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Research article

Analysis of drug-resistant tuberculosis in a two-patch environment using Caputo fractional-order modeling

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Abstract: In this study, a fractional-order mathematical model of the transmission dynamics of drug-resistant tuberculosis within a two-patch system incorporating population migration was proposed and analyzed using the Caputo operator. The positivity, boundedness, existence, and uniqueness of the solutions as well as the Ulam-Hyers stability of the model were guaranteed. Additionally, the basic reproduction numbers were derived and analyzed for sensitivity to identify the key parameters that affected the spread of drug-resistant tuberculosis. Moreover, the cure rates were used as control variables to formulate an optimal control problem, which examined the efficacy of the control measures and the influence of fractional order on the control values. The numerical results showed that controlling the cure rate can significantly reduce the number of drug-resistant tuberculosis infections, thus verifying the effectiveness of the proposed control strategy. As the fractional order decreased, the duration during which the maximum control intensity was applied in both patches increased.

Keywords: drug-resistant tuberculosis; individual migration; Caputo fractional-order derivative; Ulam-Hyers; sensitivity analysis; optimal control
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1. Introduction

Tuberculosis (TB) is an infectious disease caused by the Mycobacterium TB bacterium. When patients cough, spit, or sneeze, the disease is transmitted through the air, most affecting the lungs. It is estimated that approximately one-quarter of the global population has been infected with TB bacteria. Globally, TB is the second-leading cause of infectious disease mortality after COVID-19 (see [1]). According to the World Health Organization (WHO) 2023 reports, in 2022, an estimated 10.6 million people worldwide were afflicted with TB, with approximately 40,000 individuals developing multidrug-resistant or rifampicin-resistant TB. However, only two-fifths of drug-resistant TB (DR-TB) patients receive treatment, and improper administration can lead to even more severe forms of the

disease. Consequently, the transmission of DR-TB remains one of the most pressing challenges in global TB control efforts [2].

Transmission dynamics modeling is an important analytical tool for studying infectious diseases [3]. Dynamic transmission models can help decision-makers better understand disease transmission patterns and guide policy development and resource allocation to achieve the goal of controlling and eliminating infectious diseases. Nave et al. [4] applied the singular perturbed vector field method to the COVID-19 mathematical model. By decomposing the model into fast and slow subsystems, they compared the results of the model with actual data from China, and a fit of approximately 96% was achieved. Abidemi et al. [5] established a new nonlinear mathematical model to describe the dynamics of Lassa fever transmission between the interacting human and rodent populations, and proposed relevant measures to prevent the spread of the disease within the community. Particularly, many scholars have developed mathematical models to reveal the transmission process of TB and DR-TB. As early as 1962, Waaler et al. [6] used a mathematical model to explain the epidemiological characteristics of TB for the first time. Based on this model, subsequent generations developed various mathematical models to assess the effects of different interventions by considering multiple influencing factors [7–9]. Yu et al. [10] built on the classical model of TB to develop a framework for multidrug-resistant TB, analyzing the impact of model parameters on TB development. Ronoh et al. [11] incorporated multidrug-resistant strains of TB into a model based on the standard susceptible-exposed-infectious-recovered-susceptible epidemiologic framework. Ao et al. developed a susceptible-exposed-infectious-recovered infectious disease model that includes both drug-susceptible TB (DS-TB) and DR-TB, using real-world data from China to simulate and predict the impact of different control strategies on DR-TB [12]. As transportation continues to become more accessible, the movement of people between cities is increasing. In the process of population movement, infectious diseases can easily spread from one place to another. Numerous studies have explored the impact of migration on the spread of infectious diseases [13–16]. Several researchers have established patchy dynamic models to study the transmission mechanisms of TB. Tewa et al. [17, 18] developed a two-patch mathematical model to study TB transmission, assuming that only susceptible individuals are capable of migration. Wahid et al. [19] examined uncontrolled migration and a two-patch model of TB with age-structure.

All of the above models are defined using classical differential operators, however, these operators are localized and cannot effectively capture genetic and memory processes [20]. Given that the human immune system has a memory for viruses, there are inherent limitations in using integer-order differential equations to describe the process of disease transmission. Fractional-order calculus has become a powerful tool in the modeling of infectious diseases due to its nonlocal effects and memory properties, which enable more accurate modeling and analyses of disease transmission dynamics [21]. Several fractional-order derivatives, including Caputo, Atangana-Baleanu-Caputo, and Caputo-Fabrizio, have been proposed in the literature [22–24] and have gained significant attention from researchers. Among them, Caputo derivatives can effectively reflect the historical dependence of the system and have the advantages of physical interpretability, numerical stability, and continuity [25, 26]. Despite the Caputo-Fabrizio derivative possessing a nonsingular kernel, the clarity of its function space remains uncertain and it lacks memory effects [27]. The main advantage of the Atangana-Baleanu-Caputo operator is its nonlocal and nonsingular behavior [28], and there have also been many applications of it modeling disease dynamics [29, 30]. In comparison to alternatives such

as the Atangana-Baleanu-Caputo or Caputo-Fabrizio derivatives, the Caputo derivative stands out as the preferred and widely acknowledged option for precisely modeling the transmission of TB [25].

In recent years, numerous research results have emerged on fractional-order kinetic modeling of TB. For example, Kumar et al. [31] studied a fractional-order TB transmission model under incomplete treatment using Caputo-Fabrizio and Atangana-Baleanu derivatives. Zafar et al. [32] used different derivative operators to develop a mathematical model to study the dynamics of TB and obtained more reasonable results for the fractional-order model through numerical simulations compared to the corresponding integer-order TB model. Panchal et al. [33] developed a fractional order mathematical model in the sense of Caputo and used real data from India (2000–2020) for model fitting. Owolabi and co-workers [34] proposed and analyzed a Caputo fractional-order mathematical model of TB with control measures and showed the effect of different control parameters and fractional-order parameter values on the dynamic behavior of the fractional TB model. Some scholars have also developed fractional-order patch dynamics models to study the nature of the dynamics involved and the impact of population migration on infectious diseases [35–38]. In the literature [26, 38], many scholars have provided a reliable theoretical basis for the stability of fractional-order infectious disease models, the uniqueness and existence of solution, Ulam-Hyers stability, and other dynamical behaviors. Numerous articles describe various types of infectious diseases such as TB, measles, and COVID-19 using the theory of fractional-order differential equations. However, to the best of our knowledge, very few papers have modeled DR-TB with population migration using the Caputo fractional-order derivative operator. Therefore, the current study aims to describe the memory effect during the transmission of DR-TB using the Caputo fractional-order operator and to establish a two-patch fractional-order dynamic model for DR-TB, taking into account population migration. Theoretical analysis and numerical simulation are carried out based on the modeling and its practical significance is illustrated.

This study is organized as follows: In Section 2, we briefly review the basic concepts of fractional-order calculus. In Section 3, we outline the fractional-order model of the DR-TB epidemic in a two-patch environment. In Section 4, we prove some basic properties of the model developed. The existence, uniqueness, and Ulam-Hyers stability of the model solution are addressed in Sections 5 and 6, respectively. In Section 7, we present the numerical scheme using a two-step Lagrange interpolation method. The sensitivity analysis of the parameters and the numerical simulation of the model are detailed in Section 8. In Section 9, the control parameter "cure rate" provides an optimal control strategy for the two-patch DR-TB model. Section 10 concludes the paper.

2. System description

In this section, we introduce the formulation of a two-patch fractional-order DR-TB model. We consider a susceptible-exposed-DS-TB infectious-DR-TB infectious-recovered (SEIMR) DR-TB model that incorporates population migration between two patches. For this purpose, the populations of patches 1 and 2 have been divided into five classes, namely: the susceptible individuals $S_i(t)$, the latent individuals $E_i(t)$, the DS-TB individuals $I_i(t)$, the DR-TB individuals $M_i(t)$, and the recovered individuals $R_i(t)$, with each patch having its own total population $N_i(t)$. Furthermore,

$$N_i(t) = S_i(t) + E_i(t) + I_i(t) + M_i(t) + R_i(t) \ \ (i = 1, 2).$$

Our model is based on the model established by Ronoh et al [11]. Susceptible individuals acquire TB from DS-TB and DR-TB patients due to the transmission rate β_i. The fraction δ of newly infected individuals moves to the class $E_i(t)$, while the remaining portion 1–δ transfers to class $I_i(t)$ individuals departed from class $E_i(t)$ at a rate of ε_i to enter $I_i(t)$. Class $I_i(t)$ transitions to class $M_i(t)$ with a conversion rate of b_i, possibly due to reasons such as low treatment adherence or lower medical standards. During the treatment process, class $M_i(t)$ will transition to non-drug-resistant treatment at a rate of α_i . The cure rates for the class $I_i(t)$ and the class $M_i(t)$ are r_i and c_i, respectively. It is assumed that recruitment in each patch occurs only among susceptible individuals, and only susceptible and recovered individuals migrate between the two patches. At the same time, both natural and disease-related deaths of the population during disease transmission are taken into account in our model. Additionally, it is supposed that the total population remains unchanged and the latent individuals are noninfectious.

The flow chart of this model is shown in Figure 1, and the biological meaning of the parameters used in our proposed model is displayed in Table 1.

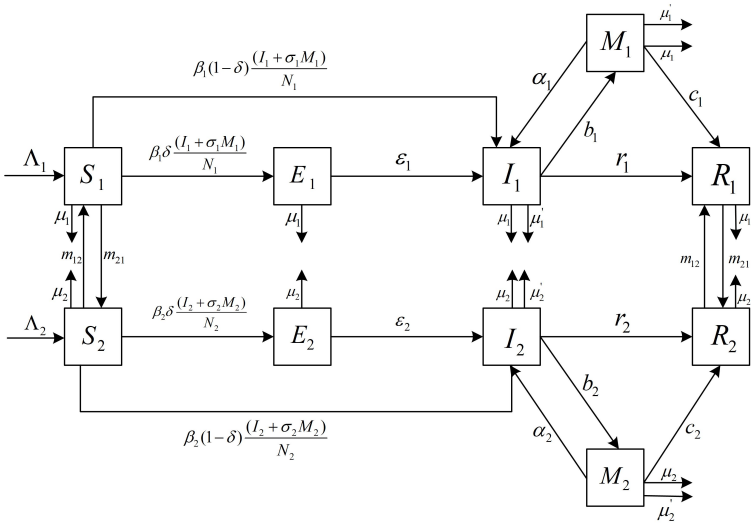


Figure 1. Transfer diagram of the two patches fractional-order DR-TB epidemic model.

Table 1. The parameters used in the model.

Parameters	Biological meanings ($i = 1, 2$; $j = 1, 2$)
δ	Rate at which susceptible individuals become exposed individuals
Λ_i	Recruitment rate
ε_i	Progression rate
β_i	Transmission rate
α_i	Rate of drug resistance loss
b_i	DS-TB to DR-TB conversion rate
r_i	Cure rate of DS-TB
c_i	Cure rate of DR-TB
σ_i	Weight coefficients of DS-TB and DR-TB
μ_i	Natural mortality rate of patch i
$\mu_i^{'}$	Disease-related mortality of patch i
m_{ij}	Migration rate from patch j to patch i ($i \neq j$)

Regarding the above hypotheses on the transmission dynamics of DR-TB, our considered Caputo fractional-order derivative model for the dynamics of the transmission DR-TB, the following dynamical system is given ($i = 1, 2$):

$$\begin{aligned}
{}_0^C D_t^q S_i &= \Lambda_i - \beta_i \frac{(I_i + \sigma_i M_i) S_i}{N_i} - \mu_i S_i + \sum_{j=1, j\neq i}^{2} (m_{ij} S_j - m_{ji} S_i), \\
{}_0^C D_t^q E_i &= \beta_i \delta \frac{(I_i + \sigma_i M_i) S_i}{N_i} - (\varepsilon_i + \mu_i) E_i, \\
{}_0^C D_t^q I_i &= \beta_i (1-\delta) \frac{(I_i + \sigma_i M_i) S_i}{N_i} + \varepsilon_i E_i + \alpha_i M_i - (r_i + b_i + \mu_i + \mu_i^{'}) I_i, \\
{}_0^C D_t^q M_i &= b_i I_i - (\alpha_i + \mu_i + c_i + \mu_i^{'}) M_i, \\
{}_0^C D_t^q R_i &= r_i I_i + c_i M_i - \mu_i R_i + \sum_{j=1, j\neq i}^{2} (m_{ij} R_j - m_{ji} R_i),
\end{aligned} \tag{1}$$

where the initial conditions satisfy the following nonnegative conditions:

$$S_i(0) > 0, \quad E_i(0) \geq 0, \quad I_i(0) \geq 0, \quad M_i(0) \geq 0, \quad R_i(0) \geq 0.$$

Here, $0 < q < 1$ is the order of derivative and ${}_0^C D_t^q$ is the Caputo derivative of order q.

3. Basic fractional results

Mathematical models play a key role in both predicting disease dynamics and assessing the effectiveness of public health interventions. Fractional order modeling has been widely used to model many dynamic processes due to its ability to capture complex and nonlinear disease processes. In this section, we will review some of the basic concepts and known theorems of the fractional calculus, which will be used in the analysis of TB models in subsequent sections.

Definition 3.1. *(Caputo fractional derivative [39]) For a given function h: $[0, T] \to \mathbb{R}$, the Caputo fractional derivative is defined as*

$${}_0^C D_t^q h(t) = \frac{1}{\Gamma(n-q)} \int_0^t \frac{h^{(n)}(s)}{(t-s)^{q-n+1}} ds, \quad (n-1 < q < n).$$

Definition 3.2. *(Mittag-Leffler function [39]) The Mittag-Leffler function is defined as*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + \beta)}, \quad (\alpha, \beta > 0)$$

and

$$E_{\alpha,1}(z) = E_\alpha(z).$$

Theorem 3.1. *(Laplace transform) The Laplace transform of the Caputo fractional operator of order q is given by*

$$L[{}_0^C D_t^q h(t)] = s^q H(s) - \sum_{i=0}^{n-1} s^{q-i-1} h^i(0), \quad n \in \mathbb{N}, q \in [n-1, n).$$

Theorem 3.2. *(Generalized mean value theorem [40]) Let $q \in (0,1]$, where function $h(t)$ is continuous on interval $[a,b]$, and if ${}_0^C D_t^q h(t) \in [a,b]$, then*

$$h(t) = h(a) + \frac{1}{\Gamma(q)}({}_0^C D_t^q)(\tau)(t-a)^q$$

with $a < \tau < t, \forall t \in [a,b]$. Thus, $\forall t \in (a,b)$, ${}_0^C D_t^q h(t) \geq 0$, $h(t)$ is increasing, and ${}_0^C D_t^q h(t) \leq 0$, $h(t)$ is decreasing.

Theorem 3.3. *[41] Assume that $f(t) \in \mathbb{R}^+$ is a differentiable function. Then, for any $t > 0$,*

$${}_0^C D_t^q \left[f(t) - f^* - f^* ln \frac{f(t)}{f^*} \right] \leq (1 - \frac{f(t)}{f^*}) {}_0^C D_t^q (f(t)), \quad f^* \in \mathbb{R}^+, \forall q \in (0,1).$$

4. Analysis of the system

Consider the dynamic of Caputo fractional-order model (1) in the biologically feasible region:

$$\Omega = \left\{ (S_1, E_1, I_1, M_1, R_1, S_2, E_2, I_2, M_2, R_2) \in \mathbb{R}_+^{10} \leq \frac{\Lambda_1 + \Lambda_2}{\mu}, \quad \mu = min\{\mu_1, \mu_2\} \right\}.$$

4.1. Nonnegativity and boundedness of the solution

Lemma 4.1. *(Positivity) Model (1) solution is nonnegative by $\forall (S_1(0), E_1(0), I_1(0), M_1(0), R_1(0), S_2(0), E_2(0), I_2(0), M_2(0), R_2(0)) \in \mathbb{R}_+^{10}$, for $t > 0$.*

Proof. We suppose that every parameter is positive. The initial conditions are

$$\begin{aligned} &S_1(0) > 0, \ \ E_1(0) \geq 0, \ \ I_1(0) \geq 0, \ \ M_1(0) \geq 0, \ \ R_1(0) \geq 0, \\ &S_2(0) > 0, \ \ E_2(0) \geq 0, \ \ I_2(0) \geq 0, \ \ M_2(0) \geq 0, \ \ R_2(0) \geq 0. \end{aligned}$$

Using the generalized mean value theorem, the proof steps presented demonstrate the nonnegativity of the proposed model $\mathbb{R}_+^{10}$. From the model (1), we obtained

$$\begin{aligned} {}_0^C D_t^q S_1 \mid_{S_1=0} &= \Lambda_1 + m_{12} S_2 > 0, \quad {}_0^C D_t^q E_1 \mid_{E_1=0} = \beta_1 \delta \frac{(I_1 + \sigma_1 M_1) S_1}{N_1} \geq 0, \\ {}_0^C D_t^q I_1 \mid_{I_1=0} &= \beta_1 (1-\delta) \frac{\sigma_1 M_1 S_1}{N_1} + \varepsilon_1 E_1 + \alpha_1 M_1 \geq 0, \\ {}_0^C D_t^q M_1 \mid_{M_1=0} &= b_1 I_1 \geq 0, \quad {}_0^C D_t^q R_1 \mid_{R_1=0} = r_1 I_1 + c_1 M_1 + m_{12} R_2 \geq 0, \\ {}_0^C D_t^q S_2 \mid_{S_2=0} &= \Lambda_2 + m_{21} S_1 > 0, \quad {}_0^C D_t^q E_2 \mid_{E_2=0} = \beta_2 \delta \frac{(I_2 + \sigma_2 M_2) S_2}{N_2} \geq 0, \\ {}_0^C D_t^q I_2 \mid_{I_2=0} &= \beta_2 (1-\delta) \frac{\sigma_2 M_2 S_2}{N_2} + \varepsilon_2 E_2 + \alpha_2 M_2 \geq 0, \\ {}_0^C D_t^q M_2 \mid_{M_2=0} &= b_2 I_2 \geq 0, \quad {}_0^C D_t^q R_2 \mid_{R_2=0} = r_2 I_2 + c_2 M_2 + m_{21} R_1 \geq 0. \end{aligned}$$

As a result, for any $t > 0$, the set Ω is nonnegative in $\mathbb{R}_+^{10}$. □

Lemma 4.2. *(Boundedness) The solution of model (1) is bounded with nonnegative initial conditions in the region* Ω.

Proof. Let

$$\Lambda = \Lambda_1 + \Lambda_2, \quad \mu = min\{\mu_1, \mu_2\}$$

and

$$N(t) = S_1(t) + E_1(t) + I_1(t) + M_1(t) + R_1(t) + S_2(t) + E_2(t) + I_2(t) + M_2(t) + R_2(t).$$

For model (1), on adding all equations, one has

$$ {}_0^C D_t^q N(t) + \mu N(t) \leq \Lambda. \tag{2}$$

Apply the Laplace transform on both sides of the Eq (2), and we get:

$$\begin{aligned} s^q L\{N(t)\} - s^{q-1}N(0) + \mu L\{N(t)\} &\leq \frac{\Lambda}{s}, \\ L\{N(t)\}(s^q + \mu) &\leq \frac{\Lambda}{s} + s^{q-1}N(0), \\ L\{N(t)\} &\leq \frac{\Lambda}{s(s^q + \mu)} + \frac{s^{q-1}N(0)}{s^q + \mu}. \end{aligned} \tag{3}$$

Taking the inverse Laplace transform on both sides of Eq (3), we obtained

$$\begin{aligned} N(t) &\leq L^{-1}\left\{\frac{\Lambda}{s(s^q + \mu)}\right\} + L^{-1}\left\{\frac{s^{q-1}N(0)}{s^q + \mu}\right\}, \\ &\leq \frac{\Lambda}{\mu}L^{-1}\left\{\frac{\mu}{s(s^q + \mu)}\right\} + N(0)L^{-1}\left\{\frac{s^{q-1}}{s^q + \mu}\right\}, \\ &\leq \frac{\Lambda}{\mu}[1 - E_q(-\mu t^q)] + N(0)E_q(-\mu t^q), \\ &\leq \frac{\Lambda}{\mu} - (\frac{\Lambda}{\mu} - N(0))E_q(-\mu t^q), \\ &\leq \frac{\Lambda}{\mu} - cE_q(-\mu t^q), \end{aligned}$$

where,

$$c = \frac{\Lambda}{\mu} - N(0).$$

So,

$$N(t) \leq \frac{\Lambda}{\mu}.$$

Therefore, the solution of model (1) is bounded within the feasible region Ω . □

4.2. Equilibrium points and basic reproduction number

The population free of both TB and DR-TB can be represented by the disease-free point, which is obtained by considering

$$ {}_0^C D_t^q \varphi_i(t) = 0 $$

for each state variable $\varphi_i (i = 1, 2, \ldots, 10)$ and substituting

$$ E_1(0) = I_1(0) = M_1(0) = R_1(0) = E_2(0) = I_2(0) = M_2(0) = R_2(0) = 0. $$

Therefore, with these substitutions, we can obtain:

$$ S_1^* = \frac{m_{12}\Lambda_2 + \Lambda_1(\mu_2 + m_{12})}{\mu_1\mu_2 + m_{12}\mu_1 + m_{21}\mu_2}, \quad S_2^* = \frac{m_{21}\Lambda_1 + \Lambda_2(\mu_1 + m_{21})}{\mu_1\mu_2 + m_{12}\mu_1 + m_{21}\mu_2}. $$

Thus, the disease-free equilibrium (DFE) point P_1^* for the model can be written as:

$$ P_1^* = (\frac{m_{12}\Lambda_2 + \Lambda_1(\mu_2 + m_{12})}{\mu_1\mu_2 + m_{12}\mu_1 + m_{21}\mu_2}, 0, 0, 0, 0, \frac{m_{21}\Lambda_1 + \Lambda_2(\mu_1 + m_{21})}{\mu_1\mu_2 + m_{12}\mu_1 + m_{21}\mu_2}, 0, 0, 0, 0). $$

Lemma 4.3. *There exists a unique DFE point* $P_1^*(S_1^*, 0, 0, 0, 0, S_2^*, 0, 0, 0, 0)$ *for model (1).*

Proof. Let

$$ S^* = (S_1^*, S_2^*), \quad \Lambda = (\Lambda_1, \Lambda_2), \quad A = \begin{pmatrix} \mu_1 + m_{21} & -m_{12} \\ -m_{21} & \mu_2 + m_{12} \end{pmatrix}. $$

Obviously, P_1^* satisfies the following equation:

$$ \begin{aligned} \Lambda_1 - \mu_1 S_1^* + m_{12} S_2^* - m_{21} S_1^* = 0, \\ \Lambda_2 - \mu_2 S_2^* + m_{21} S_1^* - m_{12} S_2^* = 0, \end{aligned} $$

then the above equation can be written as

$$ AS^* = \Lambda. $$

It can be found that the matrix A is a nonsingular M-matrix. According to [42],

$$ A^{-1} \geq 0. $$

Thus, there exists a unique solution

$$ S^* = A^{-1}\Lambda. $$

Therefore, model (1) has a unique DFE point. □

The endemic disease equilibrium point of the model (1) is denoted by $P_2^*(\hat{S}_1^*, \hat{E}_1^*, \hat{I}_1^*, \hat{M}_1^*, \hat{R}_1^*, \hat{S}_2^*, \hat{E}_2^*, \hat{I}_2^*, \hat{M}_2^*, \hat{R}_2^*)$ and considering

$$ {}_0^C D_t^q \varphi_i(t) = 0 $$

for each state variable φ_i $(i = 1, 2, \ldots, 10)$. Thus, we solve the equation as follows:

$$\begin{cases} \Lambda_1 - \beta_1 \frac{(\hat{I}_1^* + \sigma_1 \hat{M}_1^*)\hat{S}_1^*}{N_1} - \mu_1 \hat{S}_1^* + m_{12}\hat{S}_2^* - m_{21}\hat{S}_1^* = 0, \\ \beta_1 \delta \frac{(\hat{I}_1^* + \sigma_1 \hat{M}_1^*)\hat{S}_1^*}{N_1} - (\varepsilon_1 + \mu_1)\hat{E}_1^* = 0, \\ \beta_1 (1-\delta) \frac{(\hat{I}_1^* + \sigma_1 \hat{M}_1^*)\hat{S}_1^*}{N_1} + \varepsilon_1 \hat{E}_1^* + \alpha_1 \hat{M}_1^* - (r_1 + b_1 + \mu_1 + \mu_1^{'})\hat{I}_1^* = 0, \\ b_1 \hat{I}_1^* - (\alpha_1 + \mu_1 + c_1 + \mu_1^{'})\hat{M}_1^* = 0, \\ r_1 \hat{I}_1^* + c_1 \hat{M}_1^* - \mu_1 \hat{R}_1^* + m_{12}\hat{R}_2^* - m_{21}\hat{R}_1^* = 0, \\ \Lambda_2 - \beta_2 \frac{(\hat{I}_2^* + \sigma_2 \hat{M}_2^*)\hat{S}_2^*}{N_2} - \mu_2 \hat{S}_2^* + m_{21}\hat{S}_1^* - m_{12}\hat{S}_2^* = 0, \\ \beta_2 \delta \frac{(\hat{I}_2^* + \sigma_2 \hat{M}_2^*)\hat{S}_2^*}{N_2} - (\varepsilon_2 + \mu_2)\hat{E}_2^* = 0, \\ \beta_2 (1-\delta) \frac{(\hat{I}_2^* + \sigma_2 \hat{M}_2^*)\hat{S}_2^*}{N_2} + \varepsilon_2 \hat{E}_2^* + \alpha_2 \hat{M}_2^* - (r_2 + b_2 + \mu_2 + \mu_2^{'})\hat{I}_2^* = 0, \\ b_2 \hat{I}_2^* - (\alpha_2 + \mu_2 + c_2 + \mu_2^{'})\hat{M}_2^* = 0, \\ r_2 \hat{I}_2^* + c_2 \hat{M}_2^* - \mu_2 \hat{R}_2^* + m_{21}\hat{R}_1^* - m_{12}\hat{R}_2^* = 0. \end{cases}$$

We find the endemic disease equilibrium point

$$\begin{aligned} \hat{S}_1^* &= \frac{[(r_1 + b_1 + \mu_1 + \mu_1^{'})(\alpha_1 + c_1 + \mu_1 + \mu_1^{'}) - b_1\alpha_1](\varepsilon_1 + \mu_1 + \mu_1^{'})N_1}{\beta_1(\varepsilon_1 + \mu_1 - \delta\mu_1)(\alpha_1 + c_1 + \mu_1 + \mu_1^{'} + \sigma_1 b_1)}, \\ \hat{E}_1^* &= \frac{\delta\beta_1(\hat{I}_1^* + \sigma_1 \hat{M}_1^*)\hat{S}_1^*}{N_1(\varepsilon_1 + \mu_1)}, \\ \hat{I}_1^* &= \frac{N_1(\alpha_1 + c_1 + \mu_1 + \mu_1^{'})[\Lambda_1 - (\mu_1 + m_{21})\hat{S}_1^* + m_{12}\hat{S}_2^*]}{\hat{S}_1^*\beta_1(\alpha_1 + c_1 + \mu_1 + \mu_1^{'} + \sigma_1 b_1)}, \\ \hat{M}_1^* &= \frac{b_1 N_1^*[\Lambda_1 - (\mu_1 + m_{21})\hat{S}_1^* + m_{12}\hat{S}_2^*]}{\hat{S}_1^*\beta_1(\alpha_1 + c_1 + \mu_1 + \mu_1^{'} + \sigma_1 b_1)}, \\ \hat{R}_1^* &= \frac{r_1 \hat{I}_1^* + c_1 \hat{M}_1^* + m_{12}\hat{R}_2^*}{m_{21} + \mu_1}, \\ \hat{S}_2^* &= \frac{[(r_2 + b_2 + \mu_2 + \mu_2^{'})(\alpha_2 + c_2 + \mu_2 + \mu_2^{'}) - b_2\alpha_2](\varepsilon_2 + \mu_2 + \mu_2^{'})N_2}{\beta_2(\varepsilon_2 + \mu_2 - \delta\mu_2)(\alpha_2 + c_2 + \mu_2 + \mu_2^{'} + \sigma_2 b_2)}, \\ \hat{E}_2^* &= \frac{\delta\beta_2(\hat{I}_2^* + \sigma_2 \hat{M}_2^*)\hat{S}_2^*}{N_2(\varepsilon_2 + \mu_2)}, \\ \hat{I}_2 &= \frac{N_2(\alpha_2 + c_2 + \mu_2 + \mu_2^{'})[\Lambda_2 - (\mu_2 + m_{12})\hat{S}_2^* + m_{21}\hat{S}_1^*]}{\hat{S}_2^*\beta_2(\alpha_2 + c_2 + \mu_2 + \mu_2^{'} + \sigma_2 b_2)}, \\ \hat{M}_2^* &= \frac{N_2 b_2[\Lambda_2 - (\mu_2 + m_{12})\hat{S}_2^* + m_{21}\hat{S}_1^*]}{\hat{S}_2^*\beta_2(\alpha_2 + c_2 + \mu_2 + \mu_2^{'} + \sigma_2 b_2)}, \\ \hat{R}_2^* &= \frac{(m_{21} + \mu_1)(r_2 \hat{I}_2^* + c_2 \hat{M}_2^*) + m_{21}(r_1 \hat{I}_1^* + c_1 \hat{M}_1^*)}{m_{12}\mu_1 + \mu_2(\mu_1 + m_{21})}. \end{aligned}$$

To study the disease equilibrium points, we first calculate the basic reproduction number (R_0), calculating R_0 using the next generation approach method proposed by Driessche and Watmough [42]. Let

$$x = (E_1, E_2, I_1, I_2, M_1, M_2)^T,$$

and system (1) could be written as

$$_0^C D_t^q(x) = \mathcal{F} - \mathcal{V},$$

where

$$\mathcal{F} = \begin{pmatrix} \frac{\beta_1\delta(I_1+\sigma_1 M_1)S_1}{N_1} \\ \frac{\beta_2\delta(I_2+\sigma_2 M_2)S_2}{N_2} \\ \frac{\beta_1(1-\delta)(I_1+\sigma_1 M_1)S_1}{N_1} \\ \frac{\beta_2(1-\delta)(I_2+\sigma_2 M_2)S_2}{N_2} \\ 0 \\ 0 \end{pmatrix}, \quad \mathcal{V} = \begin{pmatrix} (\varepsilon_1+\mu_1)E_1 \\ (\varepsilon_2+\mu_2)E_2 \\ -\alpha_1 E_1 - \varepsilon_1 M_1 + (r_1+b_1+\mu_1+\mu_1^{'})I_1 \\ -\alpha_2 E_2 - \varepsilon_2 M_2 + (r_2+b_2+\mu_2+\mu_2^{'})I_2 \\ -b_1 I_1 + (\alpha_1+c_1+\mu_1+\mu_1^{'})M_1 \\ -b_2 I_2 + (\alpha_2+c_2+\mu_2+\mu_2^{'})M_2 \end{pmatrix}.$$

Then, take the derivative of $\mathcal{F}$ and $\mathcal{V}$ for x at the DFE point P_1^*; clearly, we can obtain

$$F = \frac{\partial \mathcal{F}_i}{\partial x_j}\,|_{P_1^*} = \begin{pmatrix} 0 & 0 & \beta_1\delta & 0 & \beta_1\delta\sigma_1 & 0 \\ 0 & 0 & 0 & \beta_2\delta & 0 & \beta_2\delta\sigma_2 \\ 0 & 0 & \beta_1(1-\delta) & 0 & \beta_1(1-\delta)\sigma_1 & 0 \\ 0 & 0 & 0 & \beta_2(1-\delta) & 0 & \beta_2(1-\delta)\sigma_2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$V = \frac{\partial \mathcal{V}_i}{\partial x_j}\,|_{P_1^*} = \begin{pmatrix} \varepsilon_1+\mu_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \varepsilon_2+\mu_2 & 0 & 0 & 0 & 0 \\ -\varepsilon_1 & 0 & r_1+b_1+\mu_1+\mu_1^{'} & 0 & -\alpha_1 & 0 \\ 0 & -\varepsilon_2 & 0 & r_2+b_2+\mu_2+\mu_2^{'} & 0 & -\alpha_2 \\ 0 & 0 & -b_1 & 0 & \alpha_1+c_1+\mu_1+\mu_1^{'} & 0 \\ 0 & 0 & 0 & -b_1 & 0 & \alpha_2+c_2+\mu_2+\mu_2^{'} \end{pmatrix}.$$

Thus, the basic reproduction number(R_0) is as follows:

$$R_0 = \rho(FV^{-1}) = max\left\{R_0^1, R_0^2\right\},$$

where

$$R_0^1 = \frac{\beta_1(b_1\sigma_1+\alpha_1+c_1+\mu_1+\mu_1^{'})(\varepsilon_1+\mu_1-\mu_1\delta)}{(\varepsilon_1+\mu_1)[(r_1+b_1+\mu_1+\mu_1^{'})(\alpha_1+c_1+\mu_1+\mu_1^{'})-b_1\alpha_1]},$$
$$R_0^2 = \frac{\beta_2(b_2\sigma_2+\alpha_2+c_2+\mu_2+\mu_2^{'})(\varepsilon_2+\mu_2-\mu_2\delta)}{(\varepsilon_2+\mu_2)[(r_2+b_2+\mu_2+\mu_2^{'})(\alpha_2+c_2+\mu_2+\mu_2^{'})-b_2\alpha_2]}.$$

4.3. The stability at the DFE points

Lemma 4.4. *If*

$$|arg(s_{F-V})| > \frac{q\pi}{2},$$

system (1) is locally asymptotically stable at DFE P_1^*.

Proof. The Jacobian matrix of model (1) at the DFE P_1^* is

$$\mathbb{J}_{p_1^*} = \begin{pmatrix} F-V & 0 \\ \mathbb{J}_3 & \mathbb{J}_4 \end{pmatrix},$$

where

$$\mathbb{J}_3 = \begin{pmatrix} 0 & 0 & r_1 & 0 & c_1 & 0 \\ 0 & 0 & 0 & r_2 & 0 & c_2 \\ 0 & 0 & -\beta_1 & 0 & -\sigma_1\beta_1 & 0 \\ 0 & 0 & 0 & -\beta_2 & 0 & -\sigma_2\beta_2 \end{pmatrix},$$

$$\mathbb{J}_4 = \begin{pmatrix} -m_{21}-\mu_1 & m_{12} & 0 & 0 \\ m_{21} & -m_{12}-\mu_2 & 0 & 0 \\ 0 & 0 & -m_{21}-\mu_1 & m_{12} \\ 0 & 0 & m_{21} & -m_{12}-\mu_2 \end{pmatrix} = -\mathbb{J}_4^{'},$$

$$\mathbb{J}_4^{'} = \begin{pmatrix} m_{21}+\mu_1 & -m_{12} & 0 & 0 \\ -m_{21} & m_{12}+\mu_2 & 0 & 0 \\ 0 & 0 & m_{21}+\mu_1 & -m_{12} \\ 0 & 0 & -m_{21} & m_{12}+\mu_2 \end{pmatrix}.$$

The stability at the DFE point is determined by all eigenvalues of the matrix $\mathbb{J}_{p_1^*}$, with F and V see Subsection 4.2. That is to say, the eigenvalues of $\mathbb{J}_{p_1^*}$ are the eigenvalues of $F-V$ and $\mathbb{J}_4$. Obviously, $\mathbb{J}_4^{'}$ is a nonsingular M-matrix. According to [42], $\mathbb{J}_4$ has all eigenvalues with negative real parts since the stability of DFE will depend on the eigenvalues of $F-V$. Due to F is a nonnegative matrix, V is a nonsingular M-matrix, then

$$|arg(s_{F-V})| > \frac{q\pi}{2} \Leftrightarrow s(F-V) < 0 \Leftrightarrow \rho(FV^{-1} < 1) \Leftrightarrow R_0 < 1.$$

If

$$|arg(s_{F-V})| > \frac{q\pi}{2},$$

all the eigenvalues of $F-V$ are negative. Therefore, all the eigenvalues of the matrix $\mathbb{J}_{p_1^*}$ are negative. Hence, if

$$|arg(s_{F-V})| > \frac{q\pi}{2},$$

P_1^* is locally asymptotically stable. □

Lemma 4.5. *If*

$$|arg(s_{F-V})| > \frac{q\pi}{2},$$

the system (1) is globally asymptotically stable at p_1^*.

Proof. To start, according to model (1), we can obtain

$$\begin{aligned} {}_0^C D_t^q S_i &= \Lambda_i - \beta_i \frac{(I_i + \sigma_i M_i) S_i}{N_i} - \mu_i S_i + \sum_{j=1, j\neq i}^{2} (m_{ij} S_j - m_{ji} S_i), \\ &\leq \Lambda_i - \mu_i S_i + \sum_{j=1, j\neq i}^{2} (m_{ij} S_j - m_{ji} S_i). \end{aligned} \tag{4}$$

Let

$$S = (S_1, S_2), \ \ S^* = (S_1^*, S_2^*), \ \ \Lambda = (\Lambda_1, \Lambda_2) \ \text{ and } \ A = \begin{pmatrix} \mu_1 + m_{21} & -m_{12} \\ -m_{21} & \mu_2 + m_{12} \end{pmatrix}.$$

Equation (4) can be written as

$${}_0^C D_t^q S \leq \Lambda - AS = AS^* - AS.$$

Thus,

$$S(t) \leq S^* - (S^* - S_0) E_q(-At^q).$$

Clearly,

$$S_k(t) \leq S_k^* \ \ (k = 1, 2).$$

Next, the following auxiliary system is considered:

$$\begin{aligned} {}_0^C D_t^q \bar{E}_i &= \beta_i \delta \frac{(\bar{I}_i + \sigma_i \bar{M}_i) \bar{S}_i}{N_i} - (\varepsilon_i + \mu_i) \bar{E}_i, \\ {}_0^C D_t^q \bar{I}_i &= \beta_i (1 - \delta) \frac{(\bar{I}_i + \sigma_i \bar{M}_i) \bar{S}_i}{N_i} + \varepsilon_i \bar{E}_i + \alpha_i \bar{M}_i - (r_i + b_i + \mu_i + \mu_i^{'}) \bar{I}_i, \\ {}_0^C D_t^q \bar{M}_i &= b_i \bar{I}_i - (\alpha_i + \mu_i + c_i + \mu_i^{'}) \bar{M}_i. \end{aligned}$$

Let

$$W = (\bar{E}, \bar{I}, \bar{M}), \quad \bar{E}_i = (\bar{E}_1, \bar{E}_2), \quad \bar{I}_i = (\bar{I}_1, \bar{I}_2), \quad \bar{M}_i = (\bar{M}_1, \bar{M}_2).$$

It is obvious to see that

$${}_0^C D_t^q W = (F - V) W.$$

Thus, if

$$|arg(s_{F-V})| > \frac{q\pi}{2},$$

the auxiliary system is locally asymptotically stable and globally asymptotically stable, and the solutions of the equation satisfy

$$\lim_{t\to\infty} \bar{E}_i = \lim_{t\to\infty} \bar{I}_i = \lim_{t\to\infty} \bar{M}_i = 0.$$

Using the comparison theory by Smith and Waltman [43] , we gained

$$\lim_{t\to\infty} E_i = \lim_{t\to\infty} I_i = \lim_{t\to\infty} M_i = 0.$$

Based on the above analysis, when $t \to \infty$, one has

$${}_0^C D_t^q S = AS^* - AS,$$

so

$$\lim_{t\to\infty} S(t) = S^*.$$

If

$$|arg(s_{F-V})| > \frac{q\pi}{2},$$

then model (1) is globally asymptotically stable at the DFE P_1^*. □

Theorem 4.1. *The epidemic equilibrium P_2^* of model (1) is globally asymptotically stable if $R_0 > 1$.*

Proof. The Lyapunov function $W(t)$ is defined as

$$W(t) = \frac{W_1(t) + W_2(t)}{m_{21}\hat{S}_1^* + m_{12}\hat{S}_2^*}$$

and

$${}_0^C D_t^q W(t) = \frac{{}_0^C D_t^q W_1(t) + {}_0^C D_t^q W_2(t)}{m_{21}\hat{S}_1^* + m_{12}\hat{S}_2^*}, \tag{5}$$

where,

$$W_1(t) = (S_1 - \hat{S}_1^* - \hat{S}_1^* ln(\frac{S_1}{\hat{S}_1^*})) + B_1(E_1 - \hat{E}_1^* - \hat{E}_1^* ln(\frac{E_1}{\hat{E}_1^*})) + B_2(I_1 - \hat{I}_1^* - \hat{I}_1^* ln(\frac{I_1}{\hat{I}_1^*})) + B_3(M_1 - \hat{M}_1^* - \hat{M}_1^* ln(\frac{M_1}{\hat{M}_1^*})),$$

$$W_2(t) = (S_2 - \hat{S}_2^* - \hat{S}_2^* ln(\frac{S_2}{\hat{S}_2^*})) + B_1^{'}(E_2 - \hat{E}_2^* - \hat{E}_2^* ln(\frac{E_2}{\hat{E}_2^*})) + B_2^{'}(I_2 - \hat{I}_2^* - \hat{I}_2^* ln(\frac{I_2}{\hat{I}_2^*})) + B_3^{'}(M_2 - \hat{M}_2^* - \hat{M}_2^* ln(\frac{M_2}{\hat{M}_2^*})).$$

Using Theorem 3.3, the fractional derivative of functions $W_1(t)$ and $W_2(t)$ is given as

$$\begin{aligned} {}_0^C D_t^q W_1(t) \leq (1 - \frac{\hat{S}_1^*}{S_1}) {}_0^C D_t^q S_1(t) + B_1(1 - \frac{\hat{E}_1^*}{E_1}) {}_0^C D_t^q E_1(t) \\ + B_2(1 - \frac{\hat{I}_1^*}{I_1}) {}_0^C D_t^q I_1(t) + B_3(1 - \frac{\hat{M}_1^*}{M_1}) {}_0^C D_t^q M_1(t). \end{aligned} \tag{6}$$

Bringing ${}_0^C D_t^q S_1(t)$, ${}_0^C D_t^q E_1(t)$, ${}_0^C D_t^q I_1(t)$, ${}_0^C D_t^q M_1(t)$ from the model(1) into Eq (6), and letting

$$\hat{\beta}_1 = \frac{\beta_1}{N_1}, \quad \hat{\beta}_2 = \frac{\beta_2}{N_2},$$

we get

$$\begin{aligned} {}_0^C D_t^q W_1(t) \leq (1 - \frac{\hat{S}_1^*}{S_1})[\Lambda_1 - \hat{\beta}_1(I_1 + \sigma_1 M_1)S_1 - \mu_1 S_1 + m_{12}S_2 - m_{21}S_1] \\ + B_1(1 - \frac{\hat{E}_1^*}{E_1})[\hat{\beta}_1\delta(I_1 + \sigma_1 M_1)S_1 - (\varepsilon_1 + \mu_1)E_1] \\ + B_2(1 - \frac{\hat{I}_1^*}{I_1})[\hat{\beta}_1(1-\delta)(I_1 + \sigma_1 M_1)S_1 + \varepsilon_1 E_1 + \alpha_1 M_1 - (r_1 + b_1 + \mu_1 + \mu_1^{'})I_1] \\ + B_3(1 - \frac{\hat{M}_1^*}{M_1})[b_1 I_1 - (\alpha_1 + \mu_1 + c_1 + \mu_1^{'})M_1]. \end{aligned} \tag{7}$$

We have model (1) in the steady state,

$$\Lambda_1 = \hat{\beta}_1(\hat{I}_1^* + \sigma_1\hat{M}_1^*)\hat{S}_1^* + \mu_1\hat{S}_1^* - m_{12}\hat{S}_2^* + m_{21}\hat{S}_1^*. \tag{8}$$

Substituting Eq (8) into Eq (7), we have

$$\begin{aligned}
{}_0^C D_t^q W_1(t) \leq\ & [\hat{\beta}_1(\hat{I}_1^* + \sigma_1\hat{M}_1^*)\hat{S}_1^* + \mu_1\hat{S}_1^* - m_{12}\hat{S}_2^* + m_{21}\hat{S}_1^* - \mu_1 S_1 + m_{12}S_2 - m_{21}S_1] \\
& - \frac{\hat{S}_1^*[\hat{\beta}_1(\hat{I}_1^* + \sigma_1\hat{M}_1^*)\hat{S}_1^* + \mu_1\hat{S}_1^* - m_{12}\hat{S}_2^* + m_{21}\hat{S}_1^* - \mu_1 S_1 + m_{12}S_2 - m_{21}S_1]}{S_1} \\
& - \frac{B_1\hat{E}_1^*[\hat{\beta}_1\delta(I_1 + \sigma_1 M_1)S_1 - (\varepsilon_1 + \mu_1)E_1]}{E_1} \\
& - \frac{B_2\hat{I}_1^*[\hat{\beta}_1(1-\delta)(I_1 + \sigma_1 M_1)S_1 + \varepsilon_1 E_1 + \alpha_1 M_1 - (r_1 + b_1 + \mu_1 + \mu_1^{'})I_1]}{I_1} \\
& - \frac{B_3\hat{M}_1^*[b_1 I_1 - (\alpha_1 + \mu_1 + c_1 + \mu_1^{'})M_1]}{M_1} + (I_1 + \sigma_1 M_1)S_1[B_1\hat{\beta}_1\delta + B_2\hat{\beta}_1(1-\delta) - 1] \\
& + E_1[B_2\varepsilon_1 - B_1(\varepsilon_1 + \mu_1)] + I_1[\hat{S}_1^*\hat{\beta}_1 - (r_1 + b_1 + \mu_1 + \mu_1^{'}) + B_3 b_1] \\
& + M_1[\hat{S}_1^*\hat{\beta}_1\sigma_1 + B_2\alpha_1 - B_3(\alpha_1 + \mu_1 + c_1 + \mu_1^{'})].
\end{aligned} \tag{9}$$

Adding all infected classes without a single star (*) from (9) to zero:

$$\begin{aligned}
& B_1\hat{\beta}_1\delta + B_2\hat{\beta}_1(1-\delta) - 1 = 0, \\
& B_2\varepsilon_1 - B_1(\varepsilon_1 + \mu_1) = 0, \\
& \hat{S}_1^*\hat{\beta}_1 - (r_1 + b_1 + \mu_1 + \mu_1^{'}) + B_3 b_1 = 0, \\
& \hat{S}_1^*\hat{\beta}_1\sigma_1 + B_2\alpha_1 - B_3(\alpha_1 + \mu_1 + c_1 + \mu_1^{'}) = 0.
\end{aligned} \tag{10}$$

Substituting the expression from (10) into (9) gives:

$$\begin{aligned}
{}_0^C D_t^q W_1(t) \leq\ & -\frac{\mu_1}{S_1}(S_1 - \hat{S}_1^*)^2 + (m_{21}\hat{S}_1^* - m_{12}\hat{S}_2^* - m_{21}S_1 + m_{12}S_2 - m_{21}\hat{S}_1^*\frac{\hat{S}_1^*}{S_1} + m_{12}\hat{S}_2^*\frac{\hat{S}_1^*}{S_1} + m_{21}\hat{S}_1^* \\
& - m_{12}S_2\frac{\hat{S}_1^*}{S_1}) + B_1\hat{\beta}_1\delta\hat{S}_1^*\hat{I}_1^*(3 - \frac{S_1}{\hat{S}_1^*}\frac{\hat{E}_1^*}{E_1}\frac{I_1}{\hat{I}_1^*} - \frac{\hat{S}_1^*}{S_1} - \frac{E_1}{\hat{E}_1^*}\frac{\hat{I}_1^*}{I_1}) \\
& + B_1\hat{\beta}_1\delta\sigma_1\hat{S}_1^*\hat{M}_1^*(4 - \frac{S_1}{\hat{S}_1^*}\frac{M_1}{\hat{M}_1^*}\frac{\hat{E}_1^*}{E_1} - \frac{\hat{S}_1^*}{S_1} - \frac{I_1}{\hat{I}_1^*}\frac{\hat{M}_1^*}{M_1} - \frac{E_1}{\hat{E}_1^*}\frac{\hat{I}_1^*}{I_1}) + B_2\hat{\beta}_1\hat{S}_1^*\hat{I}_1^*(2 - \frac{S_1}{\hat{S}_1^*} - \frac{\hat{S}_1^*}{S_1}) \\
& + B_2\hat{\beta}_1(1-\delta)\sigma_1\hat{S}_1^*\hat{M}_1^*(3 - \frac{S_1}{\hat{S}_1^*}\frac{M_1}{\hat{M}_1^*}\frac{\hat{I}_1^*}{I_1} - \frac{\hat{S}_1^*}{S_1} - \frac{I_1}{\hat{I}_1^*}\frac{\hat{M}_1^*}{M_1}) + B_2\varepsilon_1\hat{E}_1^*(1 - \frac{E_1}{\hat{E}^*}\frac{\hat{I}_1^*}{I_1} - 1 + \frac{E_1}{\hat{E}_1^*}\frac{\hat{I}_1^*}{I_1}) \\
& + B_2\alpha_1\hat{M}_1^*(2 - \frac{M_1}{\hat{M}_1^*}\frac{\hat{I}_1^*}{I_1} - \frac{I_1}{\hat{I}_1^*}\frac{\hat{M}_1^*}{M_1}) + B_3 b_1\hat{I}_1^*(1 - \frac{I_1}{\hat{I}_1^*}\frac{\hat{M}_1^*}{M_1} - 1 + \frac{I_1}{\hat{I}_1^*}\frac{\hat{M}_1^*}{M_1}).
\end{aligned} \tag{11}$$

We have an arithmetic mean that is greater than the geometric mean

$$3 - \frac{S_1}{\hat{S}_1^*}\frac{\hat{E}_1^*}{E_1}\frac{I_1}{\hat{I}_1^*} - \frac{\hat{S}_1^*}{S_1} - \frac{E_1}{\hat{E}_1^*}\frac{\hat{I}_1^*}{I_1} \leq 0, \quad 4 - \frac{S_1}{\hat{S}_1^*}\frac{M_1}{\hat{M}_1^*}\frac{\hat{E}_1^*}{E_1} - \frac{\hat{S}_1^*}{S_1} - \frac{I_1}{\hat{I}_1^*}\frac{\hat{M}_1^*}{M_1} - \frac{E_1}{\hat{E}_1^*}\frac{\hat{I}_1^*}{I_1} \leq 0,$$

$$2-\frac{S_1}{\hat{S}_1^*}-\frac{\hat{S}_1^*}{S_1}\leq 0,\quad 3-\frac{S_1}{\hat{S}_1^*}\frac{M_1}{\hat{M}_1^*}\frac{\hat{I}_1^*}{I_1}-\frac{\hat{S}_1^*}{S_1}-\frac{I_1}{\hat{I}_1^*}\frac{\hat{M}_1^*}{M_1}\leq 0,\quad 2-\frac{M_1}{\hat{M}_1^*}\frac{\hat{I}_1^*}{I_1}-\frac{I_1}{\hat{I}_1^*}\frac{\hat{M}_1^*}{M_1}\leq 0.$$

Thus,

$${}_0^C D_t^q W_1(t)\leq m_{21}\hat{S}_1^*-m_{12}\hat{S}_2^*-m_{21}S_1+m_{12}S_2-m_{21}\hat{S}_1^*\frac{\hat{S}_1^*}{S_1}+m_{12}\hat{S}_2^*\frac{\hat{S}_1^*}{S_1}+m_{21}\hat{S}_1^*-m_{12}S_2\frac{\hat{S}_1^*}{S_1}. \tag{12}$$

Using the same method of proof, we also get

$${}_0^C D_t^q W_2(t)\leq m_{12}\hat{S}_2^*-m_{21}\hat{S}_1^*-m_{12}S_2+m_{21}S_1-m_{12}\hat{S}_2^*\frac{\hat{S}_2^*}{S_2}+m_{21}\hat{S}_1^*\frac{\hat{S}_2^*}{S_2}+m_{12}\hat{S}_2^*-m_{21}S_1\frac{\hat{S}_2^*}{S_2}. \tag{13}$$

Bringing Eqs (12) and (13) into Eq (5) yields

$$\begin{aligned}
{}_0^C D_t^q W(t)&\leq \frac{m_{21}\hat{S}_1^*(\frac{\hat{S}_2^*}{S_2}-\frac{\hat{S}_1^*}{S_1}+ln\frac{\hat{S}_1^*}{S_1}-ln\frac{\hat{S}_2^*}{S_2})+m_{12}\hat{S}_2^*(\frac{\hat{S}_1^*}{S_1}-\frac{\hat{S}_2^*}{S_2}+ln\frac{\hat{S}_2^*}{S_2}-ln\frac{\hat{S}_1^*}{S_1})}{m_{21}\hat{S}_1^*+m_{12}\hat{S}_2^*}\\
&=\frac{m_{21}\hat{S}_1^*(\frac{\hat{S}_2^*}{S_2}-\frac{\hat{S}_1^*}{S_1}+ln\frac{\hat{S}_1^*}{S_1}-ln\frac{\hat{S}_2^*}{S_2})}{m_{21}\hat{S}_1^*+m_{12}\hat{S}_2^*}+\frac{m_{12}\hat{S}_2^*(\frac{\hat{S}_1^*}{S_1}-\frac{\hat{S}_2^*}{S_2}+ln\frac{\hat{S}_2^*}{S_2}-ln\frac{\hat{S}_1^*}{S_1})}{m_{21}\hat{S}_1^*+m_{12}\hat{S}_2^*}\\
&\leq\frac{m_{21}\hat{S}_1^*(\frac{\hat{S}_2^*}{S_2}-\frac{\hat{S}_1^*}{S_1}+ln\frac{\hat{S}_1^*}{S_1}-ln\frac{\hat{S}_2^*}{S_2})}{m_{21}\hat{S}_1^*}+\frac{m_{12}\hat{S}_2^*(\frac{\hat{S}_1^*}{S_1}-\frac{\hat{S}_2^*}{S_2}+ln\frac{\hat{S}_2^*}{S_2}-ln\frac{\hat{S}_1^*}{S_1})}{m_{12}\hat{S}_2^*}\\
&=\frac{\hat{S}_2^*}{S_2}-\frac{\hat{S}_1^*}{S_1}+ln\frac{\hat{S}_1^*}{S_1}-ln\frac{\hat{S}_2^*}{S_2}+\frac{\hat{S}_1^*}{S_1}-\frac{\hat{S}_2^*}{S_2}+ln\frac{\hat{S}_2^*}{S_2}-ln\frac{\hat{S}_1^*}{S_1}\\
&=0.
\end{aligned}$$

Thus,

$${}_0^C D_t^q W(t)\leq 0,$$

if $R_0>1$. Hence, if $R_0>1$, the epidemic equilibrium point P_2^* is globally asymptotically stable, according to LaSalle's invariance principle. □

5. Existence and uniqueness of solution

Here, it is important to investigate the existence of a dynamical system we are analyzing. Fixed point theory provides insights into this necessity. Assume that

$$\Theta=\mathcal{B}(s)\times\mathcal{B}(s)$$

with $\mathcal{B}(s)$ is the Banach space of continuous function $\mathbb{R}\to\mathbb{R}$ defined on s with the norm such that

$$\begin{aligned}
\|S_1,E_1,I_1,M_1,R_1,S_2,E_2,I_2,M_2,R_2\|&=\|S_1\|+\|E_1\|+\|I_1\|+\|M_1\|+\|R_1\|\\
&\quad+\|S_2\|+\|E_2\|+\|I_2\|+\|M_2\|+\|R_2\|,
\end{aligned}$$

where,

$$\|S_1\|=sup\{|S_1(t)|:t\in[0,T]\},$$

$$
\begin{aligned}
\|E_1\| &= sup\{|E_1(t)| : t \in [0,T]\},\\
\|I_1\| &= sup\{|I_1(t)| : t \in [0,T]\},\\
\|M_1\| &= sup\{|M_1(t)| : t \in [0,T]\},\\
\|R_1\| &= sup\{|R_1(t)| : t \in [0,T]\},\\
\|S_2\| &= sup\{|S_2(t)| : t \in [0,T]\},\\
\|E_2\| &= sup\{|E_2(t)| : t \in [0,T]\},\\
\|I_2\| &= sup\{|I_2(t)| : t \in [0,T]\},\\
\|M_2\| &= sup\{|M_2(t)| : t \in [0,T]\},\\
\|R_2\| &= sup\{|R_2(t)| : t \in [0,T]\}.
\end{aligned}
$$

For simplicity, the model (1) can be expressed in the form of

$$
\begin{cases}
{}_0^C D_t^q Q(t) = G(t, Q(t))\\
Q(0) = Q_0.
\end{cases}
\tag{14}
$$

The solution of Eq (14) is

$$
Q(t) = Q_0 + \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} G(\tau, Q(\tau)) d\tau,
$$

where,

$$
Q(t) = \begin{pmatrix} S_1(t)\\ E_1(t)\\ I_1(t)\\ M_1(t)\\ R_1(t)\\ S_2(t)\\ E_2(t)\\ I_2(t)\\ M_2(t)\\ R_2(t) \end{pmatrix}, \quad Q_0 = \begin{pmatrix} S_{1,0}\\ E_{1,0}\\ I_{1,0}\\ M_{1,0}\\ R_{1,0}\\ S_{2,0}\\ E_{2,0}\\ I_{2,0}\\ M_{2,0}\\ R_{2,0} \end{pmatrix}, \quad G(t, Q(t)) = \begin{pmatrix} G_1\\ G_2\\ G_3\\ G_4\\ G_5\\ G_6\\ G_7\\ G_8\\ G_9\\ G_{10} \end{pmatrix}. \tag{15}
$$

$$
\begin{aligned}
G_1(t, S_1(t)) &= \Lambda_1 - \beta_1 \frac{(I_1 + \sigma_1 M_1) S_1}{N_1} - \mu_1 S_1 + m_{12} S_2 - m_{21} S_1,\\
G_2(t, E_1(t)) &= \beta_1 \delta \frac{(I_1 + \sigma_1 M_1) S_1}{N_1} - (\varepsilon_1 + \mu_1) E_1,\\
G_3(t, I_1(t)) &= \beta_1 (1-\delta) \frac{(I_1 + \sigma_1 M_1) S_1}{N_1} + \varepsilon_1 E_1 + \alpha_1 M_1 - (r_1 + b_1 + \mu_1 + \mu_1^{'}) I_1,\\
G_4(t, M_1(t)) &= b_1 I_1 - (\alpha_1 + \mu_1 + c_1 + \mu_1^{'}) M_1,\\
G_5(t, R_1(t)) &= r_1 I_1 + c_1 M_1 - \mu_1 R_1 + m_{12} R_2 - m_{21} R_1,\\
G_6(t, S_2(t)) &= \Lambda_2 - \beta_2 \frac{(I_2 + \sigma_2 M_2) S_2}{N_2} - \mu_2 S_2 + m_{21} S_1 - m_{12} S_2,
\end{aligned}
$$

$$G_7(t, E_2(t)) = \beta_2\delta\frac{(I_2 + \sigma_2 M_2)S_2}{N_2} - (\varepsilon_2 + \mu_2)E_2,$$
$$G_8(t, I_2(t)) = \beta_2(1-\delta)\frac{(I_2 + \sigma_2 M_2)S_2}{N_2} + \varepsilon_2 E_2 + \alpha_2 M_2 - (r_2 + b_2 + \mu_2 + \mu_2^{'})I_2,$$
$$G_9(t, M_2(t)) = b_2 I_2 - (\alpha_2 + \mu_2 + c_2 + \mu_2^{'})M_2,$$
$$G_{10}(t, R_2(t)) = r_2 I_2 + c_2 M_2 - \mu_2 R_2 + m_{21}R_1 - m_{12}R_2.$$

Lemma 5.1. *All kernels G_i $(i = 1, 2, \ldots, 10)$ satisfy the Lipschitz condition in the Banach space $\mathcal{B}$ if the inequalities $0 \le L_1, L_2, L_3, L_4, L_5, L_6, L_7, L_8, L_9, L_{10} < 1$ hold.*

Proof. Let $Q(t), \bar{Q}(t)$ be two functions, $\bar{Q}(t) \in G$, where

$$Q(t) = \begin{pmatrix} S_1(t) \\ E_1(t) \\ I_1(t) \\ M_1(t) \\ R_1(t) \\ S_2(t) \\ E_2(t) \\ I_2(t) \\ M_2(t) \\ R_2(t) \end{pmatrix}, \quad \bar{Q}(t) = \begin{pmatrix} \bar{S}_1(t) \\ \bar{E}_1(t) \\ \bar{I}_1(t) \\ \bar{M}_1(t) \\ \bar{R}_1(t) \\ \bar{S}_2(t) \\ \bar{E}_2(t) \\ \bar{I}_2(t) \\ \bar{M}_2(t) \\ \bar{R}_2(t) \end{pmatrix},$$

$$\left\|G_1(t, S_1(t)) - G_1(t, \bar{S}_1(t))\right\| = \left\| \begin{array}{l} \Lambda_1 - \beta_1\frac{(I_1+\sigma_1 M_1)S_1}{N_1} - \mu_1 S_1 + m_{12}S_2 - m_{21}S_1 \\ -\Lambda_1 + \beta_1\frac{(I_1+\sigma_1 M_1)\bar{S}_1}{N_1} + \mu_1\bar{S}_1 - m_{12}S_2 + m_{21}\bar{S}_1 \end{array} \right\|$$
$$\le |e_1| \left\|S_1 - \bar{S}_1\right\|.$$

Taking $L_1 = e_1$, then

$$\left\|G_1(t, S_1(t)) - G_1(t, \bar{S}_1(t))\right\| \le |L_1| \left\|S_1 - \bar{S}_1\right\|.$$

Thus, if $0 \le L_1 < 1$, the Lipschitz condition holds for G_1. Using similar methodologies, we established the Lipschitz condition for the remaining kernels as well

$$\left\|G_2(t, E_1(t)) - G_2(t, \bar{E}_1(t))\right\| \le |L_2| \left\|E_1 - \bar{E}_1\right\|,$$
$$\left\|G_3(t, I_1(t)) - G_3(t, \bar{I}_1(t))\right\| \le |L_3| \left\|I_1 - \bar{I}_1\right\|,$$
$$\left\|G_4(t, M_1(t)) - G_4(t, \bar{M}_1(t))\right\| \le |L_4| \left\|M_1 - \bar{M}_1\right\|,$$
$$\left\|G_5(t, R_1(t)) - G_5(t, \bar{R}_1(t))\right\| \le |L_5| \left\|R_1 - \bar{R}_1\right\|,$$
$$\left\|G_6(t, S_2(t)) - G_6(t, \bar{S}_2(t))\right\| \le |L_6| \left\|S_2 - \bar{S}_2\right\|,$$
$$\left\|G_7(t, E_2(t)) - G_7(t, \bar{E}_2(t))\right\| \le |L_7| \left\|E_2 - \bar{E}_2\right\|,$$
$$\left\|G_8(t, I_2(t)) - G_8(t, \bar{I}_2(t))\right\| \le |L_8| \left\|I_2 - \bar{I}_2\right\|,$$
$$\left\|G_9(t, M_2(t)) - G_9(t, \bar{M}_2(t))\right\| \le |L_9| \left\|M_2 - \bar{M}_2\right\|,$$
$$\left\|G_{10}(t, R_2(t)) - G_{10}(t, \bar{R}_2(t))\right\| \le |L_{10}| \left\|R_2 - \bar{R}_2\right\|.$$

Next, we give the following recursive formula:

$$
\begin{aligned}
S_{1,n}(t) &= S_{1,0}(t) + \frac{1}{\Gamma(q)}\int_0^t (t-\tau)^{q-1} G_1(\tau, S_{1,n-1}(\tau))d\tau,\\
E_{1,n}(t) &= E_{1,0}(t) + \frac{1}{\Gamma(q)}\int_0^t (t-\tau)^{q-1} G_2(\tau, E_{1,n-1}(\tau))d\tau,\\
I_{1,n}(t) &= I_{1,0}(t) + \frac{1}{\Gamma(q)}\int_0^t (t-\tau)^{q-1} G_3(\tau, I_{1,n-1}(\tau))d\tau,\\
M_{1,n}(t) &= M_{1,0}(t) + \frac{1}{\Gamma(q)}\int_0^t (t-\tau)^{q-1} G_4(\tau, M_{1,n-1}(\tau))d\tau,\\
R_{1,n}(t) &= R_{1,0}(t) + \frac{1}{\Gamma(q)}\int_0^t (t-\tau)^{q-1} G_5(\tau, R_{1,n-1}(\tau))d\tau,\\
S_{2,n}(t) &= S_{2,0}(t) + \frac{1}{\Gamma(q)}\int_0^t (t-\tau)^{q-1} G_6(\tau, S_{2,n-1}(\tau))d\tau,\\
E_{2,n}(t) &= E_{2,0}(t) + \frac{1}{\Gamma(q)}\int_0^t (t-\tau)^{q-1} G_7(\tau, E_{2,n-1}(\tau))d\tau,\\
I_{2,n}(t) &= I_{2,0}(t) + \frac{1}{\Gamma(q)}\int_0^t (t-\tau)^{q-1} G_8(\tau, I_{2,n-1}(\tau))d\tau,\\
M_{2,n}(t) &= M_{2,0}(t) + \frac{1}{\Gamma(q)}\int_0^t (t-\tau)^{q-1} G_9(\tau, M_{2,n-1}(\tau))d\tau,\\
R_{2,n}(t) &= R_{2,0}(t) + \frac{1}{\Gamma(q)}\int_0^t (t-\tau)^{q-1} G_{10}(\tau, R_{2,n-1}(\tau))d\tau,
\end{aligned}
$$

to obtain the difference between the successive terms in the equations as

$$
\begin{aligned}
\mathbb{A}_n(t) &= S_{1,n}(t) - S_{1,n-1}(t) = \frac{1}{\Gamma(q)}\int_0^t (t-\tau)^{q-1}[G_1(\tau, S_{1,n-1}(\tau)) - G_1(\tau, S_{1,n-2}(\tau))]d\tau,\\
\mathbb{B}_n(t) &= E_{1,n}(t) - E_{1,n-1}(t) = \frac{1}{\Gamma(q)}\int_0^t (t-\tau)^{q-1}[G_2(\tau, E_{1,n-1}(\tau)) - G_2(\tau, E_{1,n-2}(\tau))]d\tau,\\
\mathbb{C}_n(t) &= I_{1,n}(t) - I_{1,n-1}(t) = \frac{1}{\Gamma(q)}\int_0^t (t-\tau)^{q-1}[G_3(\tau, I_{1,n-1}(\tau)) - G_3(\tau, I_{1,n-2}(\tau))]d\tau,\\
\mathbb{D}_n(t) &= M_{1,n}(t) - M_{1,n-1}(t) = \frac{1}{\Gamma(q)}\int_0^t (t-\tau)^{q-1}[G_4(\tau, M_{1,n-1}(\tau)) - G_4(\tau, M_{1,n-2}(\tau))]d\tau,\\
\mathbb{E}_n(t) &= R_{1,n}(t) - R_{1,n-1}(t) = \frac{1}{\Gamma(q)}\int_0^t (t-\tau)^{q-1}[G_5(\tau, R_{1,n-1}(\tau)) - G_5(\tau, R_{1,n-2}(\tau))]d\tau,\\
\bar{\mathbb{A}}_n(t) &= S_{2,n}(t) - S_{2,n-1}(t) = \frac{1}{\Gamma(q)}\int_0^t (t-\tau)^{q-1}[G_6(\tau, S_{2,n-1}(\tau)) - G_6(\tau, S_{2,n-2}(\tau))]d\tau,\\
\bar{\mathbb{B}}_n(t) &= E_{2,n}(t) - E_{2,n-1}(t) = \frac{1}{\Gamma(q)}\int_0^t (t-\tau)^{q-1}[G_7(\tau, E_{2,n-1}(\tau)) - G_7(\tau, E_{2,n-2}(\tau))]d\tau,\\
\bar{\mathbb{C}}_n(t) &= I_{2,n}(t) - I_{2,n-1}(t) = \frac{1}{\Gamma(q)}\int_0^t (t-\tau)^{q-1}[G_8(\tau, I_{2,n-1}(\tau)) - G_8(\tau, I_{2,n-2}(\tau))]d\tau,\\
\bar{\mathbb{D}}_n(t) &= M_{2,n}(t) - M_{2,n-1}(t) = \frac{1}{\Gamma(q)}\int_0^t (t-\tau)^{q-1}[G_9(\tau, M_{2,n-1}(\tau)) - G_9(\tau, M_{2,n-2}(\tau))]d\tau,
\end{aligned}
\tag{16}
$$

$$\bar{\mathbb{E}}_n(t) = R_{2,n}(t) - R_{2,n-1}(t) = \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} [G_{10}(\tau, R_{2,n-1}(\tau)) - G_{10}(\tau, R_{2,n-2}(\tau))] d\tau.$$

We noted that

$$\begin{aligned}
S_{1,n}(t) &= \sum_{i=1}^{n} \mathbb{A}_i(t), \quad E_{1,n}(t) = \sum_{i=1}^{n} \mathbb{B}_i(t), \quad I_{1,n}(t) = \sum_{i=1}^{n} \mathbb{C}_i(t), \\
M_{1,n}(t) &= \sum_{i=1}^{n} \mathbb{D}_i(t), \quad R_{1,n}(t) = \sum_{i=1}^{n} \mathbb{E}_i(t), \quad S_{2,n}(t) = \sum_{i=1}^{n} \bar{\mathbb{A}}_i(t), \\
E_{2,n}(t) &= \sum_{i=1}^{n} \bar{\mathbb{B}}_i(t), \quad I_{2,n}(t) = \sum_{i=1}^{n} \bar{\mathbb{C}}_i(t), \quad M_{2,n}(t) = \sum_{i=1}^{n} \bar{\mathbb{D}}_i(t), \quad R_{2,n}(t) = \sum_{i=1}^{n} \bar{\mathbb{E}}_i(t).
\end{aligned} \tag{17}$$

Next, on applying the norm on both sides of the first equation in (16), we have

$$\begin{aligned}
\|\mathbb{A}_n(t)\| &= \left\| S_{1,n}(t) - S_{1,n-1}(t) \right\|, \\
&\leq \frac{1}{\Gamma(q)} \left\| \int_0^t (t-\tau)^{q-1} [G_1(\tau, S_{1,n-1}(\tau)) - G_1(\tau, S_{1,n-2}(\tau))] d\tau \right\|, \\
&\leq \frac{L_1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} \left\| S_{1,n-1}(\tau) - S_{1,n-2}(\tau) \right\| d\tau, \\
&\leq \frac{t_{max}^q L_1}{q\Gamma(q)} \|\mathbb{A}_{n-1}\|.
\end{aligned} \tag{18}$$

In the same manner, one obtains the following results:

$$\begin{aligned}
&\|\mathbb{B}_n(t)\| \leq \frac{t_{max}^q L_2}{q\Gamma(q)} \|\mathbb{B}_{n-1}\|, \|\mathbb{C}_n(t)\| \leq \frac{t_{max}^q L_3}{q\Gamma(q)} \|\mathbb{C}_{n-1}\|, \|\mathbb{D}_n(t)\| \leq \frac{t_{max}^q L_4}{q\Gamma(q)} \|\mathbb{D}_{n-1}\|, \\
&\|\mathbb{E}_n(t)\| \leq \frac{t_{max}^q L_5}{q\Gamma(q)} \|\mathbb{E}_{n-1}\|, \left\|\bar{\mathbb{A}}_n(t)\right\| \leq \frac{t_{max}^q L_6}{q\Gamma(q)} \left\|\bar{\mathbb{A}}_{n-1}\right\|, \left\|\bar{\mathbb{B}}_n(t)\right\| \leq \frac{t_{max}^q L_7}{q\Gamma(q)} \left\|\bar{\mathbb{B}}_{n-1}\right\|, \\
&\left\|\bar{\mathbb{C}}_n(t)\right\| \leq \frac{t_{max}^q L_8}{q\Gamma(q)} \left\|\bar{\mathbb{C}}_{n-1}\right\|, \left\|\bar{\mathbb{D}}_n(t)\right\| \leq \frac{t_{max}^q L_9}{q\Gamma(q)} \left\|\bar{\mathbb{D}}_{n-1}\right\|, \left\|\bar{\mathbb{E}}_n(t)\right\| \leq \frac{t_{max}^q L_{10}}{q\Gamma(q)} \left\|\bar{\mathbb{E}}_{n-1}\right\|.
\end{aligned} \tag{19}$$

□

Lemma 5.2. *If*

$$\frac{t_{max}^q}{q\Gamma(q)} L_i < 1 \ \ (i = 1, 2, 3, \ldots, 10),$$

then model (1) has a unique solution.

Proof. It has been shown that all kernels satisfy the Lipschitz condition. Then, from (18) and (19), by recursion, we can get

$$\|\mathbb{A}_n(t)\| \leq \left(\frac{t_{max}^q}{q\Gamma(q)} L_1 \right)^2 \|\mathbb{A}_{n-2}(t)\| \leq \left(\frac{t_{max}^q}{q\Gamma(q)} L_1 \right)^3 \|\mathbb{A}_{n-3}(t)\| \leq \cdots \leq \left(\frac{t_{max}^q}{q\Gamma(q)} L_1 \right)^n \left\| S_{1,0} \right\|.$$

Repeat the same steps,

$$\|\mathbb{B}_n(t)\| \leq \left(\frac{t_{max}^q}{q\Gamma(q)}L_2\right)^n \left\|E_{1,0}\right\|, \|\mathbb{C}_n(t)\| \leq \left(\frac{t_{max}^q}{q\Gamma(q)}L_3\right)^n \left\|I_{1,0}\right\|, \|\mathbb{D}_n(t)\| \leq \left(\frac{t_{max}^q}{q\Gamma(q)}L_4\right)^n \left\|M_{1,0}\right\|,$$
$$\|\mathbb{E}_n(t)\| \leq \left(\frac{t_{max}^q}{q\Gamma(q)}L_5\right)^n \left\|R_{1,0}\right\|, \left\|\bar{\mathbb{A}}_n(t)\right\| \leq \left(\frac{t_{max}^q}{q\Gamma(q)}L_6\right)^n \left\|S_{2,0}\right\|, \left\|\bar{\mathbb{B}}_n(t)\right\| \leq \left(\frac{t_{max}^q}{q\Gamma(q)}L_7\right)^n \left\|E_{2,0}\right\|,$$
$$\left\|\bar{\mathbb{C}}_n(t)\right\| \leq \left(\frac{t_{max}^q}{q\Gamma(q)}L_8\right)^n \left\|I_{2,0}\right\|, \left\|\bar{\mathbb{D}}_n(t)\right\| \leq \left(\frac{t_{max}^q}{q\Gamma(q)}L_9\right)^n \left\|M_{2,0}\right\|, \left\|\bar{\mathbb{E}}_n(t)\right\| \leq \left(\frac{t_{max}^q}{q\Gamma(q)}L_{10}\right)^n \left\|R_{2,0}\right\|.$$

Hence, Eq (17) exists and is smooth. Next, we make the assumption that

$$\begin{aligned}
S_1(t) - S_{1,0} &= S_{1,n}(t) - \mathcal{A}_n(t),\\
E_1(t) - E_{1,0} &= E_{1,n}(t) - \mathcal{B}_n(t),\\
I_1(t) - I_{1,0} &= I_{1,n}(t) - C_n(t),\\
M_1(t) - M_{1,0} &= M_{1,n}(t) - \mathcal{D}_n(t),\\
R_1(t) - R_{1,0} &= R_{1,n}(t) - \mathcal{E}_n(t),\\
S_2(t) - S_{2,0} &= S_{2,n}(t) - \bar{\mathcal{A}}_n(t),\\
E_2(t) - E_{2,0} &= E_{2,n}(t) - \bar{\mathcal{B}}_n(t),\\
I_2(t) - I_{2,0} &= I_{2,n}(t) - \bar{C}_n(t),\\
M_2(t) - M_{2,0} &= M_{2,n}(t) - \bar{\mathcal{D}}_n(t),\\
R_2(t) - R_{2,0} &= R_{2,n}(t) - \bar{\mathcal{E}}_n(t).
\end{aligned}$$

So,

$$\begin{aligned}
\mathcal{A}_n(t) &\leq \left\| \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} [G_1(\tau, S_1(\tau)) - G_1(\tau, S_{1,n-1}(\tau))] d\tau \right\|,\\
&\leq \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} \left\| G_1(\tau, S_1(\tau)) - G_1(\tau, S_{1,n-1}(\tau)) \right\| d\tau,\\
&\leq \frac{t_{max}^q L_1}{q\Gamma(q)} \left\| S_1 - S_{1,n-1} \right\|.
\end{aligned}$$

We repeat the process recursively to get

$$\mathcal{A}_n(t) \leq \left(\frac{t_{max}^q L_1}{q\Gamma(q)}\right)^{n+1} \left\| S_{1,0} \right\|.$$

$\|\mathcal{A}_n(t)\| \to 0$ as $n \to \infty$. Similarly, we may establish that $\|\mathcal{B}_n(t)\| \to 0, \|C_n(t)\| \to 0, \|\mathcal{D}_n(t)\| \to 0, \|\mathcal{E}_n(t)\| \to 0, \left\|\bar{\mathcal{A}}_n(t)\right\| \to 0, \left\|\bar{\mathcal{B}}_n(t)\right\| \to 0, \left\|\bar{C}_n(t)\right\| \to 0, \left\|\bar{\mathcal{D}}_n(t)\right\| \to 0, \left\|\bar{\mathcal{E}}_n(t)\right\| \to 0$. Thus, the model (1) has a unique solution if

$$\frac{t_{max}^q}{q\Gamma(q)} L_i < 1 \ \ (i = 1, 2, 3, \ldots, 10).$$

The proof is completed. □

6. Ulam-Hyers stability

By proving Ulam-Hyers stability in this section, it is possible to ensure that the solution of the model remains stable in the presence of small perturbations, which enhances the reliability and usefulness of the model. Let us consider $\eta > 0$ and the inequality defined as:

$$\left\| {}_0^C D_t^q Q(t) - G(t, Q(t)) \right\| \leq \eta, \quad t \in [0, T], \tag{20}$$

where, $\eta = max\{\eta_i\}, i = 1, 2, \ldots, 10$.

Remark 6.1. *[33] Let a function $f \in \mathfrak{B}$, with $f(0) = 0$ independent of Q, be denoted as:*

(i) $|f(t)| \leq \eta$, for $\eta > 0$.

(ii) ${}_0^C D_t^q Q(t) = G(t, Q(t)) + f(t), t \in [0, T]$.

Lemma 6.1. *Let $Q(t) \in \mathfrak{B}$ be a solution of*

$$\left\| {}_0^C D_t^q Q(t) - G(t, Q(t)) \right\| \leq \eta, \quad t \in [0, T]$$

satisfying the given relation

$$\left\| Q(t) - Q_0 - \frac{1}{\Gamma(q)} \int_0^t (t - \tau)^{q-1} G(\tau, Q(\tau)) d\tau \right\| \leq \frac{\eta T^q}{q\Gamma(q)} = \eta M_T,$$

where

$$M_T = \frac{T^q}{q\Gamma(q)}.$$

Proof. The proof method references the paper [44]. Since $Q(t) \in \mathfrak{B}$ is a solution of Eq (20), under the initial condition

$$Q(0) = Q_0,$$

the solution to

$${}_0^C D_t^q Q(t) = G(t, Q(t)) + f(t)$$

is

$$Q(t) = Q_0 + \frac{1}{\Gamma(q)} \int_0^t (t - \tau)^{q-1} G(\tau, Q(\tau)) d\tau + \frac{1}{\Gamma(q)} \int_0^t (t - \tau)^{q-1} f(\tau) d\tau.$$

By using Remark 6.1, we get

$$\begin{aligned} \left\| Q(t) - Q_0 - \frac{1}{\Gamma(q)} \int_0^t (t - \tau)^{q-1} G(\tau, Q(\tau)) d\tau \right\| &= \left\| \frac{1}{\Gamma(q)} \int_0^t (t - \tau)^{q-1} f(\tau) d\tau \right\|, \\ &\leq \frac{1}{\Gamma(q)} \int_0^t (t - \tau)^{q-1} \|f(\tau)\| d\tau, \\ &\leq \frac{\eta T^q}{q\Gamma(q)} = \eta M_T. \end{aligned}$$

where

$$M_T = \frac{T^q}{q\Gamma(q)}.$$

□

Theorem 6.1. *The solution of model (1) is Ulam-Hyers stable if the following two conditions hold:*

(i) $G \in \mathfrak{B}([0,T],W)$

(ii) For all $Q(t), \bar{Q}(t) \in \mathfrak{B}$*, there exists* $\rho > 0$ *, such that* $\left\|G(t,Q(t)) - G(t,\bar{Q}(t))\right\| \leq \rho \left\|Q(t) - \bar{Q}(t)\right\|$*, for each* $t \in [0,T]$.

Proof. $Q(t)$ is the unique solution of model (1), from Lemma 5, if (i) and (ii) hold. Let $\bar{Q}(t) \in \mathfrak{B}$ be any solution of model(1), then

$$\begin{aligned}
\left\|Q(t) - \bar{Q}(t)\right\| &= \left\|Q(t) - Q_0 - \frac{1}{\Gamma(q)}\int_0^t (t-\tau)^{q-1} G(\tau,\bar{Q}(\tau))d\tau\right\|, \\
&\leq \left\|Q(t) - Q_0 - \frac{1}{\Gamma(q)}\int_0^t (t-\tau)^{q-1} G(\tau,Q(\tau))d\tau\right\| \\
&+ \frac{1}{\Gamma(q)}\int_0^t (t-\tau)^{q-1}\left\|G(\tau,Q(\tau)) - G(\tau,\bar{Q}(\tau))\right\| d\tau, \\
&\leq \left\|Q(t) - Q_0 - \frac{1}{\Gamma(q)}\int_0^t (t-\tau)^{q-1} G(\tau,Q(\tau))d\tau\right\| + \frac{\rho}{\Gamma(q)}\int_0^t (t-\tau)^{q-1}\left\|Q(\tau) - \bar{Q}(\tau)\right\| d\tau, \\
&\leq \left\|Q(t) - Q_0 - \frac{1}{\Gamma(q)}\int_0^t (t-\tau)^{q-1} G(\tau,Q(\tau))d\tau\right\| + \frac{\rho T^q}{q\Gamma(q)}\left\|Q(\tau) - \bar{Q}(\tau)\right\|, \\
&\leq M_T\eta + M_T\rho\left\|Q(\tau) - \bar{Q}(\tau)\right\|.
\end{aligned}$$

This implies that

$$\left\|Q(\tau) - \bar{Q}(\tau)\right\| \leq \eta B_\chi,$$

where

$$B_\chi = \frac{M_T}{1 - \rho M_T}.$$

So, the solution of model (1) is Ulam-Hyers stable. □

7. Numerical scheme for model (1) by Caputo

The model (1) is solved numerically using the two-step Lagrange interpolation method [45]. The differential system is as follows:

$$\begin{cases} {}_0^C D_t^q Q(t) = G(t,Q(t)), \\ Q(0) = Q_0, \end{cases}$$

and its solution is

$$Q(t) = Q_0 + \frac{1}{\Gamma(q)}\int_0^t (t-\tau)^{q-1} G(\tau,Q(\tau))d\tau,$$

where $Q(t), Q_0, G(t,Q(t))$ is consistent with Eq (15).

Let

$$h = \frac{T}{N}, \quad t_n = nh, \quad n = 0,1,2,\ldots,N \text{ and } t_0 = 0.$$

Computing at

$$t = t_{n+1},$$

we obtain

$$Q(t_{n+1}) = Q_0 + \frac{1}{\Gamma(q)} \int_0^{t_{n+1}} (t_{n+1} - \tau)^{q-1} G(\tau, Q(\tau)) d\tau,$$

which implies that

$$Q(t_{n+1}) = Q_0 + \frac{1}{\Gamma(q)} \sum_{i=0}^{n} \int_{t_i}^{t_{i+1}} (t_{n+1} - \tau)^{q-1} G(\tau, Q(\tau)) d\tau. \tag{21}$$

The function $G(\tau, Q(\tau))$ can be approximated over $[t_i, t_{i+1}]$ and

$$t_{i+1} - t_i = h,$$

using two-step Lagrange interpolation method as

$$\begin{aligned} G(\tau, Q(\tau)) &\simeq \frac{\tau - t_{i-1}}{t_i - t_{i-1}} G(\tau_i, Q(\tau_i)) + \frac{\tau - t_i}{t_{i-1} - t_i} G(\tau_{i-1}, Q(\tau_{i-1})) \\ &= \frac{\tau - t_{i-1}}{h} G(\tau_i, Q(\tau_i)) - \frac{\tau - t_i}{h} G(\tau_{i-1}, Q(\tau_{i-1})). \end{aligned} \tag{22}$$

Now, substitute (22) into formula (21), and we get

$$Q(t_{n+1}) = Q_0 + \frac{1}{\Gamma(q)} \sum_{i=0}^{n} \int_{t_i}^{t_{i+1}} (t_{n+1} - \tau)^{q-1} [\frac{\tau - t_{i-1}}{h} G(\tau_i, Q(\tau_i)) - \frac{\tau - t_i}{h} G(\tau_{i-1}, Q(\tau_{i-1}))] d\tau,$$

which can be rewritten as

$$Q(t_{n+1}) = Q_0 + \frac{1}{h\Gamma(q)} \sum_{i=0}^{n} (W_1 - W_2), \tag{23}$$

where

$$\begin{aligned} W_1 &= G(\tau_i, Q(\tau_i)) \int_{t_i}^{t_{i+1}} (\tau - t_{i-1})(t_{n+1} - \tau)^{q-1} d\tau \\ &= -\frac{G(\tau_i, Q(\tau_i))}{q} [(t_{i+1} - t_{i-1})(t_{n+1} - t_{i+1})^q - (t_i - t_{i-1})(t_{n+1} - t_i)^q] \\ &- \frac{G(\tau_i, Q(\tau_i))}{q(q+1)} [(t_{n+1} - t_{i+1})^{q+1} - (t_{n+1} - t_i)^{q+1}], \\ W_2 &= G(\tau_{i-1}, Q(\tau_{i-1})) \int_{t_i}^{t_{i+1}} (\tau - t_i)(t_{n+1} - \tau)^{q-1} d\tau \\ &= -\frac{G(\tau_{i-1}, Q(\tau_{i-1}))}{q} [(t_{i+1} - t_i)(t_{n+1} - t_{i+1})^q] - \frac{G(\tau_{i-1}, Q(\tau_{i-1}))}{q(q+1)} [(t_{n+1} - t_{i+1})^{q+1} - (t_{n+1} - t_i)^{q+1}]. \end{aligned}$$

Substituting $t_n = nh$ into W_1, W_2:

$$\begin{aligned} W_1 &= \frac{G(\tau_i, Q(\tau_i)) h^{q+1}}{q(q+1)} [(n - i + 1)^q (n - i + 2 + q) - (n - i)^q (n - i + 2 + 2q)], \\ W_2 &= \frac{G(\tau_{i-1}, Q(\tau_{i-1})) h^{q+1}}{q(q+1)} [(n - i + 1)^{q+1} - (n - i)^q (n - i + 1 + q)]. \end{aligned} \tag{24}$$

Bringing W_1 and W_2 from Eq (24) to Eq (23), thus, the approximate solution is

$$Q(t_{n+1}) = Q_0 + \frac{h^q}{\Gamma(q+2)} \sum_{i=0}^{n} [G(\tau_i, Q(\tau_i))((n-i+1)^q(n-i+2+q) - (n-i)^q(n-i+2+2q)) - G(\tau_{i-1}, Q(\tau_{i-1}))((n-i+1)^{q+1} - (n-i)^q(n-i+1+q))]. \tag{25}$$

Using the numerical scheme (25) for model (1), we get

$$S_1(t_{n+1}) = S_{1,0} + \frac{h^q}{\Gamma(q+2)} \sum_{i=0}^{n} [G_1(\tau_i, S_1(\tau_i))((n-i+1)^q(n-i+2+q) - (n-i)^q(n-i+2+2q)) - G_1(\tau_{i-1}, S_1(\tau_{i-1}))((n-i+1)^{q+1} - (n-i)^q(n-i+1+q))],$$

$$E_1(t_{n+1}) = E_{1,0} + \frac{h^q}{\Gamma(q+2)} \sum_{i=0}^{n} [G_2(\tau_i, E_1(\tau_i))((n-i+1)^q(n-i+2+q) - (n-i)^q(n-i+2+2q)) - G_2(\tau_{i-1}, E_1(\tau_{i-1}))((n-i+1)^{q+1} - (n-i)^q(n-i+1+q))],$$

$$I_1(t_{n+1}) = I_{1,0} + \frac{h^q}{\Gamma(q+2)} \sum_{i=0}^{n} [G_3(\tau_i, I_1(\tau_i))((n-i+1)^q(n-i+2+q) - (n-i)^q(n-i+2+2q)) - G_3(\tau_{i-1}, I_1(\tau_{i-1}))((n-i+1)^{q+1} - (n-i)^q(n-i+1+q))],$$

$$M_1(t_{n+1}) = M_{1,0} + \frac{h^q}{\Gamma(q+2)} \sum_{i=0}^{n} [G_4(\tau_i, M_1(\tau_i))((n-i+1)^q(n-i+2+q) - (n-i)^q(n-i+2+2q)) - G_4(\tau_{i-1}, M_1(\tau_{i-1}))((n-i+1)^{q+1} - (n-i)^q(n-i+1+q))],$$

$$R_1(t_{n+1}) = R_{1,0} + \frac{h^q}{\Gamma(q+2)} \sum_{i=0}^{n} [G_5(\tau_i, R_1(\tau_i))((n-i+1)^q(n-i+2+q) - (n-i)^q(n-i+2+2q)) - G_5(\tau_{i-1}, R_1(\tau_{i-1}))((n-i+1)^{q+1} - (n-i)^q(n-i+1+q))],$$

$$S_2(t_{n+1}) = S_{2,0} + \frac{h^q}{\Gamma(q+2)} \sum_{i=0}^{n} [G_6(\tau_i, S_2(\tau_i))((n-i+1)^q(n-i+2+q) - (n-i)^q(n-i+2+2q)) - G_6(\tau_{i-1}, S_2(\tau_{i-1}))((n-i+1)^{q+1} - (n-i)^q(n-i+1+q))],$$

$$E_2(t_{n+1}) = E_{2,0} + \frac{h^q}{\Gamma(q+2)} \sum_{i=0}^{n} [G_7(\tau_i, E_2(\tau_i))((n-i+1)^q(n-i+2+q) - (n-i)^q(n-i+2+2q)) - G_7(\tau_{i-1}, E_2(\tau_{i-1}))((n-i+1)^{q+1} - (n-i)^q(n-i+1+q))],$$

$$I_2(t_{n+1}) = I_{2,0} + \frac{h^q}{\Gamma(q+2)} \sum_{i=0}^{n} [G_8(\tau_i, I_2(\tau_i))((n-i+1)^q(n-i+2+q) - (n-i)^q(n-i+2+2q)) - G_8(\tau_{i-1}, I_2(\tau_{i-1}))((n-i+1)^{q+1} - (n-i)^q(n-i+1+q))],$$

$$M_2(t_{n+1}) = M_{2,0} + \frac{h^q}{\Gamma(q+2)} \sum_{i=0}^{n} [G_9(\tau_i, M_2(\tau_i))((n-i+1)^q(n-i+2+q) - (n-i)^q(n-i+2+2q)) - G_9(\tau_{i-1}, M_2(\tau_{i-1}))((n-i+1)^{q+1} - (n-i)^q(n-i+1+q))],$$

$$R_2(t_{n+1}) = R_{2,0} + \frac{h^q}{\Gamma(q+2)} \sum_{i=0}^{n} [G_{10}(\tau_i, R_2(\tau_i))((n-i+1)^q(n-i+2+q)$$

$$- (n-i)^q(n-i+2+2q)) - G_{10}(\tau_{i-1}, R_2(\tau_{i-1}))((n-i+1)^{q+1} - (n-i)^q(n-i+1+q))].$$

8. Numerical results and discussion

In this section, we use the numerical scheme to validate the fractional-order derivative epidemic model (1). Given factors such as potential pathogen mutations and varying degrees of human immunity, we assumed that β is variable. Since the majority of contacts between infected individuals and those susceptible to the virus lead to the latter entering a latent state, the number of latent infections is notably higher than the number of cases exhibiting direct symptoms. Consequently, we have assumed a value of δ to be 0.6. Considering that patients with DR-TB are not sensitive to therapeutic drugs and may experience a prolonged infectious cycle, thereby enhancing the chance of transmission, it can be inferred that the intensity of infection among these patients is slightly higher compared to those with DS-TB. Consequently, it is hypothesized that the value of σ is 1.5. The values of all parameters are noted in Table 2.

Table 2. Value of the parameters.

Parameters	Patch 1 values $/day^{-1}$	Patch 2 values $/day^{-1}$	Source
ε_i	0.25	0.245	[11]
β_i	Variable	Variable	Assumed
δ	0.6	0.6	Assumed
α_i	0.075	0.074	Assumed
b_i	0.05	0.05	[10]
r_i	0.1299	0.1250	[12]
c_i	0.0493	0.0490	[12]
σ_i	1.5	1.5	Assumed
μ_i	0.0199	0.0198	[11]
$\mu_i^{'}$	0.05	0.05	[10]

8.1. Effect of different fractional order on disease dynamics

In this subsection, the numerical simulation results are presented by using a step size of $h = 0.05$. We observe the impact of memory effects on disease transmission by plotting the population dynamics of different state variables in a two-patch DR-TB model across different orders.

Figures 2 and 3 respectively illustrate the changes in the population of the state variables in two patches under the influence of different fractional orders q. In all patches, we observe a significant increase in the number of susceptible groups as the fractional order q increases. Meanwhile, the number of infected individuals increases and the convergence speed of I_i and M_i decreases as the fractional order decreases. As the fractional order q decreases, the time required for recovered individuals to maintain stability increases in both patches. This implies that as the fractional order decreases, the time required for all DR-TB patients to recover also increases. The above results indicate that DR-TB does not disappear in the short term, further suggesting that the fractional order SEIMR model is more realistic.

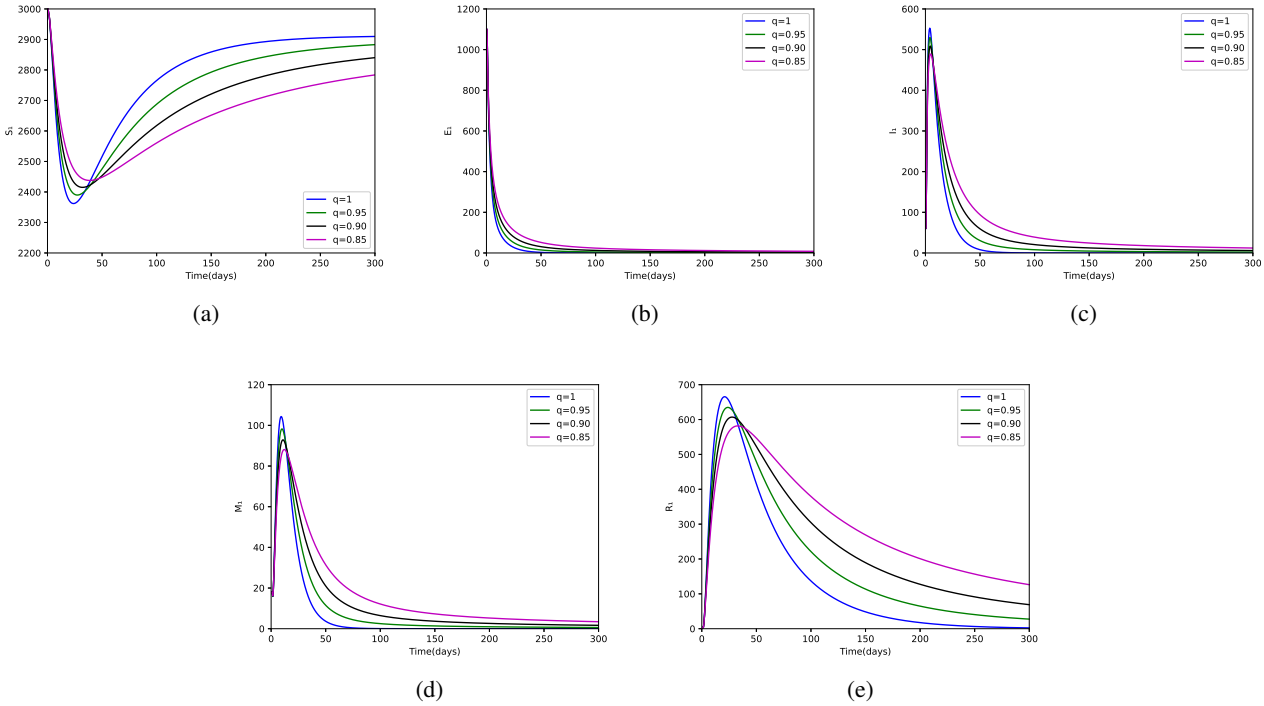


Figure 2. Dynamics of the state variables of the model (1) in patch 1 for different orders q.

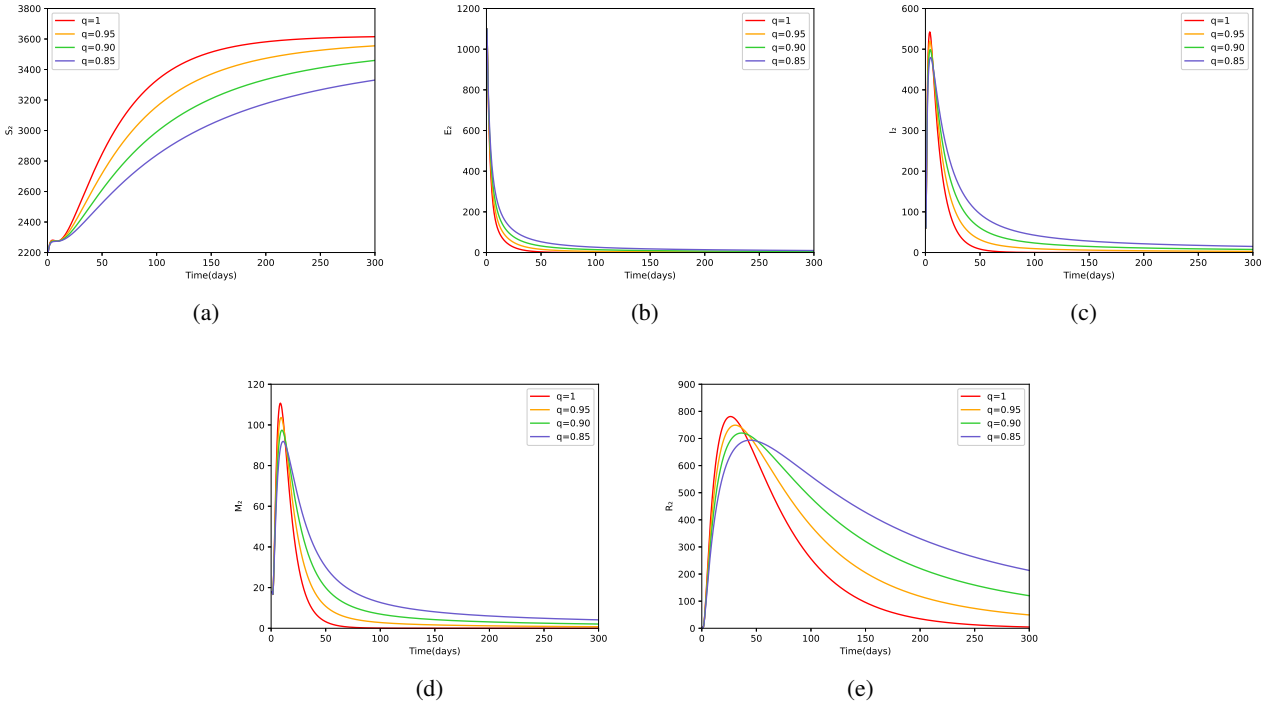


Figure 3. Dynamics of the state variables of the model (1) in patch 2 for different orders q.

8.2. Sensitivity analysis of parameters in R_0

Identifying the impact of key parameters in the model on the basic reproductive number R_0, we can propose targeted measures to effectively control the spread of DR-TB. In this section, we conduct a sensitivity analysis. The normalized index of forward sensitivity for R_0, influenced by the parameter k, is defined as [46]

$$Z_k^{R_0} = \frac{\partial R_0}{\partial k} \times \frac{k}{R_0}.$$

Accordingly,

$$\begin{aligned}
Z_{b_i}^{R_0^i} &= \frac{b_i(\alpha_i + c_i + \mu_i + \mu_i^{'})[(\sigma_i r_i - c_i) + (\sigma_i - 1)(\mu_i + \mu_i^{'})]}{[(r_i + b_i + \mu_i + \mu_i^{'})(\alpha_i + c_i + \mu_i + \mu_i^{'}) - b_i\alpha_i](b_i\sigma_i + \alpha_i + c_i + \mu_i + \mu_i^{'})}, \quad Z_{\beta_i}^{R_0^i} = 1, \\
Z_{r_i}^{R_0^i} &= -\frac{r_i(\alpha_i + c_i + \mu_i + \mu_i^{'})}{(r_i + b_i + \mu_i + \mu_i^{'})(\alpha_i + c_i + \mu_i + \mu_i^{'}) - b_i\alpha_i}, \quad Z_{\varepsilon_i}^{R_0^i} = \frac{\varepsilon_i\mu_i\delta}{(\varepsilon_i + \mu_i)(\varepsilon_i + \mu_i - \mu_i\delta)}, \\
Z_{\alpha_i}^{R_0^i} &= -\frac{b_i\alpha_i[(\sigma_i r_i - c_i) + (\mu_i + \mu_i^{'})(\sigma_i - 1)]}{[(r_i + b_i + \mu_i + \mu_i^{'})(\alpha_i + c_i + \mu_i + \mu_i^{'}) - b_i\alpha_i](b_i\sigma_i + \alpha_i + c_i + \mu_i + \mu_i^{'})}, \\
Z_{c_i}^{R_0^i} &= -\frac{b_i c_i[\alpha_i + \sigma_i(r_i + b_i + \mu_i + \mu_i^{'})]}{[(r_i + b_i + \mu_i + \mu_i^{'})(\alpha_i + c_i + \mu_i + \mu_i^{'}) - b_i\alpha_i](b_i\sigma_i + \alpha_i + c_i + \mu_i + \mu_i^{'})} \quad (i = 1, 2).
\end{aligned}$$

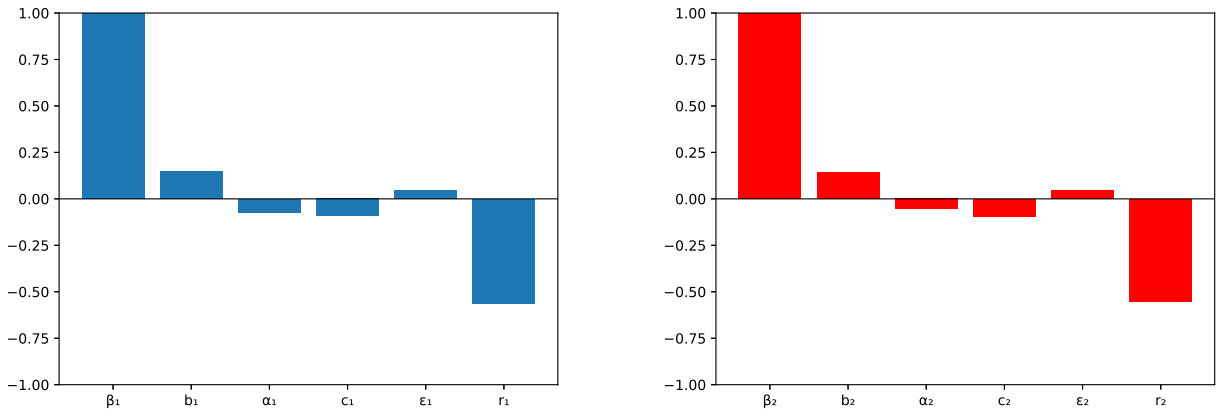
Tables 3 and 4 illustrate the sensitivity indices of R_0^i (i=1,2) in relation to the parameters of the proposed model. It can be seen that the parameters $\beta_i, b_i, \varepsilon_i$ have a positive impact on R_0^i. Clearly, β_i and b_i are the parameters more sensitive to R_0^i. Therefore, if parameters β_i and b_i increase, R_0^i will increase; if parameters β_i and b_i decrease, R_0^i will decrease accordingly. This indicates that the transmission of DR-TB can be controlled by decreasing parameters β_i and b_i. Similarly, the parameters α_i, c_i, r_i have a negative impact on R_0^i, and among these parameters, α_i and r_i have a greater negative impact on R_0^i. This means that as these two parameters increase, the value of R_0^i decreases. The sensitivity diagram is shown in Figure 4. It is evident that parameters β_i and r_i are the most sensitive.

Table 3. Sensitivity index of R_0^1.

Parameters	β_1	b_1	α_1	c_1	ε_1	r_1
Values	0.11	0.05	0.075	0.0493	0.25	0.1299
$Z_{k_i}^{R_0^1}$	1	0.1457	-0.0715	-0.0920	0.0432	-0.5638

Table 4. Sensitivity index of R_0^2.

Parameters	β_2	b_2	α_2	c_2	ε_2	r_2
Values	0.09	0.05	0.074	0.0490	0.245	0.1250
$Z_{k_j}^{R_0^2}$	1	0.1433	-0.0549	-0.0941	0.0435	-0.5540



(a) (b)

Figure 4. Sensitivity indices of R_0^i $(i = 1, 2)$ against the parameters.

From the above analysis, we conclude that if transmission rate β_i can be reduced or cure rate r_i improved through certain control measures, the disease can be effectively managed. However, controlling the transmission rate β_i is often challenging, so we can focus on increasing the cure rate r_i to reduce the impact of the disease.

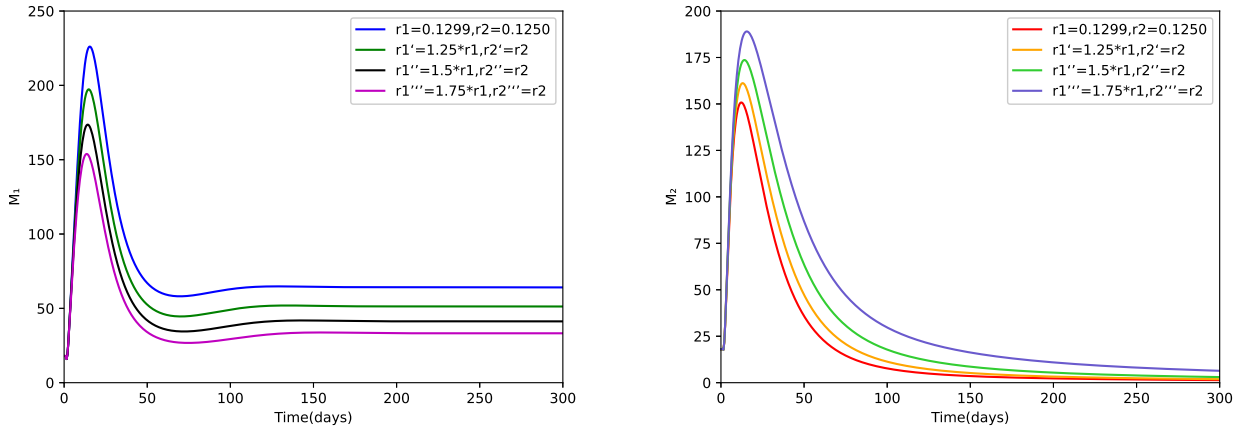
8.3. Disease control with different cure rates

In the Subsection 8.2, we obtained that the cure rate r_i of humans can be increased to reduce the impact of disease. So, in this section we quantitatively investigate the impact of different cure rates in two patches on infected individuals.

Let

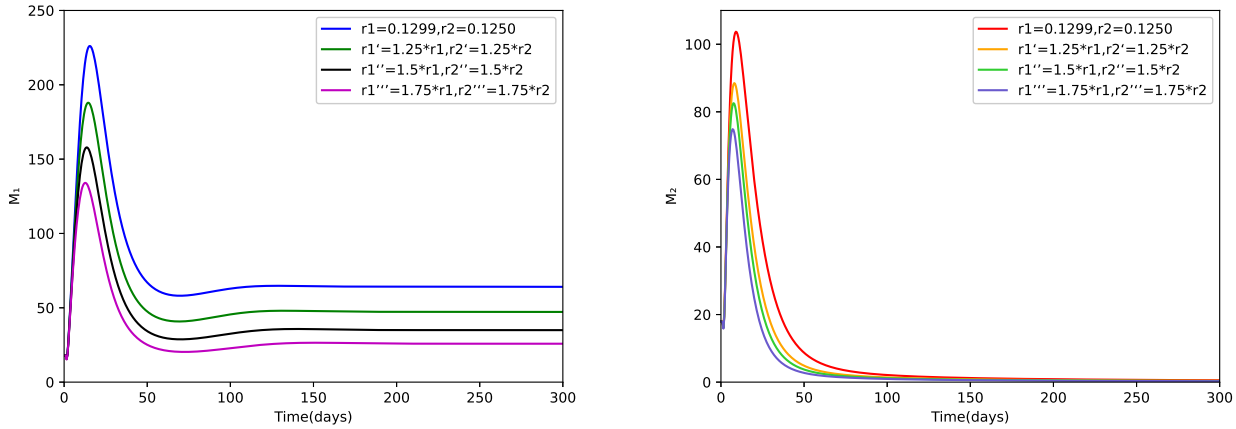
$$\beta_1 = 0.65, \quad \beta_2 = 0.09, \quad q = 0.95, \quad m_{12} = 0.01, \quad m_{21} = 0.02,$$

and suppose that r_1 continues to increase, while r_2 remains unchanged. Figure 5a,b indicates that with the increase of r_1, the number of DR-TB patients in patch 1 gradually decreases, while the number of DR-TB patients in patch 2 increases. Due to population migration between the two patches, if only the cure rate (r_1) of patch 1, without improving the cure rate (r_2) of patch 2, the number of DR-TB in patch 1 increases during the epidemic. This would control the spread of the disease in patch 1 while putting pressure on disease control in patch 2, resulting in a longer time required for DR-TB to disappear in patch 2. Suppose that the cure rate of DS-TB is simultaneously increased in both patch 1 and patch 2. Figure 6a,b shows that with the continuous increase in r_1 and r_2, the DR-TB in both patches will be controlled and will lead to an earlier disappearance of the disease in patch 2.



(a) (b)

Figure 5. Impact of M_i at $\alpha = 0.95$ with r_1 in patch i ($i = 1, 2$).



(a) (b)

Figure 6. Impact of M_i at $\alpha = 0.95$ with r_1 and r_2 in patch i ($i = 1, 2$).

9. Optimization of the DR-TB model

The sensitivity analysis of R_0^i in Subsection 8.2 shows that the parameter r_i has the largest negative effect on the basic reproductive number. Thus, in this section, we consider the cure rates for TB of the two patches, denoted by r_1 and r_2 as control variables $u_1(t)$ and $u_2(t)$, respectively, and derive optimality conditions from the Hamiltonian function by using Pontryagin's principle [26, 28, 47]. Meanwhile, we compute the optimal function of control to determine the best measure of the treatment aspect in the two patches in order to maximize the number of recovered individuals in each patch while minimizing the number of infected individuals.

9.1. Optimal control problem and optimality conditions

Our goal is to determine the optimal control u_1^* and u_2^* by controlling the cure rates of two patches to minimize the cost function $J(I_1, I_2, u_1, u_2)$ of the control strategy. Let

$$\hat{\beta}_1 = \frac{\beta_1}{N_1}, \quad \hat{\beta}_2 = \frac{\beta_2}{N_2}.$$

Under the control measures, the proposed model (1) is modified as

$$\begin{aligned}
{}_0^C D_t^q S_1 &= \Lambda_1 - \hat{\beta}_1(I_1 + \sigma_1 M_1)S_1 - \mu_1 S_1 + m_{12}S_2 - m_{21}S_1,\\
{}_0^C D_t^q E_1 &= \hat{\beta}_1\delta(I_1 + \sigma_1 M_1)S_1 - (\varepsilon_1 + \mu_1)E_1,\\
{}_0^C D_t^q I_1 &= \hat{\beta}_1(1-\delta)(I_1 + \sigma_1 M_1)S_1 + \varepsilon_1 E_1 + \alpha_1 M_1 - (u_1(t) + b_1 + \mu_1 + \mu_1^{'})I_1,\\
{}_0^C D_t^q M_1 &= b_1 I_1 - (\alpha_1 + \mu_1 + c_1 + \mu_1^{'})M_1,\\
{}_0^C D_t^q R_1 &= u_1(t)I_1 + c_1 M_1 - \mu_1 R_1 + m_{12}R_2 - m_{21}R_1,\\
{}_0^C D_t^q S_2 &= \Lambda_2 - \hat{\beta}_2(I_2 + \sigma_2 M_2)S_2 - \mu_2 S_2 + m_{21}S_1 - m_{12}S_2,\\
{}_0^C D_t^q E_2 &= \hat{\beta}_2\delta(I_2 + \sigma_2 M_2)S_2 - (\varepsilon_2 + \mu_2)E_2,\\
{}_0^C D_t^q I_2 &= \hat{\beta}_2(1-\delta)(I_2 + \sigma_2 M_2)S_2 + \varepsilon_2 E_2 + \alpha_2 M_2 - (u_2(t) + b_2 + \mu_2 + \mu_2^{'})I_2,\\
{}_0^C D_t^q M_2 &= b_2 I_2 - (\alpha_2 + \mu_2 + c_2 + \mu_2^{'})M_2,\\
{}_0^C D_t^q R_2 &= u_2(t)I_2 + c_2 M_2 - \mu_2 R_2 + m_{21}R_1 - m_{12}R_2,
\end{aligned} \tag{1'}$$

and with the nonnegative initial conditions

$$S_i(0) > 0, \quad E_i(0) \geq 0, \quad I_i(0) \geq 0, \quad M_i(0) \geq 0, \quad R_i(0) \geq 0 \ (i = 1, 2).$$

The control is completely effective when $u_1 = 0, u_2 = 0$. We can be done by considering the following fractional optimal control problem to minimize the objective functional given by

$$J(I_1, I_2, u_1, u_2) = \int_0^{T_f} [T_1 I_1 + T_2 I_2 + \frac{1}{2}B_1 u_1^2(t) + \frac{1}{2}B_2 u_2^2(t)]dt,$$

where T_f is the fixed terminal time, the quantities T_1 and T_2 are the positive weight constants on the advantage of the cost, and $u_1(t), u_2(t)$ are the control variable. $\frac{1}{2}B_1 u_1^2(t)$ and $\frac{1}{2}B_2 u_2^2(t)$ are the cost functions of the control methods for the cure rate of patients with DS-TB in the considered patch 1 and patch 2 in the following cases, respectively. The optimal control problem is then defined as

$$J(u_1^*, u_2^*) = \min_{u_1(t), u_2(t) \in u} J(u_1, u_2).$$

The Hamiltonian of optimal problem is defined by

$$\begin{aligned}
\mathcal{H} &= T_1 I_1 + T_2 I_2 + \frac{1}{2}B_1 u_1^2 + \frac{1}{2}B_2 u_2^2\\
&+ \lambda_1[\Lambda_1 - \hat{\beta}_1(I_1 + \sigma_1 M_1)S_1 - \mu_1 S_1 + m_{12}S_2 - m_{21}S_1]\\
&+ \lambda_2[\hat{\beta}_1\delta(I_1 + \sigma_1 M_1)S_1 - (\varepsilon_1 + \mu_1)E_1]\\
&+ \lambda_3[\hat{\beta}_1(1-\delta)(I_1 + \sigma_1 M_1)S_1 + \varepsilon_1 E_1 + \alpha_1 M_1 - (u_1(t) + b_1 + \mu_1 + \mu_1^{'})I_1]
\end{aligned}$$

$$
\begin{aligned}
&+ \lambda_4[b_1 I_1 - (\alpha_1 + \mu_1 + c_1 + \mu_1^{'})M_1] \\
&+ \lambda_5[u_1(t)I_1 + c_1 M_1 - \mu_1 R_1 + m_{12}R_2 - m_{21}R_1] \\
&+ \lambda_6[\Lambda_2 - \hat{\beta}_2(I_2 + \sigma_2 M_2)S_2 - \mu_2 S_2 + m_{21}S_1 - m_{12}S_2] \\
&+ \lambda_7[\hat{\beta}_2\delta(I_2 + \sigma_2 M_2)S_2 - (\varepsilon_2 + \mu_2)E_2] \\
&+ \lambda_8[\hat{\beta}_2(1-\delta)(I_2 + \sigma_2 M_2)S_2 + \varepsilon_2 E_2 + \alpha_2 M_2 - (u_2(t) + b_2 + \mu_2 + \mu_2^{'})I_2] \\
&+ \lambda_9[b_2 I_2 - (\alpha_2 + \mu_2 + c_2 + \mu_2^{'})M_2] \\
&+ \lambda_{10}[u_2(t)I_2 + c_2 M_2 - \mu_2 R_2 + m_{21}R_1 - m_{12}R_2],
\end{aligned}
$$

where $\lambda_1, \lambda_2, \ldots, \lambda_{10}$ are the adjoint variables.

Theorem 9.1. *Let $S_1^*, E_1^*, I_1^*, M_1^*, R_1^*$ and $S_2^*, E_2^*, I_2^*, M_2^*, R_2^*$ be optimal state solutions with associated optimal control variables for the optimal control problems of model (1'). Then, there exist adjoint variables u_1^* and u_2^* satisfying*

$$
\begin{aligned}
{}_0^C D_{T_f}^q \lambda_1 &= (\lambda_1 - \lambda_3)\hat{\beta}_1(I_1 + \sigma_1 M_1) + (\lambda_3 - \lambda_2)\hat{\beta}_1\delta(I_1 + \sigma_1 M_1) + (\lambda_1 - \lambda_6)m_{21} + \lambda_1\mu_1, \\
{}_0^C D_{T_f}^q \lambda_2 &= (\lambda_2 - \lambda_3)\varepsilon_1 + \lambda_2\mu_1, \\
{}_0^C D_{T_f}^q \lambda_3 &= -T_1 + (\lambda_1 - \lambda_3)\hat{\beta}_1 S_1 + (\lambda_3 - \lambda_2)\hat{\beta}_1\delta S_1 + \lambda_3(\mu_1 + \mu_1^{'}) + (\lambda_3 - \lambda_4)b_1 + (\lambda_3 - \lambda_5)u_1(t), \\
{}_0^C D_{T_f}^q \lambda_4 &= (\lambda_1 - \lambda_3)\hat{\beta}_1\sigma_1 S_1 + (\lambda_3 - \lambda_2)\hat{\beta}_1\delta\sigma_1 S_1 + (\lambda_4 - \lambda_3)\alpha_1 + (\lambda_4 - \lambda_5)c_1 + \lambda_4(\mu_1 + \mu_1^{'}), \\
{}_0^C D_{T_f}^q \lambda_5 &= (\lambda_5 - \lambda_{10})m_{21} + \lambda_5\mu_1, \\
{}_0^C D_{T_f}^q \lambda_6 &= (\lambda_6 - \lambda_8)\hat{\beta}_2(I_2 + \sigma_2 M_2) + (\lambda_8 - \lambda_7)\hat{\beta}_2\delta(I_2 + \sigma_2 M_2) + (\lambda_6 - \lambda_1)m_{12} + \lambda_6\mu_2, \\
{}_0^C D_{T_f}^q \lambda_7 &= (\lambda_7 - \lambda_8)\varepsilon_2 + \lambda_7\mu_2, \\
{}_0^C D_{T_f}^q \lambda_8 &= -T_2 + (\lambda_6 - \lambda_8)\hat{\beta}_2 S_2 + (\lambda_8 - \lambda_7)\hat{\beta}_2\delta S_2 + \lambda_8(\mu_2 + \mu_2^{'}) + (\lambda_8 - \lambda_9)b_2 + (\lambda_8 - \lambda_{10})u_2(t), \\
{}_0^C D_{T_f}^q \lambda_9 &= (\lambda_6 - \lambda_8)\hat{\beta}_2\sigma_2 S_2 + (\lambda_8 - \lambda_7)\hat{\beta}_2\delta\sigma_2 S_2 + (\lambda_9 - \lambda_8)\alpha_2 + (\lambda_9 - \lambda_{10})c_2 + \lambda_9(\mu_2 + \mu_2^{'}), \\
{}_0^C D_{T_f}^q \lambda_{10} &= (\lambda_{10} - \lambda_5)m_{12} + \lambda_{10}\mu_2,
\end{aligned}
$$

with transversally conditions or boundary conditions

$$\lambda_1(T_f) = \lambda_2(T_f) = \lambda_3(T_f) = \lambda_4(T_f) = \lambda_5(T_f) = \lambda_6(T_f) = \lambda_7(T_f) = \lambda_8(T_f) = \lambda_9(T_f) = \lambda_{10}(T_f) = 0.$$

Furthermore, the control functions u_1^* and u_2^* are given by

$$
\begin{aligned}
u_1^*(t) &= min\left\{1, max\left\{\frac{\lambda_3 - \lambda_5}{B_1} I_1(t), 0\right\}\right\}, \\
u_2^*(t) &= min\left\{1, max\left\{\frac{\lambda_8 - \lambda_{10}}{B_2} I_2(t), 0\right\}\right\}.
\end{aligned}
$$

Proof. The adjoint system is obtained from the Hamiltonian $\mathcal{H}$ as

$$
\begin{aligned}
&{}_0^C D_{T_f}^q \lambda_1(t) = -\frac{\partial \mathcal{H}}{\partial S_1}, \quad {}_0^C D_{T_f}^q \lambda_2(t) = -\frac{\partial \mathcal{H}}{\partial E_1}, \quad {}_0^C D_{T_f}^q \lambda_3(t) = -\frac{\partial \mathcal{H}}{\partial I_1}, \\
&{}_0^C D_{T_f}^q \lambda_4(t) = -\frac{\partial \mathcal{H}}{\partial M_1}, \quad {}_0^C D_{T_f}^q \lambda_5(t) = -\frac{\partial \mathcal{H}}{\partial R_1}, \quad {}_0^C D_{T_f}^q \lambda_6(t) = -\frac{\partial \mathcal{H}}{\partial S_2},
\end{aligned}
$$

$$ {}_0^C D_{T_f}^q \lambda_7(t) = -\frac{\partial \mathcal{H}}{\partial E_2}, \quad {}_0^C D_{T_f}^q \lambda_8(t) = -\frac{\partial \mathcal{H}}{\partial I_2}, \quad {}_0^C D_{T_f}^q \lambda_9(t) = -\frac{\partial \mathcal{H}}{\partial M_2}, \quad {}_0^C D_{T_f}^q \lambda_{10}(t) = -\frac{\partial \mathcal{H}}{\partial R_2}, $$

with zero final time conditions

$$ \lambda_1(T_f) = \lambda_2(T_f) = \lambda_3(T_f) = \lambda_4(T_f) = \lambda_5(T_f) = \lambda_6(T_f) = \lambda_7(T_f) = \lambda_8(T_f) = \lambda_9(T_f) = \lambda_{10}(T_f) = 0. $$

Using the first condition of Pontryagin's principle, we obtain two equations for the control,

$$ \frac{\partial \mathcal{H}}{\partial u_1(t)} = 0 \Rightarrow B_1 u_1 - \lambda_3 I_1 + \lambda_5 I_1 = 0 \Rightarrow u_1(t) = \frac{\lambda_3 - \lambda_5}{B_1} I_1, $$
$$ \frac{\partial \mathcal{H}}{\partial u_2(t)} = 0 \Rightarrow B_2 u_2 - \lambda_8 I_2 + \lambda_{10} I_2 = 0 \Rightarrow u_2(t) = \frac{\lambda_8 - \lambda_{10}}{B_2} I_2. $$

Thus, the optimal control characterization for $u_1^*(t)$ and $u_2^*(t)$ with bounds are given as follows:

$$ u_1^*(t) = min\left\{1, max\left\{\frac{\lambda_3 - \lambda_5}{B_1} I_1(t), 0\right\}\right\}, $$
$$ u_2^*(t) = min\left\{1, max\left\{\frac{\lambda_8 - \lambda_{10}}{B_2} I_2(t), 0\right\}\right\}. $$

This completes the proof. □

9.2. Numerical simulation with control measures

In this subsection, we undertake simulations of the fractional optimal control of the model and examine the impact of the controls incorporated into the model on the dissemination of the epidemic.

Figure 7 compares the changes in the number of $I_1, I_2, M_1, M_2, R_1, R_2$ in two patches with control and without control at different orders

$$ q = 1, 0.95, 0.90, 0.85, $$

respectively. With the optimal cure rates implemented for different fractional orders, the optimized curves for infected individuals exhibit a decline with an increasing fractional order q. The number of DS-TB and DR-TB patients corresponding to each order decreases over time, while the number of recoveries in both compartments increases. This demonstrates that, with effective treatment, infected individuals are more likely to recover than those who do not receive any treatment. Even after implementing the same controls, the fractional order still affects the dynamical behavior of the disease, with smaller orders resulting in a longer duration of disease presence. Thus, the fractional order model allows for a better determination of the optimal treatment rate for different orders according to reality, which is not possible with the integer-order model.

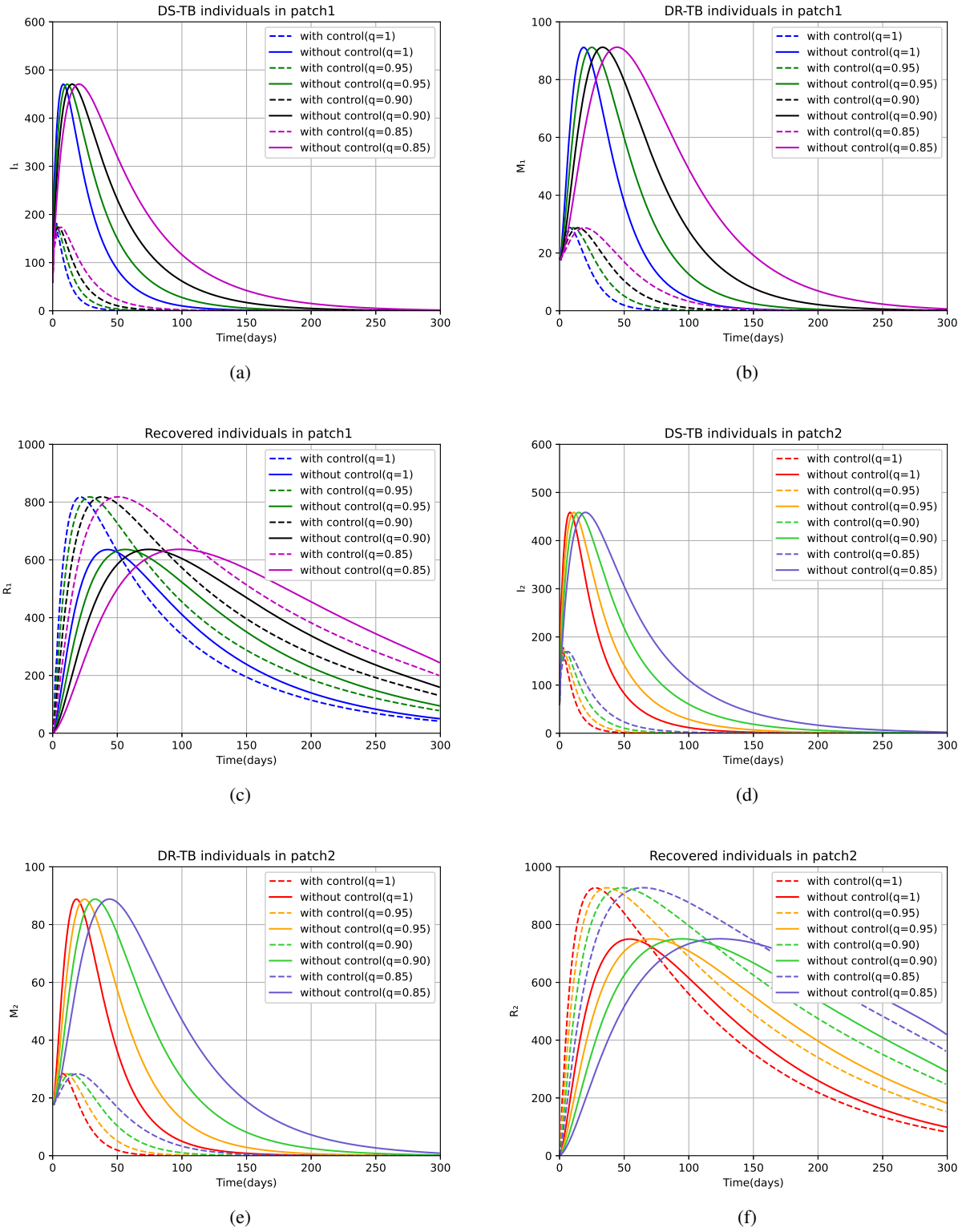
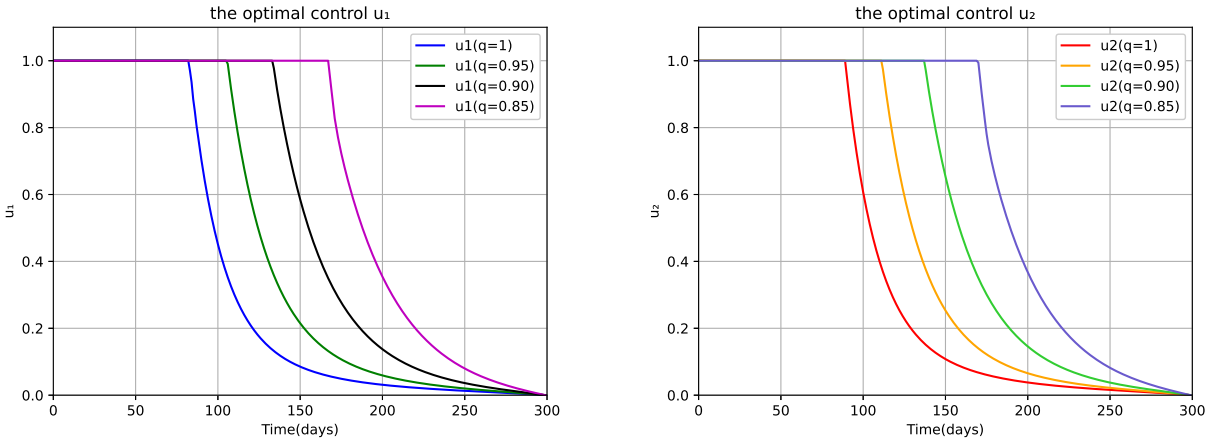


Figure 7. The respective state variables (I_i, M_i, R_i) of the two patches with and without control at different q-values.

Finally, Figure 8a,b shows the trend plots of optimal control u_1^* and u_2^* over time for two patches. It can be seen that in the early stage of the disease outbreak, for the treatment control rate in two patches to be maintained at the maximum value, the treatment measures are effective and can effectively prevent the further spread of the disease. As the value of fractional order q gradually decreases from 1, the time to stay at the maximum level of controls $u_1(t)$ and $u_2(t)$ increases in order to control the spread of the disease. Consequently, the presence of the fractional derivative order q increases the control of effective treatment in both patches.



(a) (b)

Figure 8. Time series of optimal control variable u_1 and u_2.

10. Conclusions

In this study, we introduced the two-patch Caputo fractional-order derivative model for DR-TB to investigate the disease dynamics for an optimal control analysis. To verify the effectiveness of the proposed Caputo fractional-order model, the positivity and boundedness of the model (1) are proved and the equilibrium point and the basic reproduction number R_0 are computed, and we prove the local and global asymptotic stability of the DFE point and the epidemic equilibrium piont. The existence and uniqueness of the solution were proved and investigated for the Ulam-Hyers stability of model (1). Additionally, sensitivity analysis of R_0 was conducted using the normalized forward sensitivity index method, identifying the sensitive parameters in R_0. The Caputo fractional-order model was numerically solved using the two-step Lagrange interpolation technique. Numerical simulations were carried out to investigate the effect of different fractional orders on disease transmission and to quantify the effect of different cure rates on the number of infected persons in two patches. We discovered that in a two-patch fractional-order model of DR-TB, increasing the disease cure rate in one patch alone leads to an increase in the number of DR-TB patients in the other patch. Therefore, it is crucial to simultaneously increase the treatment rates in both patches, strive for an even distribution of medical resources, and avoid favoring a particular patch in the allocation of medical resources.

The main aim of this study is to control DR-TB by minimizing the number of infected people. We considered the cure rates for diseases of the two patches as control variables and employed the Pontryagin's maximum principle to provide necessary conditions for the existence of the optimal

solution to the optimal control problem. Simulation results show that the implementation of controls can be effective in both patches to reduce the number of people infected with the disease in both patches and make the number of people recovered increase, which verifies the effectiveness of our proposed control measures. It is also concluded that reducing the fractional order q will lead to an increase in the time required to implement effective controls.

Author contributions

Shaoping Jiang: conceptualization and modeling, writing—review and editing; Hongyan Wang: theoretical analysis, writing—original draft; Yudie Hu: numerical simulation; Supaporn Lonapalawong: writing—review and editing. Each author equally contributed to writing and finalizing the article. All authors have read and agreed to the published version of the manuscript.

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Conflict of interest

The authors declare no conflicts of interest.

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