac{C_0}{|\lambda_n|}\Big(|\rho_n|t^{-\xi_0} + \sum_{j=1}^{m-1} a_j|\rho_n|t^{-\xi_j} + |\nu_n|t^{1-\xi_0} + \sum_{j=1}^{m-1} a_j|\nu_n|t^{1-\xi_j} + |f_n|\Big).$$

Due to Lemma 6, we obtain

$$t^{\xi_0+\xi_j-1}|u(x,t)| \leq \sum_{n=1}^{\infty} \frac{C_0}{|\lambda_n|^3}\Big(\|\rho''\|t^{\xi_j-1} + \sum_{j=1}^{m-1} a_j\|\rho''\|t^{\xi_0-1} + \|\nu''\|t^{\xi_j} + \sum_{j=1}^{m-1} a_j\|\nu''\|t^{\xi_0} + \|f''\|t^{\xi_0+\xi_j-1}\Big). \tag{3.8}$$

Based on (3.8), the uniform convergence of the $t^{\xi_0+\xi_j-1}|u(x,t)|$ is guaranteed due to the Weierstrass M-test. Hence, we can say that the series $t^{\xi_0+\xi_j-1}|u(x,t)|$ represents a continuous function.

Next, we will discuss the convergence of $Q_{\Omega}^{\eta_1,\eta_2}u(x,t)$. Due to (1.6), we have

$$Q_{\Omega}^{\eta_1,\eta_2}u(x,t) = -\sum_{n=1}^{\infty} \overline{\lambda}_n^{\,\rho/2} T_n(t)X_n(x), \quad \lambda_n = \lambda_n + \bar{\lambda}_n^{\,\eta_2/2}. \tag{3.9}$$

From (3.8) and under the assumption of Theorem 4, we can concluded that the uniform convergence of $Q_{\Omega}^{\eta_1,\eta_2}u(x,t)$ is ensured. Next, we will investigate the uniform convergence of the corresponding infinite series $t^{2\xi_0+\xi_j-1}|{}^{C}D_{0_+,t}^{\xi_0}u(x,t)|$ and $t^{2\xi_0+\xi_j-1}|{}^{C}D_{0_+,t}^{\xi_j}u(x,t)|$. Due to Lemma 7, we have

$${}^{C}D_{0_+,t}^{\xi_0}u(x,t) = \sum_{n=1}^{\infty} {}^{C}D_{0_+,t}^{\xi_0}T_n(t)\psi_n(x).$$

In order to prove the uniform convergence of $t^{2\xi_0+\xi_j-1}|{}^{C}D_{0_+,t}^{\xi_0}u(x,t)|$, we need to show that $t^{2\xi_0+\xi_j-1}\ |{}^{C}D_{0_+,t}^{\xi_0}T_n(t)|$ is uniformly convergent. By Lemma 3 and Eq (3.2), we obtain the following expression:

$$\begin{aligned}{}^{C}D_{0_+,t}^{\xi_0}T_n(t) = {} & \mathcal{E}_{\xi,1-\xi_0}(..r..)\Big\langle \rho(x),\psi_n(x)\Big\rangle + \sum_{j=1}^{m-1} a_j\mathcal{E}_{\xi,1-\xi_j}(..r..)\Big\langle \rho(x),\psi_n(x)\Big\rangle \\ & + \mathcal{E}_{\xi,2-\xi_0}(..r..)\Big\langle \nu(x),\psi_n(x)\Big\rangle + \sum_{j=1}^{m-1} a_j\mathcal{E}_{\xi,2-\xi_j}(..r..)\Big\langle \nu(x),\psi_n(x)\Big\rangle \\ & + \mathcal{E}_{\xi,1}(..r..)\Big\langle f(x),\psi_n(x)\Big\rangle.\end{aligned}$$

By using Lemmas 1 and 6, we obtain

$$\begin{aligned}t^{2\xi_0+\xi_j-1}|{}^{C}D_{0_+,t}^{\xi_0}T_n(t)| \le {} & \sum_{n=1}^{\infty}\frac{C_0}{|\lambda_n|^3}\Bigg(\|\rho''(x)\|t^{\xi_j-1} + \sum_{j=1}^{m-1} a_j\|\rho''(x)\|t^{\xi_0-1} + \|\nu''(x)\|t^{\xi_j} \\ & + \sum_{j=1}^{m-1} a_j\|\nu''(x)\|t^{\xi_0} + \|f''(x)\|t^{2\xi_0+\xi_j-1}\Bigg).\end{aligned}$$

The series $t^{2\xi_0+\xi_j-1}|{}^{C}D_{0_+,t}^{\xi_0}T_n(t)|$ is bounded above. Hence, $t^{2\xi_0+\xi_j-1}|{}^{C}D_{0_+,t}^{\xi_0}u(x,t)|$ is uniformly continuous by the Weierstrass M-test. In a similar way, we can find that the series corresponding to $t^{2\xi_0+\xi_j-1}|{}^{C}D_{0_+,t}^{\xi_j}u(x,t)|$, $j = 1,2,3,...,m-1$, represents a continuous function. □

3.3. Uniqueness of the solution of the ISP-I

In this subsection, the uniqueness of the solution of the ISP-I is discussed.

Theorem 3. *Let $\{u(x,t), f(x)\}$ and $\{\tilde{u}(x,t), \tilde{f}(x)\}$ be two regular solution sets of the ISP-I. If $u(x_0,t) = \tilde{u}(x_0,t)$ for some $x_0 \in (-\pi,\pi)$, then*

$$u(x,t) = \tilde{u}(x,t) \quad \Rightarrow f(x) = \tilde{f}(x), \quad x \in (-\pi,\pi) \quad \textit{and} \quad (x,t) \in \Omega_T.$$

Proof. Consider the following functions:

$$T_n(t) = \int_{-1}^{1} u(x,t)\psi_n(x)dx, \quad \text{and} \quad \tilde{T}_n(t) = \int_{-1}^{1} \tilde{u}(x,t)\psi_n(x)dx. \tag{3.10}$$

Applying ${}^{C}D^{\xi_j}_{0_+,t}$, $j = 0, 1, 2, ..., m-1$, to both sides of the second equation in (3.10), we obtain

$${}^{C}D^{\xi_i}_{0_+,t}\tilde{T}_n(t) = \int_{-1}^{1} {}^{C}D^{\xi_i}_{0_+,t}\tilde{u}(x,t)\psi_n(x)dx.$$

From (1.1), we obtain the following fractional differential equations:

$${}^{C}D^{\xi_i}_{0_+,t}\tilde{T}_n(t) = -\lambda_n\tilde{T}_n(t) + \tilde{f}_n. \tag{3.11}$$

Using the Laplace transform and initial conditions (1.3) and (1.4), we have

$$\tilde{T}_n(t) = \tilde{T}_n(0)\Big(\mathcal{E}_{\xi,1}(..r..) + \sum_{j=1}^{m-1} a_j\mathcal{E}_{\xi,\xi_0-\xi_j+1}(..r..)\Big) + \tilde{T}_n'(0)\Big(\mathcal{E}_{\xi,2}(..r..) + \sum_{j=1}^{m-1} a_j\mathcal{E}_{\xi,\xi_0-\xi_j+2}(..r..)\Big)$$
$$+ \tilde{f}_n\mathcal{E}_{\xi,\xi_0+1}(..r..).$$

Similarly, the following expressions of $T_n(t)$ from the first equations in (3.10) is obtained:

$$T_n(t) = T_n(0)\Big(\mathcal{E}_{\xi,1}(..r..) + \sum_{j=1}^{m-1} a_j\mathcal{E}_{\xi,\xi_0-\xi_j+1}(..r..)\Big) + T_n'(0)\Big(\mathcal{E}_{\xi,2}(..r..) + \sum_{j=1}^{m-1} a_j\mathcal{E}_{\xi,\xi_0-\xi_j+2}(..r..)\Big)$$
$$+ f_n\mathcal{E}_{\xi,\xi_0+1}(..r..).$$

By using the assumption $u(x,t) = \tilde{u}(x,t)$, we have $T_n(t) = \tilde{T}_n(t)$ and hence

$$f_n\mathcal{E}_{\xi,\xi_0+1}(..r..) = \tilde{f}_n\mathcal{E}_{\xi,\xi_0+1}(..r..), \quad \Rightarrow \quad \Big(f_n - \tilde{f}_n\Big)\mathcal{E}_{\xi,\xi_0+1}(..r..) = 0.$$

Taking the Laplace transform, we get

$$\frac{\Big(f_n - \tilde{f}_n\Big)}{s^{\xi_0} + \sum_{j=1}^{m-1}\mu_i s^{\alpha_i} - \lambda_n} = 0, \quad Re\ s > 0, \quad \Rightarrow \quad \frac{f_n - \tilde{f}_n}{\omega + \lambda_n} = 0, \tag{3.12}$$

where $s^{\alpha_0} + \sum_{i=1}^{m}\mu_i s^{\alpha_i} = \omega$. By taking a suitable disk D_1 which includes only $\lambda_{1,1}$ and using the Cauchy integral theorem, integrating (3.12) along the disk, we have

$$f_{k,1} = \tilde{f}_{k,1}, \quad \text{for} \quad k = 0.$$

In a similar way, by taking different disks, we can find that

$$f_{k,1} = \tilde{f}_{k,1}, \quad \text{for all} \quad k \in \mathbb{N}.$$

Similarly, we can find that

$$f_{k,2} = \tilde{f}_{k,2}, \quad \text{for all} \quad k \in \mathbb{N}.$$

Hence, we have $f(x) = \tilde{f}(x)$. □

3.4. Ill-posedness of the ISP-I

In this subsection, we present an example to demonstrate the ill-posedness of ISP-I. Before delving into the example, it is important to summarize a few pertinent observations.

Lemma 8. *For $\lambda_n > 0$, the following result holds:*

$$\frac{d}{dt}\mathcal{E}_{\xi,1}(..r..) = -\lambda_n t^{\xi_0-1}\mathcal{E}_{\xi,\xi_0}(..r..). \tag{3.13}$$

Moreover, for $T > 0$ the following estimate holds true:

$$\int_0^T t^{\xi_0-1}\mathcal{E}_{\xi,\xi_0}(..r..)|_{r=\tau}d\tau \leq \frac{C_1}{\lambda_n}, \tag{3.14}$$

where C_1 is a positive constant.

Proof. From Lemma 2, Eq (3.13) can be proved. To obtain the estimate of (3.14), we consider

$$\begin{aligned}\int_0^T \mathcal{E}_{\xi,\xi_0}(..r..)|_{r=\tau}d\tau &= -\tfrac{1}{\lambda_n}\int_0^T \tfrac{d}{d\tau}\mathcal{E}_{\xi,1}(..r..)|_{r=\tau}d\tau \\ &= \tfrac{1}{\lambda_n}(1-\mathcal{E}_{\xi,1}(..r..)|_{r=T}).\end{aligned}$$

This implies

$$\int_0^T \mathcal{E}_{\xi,\xi_0}(..r..)|_{r=\tau}d\tau = \frac{1}{\lambda_n}(1-\lambda_n t^{\xi_0}\mathcal{E}_{\xi,\xi_0+1}(..r..)|_{r=T}).$$

Due to Lemma 1, we obtain

$$\int_0^T \mathcal{E}_{\xi,\xi_0}(..r..)|_{r=\tau}d\tau \leq \frac{C_1}{\lambda_n}.$$

The above inequality can be written as

$$\mathcal{E}_{\xi,\xi_0+1}(..r..)|_{t=T} \leq \frac{C_1}{\lambda_n}.$$

□

The following example addresses the result concerning the ill-posedness of ISP-I. Considering the initial conditions to be $\tilde{\rho}(x) = 0$ and $\tilde{\nu}(x) = 0$, and the final condition as

$$\tilde{\Phi}(x) = \frac{1}{\sqrt{\lambda_k}}\sin\Big(\frac{k\pi}{2}(x+1)\Big),$$

where $k \in \mathbb{N}$, the following expression for $\tilde{f}(x)$ is obtained:

$$\tilde{f}(x) = \frac{1}{\sqrt{\lambda_k}\mathcal{E}_{\xi,\xi_0+1}(..r..)|_{t=T}}\sin\Big(\frac{k\pi}{2}(x+1)\Big).$$

By considering another final data $\Phi(x) = 0$ and fixing the initial conditions as $\tilde{\rho}(x) = 0$ and $\tilde{\nu}(x) = 0$, we obtain $f(x) = 0$. The two input final data have the following error in the L^2-norm:

$$\|\tilde{\Phi} - \Phi\|_{L^2((-1,1))} = \Big\|\frac{1}{\sqrt{\lambda_k}}\sin\Big(\frac{k\pi}{2}(x+1)\Big)\Big\|_{L^2((-1,1))} = \frac{1}{\sqrt{\lambda_k}}.$$

Hence,

$$\lim_{k\to+\infty} \|\tilde{\Phi} - \Phi\|_{L^2((-1,1))} = \lim_{k\to+\infty} \frac{1}{\sqrt{\lambda_k}} = 0. \tag{3.15}$$

Additionally, the difference between corresponding source terms in the L^2-norm is

$$\|\tilde{f} - f\|_{L^2((-1,1))} = \left\| \frac{1}{\sqrt{\lambda_k}\mathcal{E}_{\xi,\xi_0+1}(..r..)|_{t=T}} \sin\left(\frac{k\pi}{2}(x+1)\right) \right\|_{L^2((-1,1))} = \frac{1}{\sqrt{\lambda_k}\mathcal{E}_{\xi,\xi_0+1}(..r..)|_{t=T}}.$$

Using estimate (3.14), we obtain

$$\|\tilde{f} - f\|_{L^2((-1,1))} \geq \frac{\sqrt{\lambda_k}}{C_1},$$

which leads us to

$$\lim_{k\to+\infty} \|\tilde{f} - f\|_{L^2((-1,1))} > \lim_{k\to+\infty} \frac{\sqrt{\lambda_k}}{C_1} = +\infty. \tag{3.16}$$

Hence, based on Eqs (3.15) and (3.16), we conclude that ISP-I is ill-posed.

4. Inverse Source Problem-II

The solution of the inverse problems (1.1)–(1.4) and (1.8) can be written by using Fourier's method:

$$u(t,x) = \sum_{n=1}^{\infty} T_n(t)\psi_n(x),$$

where $T_n(t)$ are the unknowns and satisfy the following fractional differential equation:

$${}^C D_{0_+,t}^{\xi_1} T_n(t) + \sum_{j=1}^{m-1} a_j {}^C D_{0_+,t}^{\xi_j} T_n(t) = -\lambda_n T_n(t) + \langle q(t) f(x,t), \psi_n(x) \rangle. \tag{4.1}$$

By using the Laplace transform and the initial conditions (1.3) and (1.4), we get

$$\begin{aligned}
\mathcal{L}\{T_n(t); s\} = {} & \frac{s^{(\xi_0-1)}\left\langle \rho(x), \psi_n(x) \right\rangle}{s^{\xi_0} + \sum_{j=1}^{m-1} a_j s^{\xi_j} + \lambda_n} + \frac{\sum_{j=1}^{m-1} a_j s^{(\xi_j-1)}\left\langle \rho(x), \psi_n(x) \right\rangle}{s^{\xi_0} + \sum_{j=1}^{m-1} a_j s^{\xi_j} + \lambda_n} + \frac{s^{(\xi_0-1)}\left\langle \nu(x), \psi_n(x) \right\rangle}{s^{\xi_0} + \sum_{j=1}^{m-1} \mu_j s^{\xi_j} + \lambda_n} \\
& + \frac{\sum_{j=1}^{m-1} a_j s^{(\xi_j-1)}\left\langle \nu(x), \psi_n(x) \right\rangle}{s^{\xi_0} + \sum_{j=1}^{m-1} \mu_j s^{\xi_j} + \lambda_n} + \frac{\mathcal{L}\left\{\left\langle q(t) f(x,t), \psi_n(x) \right\rangle; s\right\}}{s^{\xi_0} + \sum_{j=1}^{m-1} \mu_j s^{\xi_j} + \lambda_n}.
\end{aligned}$$

Due to Lemma 2, we obtain

$$T_n(t) = \mathcal{E}_{\xi,1}(..r..)\left\langle \rho(x), \psi_n(x) \right\rangle + \sum_{j=1}^{m-1} a_j \mathcal{E}_{\xi,\xi_0-\xi_j+1}(..r..)\left\langle \rho(x), \psi_n(x) \right\rangle + \mathcal{E}_{\xi,2}(..r..)\left\langle \nu(x), \psi_n(x) \right\rangle$$

$$+\sum_{j=1}^{m-1} a_j \mathcal{E}_{\boldsymbol{\xi},\xi_0-\xi_j+2}(..r..)\Big\langle \nu(x),\psi_n(x)\Big\rangle + \mathcal{E}_{\boldsymbol{\xi},\xi_0}(..r..) * \Big\langle q(t)f(x,t),\psi_n(x)\Big\rangle. \tag{4.2}$$

Hence, the solution $u(x,t)$ is given by

$$\begin{aligned} u(x,t) = \sum_{n=1}^{\infty}\Big(& \mathcal{E}_{\xi,1}(..r..)\Big\langle \rho(x),\psi_n(x)\Big\rangle + \sum_{j=1}^{m-1} a_j \mathcal{E}_{\boldsymbol{\xi},\xi_0-\xi_j+1}(..r..)\Big\langle \rho(x),\psi_n(x)\Big\rangle \\ & + \mathcal{E}_{\boldsymbol{\xi},2}(..r..)\Big\langle \nu(x),\psi_n(x)\Big\rangle + \sum_{j=1}^{m-1} a_j \mathcal{E}_{\boldsymbol{\xi},\xi_0-\xi_j+2}(..r..)\Big\langle \nu(x),\psi_n(x)\Big\rangle \\ & + \mathcal{E}_{\boldsymbol{\xi},\xi_0}(..r..) * \Big\langle q(t)f(x,t),\psi_n(x)\Big\rangle\Big)\psi_n(x), \end{aligned} \tag{4.3}$$

where $q(t)$ is still to be determined.

5. The main results of the ISP-II

In this section, we will discuss the main results for the solution of the ISP-II.

5.1. Existence of the solution of the ISP-II

In this subsection, we will present the existence of the solution of the ISP-II under certain assumptions of the following theorem.

Theorem 4. *Let the following conditions hold:*

(1) $\rho \in C^2)(\Omega)$ *such that* $\rho(-1) = 0 = \rho(1)$.
(2) $\nu \in C^2(\Omega)$ *such that* $\nu(-1) = 0 = \nu(1)$.
(3) $f(\cdot,t) \in C^2(\Omega)$ *such that* $f(-1,t) = 0 = f(1,t)$. *Furthermore* $\int_{-1}^{1} f(x,t)dx \neq 0$, *and*

$$\left(\int_{-1}^{1} f(x,t)dx\right)^{-1} \leq M_1,$$

for some positive constant M_1,
(4) $E \in AC([0,T])$ *and* $E(t)$ *satisfies the following consistency condition:*

$$\int_{-1}^{1} \rho(x)dx = E(t).$$

Then, there exists a unique regular solution of the ISP-II.

Proof. To prove the unique existence of the time-dependent source term $q(t)$, we will use the over-specified condition (1.8), and then we have the following relation:

$$\int_{-1}^{1}\left({}^{C}D^{\xi_1}_{0_+,t}u(x,t) + \sum_{j=1}^{m-1} a_j\, {}^{C}D^{\xi_j}_{0_+,t}u(x,t)\right)dx = \left({}^{C}D^{\xi_1}_{0_+,t} + \sum_{j=1}^{m-1} a_j\, {}^{C}D^{\xi_j}_{0_+,t}\right)E(t).$$

From (1.1), we have

$$\int_{-1}^{1}\Big(-(-\Delta)^{\eta_1/2}u(x,t)+-(-\Delta)^{\eta_2/2}u(x,t)+q(t)f(x,t)\Big)dx \quad = \quad {}^{C}D^{\xi_1}_{0_+,t}E(t)+\sum_{j=1}^{m-1}a_j{}^{C}D^{\xi_j}_{0_+,t}E(t),$$

which yields the following expression:

$$\begin{aligned} q(t) = &\left[\int_{-1}^{1}f(x,t)dx\right]^{-1}\Big[{}^{C}D^{\xi_1}_{0_+,t}E(t)+\sum_{j=1}^{m-1}a_j{}^{C}D^{\xi_j}_{0_+,t}E(t)+\sum_{n=1}^{\infty}n\pi\Big\{\mathcal{E}_{\xi,1}(..r..)\Big\langle\rho(x),\psi_n(x)\Big\rangle \\ &+\sum_{j=1}^{m-1}a_j\mathcal{E}_{\xi,\xi_0-\xi_j+1}(..r..)\Big\langle\rho(x),\psi_n(x)\Big\rangle+\mathcal{E}_{\xi,2}(..r..)\Big\langle\nu(x),\psi_n(x)\Big\rangle+\sum_{j=1}^{m-1}a_j\mathcal{E}_{\xi,\xi_0-\xi_j+2}(..r..) \\ &\Big\langle\nu(x),\psi_n(x)\Big\rangle+\int_0^t(t-\tau)^{\xi_1-1}E_{\xi,\xi_1}(..r..)|_{t=t-\tau}\Big\langle q(\tau)f(x,\tau),\psi_n(x)\Big\rangle d\tau\Big\}(\cos(n\pi)-1)\Big]. \end{aligned} \tag{5.1}$$

We let

$$\begin{aligned} \mathcal{T}(t) = &\sum_{n=1}^{\infty}n\pi\Big\{\mathcal{E}_{\xi,1}(..r..)\Big\langle\rho(x),\psi_n(x)\Big\rangle+\sum_{j=1}^{m-1}a_j\mathcal{E}_{\xi,\xi_0-\xi_j+1}(..r..)\Big\langle\rho(x),\psi_n(x)\Big\rangle \\ &+\mathcal{E}_{\xi,2}(..r..)\Big\langle\nu(x),\psi_n(x)\Big\rangle+\sum_{j=1}^{m-1}a_j\mathcal{E}_{\xi,\xi_0-\xi_j+2}(..r..)\Big\langle\nu(x),\psi_n(x)\Big\rangle\Big\}(\cos(n\pi)-1), \end{aligned} \tag{5.2}$$

and

$$K(t,\tau)=\sum_{n=1}^{\infty}n\pi\mathcal{E}_{\xi,\xi_1}(..r..)|_{t=t-\tau}\Big\langle f(x,\tau),\psi_n(x)\Big\rangle(\cos(n\pi)-1). \tag{5.3}$$

Hence, (5.1) becomes

$$q(t)=\left[\int_{-1}^{1}f(x,t)dx\right]^{-1}\Big[{}^{C}D^{\xi_0}_{0_+,t}E(t)+\sum_{j=1}^{m-1}a_j{}^{C}D^{\xi_j}_{0_+,t}E(t)+\mathcal{T}(t)+\int_0^t q(\tau)K(t,\tau)d\tau\Big]. \tag{5.4}$$

Define the mapping $\mathcal{S}: C([0,T])\to C([0,T])$ by

$$\mathcal{S}(q(t)) := q(t), \tag{5.5}$$

where $q(t)$ is given by (5.1). First, we will show that for $q(t)\in C([0,T])$, the mapping $\mathcal{S}(q(t))$ is well-defined and then, second, we will show that the mapping is a contraction.

Due to Lemmas 1 and 6, and Eqs (5.2) and (5.3), we have

$$t^{\xi_0+\xi_j-1}|\mathcal{T}(t)|\leq\sum_{n=1}^{\infty}\frac{C_0}{|\lambda_n|^3}\Big(\|\rho''(x)\|t^{\xi_j-1}+\sum_{j=1}^{m-1}a_j\|\rho''(x)\|t^{\xi_0-1}+\|\nu''(x)\|t^{\xi_j}+\sum_{j=1}^{m-1}a_j\|\nu''(x)\|t^{\xi_0}\Big), \tag{5.6}$$

$$(t-\tau)|K(t,\tau)|\leq\sum_{n=1}^{\infty}\frac{C_0}{|\lambda_n|^3}\|f''(\tau)\|. \tag{5.7}$$

From Eqs (5.6) and (5.7), we conclude that the series $t^{\xi_0+\xi_j-1}|\mathcal{T}(t)|$ and $(t-\tau)|K(t,\tau)|$ are bounded above. Hence, by virtue of the Weierstrass M-test, $t^{\xi_0+\xi_j-1}|\mathcal{T}(t)|$ and $(t-\tau)K(t,\tau)$ represent continuous functions. Furthermore, we can have $M_2 > 0$ such that

$$|K(t,\tau)| \leq M_2.$$

Hence, the mapping defined by (5.5) is well-defined.

Next, we will show that the mapping $\mathcal{S}(q(t))$ is a contraction. By virtue of Eq (5.4), we have

$$|q_1(\tau) - q_2(\tau)| \;\; = \;\; \left[\int_{-1}^{1} (x,t)dx\right]^{-1} \left\{\int_0^t K(t,\tau)|q_1(\tau) - q_2(\tau)|d\tau\right\}. \tag{5.8}$$

By the assumptions of Theorem 4, we obtain

$$\max_{0\leq t\leq T} |\mathcal{S}(q_1(t)) - \mathcal{S}(q_2(t))| \;\; \leq \;\; M_1M_2T \max_{0\leq t\leq T} |q_1(\tau) - q_2(\tau)|.$$

For $T < \dfrac{1}{M_1M_2}$, where M_1 and M_2 are positive constant independent of n,

$$\max_{0\leq t\leq T} \|\mathcal{S}(q_1(t)) - \mathcal{S}(q_2(t))\|_{C([0,T])} \;\; \leq \;\; M_1M_2T \max_{0\leq t\leq T} \|q_1 - q_2\|_{C([0,T])},$$

which implies that the mapping $\mathcal{S}(.)$ is a contraction. Hence, the unique existence is guarenteed by using the Banach fixed point theorem.

Next, we will prove that the regular solution $u(x,t)$ given by (4.3), $Q_{\Omega}^{\eta_1,\eta_2}u(x,t)$, $t^{2\xi_0+\xi_j-1}|^{C}D_{0_+,t}^{\xi_0}u(x,t)|$, and $t^{2\xi_0+\xi_j-1}|^{C}D_{0_+,t}^{\xi_j}u(x,t)|$, $\;j = 1,2,...,m-1,\;\; m \in \mathbb{N}$, represent continuous functions. From Eq (4.3), we have

$$\begin{aligned}|u(x,t)| \leq \sum_{n=1}^{\infty}\Bigg(&\Big|\mathcal{E}_{\xi,1}(..r..)\Big\langle\rho(x),\psi_n(x)\Big\rangle\Big| + \Big|\sum_{j=1}^{m-1} a_j\mathcal{E}_{\xi,\xi_0-\xi_j+1}(..r..)\Big\langle\rho(x),\psi_n(x)\Big\rangle\Big| \\ &+ \Big|\mathcal{E}_{\xi,2}(..r..)\Big\langle\nu(x),\psi_n(x)\Big\rangle\Big| + \Big|\sum_{j=1}^{m-1} a_j\mathcal{E}_{\xi,\xi_0-\xi_j+2}(..r..)\Big\langle\nu(x),\psi_n(x)\Big\rangle\Big| \\ &+ \Big|\mathcal{E}_{\xi,\xi_0}(..r..) * \Big\langle q(t)f(x,t),\psi_n(x)\Big\rangle\Big|\Bigg)|\psi_n(x)|.\end{aligned}$$

By Lemmas 1 and 4, we have

$$\begin{aligned}|u(x,t)| \leq \sum_{n=1}^{\infty}\frac{C_0}{|\lambda_n|}\Bigg(&\|\rho''(x)\|t^{-\xi_0} + \sum_{j=1}^{m-1} a_j\|\rho''(x)\|t^{-\xi_j} + \|\nu''(x)\|t^{1-\xi_0} \\ &+ \sum_{j=1}^{m-1} a_j\|\nu''(x)\|t^{1-\xi_j} + \|qf\|\Bigg),\end{aligned}$$

which yields to

$$t^{\xi_0+\xi_j-1}|u(x,t)| \leq \sum_{n=1}^{\infty}\frac{C_0}{|\lambda_n|}\Bigg(\|\rho''(x)\|t^{\xi_j-1} + \sum_{j=1}^{m-1} a_j\|\rho''(x)\|t^{\xi_0-1} + \|\nu''(x)\|t^{\xi_j}$$

$$+\sum_{j=1}^{m-1} a_j\|\rho''(x)\|t^{\xi_0} + M_3\|f\|t^{\xi_0+\xi_j-1}\Big). \tag{5.9}$$

The uniform convergence of the series involved in (5.9) is ensured by using the Weierstrass M-test. Hence, we deduce that the series $t^{\xi_0+\xi_j-1}|u(x,t)$ represents a continuous function.

Next, due to (1.6), (3.8), and under the assumption of Theorem 4, we can show the uniform convergence of $Q_{\Omega}^{\eta_1,\eta_2}u(x,t)$.

Similarly, we will show that the convergence of $t^{2\xi_0+\xi_j-1}|{}^{C}D_{0_+,t}^{\xi_0}u(x,t)|$ and $t^{2\xi_0+\xi_j-1}|{}^{C}D_{0_+,t}^{\xi_j}u(x,t)|,\quad j = 1,2,...,m-1,\quad m\in\mathbb{N}$, represent continuous functions. □

5.2. Uniqueness of the solution

In this subsection, we will discuss the uniqueness of the solution of the ISP-II (1.1)–(1.4) and (1.8).

Theorem 5. *The regular solution of the ISP-II is unique by satisfying the assumptions of Theorem 4.*

Proof. We have already proved the uniqueness of the time-dependent source term $q(t)$ by using the Banach fixed point theorem. It remains to prove the uniqueness of $u(x,t)$. Let $\bar{u}(x,t) = u_1(x,t)-u_2(x,t)$, where $u_1(x,t)$ and $u_2(x,t)$ are two solution sets of the ISP-II (1.1)–(1.4) and (1.8). Then, we have the following relation:

$${}^{C}D_{0_+,t}^{\xi_0}\tilde{u}(x,t) + \sum_{j=1}^{m-1} a_j{}^{C}D_{0_+,t}^{\xi_j}\tilde{u}(x,t) = -(-\Delta)^{\eta_1/2}\tilde{u}(x,t) + -(-\Delta)^{\eta_2/2}\tilde{u}(x,t), \quad x\in\Omega_T,$$

with boundary conditions (1.2) and initial conditions:

$$\tilde{u}(x,0) = 0, \qquad \tilde{u}_t(x,0) = 0, \qquad x\in(-1,1). \tag{5.10}$$

Consider the following function:

$$\tilde{T}_n(t) = \int_{-1}^{1}\tilde{u}(x,t)\psi_n(x)dx. \tag{5.11}$$

Taking the Caputo fractional derivatives ${}^{C}D_{0_+,t}^{\xi}(.)$, we obtain the following fractional differential equation:

$${}^{C}D_{0_+,t}^{\xi_0}\tilde{T}_n(t) + \sum_{j=1}^{m-1} a_j{}^{C}D_{0_+,t}^{\xi_j}\tilde{T}_n(t) = -\lambda_n\tilde{T}_n(t) + \bar{q}(t)f_n(t).$$

By using the Laplace transform technique, we obtain

$$\tilde{T}_n(t) = \tilde{T}_n(0)\Big(\mathcal{E}_{\xi,1}(..r..) + \sum_{j=1}^{m-1} a_j\mathcal{E}_{\xi,\xi_0-\xi_j+1}(..r..)\Big) + \tilde{T}'_n(0)\Big(\mathcal{E}_{\xi,2}(..r..) + \sum_{j=1}^{m-1} a_j\mathcal{E}_{\xi,\xi_0-\xi_j+2}(..r..)\Big)$$
$$+ \mathcal{E}_{\xi,\xi_0}(..r..) * \Big\langle\tilde{q}(t)f(x,t),\psi_n(x)\Big\rangle.$$

By using the uniqueness of $q(t)$ and the initial conditions (5.10), we get

$$\tilde{T}_n(t) = 0, \quad t\in[0,T].$$

Hence, we have

$$u_1(x,t) = u_2(x,t).$$

□

5.3. Ill-posedness of the ISP-II

In this subsection, we will demonstrate the ill-posedness of the ISP-II. We present the following example to illustrate the ill-posedness of ISP-II. In Equation (1.1), we consider two fractional derivatives, that is, $a_i = 0,\ \ i = 2, 3, ..., m-1$, and

$$\tilde{f}(x,t) = \lambda_k\left(\frac{\Gamma(3-\xi_1)}{\Gamma(3-\xi_0-\xi_1)} + a_1\frac{\Gamma(3-\xi_1)}{\Gamma(3-2\xi_1)}t^{\xi_0-\xi_1} + ((\pi/2)^{\eta_1} + (\pi/2)^{\eta_2})t^{\xi_0}\right)\sin\left(\frac{\pi}{2}(x+1)\right).$$

The initial conditions (1.3) and (1.4) are taken to be zero and the over-specified condition is given by

$$\int_{-1}^{1}\tilde{u}(x,t)dx = \frac{4\lambda_k t^{2-\xi_1}}{\pi}.$$

Using Lemma 5 and Proposition 1, we obtain

$$\tilde{u}(x,t) = \lambda_k t^{2-\xi_1}\sin\left(\frac{\pi}{2}(x+1)\right).$$

The implicit expression for $q(t)$ has the following form:

$$q(t) = \left[\int_{-1}^{1} f(x,t)dx\right]^{-1}\left[{}^{C}D^{\xi_0}_{0_+,t}E(t) + a_1{}^{C}D^{\xi_1}_{0_+,t}E(t) + \mathcal{T}(t) + \int_0^t q(\tau)K(t,\tau)d\tau\right],$$

where

$$\int_0^1 f(x,t)dx = \frac{4\lambda_k}{\pi}\left(\frac{\Gamma(3-\xi_1)}{\Gamma(3-\xi_0-\xi_1)} + a_1\frac{\Gamma(3-\xi_1)}{\Gamma(3-2\xi_1)}t^{\xi_0-\xi_1} + ((\pi/2)^{\eta_1} + (\pi/2)^{\eta_2})t^{\xi_0}\right),$$

$${}^{C}D^{\xi_0}_{0_+,t}E(t) = \frac{4\lambda_k\Gamma(3-\xi_1)}{\Gamma(3-\xi_0-\xi_1)}t^{2-\xi_0-\xi_1}, \qquad {}^{C}D^{\xi_1}_{0_+,t}E(t) = \frac{4\lambda_k\Gamma(3-\xi_1)}{\pi\Gamma(3-2\xi_1)}t^{2-2\xi_1}, \qquad \mathcal{T}(t) = 0,$$

$$K(t,\tau) = \frac{4\lambda_k}{\pi}\left(\frac{\Gamma(3-\xi_1)}{\Gamma(3-\xi_0-\xi_1)} + a_1\frac{\Gamma(3-\xi_1)}{\Gamma(3-2\xi_1)}t^{\xi_0-\xi_1} + ((\pi/2)^{\eta_1} + (\pi/2)^{\eta_2})t^{\xi_0}\right)\mathcal{E}_{\xi,\xi_1}(..r..)|_{t=t-\tau}.$$

Consequently, we get

$$q(t) = t^{3-\xi_0-\xi_1}.$$

Now, assume that we have another source term, i.e., $f(x,t)$. Hence, the solution related to $f(x,t)$ is $u(x,t)$.

An error in $L^2(\Omega_T)$ between two corresponding solutions is:

$$\|\tilde{u} - u\|_{L^2(\Omega_T)} = \int_0^T \|\tilde{u} - u\|_{L^2(-1,1)}dt \ \ \leq \ \ \frac{4\lambda_k}{\pi}\int_0^T t^{2-\xi_1}\ dt.$$

This yields

$$\|\tilde{u} - u\|_{L^2(\Omega_T)} = \frac{4\lambda_k T^{3-\xi_1}}{\pi(3-\xi_1)}.$$

Taking the limit as $k \to \infty$, one gets the ill-posedness of the ISP-II:

$$\lim_{k\to\infty}\|\tilde{u} - u\|_{L^2(\Omega_T)} = \frac{4T^{3-\xi_1}}{\pi(3-\xi_1)}\lim_{k\to\infty}\lambda_k = +\infty.$$

6. Examples

In this section, we will present some examples for the ISPs.

Example 1. *As a specific example of ISP-I, we consider*

$$\rho(x) = \cos\left(\frac{x}{2}\right), \quad \nu(x) = \frac{t^{\xi_0}}{\Gamma(5+\xi_0)}\cos\left(\frac{\pi x}{2}\right), \quad \textit{and} \quad \Phi(x) = T^{\xi_0}\sin\Big(\frac{\pi}{2}(x+1)\Big).$$

The coefficients of the series expansion of $\rho(x)$, $\nu(x)$, and $\Phi(x)$ for $n = 1$ are given by

$$\rho_1 = \frac{4\pi\cos\left(\frac{x}{2}\right)}{\pi^2-1}, \quad \nu_1 = \frac{t^{\xi_0}}{\Gamma(5+\xi_0)}, \quad \textit{and} \quad \Phi_1 = T^{\xi_0}.$$

By plugging $a_i = 0$ for $i = 1, 2, 3, ..., m-1$ and using (3.3)*, we obtain*

$$f_1 = \frac{1}{\mathcal{E}_{\xi_0,\xi_0+1}(T;\lambda_1)}\Big(T^{\xi_0} - \frac{4\pi\cos\left(\frac{x}{2}\right)}{\pi^2-1}\mathcal{E}_{\xi_0,1}(T;\lambda_1) - \frac{t^{\xi_0}}{\Gamma(5+\xi_0)}\mathcal{E}_{\xi_0,2}(T;\lambda_1)\Big). \tag{6.1}$$

Equation (3.2) *yields the following expressions:*

$$T_1(t) = \frac{4\pi\cos\left(\frac{x}{2}\right)}{\pi^2-1}\mathcal{E}_{\xi_0,1}(T;\lambda_1) + \frac{t^{\xi_0}}{\Gamma(5+\xi_0)}\mathcal{E}_{\xi_0,2}(T;\lambda_1) + f_1\mathcal{E}_{\xi_0,\xi_0+1}(T;\lambda_1), \tag{6.2}$$

where f_1 is given in Eq (6.1)*. By substituting the previously derived expressions for f_1 and $T_1(t)$, we obtain the solution to the inverse source problem, namely $f(x)$ and $u(x,t)$, as follows:*

$$f(x) = \frac{\sin\Big(\frac{\pi}{2}(x+1)\Big)}{\mathcal{E}_{\xi_0,\xi_0+1}(T;\lambda_1)}\Big(T^{\xi_0} - \frac{4\pi\cos\left(\frac{x}{2}\right)}{\pi^2-1}\mathcal{E}_{\xi_0,1}(T;\lambda_1) - \frac{t^{\xi_0}}{\Gamma(5+\xi_0)}\mathcal{E}_{\xi_0,2}(T;\lambda_1)\Big),$$

$$u(x,t) = \Big(\frac{4\pi\cos\left(\frac{x}{2}\right)}{\pi^2-1}\mathcal{E}_{\xi_0,1}(T;\lambda_1) + \frac{t^{\xi_0}}{\Gamma(5+\xi_0)}\mathcal{E}_{\xi_0,2}(T;\lambda_1) + f_1\mathcal{E}_{\xi_0,\xi_0+1}(T;\lambda_1)\Big)\sin\Big(\frac{\pi}{2}(x+1)\Big).$$

Example 2. *In this second example, the solution set $u(x,t)$ and $f(x)$ of the ISP-I is obtained by setting $a_1 = 1$ and $a_i = 0$, $i = 2, 3, ..., m-1$:*

$$f(x) = \frac{\sin\Big(\frac{\pi}{2}(x+1)\Big)}{\mathcal{E}_{(\xi_0,\xi_0-\xi_1),\xi_0+1}(T;\lambda_1)}\Big(T^{\xi_0} - \frac{4\pi\cos\left(\frac{x}{2}\right)}{\pi^2-1}\mathcal{E}_{(\xi_0,\xi_0-\xi_1),1}(T;\lambda_1) - \frac{t^{\xi_0}}{\Gamma(5+\xi_0)}\mathcal{E}_{(\xi_0,\xi_0-\xi_1),2}(T;\lambda_1)\Big),$$

$$\begin{aligned} u(x,t) = \Big(&\frac{4\pi\cos\left(\frac{x}{2}\right)}{\pi^2-1}\mathcal{E}_{(\xi_0,\xi_0-\xi_1),1}(T;\lambda_1) + \frac{t^{\xi_0}}{\Gamma(5+\xi_0)}\mathcal{E}_{(\xi_0,\xi_0-\xi_1),2}(T;\lambda_1) \\ &+ f_1\mathcal{E}_{(\xi_0,\xi_0-\xi_1),\xi_0+1}(T;\lambda_1)\Big)\sin\Big(\frac{\pi}{2}(x+1)\Big). \end{aligned}$$

Example 3. *For the specific case of ISP-II, we consider only one fractional derivative in Eq* (1.1)*, where $a_i = 0$ for $i = 1, 2, ..., m-1$. The function $f(x,t)$ is given by*

$$f(x,t) = \Big(\frac{\Gamma(3)}{\Gamma(3-\xi_0)} + ((\pi/2)^{\eta_1} + (\pi/2)^{\eta_2})t^{\xi_0}\Big)\sin\Big(\frac{\pi}{2}(x+1)\Big), \qquad n = 1,$$

and the initial conditions in (1.3) *and* (1.4) *are set to zero. The over-specified condition is $\int_{-1}^{1} u(x,t)dx = \frac{4t^2}{\pi}$. Using Lemma 5 and Proposition 1, we obtain*

$$u(x,t) = t^2 \sin\Big(\frac{\pi}{2}(x+1)\Big).$$

The expression for $q(t)$ given by (5.1) *takes the form*

$$q(t) = \Big[\int_{-1}^{1} f(x,t)dx\Big]^{-1}\Big[{}^{C}D^{\xi_0}_{0_+,t}E(t) + \mathcal{T}(t) + \int_0^t q(\tau)K(t,\tau)d\tau\Big],$$

where

$$\begin{aligned}\int_{-1}^{1} f(x,t)dx &= \frac{4}{\pi}\Big(\frac{\Gamma(3)}{\Gamma(3-\xi_0)} + ((\pi/2)^{\eta_1} + (\pi/2)^{\eta_2})t^{\xi_0}\Big),\\ {}^{C}D^{\xi_0}_{0_+,t}E(t) &= \frac{4\Gamma(3)}{\pi\Gamma(3-\xi_0)}t^{2-\xi_0}, \qquad \mathcal{T}(t) = 0,\\ K(t,\tau) &= \Big(\frac{\Gamma(3)}{\Gamma(3-\xi_0)} + ((\pi/2)^{\eta_1} + (\pi/2)^{\eta_2})\tau^{\xi_0}\Big)\mathcal{E}_{\xi_0,\xi_0}(t-\tau;\lambda_1).\end{aligned}$$

In this case, we can find an explicit expression for $q(t)$ given by

$$q(t) = t^{2-\xi_0}.$$

Example 4. *In this example of ISP-II, we take two fractional derivatives in Eq* (1.1)*, where $a_i = 0$, $i = 2, 3, ..., m-1$. Consider*

$$f(x,t) = \left(\frac{\Gamma(3-\xi_1)}{\Gamma(3-\xi_0-\xi_1)} + a_1\frac{\Gamma(3-\xi_1)}{\Gamma(3-2\xi_1)}t^{\xi_0-\xi_1} + ((\pi/2)^{\eta_1} + (\pi/2)^{\eta_2})t^{\xi_0}\right)\sin\Big(\frac{\pi}{2}(x+1)\Big).$$

The initial conditions (1.3) *and* (1.4) *are taken to be zero and the over-specified condition is $\int_{-1}^{1} u(x,t)dx = \frac{4t^{2-\xi_1}}{\pi}$. Due to Lemma 5 and Proposition 1, we obtain*

$$u(x,t) = t^{2-\xi_1}\sin\Big(\frac{\pi}{2}(x+1)\Big).$$

The implicit expression for $q(t)$ has the following form:

$$q(t) = \Big[\int_{-1}^{1} f(x,t)dx\Big]^{-1}\Big[{}^{C}D^{\xi_0}_{0_+,t}E(t) + a_1{}^{C}D^{\xi_1}_{0_+,t}E(t) + \mathcal{T}(t) + \int_0^t q(\tau)K(t,\tau)d\tau\Big],$$

where

$$\int_0^1 f(x,t)dx = \frac{4}{\pi}\left(\frac{\Gamma(3-\xi_1)}{\Gamma(3-\xi_0-\xi_1)} + a_1\frac{\Gamma(3-\xi_1)}{\Gamma(3-2\xi_1)}t^{\xi_0-\xi_1} + ((\pi/2)^{\eta_1} + (\pi/2)^{\eta_2})t^{\xi_0}\right),$$

$$ {}^{C}D^{\xi_0}_{0_+,t}E(t) = \frac{4\Gamma(3-\xi_1)}{\Gamma(3-\xi_0-\xi_1)}t^{2-\xi_0-\xi_1}, \quad {}^{C}D^{\xi_1}_{0_+,t}E(t) = \frac{4\Gamma(3-\xi_1)}{\pi\Gamma(3-2\xi_1)}t^{2-2\xi_1}, \quad \mathcal{T}(t) = 0, $$

$$ K(t,\tau) = \frac{4}{\pi}\left(\frac{\Gamma(3-\xi_1)}{\Gamma(3-\xi_0-\xi_1)} + a_1\frac{\Gamma(3-\xi_1)}{\Gamma(3-2\xi_1)}t^{\xi_0-\xi_1} + ((\pi/2)^{\eta_1} + (\pi/2)^{\eta_2})t^{\xi_0}\right)\mathcal{E}_{\xi,\xi_1}(..r..)|_{t=t-\tau}. $$

Consequently, we get

$$ q(t) = t^{3-\xi_0-\xi_1}. $$

7. Conclusions

In this article, two ISPs for a multi-term space-time fractional differential equation (FDE) incorporating Caputo fractional derivatives with respect to time and a bi-fractional Laplacian operator with respect to space are considered. The series solution for the ISPs is constructed by using the eigenfunction expansion method. First, ISP-I is addressed, which determines a space-varying source term from the over-specified condition, i.e., the given data at a specific time T, along with the solution. The series solutions, obtained through the eigenfunction expansion method using an orthonormal set of eigenfunctions of the associated spectral problem, involve multinomial Mittag-Leffler functions. The regularity of the solution is ensured under certain assumptions about the given data and by utilizing results related to multinomial Mittag-Leffler functions. Additionally, the uniqueness and stability of the solution concerning the given data are guaranteed in a similar manner. These ISPs are shown to be ill-posed in the sense of Hadamard. The ISP-II is focused on recovering a time-dependent source term from the over-determined condition of integral type, along with the solution. The unique existence of a continuous source term in ISP-II is established using the Banach fixed point theorem. Furthermore, the uniqueness of the solution is demonstrated using the completeness of the eigenfunctions.

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Conflict of interest

The author declares that there is no conflict of interest.

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