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Research article

Bayesian and non-Bayesian estimation of some entropy measures for a Weibull distribution

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Abstract: Entropy measures have been employed in various applications as a helpful indicator of information content. This study considered the estimation of Shannon entropy, ζ-entropy, Arimoto entropy, and Havrda and Charvat entropy measures for the Weibull distribution. The classical and Bayesian estimators for the suggested entropy measures were derived using generalized Type II hybrid censoring data. Based on symmetric and asymmetric loss functions, Bayesian estimators of entropy measurements were developed. Asymptotic confidence intervals with the help of the delta method and the highest posterior density intervals of entropy measures were constructed. The effectiveness of the point and interval estimators was evaluated through a Monte Carlo simulation study and an application with actual data sets. Overall, the study's results indicate that with longer termination times, both maximum likelihood and Bayesian entropy estimates were effective. Furthermore, Bayesian entropy estimates using the linear exponential loss function tended to outperform those using other loss functions in the majority of scenarios. In conclusion, the analysis results from real-world examples aligned with the simulated data. Drawing insights from the analysis of glass fiber, we can assert that this research holds practical applications in reliability engineering and financial analysis.

Keywords: Weibull distribution; ζ-entropy; Havrda and Charvat's entropy; Bayesian estimation
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1. Introduction

The Weibull distribution is of particular importance since it naturally follows the extreme value theorem [1] and has a useful physical interpretation in numerous practical applications. In various engineering applications, such as independent component analysis, image analysis, genetic analysis, and time delay estimation, it is useful to estimate the entropy of a system or process given some observations (see [2–5]). The cumulative distribution function (CDF) and probability density function (PDF) of this distribution, for $x > 0$, are, respectively, as follows:

$$F(x) = 1 - e^{-\lambda x^{\beta}}, \tag{1.1}$$

and

$$f(x) = \beta\lambda x^{\beta-1} e^{-\lambda x^{\beta}}, \tag{1.2}$$

where $\beta > 0$ is the shape parameter and $\lambda > 0$ is the scale parameter.

The entropy in the Weibull distribution for progressive censoring was analyzed by Cramer and Bag [6], and Cho et al. [7] using generalized progressive Type II hybrid censored samples to develop estimators for the entropy function of a Weibull distribution. Entropy estimation for an inverse Weibull distribution using multiple censoring samples has been discussed by Hassan and Zaky [8]. The estimation of entropy for the Weibull distribution based on record values was considered by Chacko and Asha [9].

It is advisable to end the test before all of the items fail because most trials in life are time- and money-constrained. The observations arising from that situation are referred to as censored samples, and many censoring methods exist. Two of the most prevalent forms of censorship are Type II (T-II) and Type I (T-I). Childs et al. [10] combined T-I and T-II censoring to create a hybrid censoring scheme (HCS), which is divided into two categories: T-I HCS and T-II HCS. These two variants have been widely implemented in various research studies. Chandrasekar et al. [11] expanded these methods by introducing two new forms, termed generalized T-I HCS (GT-I HCS) and generalized T-II HCS (GT-II HCS). The GT-II HCS allows for a flexible censoring scheme that combines T-I and T-II censoring, accommodating various censoring patterns observed in real-world data. In many practical scenarios, data may exhibit a combination of T-I and T-II censorship due to different reasons, such as administrative constraints, equipment failures, or study design considerations. For some recent studies, see [12–14].

Entropy is a measure of uncertainty in a random variable and is used in information theory to determine the expected value of the information contained in that random variable. In many fields, including statistics, physics, chemistry, economics, insurance, financial analysis, and biological phenomena, measuring entropy is important. Less information in a sample is referred to as having more entropy. One of the most popular ways to estimate entropy is Shannon's entropy. This measure has proven to be successful in the research of many applications. One of Shannon's measure's biggest drawbacks is that it could be negative for specific probability distributions, making it useless as a measure of uncertainty. Measures of uncertainty, including ζ-entropy, Arimoto entropy, and Havrda and Charvat entropy, which are the subject of our attention, are explained in Section 2.

Numerous academics have researched entropy estimates for various life distributions. Cui and Ding [15] studied the convergence of Rényi entropy of the normalized sums of independent, identically distributed random variables. Entropy estimators for a double exponential distribution were created by

Kang et al. [16], and Cho et al. [17] addressed entropy estimates for the Rayleigh distribution using double G-II HCS. Cho et al. [18] used generalized progressive T-II HCS to derive estimators for the entropy measure of a Weibull distribution. Ahmadini et al. [19] examined a Bayesian estimator of dynamic cumulative residual entropy based on the Parto II distribution. Entropy estimators for the Lomax distribution have been considered, respectively, by Al-Babtain et al. [20], and Hassan and Zaki [21]. Al-Omari et al. [22] used record data to investigate an entropy Bayesian estimator for an extended inverse exponential distribution. For more recent studies, see [23–27].

The goal of this work is to examine the challenges associated with estimating uncertainty measures of the Weibull distribution using the GT-II HCS. We were motivated to investigate this issue due to the significance of the Weibull distribution in various fields, including survival analysis and reliability engineering. Uncertainty measures are essential as they reflect the degree of confidence in the estimates and help in making informed decisions in practical applications. Moreover, the GT-II HCS provides flexibility, realism, efficiency, and a comprehensive analysis framework for studies in these areas. Notably, there are no existing studies that have utilized the GT-II HCS in conjunction with various entropy measures in this context. The current work will now be summarized as follows:

• The maximum likelihood (ML) and Bayesian estimators of Shannon entropy, ζ-entropy, Arimoto entropy, and Havrda and Charvat entropy are addressed using the GT-II HCS.

• The asymmetric loss function (ASLOF) and the symmetric loss function (SLOF) are both used in the formulation of the Bayesian estimator.

• Asymptotic confidence intervals (ACIs), based on the delta method, and the highest posterior density (HPD) intervals are established.

• The complicated forms of different entropy estimates and how to construct the HPD intervals need the use of the Metropolis-Hastings (M-H) algorithm in the Markov chain Monte Carlo (MCMC) approach.

• To evaluate the performance of different entropy metrics, Monte Carlo simulations were conducted using accuracy measures such as mean squared errors (MSEs), average lengths (ALs), and coverage probabilities (CPs). Additionally, the inferential approaches presented in this paper were applied to real-world data to demonstrate their effectiveness.

This paper is structured as follows: Section 2 provides the derivation of the formulae for the entropy measures. Section 3 examines the four various kinds of entropy measurements under the GT-II HCS using both classical and Bayesian techniques. Also, in Section 3, the MCMC procedure is used to get the Bayesian estimates based on the Metropolis-Hastings algorithm. The discussion of the simulation issue and its application using actual data sets is covered in Section 4. The paper concludes with a series of final remarks and observations in Section 5.

2. Expressions of entropies

In this section, we derive analytical formulas for Shannon, Arimoto, Havrda and Charvat, and ζ-entropy measures for the Weibull distribution.

Shannon entropy is defined as follows:

$$E_1 = -\int_{-\infty}^{\infty} f(x) \ln f(x) dx. \tag{2.1}$$

Let X be a random variable following the Weibull distribution. Then from Eqs (1.2) and (2.1), and according to Cho et al. [7], Shannon entropy takes the following form:

$$\begin{aligned} E_1 &= -\int_0^{\infty} (\beta\lambda x^{\beta-1} e^{-\lambda x^{\beta}}) \ln (\beta\lambda x^{\beta-1} e^{-\lambda x^{\beta}}) dx \\ &= -\ln(\beta\lambda) + \frac{(\beta-1)}{\beta}[\gamma + \ln(\lambda)] + 1 \end{aligned} \tag{2.2}$$

where γ is the Euler constant.

ζ-entropy is a parametric extension of Shannon entropy. It was introduced by the physicist Tsallis [28], and this type has several applications in physics, statistical mechanics econophysics, and finance. For a random variable X with a PDF $f(x)$, for $\zeta > 0$, $\zeta \neq 1$, the ζ-entropy (E_2) measure is defined as follows:

$$E_2 = \frac{1}{\zeta - 1}\Big(1 - \int_{-\infty}^{\infty} f(x)^{\zeta} dx\Big).$$

The expression of ζ-entropy of the Weibull distribution can be calculated from Eq (2) as follows:

$$E_2 = \frac{1}{\zeta - 1}\Big(1 - \int_0^{\infty} (\beta\lambda x^{\beta-1} e^{-\lambda x^{\beta}})^{\zeta} dx\Big),$$

where the value of the integral is given by:

$$\begin{aligned} I &= \int_0^{\infty} (\beta\lambda x^{\beta-1} e^{-\lambda x^{\beta}})^{\zeta} dx \\ &= (\beta\lambda)^{\zeta} \int_0^{\infty} (x^{\beta-1} e^{-\lambda x^{\beta}})^{\zeta} dx. \end{aligned}$$

Let $u = x^{\beta}$, $x = u^{\frac{1}{\beta}}$, and $dx = \frac{1}{\beta} u^{\frac{1}{\beta}-1} du$, and then

$$I = (\beta\lambda)^{\zeta} \int_0^{\infty} (u^{\frac{\beta-1}{\beta}} e^{-\lambda u})^{\zeta} (\frac{1}{\beta} u^{\frac{1}{\beta}-1}).$$

After simplification, I is as follows:

$$I = (\frac{\beta}{\lambda^{\frac{-1}{\beta}}})^{\zeta-1} \frac{\Gamma(\zeta(1-\frac{1}{\beta}) + \frac{1}{\beta})}{\zeta^{\zeta(1-\frac{1}{\beta})+\frac{1}{\beta}}},$$

where $\Gamma(.)$ is the gamma function. Then the value of ζ-entropy is given by:

$$E_2 = \frac{1}{\zeta - 1}\Big(1 - (\frac{\beta}{\lambda^{\frac{-1}{\beta}}})^{\zeta-1} \frac{\Gamma(\zeta(1-\frac{1}{\beta}) + \frac{1}{\beta})}{\zeta^{\zeta(1-\frac{1}{\beta})+\frac{1}{\beta}}}\Big). \tag{2.3}$$

Arimoto entropy is a generalized form of the well-known Shannon entropy and has several applications in clustering, image processing, and data analysis. The characteristics of Arimoto's (E_3) entropy [29] measure is given by:

$$E_3 = \frac{\zeta}{1 - \zeta}\Big[\big(\int_{-\infty}^{\infty} f(x)^{\zeta} dx\big)^{\frac{1}{\zeta}} - 1\Big].$$

The value of Arimoto's entropy is

$$E_3 = \frac{\zeta}{1-\zeta}\Big[\Big((\frac{\beta}{\lambda^{\frac{-1}{\beta}}})^{\zeta-1}\frac{\Gamma(\zeta(1-\frac{1}{\beta})+\frac{1}{\beta})}{\zeta^{\zeta(1-\frac{1}{\beta})+\frac{1}{\beta}}}\Big)^{\frac{1}{\zeta}}-1\Big]. \tag{2.4}$$

Havrda and Charvat (HC) entropy [30] represents an extension of Shannon entropy. This particular extension is denoted as E_4 entropy of degree ζ, $\zeta \neq 1$, and is characterized by the following properties:

$$E_4 = \frac{1}{2^{1-\zeta}-1}\Big[\int_{-\infty}^{\infty} f(x)^{\zeta}dx - 1\Big].$$

In the same way as Arimoto entropy, the value of E_4 entropy is calculated as follows:

$$E_4 = \frac{1}{2^{1-\zeta}-1}\Big[\Big((\frac{\beta}{\lambda^{\frac{-1}{\beta}}})^{\zeta-1}\frac{\Gamma(\zeta(1-\frac{1}{\beta})+\frac{1}{\beta})}{\zeta^{\zeta(1-\frac{1}{\beta})+\frac{1}{\beta}}}\Big)^{\zeta}-1\Big]. \tag{2.5}$$

3. Estimation methods

In this section, we examine the entropies measures of the Weibull distribution using the ML and Bayesian methods. When using the Bayesian method, we acquire the entropy measure estimators for SLOF and ASLOF, and compute these estimators using the Metropolis-Hastings (M-H) algorithm.

3.1. ML estimator

The ML estimators for the Weibull distribution are obtained based on the GT-II HCS. The GT-II HCS is explained as follows:

In the GT-II HCS, one sets $r \in (1, 2, ..., n)$ and time $T_1, T_2 \in (0, \infty)$, where $T_1 < T_2$. If the r^{th} failure occurs before T_1, then the termination time is $T^* = T_1$, if the r^{th} failure occurs between T_1 and T_2, then the termination time is $T^* = x_{r:n}$, and if the r^{th} failure occurs after T_2, the termination time is $T^* = T_2$.

Therefore under the GT-II HCS, there are three forms of data:

$$\begin{aligned}
&Case\ 1: x_{1:n} < ... < x_{d_1:n} \quad if\ x_{r:n} < T_1;\\
&Case\ 2: x_{1:n} < ... < x_{d_1:n}, ... < x_{r:n} \quad if\ T_1 < x_{r:n} < T_2;\\
&Case\ 3: x_{1:n} < ... < x_{d_2:n}, .. < T_2 \quad if\ x_{r:n} \geq T_2.
\end{aligned}$$

Suppose in a life-testing study, there are n identical items, and let $x_{1:n}, x_{2:n}, ..., x_{n:n}$ represent the ordered failure times of these items, $T_1, T_2 \in (0, \infty)$. Then the likelihood function of β and λ is as follows:

$$L(\underline{x}|\beta;\lambda) = \frac{n!}{(n-D)!}[\prod_{i=1}^{D} f(x_{i:n})][1-F(C)]^{n-D}, \tag{3.1}$$

where D is the number of total failures in the experiment up to time C and its value is given by:

$$(D;C) = \begin{cases} (d_1, T_1) & \text{for Case 1}\\ (r, x_{r:n}) & \text{for Case 2}\\ (d_2, T_2) & \text{for Case 3}\end{cases},$$

where d_i denotes the number of failures that occurred until time T_i. Then inserting (1.1) and (1.2) in (3.1) gives:

$$L(\underline{x}|\beta;\lambda) = \frac{n!}{(n-D)!}[\prod_{i=1}^{D}\beta\lambda x_i^{\beta-1}e^{-\lambda x_i^{\beta}}][e^{-\lambda C^{\beta}}]^{n-D}. \tag{3.2}$$

For a simplified form, replace $x_{i:n}$ with x_i in Equation (3.2). By taking the logarithm of each side, indicated by l, we have

$$l \propto D\ln(\beta) + D\ln(\lambda) + (\beta-1)\sum_{i=1}^{D}\ln(x_i) - \lambda\sum_{i=1}^{D}x_i^{\beta} - (n-D)\lambda C^{\beta}. \tag{3.3}$$

The derivatives of (3.3), owing to β and λ, allow us to obtain

$$\frac{\partial l}{\partial\beta} = \frac{D}{\beta} + \sum_{i=1}^{D}\ln(x_i) - \lambda\sum_{i=1}^{D}x_i^{\beta}\ln(x_i) - (n-D)\lambda C^{\beta}\ln(C), \tag{3.4}$$

and

$$\frac{\partial l}{\partial\lambda} = \frac{D}{\lambda} - \sum_{i=1}^{D}x_i^{\beta} - (n-D)C^{\beta}. \tag{3.5}$$

To obtain the ML estimators of the two parameters, set (3.4) and (3.5) to zero and solve the resulting system of equations. Equating (3.5) with zero, we have

$$\frac{D}{\hat{\lambda}} - \sum_{i=1}^{D}x_i^{\hat{\beta}} - (n-D)C^{\hat{\beta}} = 0,$$

and this can be written as

$$\hat{\lambda} = \frac{D}{\sum_{i=1}^{D}x_i^{\hat{\beta}} + (n-D)C^{\hat{\beta}}} = A(\hat{\beta}). \tag{3.6}$$

Substituting from (3.6) into (3.4) and setting it to zero, we have

$$\frac{D}{\hat{\beta}} + \sum_{i=1}^{D}\ln(x_i) - A(\hat{\beta})\sum_{i=1}^{D}x_i^{\hat{\beta}}\ln(x_i) - A(\hat{\beta})(n-D)C^{\hat{\beta}}\ln(C) = 0. \tag{3.7}$$

The ML estimator of β may be obtained iteratively by calculating the ML estimator from (3.7) and then substituting it into (3.6) to compute the ML estimator of λ. Hence, based on the invariance property, the ML estimator of E_1, E_2, E_3, and E_4 are produced by inserting $\hat{\beta}$ and $\hat{\lambda}$ in Eqs (2.2)–(2.5), respectively, as follows:

$$\hat{E}_1 = -\ln(\hat{\beta}\hat{\lambda}) + \frac{(\hat{\beta}-1)}{\hat{\beta}}[\gamma + \ln(\hat{\lambda})] + 1, \tag{3.8}$$

$$\hat{E}_2 = \frac{1}{\zeta-1}\Big(1 - (\frac{\hat{\beta}}{\hat{\lambda}^{\frac{-1}{\hat{\beta}}}})^{\zeta-1}\frac{\Gamma(\zeta(1-\frac{1}{\hat{\beta}})+\frac{1}{\hat{\beta}})}{\zeta^{\zeta(1-\frac{1}{\hat{\beta}})+\frac{1}{\hat{\beta}}}}\Big), \tag{3.9}$$

$$\hat{E}_3 = \frac{\zeta}{1-\zeta}\Big[\Big((\frac{\hat{\beta}}{\hat{\lambda}^{\frac{-1}{\hat{\beta}}}})^{\zeta-1}\frac{\Gamma(\zeta(1-\frac{1}{\hat{\beta}})+\frac{1}{\hat{\beta}})}{\zeta^{\zeta(1-\frac{1}{\hat{\beta}})+\frac{1}{\hat{\beta}}}}\Big)^{\frac{1}{\zeta}} - 1\Big], \tag{3.10}$$

and

$$\hat{E}_4 = \frac{1}{2^{1-\zeta} - 1}\Big[\Big((\frac{\hat{\beta}}{\hat{\lambda}^{\frac{-1}{\hat{\beta}}}})^{\zeta-1}\frac{\Gamma(\zeta(1-\frac{1}{\hat{\beta}})+\frac{1}{\hat{\beta}})}{\zeta^{\zeta(1-\frac{1}{\hat{\beta}})+\frac{1}{\hat{\beta}}}}\Big)^{\zeta} - 1\Big]. \tag{3.11}$$

To compute the ACIs, the asymptotic variance-covariance matrix (AV-CM) of $\hat{\beta}$ and $\hat{\lambda}$ can be obtained by inverting the Fisher information matrix (FM) defined as the negative expected value of the second derivative of the log-likelihood function.

$$\hat{I}(\hat{\beta}, \hat{\lambda}) = -\mathbb{E}\begin{bmatrix} \frac{\partial^2 l}{\partial \beta^2} & \frac{\partial^2 l}{\partial \beta \partial \lambda} \\ \frac{\partial^2 l}{\partial \lambda \partial \beta} & \frac{\partial^2 l}{\partial \lambda^2} \end{bmatrix}_{(\hat{\beta},\hat{\lambda})}.$$

It is difficult to find exact closed-form solutions for the given requirements. Therefore, the observed Fisher information matrix $\hat{I}(\hat{\beta}, \hat{\lambda})$, obtained by removing the expectation operator $\mathbb{E}$, will be used to construct ACIs for the parameters, see [31]. The second partial derivative of the log-likelihood function from the entries of the observed matrix is represented by

$$\hat{I}(\hat{\beta}, \hat{\lambda}) = -\begin{bmatrix} \frac{\partial^2 l}{\partial \beta^2} & \frac{\partial^2 l}{\partial \beta \partial \lambda} \\ \frac{\partial^2 l}{\partial \lambda \partial \beta} & \frac{\partial^2 l}{\partial \lambda^2} \end{bmatrix}_{(\hat{\beta},\hat{\lambda})}.$$

The elements of the FM are obtained as follows:

$$\frac{\partial^2 l}{\partial \beta^2} = \frac{-D}{\beta^2} - (n-D)\lambda C^{\beta} ln(C)^2 - \lambda \sum_{i=1}^{D} x_i^{\beta}(\ln(x_i))^2,$$

$$\frac{\partial^2 l}{\partial \lambda^2} = \frac{-D}{\lambda^2},$$

$$\frac{\partial^2 l}{\partial \lambda \partial \beta} = -\sum_{i=1}^{D} x_i^{\beta}\ln(x_i) - (n-D)\lambda C^{\beta}\ln(C).$$

To construct the AV-CM for the ML estimators, the observed FM is inverted as follows:

$$\big[\hat{V}\big] = \hat{I}^{-1}(\hat{\beta}, \hat{\lambda}) = \begin{bmatrix} -\frac{\partial^2 l}{\partial \beta^2} & -\frac{\partial^2 l}{\partial \beta \partial \lambda} \\ -\frac{\partial^2 l}{\partial \lambda \partial \beta} & -\frac{\partial^2 l}{\partial \lambda^2} \end{bmatrix}^{-1}_{(\hat{\beta},\hat{\lambda})} = \begin{bmatrix} -var(\hat{\beta}) & -cov(\hat{\beta},\hat{\lambda}) \\ -cov(\hat{\beta},\hat{\lambda}) & -var(\hat{\lambda}) \end{bmatrix}.$$

The two-sided $100(1-\omega)\%$ ACI for β and λ can be constructed based on the asymptotic normality conditions of the ML estimators as:

$$(\hat{\beta} \pm Z_{\frac{\omega}{2}}\sqrt{var(\hat{\beta})}) \qquad (\hat{\lambda} \pm Z_{\frac{\omega}{2}}\sqrt{var(\hat{\lambda})}),$$

where $Z_{\frac{\omega}{2}}$ is an upper $\frac{\omega}{2}$ % of the standard normal distribution.

Additionally, we must ascertain the variations of the entropy measures to derive the ACI. We employ the delta method described in [32] to obtain a rough estimate of the entropy measures. This method is a statistical technique used to approximate the distribution of a nonlinear function of random variables using derivatives. This method is based on the principle that a nonlinear function can be approximated using its first derivative, allowing for the estimation of the variance of complex statistics.

This methodology allows us to approximate the variance of E_1, E_2, E_3, and E_4 as follows:

$$\text{var}(\hat{E}_1) = [\nabla_1 \hat{E}_1]^T [\hat{V}][\nabla_1 \hat{E}_1], \quad \text{var}(\hat{E}_2) = [\nabla_2 \hat{E}_2]^T [\hat{V}][\nabla_2 \hat{E}_2],$$

$$\text{var}(\hat{E}_3) = [\nabla_3 \hat{E}_3]^T [\hat{V}][\nabla_3 \hat{E}_3], \quad \text{var}(\hat{E}_4) = [\nabla_4 \hat{E}_4]^T [\hat{V}][\nabla_4 \hat{E}_4],$$

where $\nabla_i \hat{E}_i = \left(\frac{\partial E_i}{\partial \beta}, \frac{\partial E_i}{\partial \lambda}\right)$.

$$\frac{\partial E_1}{\partial \beta} = \frac{-1}{\beta} + \frac{1}{\beta^2}[\gamma + \ln(\lambda)], \qquad \frac{\partial E_1}{\partial \lambda} = \frac{-1}{\beta \lambda},$$

$$\frac{\partial E_2}{\partial \lambda} = \lambda^{\frac{1}{\beta}-1} \left(\frac{\beta}{\lambda^{-\frac{1}{\beta}}}\right)^{\zeta-2} \frac{\Gamma\left(\zeta\left(1-\frac{1}{\beta}\right)+\frac{1}{\beta}\right)}{\zeta^{\zeta\left(1-\frac{1}{\beta}\right)+\frac{1}{\beta}}},$$

$$\frac{\partial E_2}{\partial \beta} = \frac{\zeta^{\frac{\zeta(1-\beta)-1}{\beta}} \Gamma\left(\frac{1+(-1+\beta)\zeta}{\beta}\right)\left(\beta - \log(\zeta) - \log(\lambda) + \psi\left(\frac{1+(-1+\beta)\zeta}{\beta}\right)\right)}{\beta^{3-\zeta} \lambda^{\frac{1-\zeta}{\beta}}},$$

$$\frac{\partial E_3}{d\lambda} = \lambda^{\frac{1}{\beta}-1} \frac{\zeta(\zeta-1}{1-\zeta} \left(\frac{\beta}{\lambda^{-\frac{1}{\beta}}}\right)^{\zeta-2} \frac{\Gamma\left(\zeta\left(1-\frac{1}{\beta}\right)+\frac{1}{\beta}\right)^{\frac{1}{\zeta}}}{\zeta^{\zeta\left(1-\frac{1}{\beta}\right)+\frac{1}{\beta}}},$$

$$\frac{\partial E_3}{d\beta} = \frac{\zeta^{\frac{-1+\zeta-\beta\zeta}{\beta\zeta}} \Gamma\left(\frac{1+(-1+\beta)\zeta}{\beta}\right)^{\frac{1}{\zeta}} \left(-\beta\zeta + \zeta \log(\lambda) + \log(\zeta) - \psi\left(\frac{1+(-1+\beta)\zeta}{\beta}\right)\right)}{\beta^{3-\zeta} \lambda^{\frac{1-\zeta}{\beta}}},$$

$$\frac{\partial E_4}{d\lambda} = \lambda^{\frac{1}{\beta}-1} \frac{\zeta-1}{2^{1-\zeta}-1} \left(\frac{\beta}{\lambda^{-\frac{1}{\beta}}}\right)^{\zeta-2} \frac{\Gamma\left(\zeta\left(1-\frac{1}{\beta}\right)+\frac{1}{\beta}\right)^{\zeta}}{\zeta^{\zeta\left(1-\frac{1}{\beta}\right)+\frac{1}{\beta}}},$$

$$\frac{\partial E_4}{\partial \beta} = \frac{2^{\zeta}(-1+\zeta)\zeta^{\frac{-\zeta+\zeta^2-\beta\zeta^2}{\beta}} \Gamma\left(\frac{1+(-1+\beta)\zeta}{\beta}\right)^{\zeta} \left(-\beta + \log(\lambda) + \zeta \log(\zeta) - \zeta\psi\left(\frac{1+(-1+\beta)\zeta}{\beta}\right)\right)}{(-2+2^{\zeta})\beta^{3-\zeta} \lambda^{\frac{1-\zeta}{\beta}}},$$

where $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$ is the digamma function.

The delta method is employed because it effectively approximates the distribution, simplifying the variance computation for the entropy measures. It also facilitates estimating uncertainty and constructing confidence intervals, especially when using ML estimates. Thus, the two-sided $100(1-\omega)\%$ ACI for E_1, E_2, E_3, and E_4 can be constructed as follows:

$$\left(\hat{E}_i \pm Z_{\frac{\omega}{2}} \sqrt{\text{var}(\hat{E}_i)}\right), \quad i = 1, 2, 3, 4.$$

3.2. Bayesian estimator

Since both β and λ are unknown and lack a natural conjugate bivariate prior distribution, independent gamma distributions are assumed for each. Specifically, β is assigned a gamma distribution with parameters (a_1, b_1) and λ with parameters (a_2, b_2). The means of these distributions are given by $\frac{a_1}{b_1}$ for β and $\frac{a_2}{b_2}$ for λ.

The joint prior distribution is as follows:

$$\pi(\beta,\lambda) = \frac{1}{\Gamma(\beta)\Gamma(\lambda)}\beta^{a_1-1}\lambda^{a_2-1}e^{-(b_1\beta+b_2\lambda)}, \tag{3.12}$$

where a_1, b_1, a_2, and b_2 are positive hyperparameters that represent prior knowledge.

The posterior distribution is given by

$$\pi^*(\beta,\lambda|\underline{x}) = M_1\beta^{D+a_1-1}\lambda^{D+a_2-1}e^{-(b_1\beta+b_2\lambda)}[\prod_{i=1}^{D} x_i^{\beta-1}e^{-\lambda x_i^{\beta}}][e^{-(n-D)\lambda C^{\beta}}], \tag{3.13}$$

which can be written as:

$$\pi^*(\beta,\lambda|\underline{x}) = M_1\beta^{D+a_1-1}\lambda^{D+a_2-1}e^{-b_1\beta}e^{-b_2\lambda}\times[e^{(\beta-1)\sum_{i=1}^{D}\ln(x_i)-\lambda\sum_{i=1}^{D}x_i^{\beta})}][e^{-(n-D)\lambda C^{\beta}}], \tag{3.14}$$

and further simplified as:

$$\pi^*(\beta,\lambda|\underline{x}) = M_1\beta^{D+a_1-1}\lambda^{D+a_2-1}e^{-\sum_{i=1}^{D}\ln(x_i)}e^{-\beta(b_1-\sum_{i=1}^{D}\ln(x_i))}e^{-\lambda[b_2+\sum_{i=1}^{D}x_i^{\beta}+(n-D)C^{\beta}]}, \tag{3.15}$$

where

$$M_1^{-1} = \int_0^{\infty}\int_0^{\infty} L(\underline{x}|\beta,\lambda)\pi(\beta,\lambda)d\beta d\lambda$$

is the normalizing constant.

The marginal posterior distributions of β and λ are given by:

(1) Marginal posterior distribution of β:

$$\pi_1^*(\beta|\underline{x}) \propto \beta^{D+a_1-1}e^{-\beta(b_1-\sum_{i=1}^{D}\ln(x_i))}\times\int_0^{\infty}\lambda^{D+a_2-1}e^{-\lambda[b_2+\sum_{i=1}^{D}x_i^{\beta}+(n-D)C^{\beta}]}d\lambda, \tag{3.16}$$

(2) Marginal posterior distribution of λ:

$$\pi_2^*(\lambda|\underline{x}) \propto \lambda^{D+a_2-1}e^{-\lambda b_2}\times\int_0^{\infty}\beta^{D+a_1-1}e^{-\lambda[\sum_{i=1}^{D}x_i^{\beta}+(n-D)C^{\beta}]}e^{-\beta(b_1-\sum_{i=1}^{D}\ln(x_i))}d\beta. \tag{3.17}$$

From the expressions in (3.15)–(3.17), the conditional posterior distribution of λ given β is:

$$\pi_1^*(\lambda|\beta,\underline{x}) \propto \lambda^{D+a_2-1}e^{-\lambda[b_2+\sum_{i=1}^{D}x_i^{\beta}+(n-D)C^{\beta}]}. \tag{3.18}$$

As a result, the gamma distribution with shape parameter $(D + a_2 - 1)$ and scale parameter $(b_2 + \sum_{i=1}^{D}x_i^{\beta} + (n-D)C^{\beta})$ is the posterior density function of $\pi_1^*(\lambda|\beta,\underline{x})$. So, any gamma-producing technique can be used to generate λ samples with ease. One cannot sample directly from $\pi_2^*(\beta|\lambda,\underline{x})$ as it cannot be analytically reduced to well-known distributions. The MCMC method-based M-H algorithm is employed to get an estimate.

3.3. Loss functions

One of the accuracy metrics used in the Bayesian estimating process is the loss function, which is defined as the amount of loss incurred while making a Bayesian judgment for an unknown parameter. It is a measurement of the discrepancy between this parameter's estimated and actual values. Generally speaking, loss functions may be divided into two primary categories based on symmetry criteria: First, there are SLOFs, which assume that the loss incurred in a positive direction is equal to the loss incurred in a negative direction. The second class of loss functions is known as ASLOFs; in this class, it is assumed that the amount of loss under the Bayes decision in both the positive and negative directions need not be equal.

This sub-section examines the Bayesian estimators of different entropy measures for both SLOFs and ASLOFs. The squared error (SE) LOF is one of the most extensively utilized SLOFs. This kind is appropriate for reducing the mean squared error since it penalizes greater mistakes more severely than smaller ones. The SE LOF is provided as below:

$$L_1(\phi, \delta) = (\delta - \phi)^2,$$

where δ is an estimator of ϕ. In this situation, the Bayesian estimator is calculated as follows:

$$\hat{\phi}_{SE} = E(\phi|data). \tag{3.19}$$

In the context of ASLOFs, the linear-exponential (LINEX) LOF exhibits less sensitivity to outliers than the SE LOF, striking a compromise between bias and variance. The LINEX LOF is defined as follows:

$$L_2(\phi, \delta) = e^{-q(\delta-\phi)} - q(\delta - \phi) - 1,$$

where q represents the sign that indicates the direction of asymmetry. Under the LINEX LOF, the Bayesian estimator is provided by

$$\hat{\phi}_{LINEX} = \frac{-1}{q} \ln[E(e^{-q\phi}|data)]. \tag{3.20}$$

Another ASLOF is the general entropy (GE) LOF which provides a measure of dissimilarity between probability distributions and is used to focus on maximizing the similarity between predicted and actual distributions rather than minimizing prediction errors. The GE LOF has the following formula:

$$L_3(\phi, \delta) = (\frac{\delta}{\phi})^q - q \log(\frac{\delta}{\phi}) - 1.$$

The Bayesian estimator via GE LOF is:

$$\hat{\phi}_{GE} = [E(\phi^{-q}|data)]^{-\frac{1}{q}}. \tag{3.21}$$

Now the entropy Bayesian estimators via SE, LINEX, and GE LOFs, are as follows:

$$\hat{J}_{SE} = M_1 \int_0^\infty \int_0^\infty J\beta^{D+a_1-1}\lambda^{D+a_2-1}e^{-b_1\beta} \times e^{-b_2\lambda+K_i(\beta,\lambda,x_i)}[e^{-(n-D)\lambda c^\beta}]d\beta d\lambda, \tag{3.22}$$

$$\hat{J}_{LINEX} = \frac{-1}{q}\ln[M_1\int_0^{\infty}\int_0^{\infty} e^{-qJ}\beta^{D+a_1-1}\lambda^{D+a_2-1}e^{-b_1\beta}\times e^{-b_2\lambda+K_i(\beta,\lambda,x_i)}(e^{-(n-D)\lambda c^{\beta}})d\beta d\lambda], \tag{3.23}$$

$$\hat{J}_{GE} = [M_1\int_0^{\infty}\int_0^{\infty}(J)^{-q}\beta^{D+a_1-1}\lambda^{D+a_2-1}e^{-b_1\beta}\times e^{-b_2\lambda+K_i(\beta,\lambda,x_i)}(e^{-(n-D)\lambda c^{\beta}})d\beta d\lambda]^{\frac{-1}{q}}, \tag{3.24}$$

where

$$K_i(\beta,\lambda,x_i) = ((\beta-1)\sum_{i=1}^{D}\ln(x_i) - \lambda\sum_{i=1}^{D}x_i^{\beta}),$$

M_1 is the normalizing constant, and to calculate the different entropy measures, we put $J = E_1, E_2, E_3$, and E_4. It is important to note that all Bayesian entropy estimators are formulated as a ratio of two integrals. These integrals cannot be simplified or calculated directly. Therefore, to compute these estimators and to construct their HPD intervals, the MCMC method is employed.

3.4. MCMC method

The behavior of the ML estimates (MLEs) and Bayesian estimates (BEs) for the different measures of entropy for the Weibull distribution was investigated numerically using various LOFs. Bayesian estimators were calculated using the M-H algorithm under the SE, LINEX, and GE LOFs. Samples were created from the posterior distributions using the MCMC method. The M-H algorithm proceeds as follows:

(1) Put $\beta_0 = \hat{\beta}$.
(2) Let $l = 1$.
(3) $\lambda^{(l)}$ is obtained from gamma $\pi_1^*(\lambda|\beta_{l-1}, \underline{x})$.
(4) Generate $\beta^{(l)}$ from $\pi_2^*(\beta|\lambda^l, \underline{x})$ using the same procedure of Metropolis-Hastings [33] and use the normal distribution as a proposal distribution.
(5) Put $l = l + 1$.
(6) Calculate $\beta^{(t)}$ and $\lambda^{(t)}$.
(7) Repeat Steps $3-6$ N times.
(8) Acquire the BEs of β and λ and obtain the entropy measure concerning the LOFs.

To compute the BEs, we implemented the MCMC algorithm with a dataset of $N = 10000$ observations. Initially, we used the MLEs for the unknown parameters λ and β as starting values for the MCMC algorithm. However, it is important to note that the initial values may differ from the final converged values. Therefore, we discarded the first $M = 1000$ values to account for this discrepancy. To verify the convergence of the MCMC samples and determine the burn-in period, we conducted diagnostic tests, examined trace plots, and assessed posterior density plots for various parameters and censoring schemes. These analyses helped us identify the burn-in period and ensure the convergence of the MCMC algorithm before analyzing the data further.

The method proposed by Chen and Shao [34] is employed to construct the $100(1-\omega)\%$ HPD credible intervals for entropy measures.

From Figures 1 and 2, the estimation demonstrates that all of the generated posteriors closely match the theoretical posterior density functions, and it is evident that a big MCMC loop yields results that are comparable and more effective than smaller loops. These plots have no significant lengthy upward or downward trends, which are convergence markers.

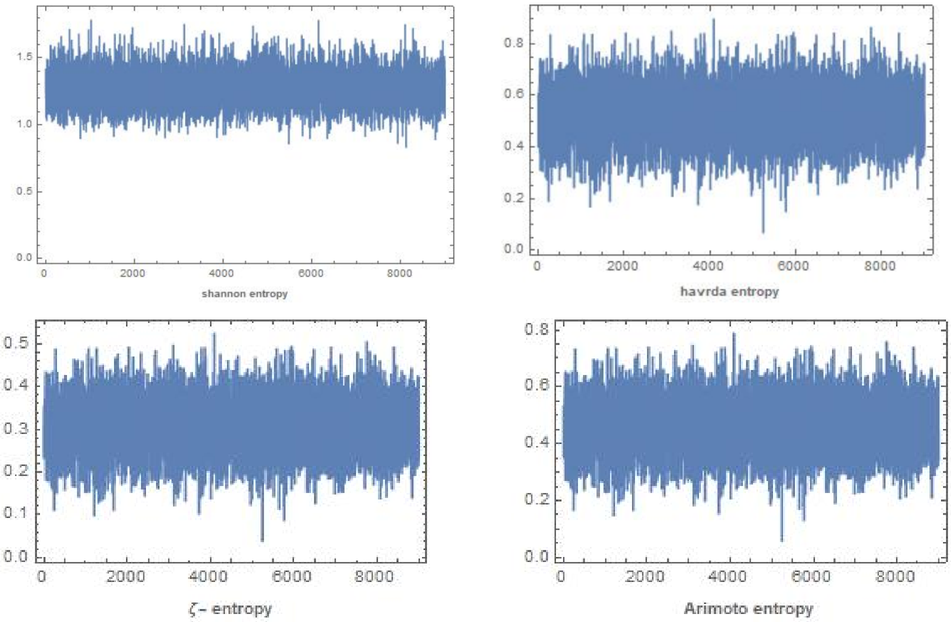

Figure 1. The posterior sample trace plots for different measures of entropy.

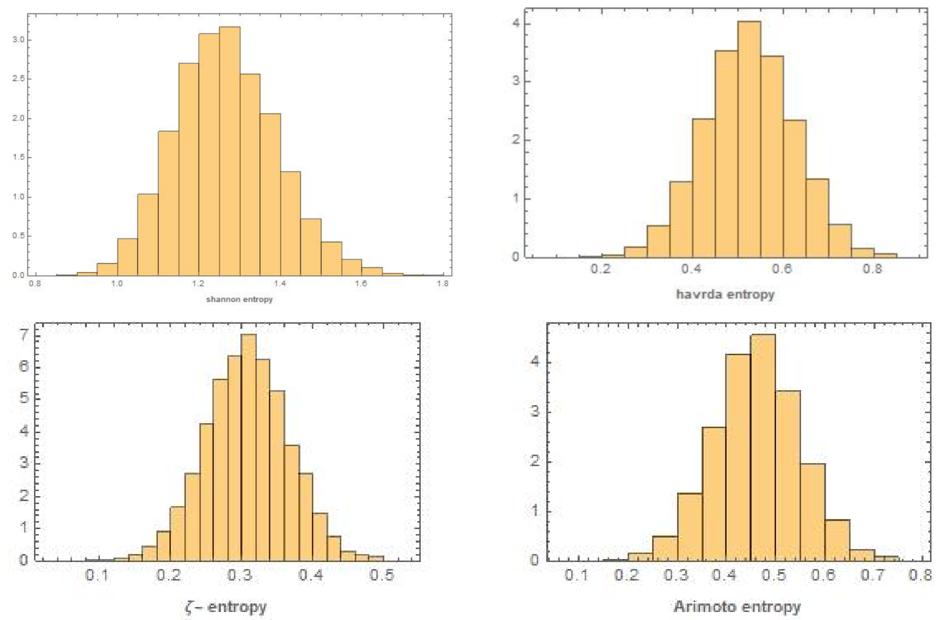

Figure 2. The posterior sample histograms for different measures of entropy.

4. Simulation analysis and outcomes

This section is dedicated to evaluating the performance of all previously suggested estimators for entropy measures. To achieve this, a simulation study is conducted for estimation purposes. Additionally, an analysis of actual data is provided to further support the study.

4.1. Simulation study

The MCMC simulations were performed to compare the estimates using Mathematica 12. Using the following procedure, the simulation research is carried out.

(1) A random sample of sizes $n = 150$ and 250, with true parameter values

$$\lambda = 2.5, \beta = 1.5,$$

and entropy values

$$E_1 = 0.267853, E_2 = 0.183147, E_3 = 0.27472, \text{ and } E_4 = 0.312651,$$

was generated from the Weibull distribution using the quantile function. The MLE of β was obtained using an iterative technique based on Eq (3.7). Subsequently, the MLE of λ was derived by substituting the estimated $\hat{\beta}$ into Eq (3.6).

(2) Using the invariance property, the MLEs for the entropy values E_1, E_2, E_3, and E_4 were calculated by inserting $\hat{\beta}$ and $\hat{\lambda}$ into Eqs (3.8)–(3.11), respectively. After obtaining these estimates, the 95% ACIs, ALs, and CPs were computed.

(3) The BEs were then calculated using the proposed LOF through the M-H algorithm, as described in subsection (3.4). The values of q were assumed to be $(-4, 4)$. For the different measures of entropy, the 95% HPD intervals, ALs, and CPs were calculated at $\zeta = 1.5$ and 0.5.

(4) To enhance the stability of the model and simplify its complexity, fixed values for the hyperparameters were chosen as

$$a_1 = 0.6, b_1 = 1.2, a_2 = 2, \text{ and } b_2 = 0.4,$$

based on prior evidence supporting these values. The parameters n, r, T_1, and T_2 were selected according to Table 1, and steps (1–4) were repeated 1000 times. The MSEs of the various entropy estimates were then computed. The outcomes of the simulation study are recorded in Tables 2–9.

Table 1. Selected values of n, r, T_1, and T_2.

n	r	T_1	T_2
250	200	(2, 5)	7
150	120	(2, 5)	7
250	(200, 170)	0.2	1.2
150	(120, 80)	0.2	1.2
250	200	1.5	(3, 7)
150	120	1.5	(3, 7)

Table 2. Different entropy estimates and associated MSE at $T_1 = 0.2$ and $T_2 = 1.2$ under different values of r at $\zeta = 1.5$.

Entropy	n	r	MLE	SE	LINEX		GE	
					$q = (-4)$	$q = (4)$	$q = (-4)$	$q = (4)$
E_1	150	120	1.36712	1.51529	1.57698	1.4603	1.54341	1.46821
			0.60911	0.61993	0.78749	0.47828	0.69251	0.5022
E_2			0.19348	0.14388	0.15158	0.13615	0.17786	0.01169
			0.0026	0.00428	0.00363	0.00507	0.00157	0.03013
E_3			0.29022	0.21582	0.23314	0.19842	0.26679	0.01753
			0.0059	0.00964	0.00754	0.01239	0.00353	0.0678
E_4			0.33029	0.24562	0.26804	0.22307	0.30362	0.01995
			0.0077	0.01249	0.00946	0.0166	0.00457	0.08781
E_1	150	80	1.45558	1.59925	1.70715	1.51007	1.64329	1.52503
			.43429	0.88746	1.21708	0.63845	1.01106	0.68956
E_2			0.14164	0.12597	0.13823	0.1136	0.18205	0.00373
			0.00183	0.00796	0.00644	0.00983	0.00175	0.03244
E_3			0.21247	0.18895	0.2165	0.16107	0.27307	0.00559
			0.0041	0.01791	0.01305	0.02451	0.00394	0.07299
E_4			0.2418	0.21504	0.2507	0.17889	0.31078	0.00636
			0.00532	0.0232	0.0162	0.03311	0.0051	0.09454
E_1	250	200	1.41556	1.47743	1.51066	1.44628	1.49357	1.45048
			0.42562	0.49233	0.57613	0.41596	0.53207	0.42718
E_2			0.18265	0.16257	0.1671	0.15803	0.1816	0.05219
			0.0188	0.0166	0.0147	0.019	0.0093	0.02149
E_3			0.27397	0.24386	0.25404	0.23364	0.2724	0.07828
			0.0423	0.0374	0.0312	0.0457	0.021	0.04836
E_4			0.3118	0.27753	0.29072	0.26429	0.31001	0.08909
			0.00547	0.0484	0.0395	0.0609	0.0272	0.06263
E_1	250	170	1.43521	1.02243	1.03456	1.01037	1.03126	1.00721
			0.7071	0.57443	0.59289	0.55637	0.58775	0.55186
E_2			0.17722	0.46139	0.46597	0.45683	0.46873	0.44847
			0.00349	0.0779	0.08047	0.07539	0.08202	0.07092
E_3			0.26582	0.69208	0.70242	0.68185	0.7031	0.67271
			0.00784	0.17527	0.184	0.16684	0.18454	0.15957
E_4			0.30253	0.78764	0.80104	0.77439	0.80017	0.76559
			0.01016	0.22701	0.23991	0.21462	0.23902	0.20668

Table 3. Different entropy estimates and associated MSE at $T_2 = 7$ under different values of T_1 at $\zeta = 1.5$.

Entropy	n	T_1	MLE	SE	LINEX		GE	
					$q = (-4)$	$q = (4)$	$q = (-4)$	$q = (4)$
E_1	150	2	1.42207	1.10074	1.12712	1.07561	1.11805	1.07101
			0.74852	0.71389	0.75988	0.67162	0.74317	0.66525
E_2			0.18175	0.33537	0.3417	0.32909	0.34903	0.304
			0.00373	0.02404	0.02599	0.02218	0.02829	0.01683
E_3			0.27262	0.50306	0.51733	0.48895	0.52355,	0.45599
			0.00838	0.0541	0.06077	0.0479	0.06366	0.03786
E_4			0.31026	0.57252	0.59102	0.55425	0.59583	0.51895
			0.01086	0.07006	0.07995	0.06098	0.08245	0.04904
E_1	150	5	1.4257	1.09124	1.11732	1.06638	1.10857	1.06145
			0.35491	0.696	0.74062	0.65491	0.72485	0.64805
E_2			0.1727	0.33467	0.34102	0.32838	0.34835	0.30641
			0.00347	0.02359	0.02554	0.02173	0.02785	0.01618
E_3			0.25905	0.50201	0.51631	0.48787	0.52252	0.45961
			0.0078	0.05307	0.05974	0.04687	0.06266	0.0364
E_4			0.29482	0.57132	0.58986	0.55302	0.59467	0.52307
			0.01011	0.06874	0.07862	0.05965	0.08116	0.04714
E_1	250	2	1.42515	1.08207	1.09661	1.06792	1.09196	1.06535
			0.5497	0.67419	0.69842	0.65104	0.6904	0.64724
E_2			0.1791	0.35346	0.35724	0.34969	0.36132	0.33912
			0.00256	0.02935	0.03065	0.02808	0.03207	0.02472
E_3			0.26866	0.53018	0.53871	0.52171	0.54198	0.50868
			0.00575	0.06603	0.07045	0.06178	0.07215	0.05562
E_4			0.30575	0.60339	0.61444	0.59241	0.61682	0.57892
			0.00745	0.08552	0.09206	0.07928	0.09345	0.07204
E_1	250	5	1.43201	1.074	1.08855	1.05981	1.08398	1.0570
			0.36655	0.66264	0.68669	0.63962	0.67882	0.63567
E_2			0.16956	0.35109	0.35487	0.34733	0.35901	0.33668
			0.00258	0.02859	0.02987	0.02734	0.03129	0.02401
E_3			0.25435	0.52664	0.53516	0.51817	0.53851	0.50502
			0.00581	0.06432	0.06868	0.06014	0.07039	0.05403
E_4			0.28946	0.59935	0.6104	0.58839	0.61286	0.57474
			0.00752	0.08331	0.08975	0.07717	0.09117	0.06998

Table 4. Different entropy estimates and associated MSE at $T_1 = 1.5$ under different values of T_2 at $\zeta = 1.5$.

Entropy	n	T_2	MLE	SE	LINEX		GE	
					$q = (-4)$	$q = (4)$	$q = (-4)$	$q = (4)$
E_1	150	3	1.44174	1.10164	1.12749	1.07697	1.11865	1.07244
			0.69463	0.71086	0.7555	0.66966	0.73956	0.66315
E_2			0.17451	0.34042	0.34679	0.33409	0.35394	0.31117
			0.00411	0.0254	0.02744	0.02346	0.02978	0.01788
E_3			0.26176	0.51063	0.525	0.49642	0.53092	0.46676
			0.00925	0.05716	0.06412	0.05069	0.067	0.04023
E_4			0.2979	0.58113	0.59976	0.56274	0.60422	0.5312
			0.01198	0.07404	0.08435	0.06454	0.08678	0.05211
E_1	150	7	1.45241	1.09892	1.12479	1.07423	1.116	1.06958
			0.41651	0.70499	0.74941	0.664	0.73368	0.65728
E_2			0.16415	0.3389	0.34527	0.33257	0.35247	0.31123
			0.0043	0.02485	0.02687	0.02293	0.02921	0.01722
E_3			0.24623	0.50835	0.52272	0.49412	0.5287	0.46684
			0.00967	0.05592	0.06282	0.0495	0.06572	0.03875
E_4			0.28022	0.57853	0.59716	0.56012	0.6017	0.5313
			0.01253	0.07243	0.08265	0.0630	0.08512	0.05019
E_1	250	3	1.43076	1.0857	1.10019	1.07159	1.09552	1.06909
			0.36052	0.67711	0.70126	0.654	0.69327	0.65021
E_2			0.1783	0.35307	0.35684	0.34933	0.36091	0.33882
			0.00213	0.02927	0.03056	0.02801	0.03197	0.02468
E_3			0.26745	0.52961	0.5381	0.52119	0.54137	0.50823
			0.00479	0.06585	0.07024	0.06164	0.07194	0.05553
E_4			0.30438	0.60274	0.61373	0.59183	0.61612	0.5784
			0.00621	0.0853	0.09178	0.0791	0.09317	0.07192
E_1	250	7	1.43755	1.09109	1.10581	1.07676	1.10102	1.07429
			0.35688	0.68555	0.71023	0.66196	0.702	0.6582
E_2			0.17673	0.35121	0.35498	0.34746	0.35909	0.33684
			0.00222	0.02865	0.02993	0.02741	0.03134	0.02408
E_3			0.2651	0.52681	0.53531	0.51838	0.53864	0.50526
			0.00499	0.06447	0.06882	0.0603	0.07052	0.05418
E_4			0.3017	0.59955	0.61056	0.58863	0.61301	0.57502
			0.00646	0.0835	0.08993	0.07737	0.09134	0.07018

Table 5. Different entropy estimates and associated MSE at $T_1 = 0.2$ and $T_2 = 1.2$ under different values of r at $\zeta = 0.5$.

Entropy	n	r	MLE	SE	LINEX		GE	
					$q = (-4)$	$q = (4)$	$q = (-4)$	$q = (4)$
E_1	150	80	1.44582	1.603	1.73021	1.50082	1.65364	1.51755
			0.40585	0.92552	1.32941	0.63469	1.06945	0.69625
E_2			0.44828	0.39371	0.41826	0.37158	0.43654	0.25012
			0.00698	0.01149	0.00913	0.01474	0.00655	0.06768
E_3			0.22414	0.19686	0.20282	0.19119	0.21827	0.12506
			0.00174	0.00287	0.00255	0.00326	0.00164	0.01692
E_4			0.54112	0.47525	0.51149	0.4433	0.52695	0.30192
			0.01017	0.01675	0.0128	0.02258	0.00955	0.09861
E_1	150	120	1.43487	1.5157	1.57724	1.46066	1.54378	1.46853
			0.37627	0.61997	0.78671	0.47843	0.69239	0.50217
E_2			0.44775	0.41609	0.42997	0.40302	0.43997	0.36094
			0.00522	0.00589	0.00495	0.00714	0.00406	0.01896
E_3			0.22388	0.20805	0.21146	0.20473	0.21998	0.18047
			0.00131	0.00147	0.00134	0.00162	0.00101	0.00474
E_4			0.54049	0.50227	0.52262	0.48332	0.53109	0.43569
			0.00761	0.00859	0.00699	0.01083	0.00591	0.02762
E_1	250	170	1.44429	1.01809	1.03028	1.00595	1.027	1.0027
			0.39576	0.5683	0.58679	0.5502	0.58168	0.5456
E_2			0.43933	0.99617	1.02445	0.97078	1.0163	0.96342
			0.00699	0.29079	0.32224	0.26396	0.31289	0.2566
E_3			0.21966	0.49809	0.50495	0.49158	0.50815	0.48171
			0.00175	0.0727	0.07646	0.06922	0.07822	0.06415
E_4			0.53032	1.20248	1.24422	1.16586	1.22679	1.16296
			0.01019	0.42372	0.48007	0.37722	0.45592	0.3739
E_1	250	200	1.42666	1.07694	1.09132	1.06292	1.08678	1.06027
			0.5947	0.46296	0.48666	0.44027	0.47897	0.43631
E_2			0.4528	0.75476	0.76948	0.74095	0.76889	0.73139
			0.00397	0.08851	0.09748	0.08049	0.09706	0.07525
E_3			0.2264	0.37738	0.381	0.37387	0.38445	0.3657
			0.00099	0.02213	0.02322	0.0211	0.02427	0.01881
E_4			0.54658	0.91107	0.93268	0.89108	0.92814	0.88287
			0.00578	0.12897	0.14495	0.11502	0.14143	0.10965

Table 6. Different entropy estimates and associated MSE at $T_2 = 7$ under different values of T_1at $\zeta = 0.5$.

Entropy	n	T_1	MLE	SE	LINEX		GE	
					$q = (-4)$	$q = (4)$	$q = (-4)$	$q = (4)$
E_1	150	2	1.43621	1.08769	1.11309	1.0634	1.10461	1.05858
			0.38391	0.68731	0.73053	0.64734	0.71542	0.64047
E_2			0.44669	0.74201	0.76687	0.71968	0.76569	0.70293
			0.00752	0.08279	0.09771	0.07048	0.09677	0.06223
E_3			0.22335	0.37101	0.37704	0.36528	0.38284	0.35146
			0.00188	0.0207	0.02247	0.01909	0.02419	0.01556
E_4			0.5392	0.89569	0.93237	0.86347	0.92427	0.84851
			0.01096	0.12064	0.14747	0.09936	0.141	0.09068
E_1	150	5	1.45913	1.11113	1.13795	1.08562	1.12859	1.08114
			0.33642	0.63146	0.67869	0.58809	0.66129	0.58194
E_2			0.44216	0.7227	0.74673	0.70098	0.74627	0.68352
			0.00626	0.07177	0.08517	0.06067	0.08473	0.05273
E_3			0.22108	0.36134	0.36719	0.35579	0.37313	0.34175
			0.00157	0.01794	0.01953	0.01649	0.02118	0.01318
E_4			0.53374	0.87237	0.90781	0.84102	0.90082	0.82508
			0.00913	0.10457	0.12865	0.08539	0.12345	0.07684
E_1	250	2	1.42519	1.08909	1.10364	1.07491	1.09892	1.07243
			0.44944	0.68529	0.70973	0.66191	0.70155	0.65823
E_2			0.46296	0.75583	0.77058	0.74198	0.76997	0.73241
			0.00409	0.08983	0.0989	0.08172	0.09843	0.07648
E_3			0.23148	0.37791	0.38154	0.3744	0.38498	0.3662
			0.00102	0.02246	0.02356	0.02142	0.02461	0.01912
E_4			0.55884	0.91237	0.93402	0.89231	0.92943	0.88409
			0.00996	0.13089	0.14704	0.11679	0.14342	0.11143
E_1	250	5	1.43915	1.08467	1.09935	1.07035	1.09464	1.06776
			0.3813	0.67743	0.70194	0.65398	0.69382	0.65014
E_2			0.44599	0.74805	0.76262	0.73437	0.76216	0.72469
			0.0048	0.08519	0.09392	0.0774	0.09356	0.07223
E_3			0.22299	0.37402	0.37761	0.37055	0.38108	0.36234
			0.0012	0.0213	0.02236	0.0203	0.02339	0.01806
E_4			0.53835	0.90297	0.92436	0.88316	0.92001	0.87478
			0.007	0.12414	0.13968	0.11058	0.13633	0.10525

Table 7. Different entropy estimates and associated MSE at $T_1 = 1.5$ under different values of T_2 at $\zeta = 0.5$.

Entropy	n	T_2	MLE	SE	LINEX		GE	
					$q = (-4)$	$q = (4)$	$q = (-4)$	$q = (4)$
E_1	150	3	1.4405	1.10344	1.12943	1.07871	1.12043	1.07437
			0.924	0.71613	0.76153	0.67439	0.745	0.66824
E_2			0.17586	0.34206	0.34844	0.33572	0.35556	0.31302
			0.00344	0.02604	0.02809	0.02408	0.03042	0.01859
E_3			0.26378	0.5131	0.52749	0.49886	0.53337	0.46953
			0.00774	0.0586	0.06561	0.05207	0.06846	0.04182
E_4			0.3002	0.58394	0.60258	0.56552	0.607	0.53439
			0.01002	0.07589	0.08627	0.06632	0.08866	0.05418
E_1	150	7	1.44209	1.10927	1.13531	1.08448	1.12627	1.08014
			0.79386,	0.62524	0.67072	0.58337	0.65418	0.57719
E_2			0.17903	0.34127	0.34767	0.33492	0.35482	0.31269
			0.00275	0.02572	0.02778	0.02376	0.03012	0.01796
E_3			0.26854	0.51191	0.52633	0.49764	0.53222	0.46903
			0.00844	0.05787	0.0649	0.05134	0.06777	0.04041
E_4			0.30562	0.58259	0.60129	0.56412	0.60571	0.53379
			0.01093	0.07496	0.08536	0.06537	0.08778	0.05234
E_1	250	3	1.42874	1.08905	1.10354	1.07494	1.09884	1.07249
			0.55561	0.78152	0.77575	0.67835	0.79771	0.6946
E_2			0.18196	0.3545	0.35827	0.35075	0.36231	0.34031
			0.00272	0.02977	0.03107	0.0285	0.03248	0.02516
E_3			0.27294	0.53175	0.54025	0.52332	0.54347	0.51047
			0.00611	0.06697	0.0714	0.06272	0.07309	0.0566
E_4			0.31062	0.60517	0.61618	0.59426	0.61851	0.58095
			0.00792	0.08675	0.09329	0.08049	0.09466	0.07331
E_1	250	7	1.43871	1.094	1.10877	1.07963	1.10393	1.07719
			0.38096	0.6932	0.71814	0.66939	0.70972	0.66574
E_2			0.17714	0.3506	0.35439	0.34684	0.35853	0.33616
			0.00238	0.02842	0.0297	0.02718	0.03111	0.02385
E_3			0.26571	0.5259	0.53443	0.51744	0.53779	0.50424
			0.00535	0.06395	0.06829	0.05977	0.07001	0.05365
E_4			0.3024	0.59852	0.60956	0.58756	0.61204	0.57386
			0.00693	0.08282	0.08925	0.07669	0.09067	0.06949

Table 8. The 95% ACI and HPD intervals for entropy measures at $\zeta = 1.5$.

Entropy	n		ACI			HPD		
			Interval	AL	CP	Interval	AL	CP
E_1	150	$r = 120$	1.32397(1.54377)	0.2198	0.92	1.204 (1.8658)	0.6618	0.9
E_2			0.07005 (0.27138)	0.20133	0.96	0.02202 (0.26484)	0.24282	0.94
E_3			0.10508 (0.40707)	0.30199	0.96	0.03303 (0.39726)	0.36423	0.94
E_4			0.11958 (0.46327)	0.34369	0.96	0.03759 (0.45212)	0.41452	0.94
E_1	150	$r = 80$	1.31453 (1.58589)	0.27136	0.953	1.2048 (2.05354)	0.84873	0.935
E_2			0.04964 (0.29784)	0.2482	0.97	-0.02797 (0.27811)	0.30608	0.96
E_3			0.07447 (0.44676)	0.3723	0.97	-0.04195 (0.41717)	0.45912	0.96
E_4			0.08475 (0.50845)	0.4237	0.97	-0.04774 (0.47477)	0.52251	0.96
E_1	250	$r = 200$	1.33008 (1.50104)	0.17096	0.912	1.24128 (1.73582)	0.49454	0.89
E_2			0.10495 (0.26035)	0.1554	0.93	0.06898 (0.25542)	0.18644	0.925
E_3			0.15742 (0.39053)	0.23311	0.93	0.10347 (0.38313)	0.27967	0.925
E_4			0.17916 (0.44445)	0.26529	0.93	0.11776 (0.43603)	0.31828	0.925
E_1	250	$r = 170$	1.30827 (1.56216)	0.25389	0.95	0.87087 (1.17569)	0.30482	0.930
E_2			0.0614 (0.29303)	0.23163	0.97	0.36874 (0.5565)	0.18775	0.942
E_3			0.0921 (0.43955)	0.34745	0.97	0.55311 (0.83474)	0.28163	0.943
E_4			0.10481 (0.50024)	0.39543	0.97	0.62948(0.95)	0.32051	0.942
E_1	150	$T_1 = 5$	1.28738(1.55676)	0.26939	0.952	0.88645 (1.33038)	0.44393	0.92
E_2			0.05928 (0.30422)	0.24494	0.971	0.22668(0.4469)	0.22022	0.91
E_3			0.08892 (0.45633)	0.36741	0.971	0.34002(0.67035)	0.33033	0.912
E_4			0.10119 (0.51934)	0.41814	0.971	0.38697(0.76291)	0.37594	0.912
E_1	150	$T_1 = 2$	1.29124 (1.56016)	0.26892	0.94	0.87814 (1.32003)	0.44189	0.94
E_2			0.04976 (0.29565)	0.24589	0.98	0.22572 (0.44618)	0.22046	0.95
E_3			0.07464 (0.44347)	0.36883	0.98	0.33858 (0.66927)	0.3307	0.95
E_4			0.08494 (0.5047)	0.41976	0.98	0.38532 (0.76168)	0.37636	0.95
E_1	250	$T_1 = 5$	1.32001 (1.53028)	0.21027	0.88	0.92047 (1.25225)	0.33179	0.92
E_2			0.08331 (0.2749)	0.19158	0.92	0.26901 (0.43942)	0.17041	0.92
E_3			0.12497 (0.41235)	0.28738	0.92	0.40351 (0.65913)	0.25562	0.92
E_4			0.14222 (0.46928)	0.32705	0.92	0.45923(0.75014)	0.29091	0.92
E_1	250	$T_1 = 2$	1.32681 (1.53722)	0.21041	0.931	0.91238 (1.24443)	0.33205	0.94
E_2			0.07314 (0.26599)	0.19285	0.96	0.26679 (0.43719)	0.1704	0.94
E_3			0.10971 (0.39898)	0.28927	0.96	0.40018 (0.65578)	0.25561	0.94
E_4			0.12486 (0.45407)	0.32921	0.96	0.45543 (0.74633)	0.2909	0.94
E_1	150	$T_2 = 7$	1.30623 (1.57724)	0.27101	0.93	0.88947 (1.3293)	0.43983	0.92
E_2			0.05067 (0.29834)	0.24767	0.95	0.23143 (0.45226)	0.22083	0.94
E_3			0.076 (0.44751)	0.37151	0.95	0.34715 (0.67839)	0.33124	0.94
E_4			0.0865 (0.5093)	0.42281	0.95	0.39508 (0.77205)	0.37697	0.94
E_1	150	$T_2 = 3$	1.31687 (1.58794)	0.27108	0.93	0.88647 (1.32694)	0.44047	0.93
E_2			0.03946 (0.28884)	0.24937	0.94	0.22951 (0.45066)	0.22115	0.92
E_3			0.0592 (0.43326)	0.37406	0.94	0.34427 (0.67599)	0.33172	0.92
E_4			0.06737 (0.49308)	0.42571	0.94	0.39181 (0.76933)	0.37752	0.92
E_1	250	$T_2 = 7$	1.3256 (1.53593)	0.21033	0.93	0.92428 (1.25566)	0.33138	0.94
E_2			0.08241 (0.27419)	0.19178	0.95	0.26889 (0.43892)	0.17003	0.951
E_3			0.12362 (0.41129)	0.28767	0.95	0.40334 (0.65838)	0.25504	0.951
E_4			0.14068 (0.46808)	0.32739	0.95	0.45903(0.74928)	0.29025	0.951
E_1	250	$T_2 = 3$	1.33245 (1.54265)	0.2102	0.952	0.92852 (1.26253)	0.33402	0.972
E_2			0.08078 (0.27268)	0.1919	0.98	0.26694 (0.437)	0.17005	0.973
E_3			0.12118 (0.40902)	0.28784	0.98	0.40042 (0.6555)	0.25508	0.97
E_4			0.13791 (0.4655)	0.32759	0.98	0.4557 (0.746)	0.2903	0.972

Table 9. The 95% ACI and HPD intervals for entropy measures at $\zeta = 0.5$.

Entropy	n		ACI			HPD		
			Interval	AL	CP	Interval	AL	CP
E_1	150	$r = 120$	1.32451 (1.54522)	0.22071	0.912	1.20353 (1.86439)	0.66086	0.94
E_2			0.31014 (0.58536)	0.27522	0.93	0.26572 (0.58693)	0.3212	0.952
E_3			0.15507 (0.29268)	0.13761	0.93	0.13286 (0.29346)	0.1606	0.951
E_4			0.37438 (0.7066)	0.33222	0.93	0.32076 (0.70848)	0.38773	0.952
E_1	150	$r = 80$	1.30054 (1.59109)	0.29055	0.92	1.18112 (2.09479)	0.91367	0.94
E_2			0.2672 (0.62936)	0.36215	0.95	0.19976 (0.62205)	0.42229	0.96
E_3			0.1336(0.31468)	0.18108	0.95	0.09988 (0.31103)	0.21114	0.96
E_4			0.32254 (0.7597)	0.43716	0.95	0.24114 (0.75088)	0.50975	0.960
E_1	250	$r = 200$	1.32132 (1.53199)	0.21067	0.90	0.91591 (1.24612)	0.33021	0.938
E_2			0.32109 (0.58451)	0.26342	0.96	0.60057 (0.93122)	0.33065	0.941
E_3			0.16055 (0.29225)	0.13171	0.96	0.30029 (0.46561)	0.16532	0.94
E_4			0.38759 (0.70556)	0.31797	0.96	0.72496 (1.12409)	0.39913	0.941
E_1	250	$r = 170$	1.31742 (1.57116)	0.25374	0.951	0.86563 (1.17167)	0.30604	0.95
E_2			0.28194 (0.59672)	0.31479	0.98	0.78845 (1.24102)	0.45257	0.95
E_3			0.14097 (0.29836)	0.15739	0.98	0.39422(0.62051)	0.22629	0.951
E_4			0.34033 (0.72031)	0.37998	0.98	0.95174 (1.49804)	0.5463	0.951
E_1	150	$T_1 = 5$	1.32346(1.5948)	0.27135	0.292	0.89618 (1.34279)	0.44661	0.90
E_2			0.27375 (0.61058)	0.33683	0.95	0.53076 (0.94919)	0.41843	0.94
E_3			0.13687(0.30529)	0.16842	0.95	0.26536(0.47461)	0.20924	0.94
E_4			0.33044 (0.73704)	0.40659	0.95	0.64065 (1.14576)	0.5051	0.94
E_1	150	$T_1 = 3$	1.30039 (1.57202)	0.27163	0.891	0.87685 (1.31329)	0.43644	0.88
E_2			0.27726 (0.61612)	0.33886	0.94	0.54827 (0.9726)	0.42433	0.93
E_3			0.13863 (0.30806)	0.16943	0.94	0.27414 (0.4863)	0.21217	0.93
E_4			0.33468 (0.74373)	0.40905	0.94	0.66182 (1.17403)	0.51221	0.93
E_1	250	$T_1 = 5$	1.31995 (1.53043)	0.21048	0.94	0.92732 (1.25933)	0.33201	0.97
E_2			0.33064 (0.59527)	0.26462	0.97	0.60102 (0.93213)	0.33111	0.96
E_3			0.16532 (0.29763)	0.13231	0.97	0.30051 (0.46607)	0.16556	0.96
E_4			0.39912 (0.71855)	0.31943	0.97	0.72549 (1.12518)	0.39969	0.96
E_1	250	$T_1 = 2$	1.33397 (1.54433)	0.21036	0.88	0.92234 (1.25581)	0.33347	0.89
E_2			0.31502 (0.57695)	0.26194	0.91	0.59443 (0.92381)	0.32939	0.92
E_3			0.15751 (0.28848)	0.13097	0.91	0.29721 (0.46191)	0.16469	0.92
E_4			0.38026 (0.69645)	0.31619	0.91	0.71753 (1.11514)	0.3976	0.92
E_1	150	$T_2 = 7$	1.30473 (1.57627)	0.27154	0.93	0.89139 (1.33151)	0.44012	0.94
E_2			0.05189 (0.29982)	0.24793	0.96	0.23277 (0.45373)	0.22096	0.95
E_3			0.07784 (0.44973)	0.37189	0.96	0.34912 (0.68059)	0.33146	0.95
E_4			0.08858 (0.51183)	0.42324	0.96	0.39734 (0.77451)	0.37717	0.95
E_1	150	$T_2 = 3$	1.30623 (1.57796)	0.27172	0.94	0.89652 (1.33842)	0.44189	0.93
E_2			0.05519 (0.30286)	0.24767	0.97	0.23189 (0.45337)	0.22149	0.94
E_3			0.08279 (0.45429)	0.3715	0.97	0.34783 (0.68006)	0.33223	0.94
E_4			0.09422 (0.51701)	0.42279	0.97	0.39585 (0.77396)	0.37811	0.94
E_1	250	$T_2 = 7$	1.32349 (1.53398)	0.21048	0.92	0.92772 (1.25923)	0.33151	0.956
E_2			0.08622 (0.2777)	0.19149	0.96	0.27025 (0.44047)	0.17022	0.981
E_3			0.12932 (0.41655)	0.28723	0.96	0.40537 (0.6607)	0.25533	0.981
E_4			0.14718 (0.47407)	0.32689	0.96	0.46134 (0.75193)	0.29058	0.951
E_1	250	$T_2 = 3$	1.33356 (1.54387)	0.21031	0.91	0.93109 (1.26572)	0.33463	0.899
E_2			0.08117 (0.27311)	0.19194	0.945	0.26625 (0.43669)	0.17044	0.91
E_3			0.12176 (0.40967)	0.28791	0.945	0.39937 (0.65503)	0.25566	0.912
E_4			0.13857 (0.46623)	0.32766	0.945	0.45451 (0.74547)	0.29096	0.912

4.2. Simulation results

Here are some observations on the MLEs and BEs of entropy measure performance as shown in Tables 2–9 above.

(1) The MSEs of MLEs and BEs decrease when T_1 increases for all entropy measures in most cases and this satisfies Case 1 in the GT-II HCS.
(2) The MSEs of MLEs and BEs decrease when r increases for all entropy measures in most cases and this satisfies Case 2 in the GT-II HCS.
(3) The MSEs of MLEs and BEs decrease when T_2 increases for all entropy measures in most cases and this satisfies Case 3 in the GT-II HCS (see Tables 2–7). Additional clarification is available in Figures 3–6.

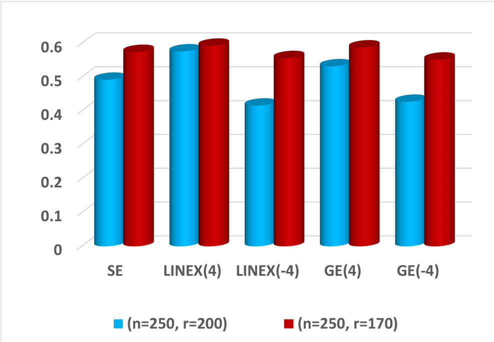

Figure 3. The MSEs of Shannon entropy at $n = 250$ and different values of r.

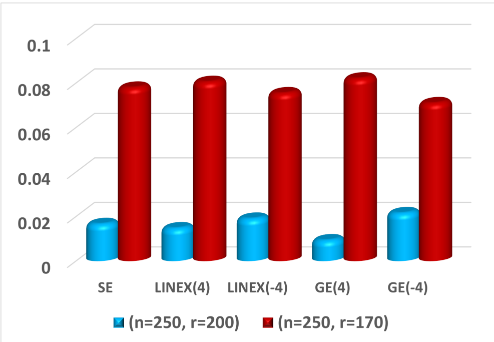

Figure 4. The MSEs of ζ-entropy at $n = 250$ and different values of r.

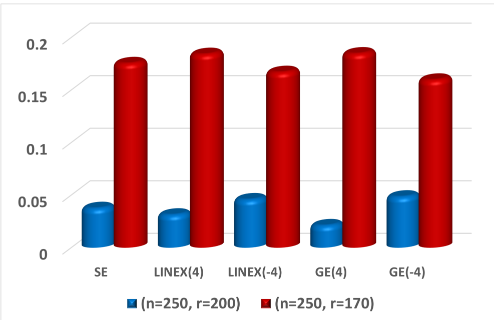

Figure 5. The MSEs of Arimoto entropy at $n = 250$ and different r values.

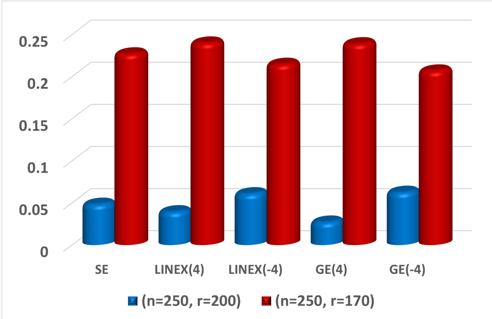

Figure 6. The MSEs of HC entropy at $n = 250$ and different values of r.

(4) In most circumstances, the BEs of all entropy measurements under LINEX LOF provide the best values and are greater than data gathered under other LOFs (see Tables 2–7).
(5) The MSE at $\zeta = 1.5$ is smaller than the MSE at $\zeta = 0.5$ for most entropy measurements, and as the value of ζ increases, the BEs of all entropy measurements improve (see Tables 2–7).
(6) The BEs of all entropy measurements under LINEX LOF and GE LOF ($q = -4$) provide more information, exhibiting smaller MSE values and consequently less uncertainty in most cases (see Tables 2–7). Further explanation is available in Figure 7.

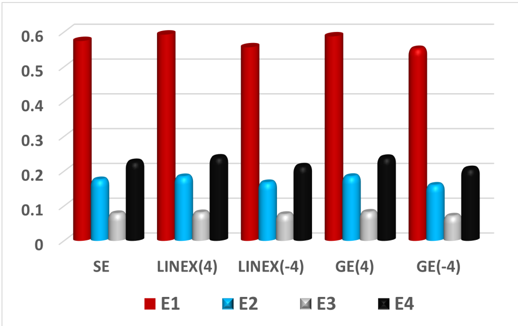

Figure 7. The MSEs of BEs at $n = 150$ and $r = 120$ for all entropy measures under three LOFs.

(7) Estimating entropy measures for the Weibull distribution aids in analyzing reliability data, enabling better decision-making regarding product reliability and risk assessment. Bayesian entropy estimates can also be utilized to analyze financial data involving Weibull distributions, helping evaluate financial risk and time-to-event outcomes.
(8) The developed estimators can be applied to analyze survival data in medical studies, contributing to understanding disease progression and patient outcomes.
(9) As seen in Tables 8 and 9, the length of the interval decreases and the CP values drop as the values of r, T_1, and T_2 increase. The CPs of the BEs for the entropy measures are smaller than those corresponding to the MLEs.

4.3. Data analysis

This paper is structured as a case study in which we look at some fiber strength data. The sample consists of experimental data from the National Physical Laboratory in England on the strength of 1.5-cm-long glass fibers. The data set is obtained from Alizadeh et al. [35]:

0.55, 0.74, 0.77, 0.81, 0.84, 0.93, 1.04, 1.11, 1.13, 1.24, 1.25, 1.27, 1.28, 1.29, 1.30, 1.36, 1.39, 1.42, 1.48, 1.48, 1.49, 1.49, 1.50, 1.50, 1.51, 1.52, 1.53, 1.54, 1.55, 1.55, 1.58, 1.59, 1.60, 1.61, 1.61, 1.61, 1.61, 1.62, 1.62, 1.63, 1.64, 1.66, 1.66, 1.66, 1.67, 1.68, 1.68, 1.69, 1.70, 1.70, 1.73, 1.76, 1.76, 1.77, 1.78, 1.81, 1.82, 1.84, 1.84, 1.89, 2.00, 2.01, 2.24.

According to the Kolmogorov-Smirnov goodness of fit test applied to this genuine data, the Weibull distribution matches the data where the p-value = 0.53 and the statistic value = 0.0311. Figure 8 illustrates the estimated PDF and CDF of the Weibull distribution.

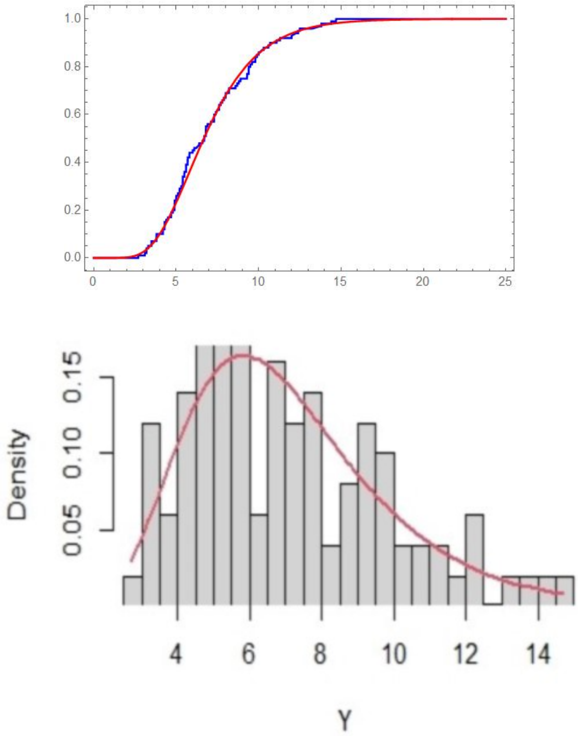

Figure 8. Estimated PDF and CDF of the Weibull distribution.

We will now examine what occurs if the data are censored. Using this data set, we produce three artificial GT-II HCS sets in the manner described below:

$$Case \quad 1 : T_1 = 1.5, T_2 = 2, r = 30 \quad where \quad D = 32, C = T_1 = 1.5.$$
$$Case \quad 2 : T_1 = 1.5, T_2 = 2, r = 40 \quad where \quad D = 40, C = x_r = 1.63.$$
$$Case \quad 3 : T_1 = 1.5, T_2 = 2, r = 60 \quad where \quad D = 55, C = T_2 = 2.$$

For entropy measurements in these situations, we applied ML and Bayesian techniques. We employed the MCMC algorithm with a dataset of $N = 10000$ observations and $M = 1000$ as burn-in at various LOFs. To compute the BEs, we utilize a non-informative prior because we do not know anything about the priors. We take $a_1 = b_1 = a_2 = b_2 = 0.0001$, which are almost identical to Jeffrey's prior as mentioned by Congdon [36]. The value of ζ is selected as $\zeta = 1.5$.

The BE of entropy via LINEX LOF and the GE LOF at $q = -4$ have a large value as shown in Table 10. In the end, it is concluded that the actual data matches the simulated research findings.

Table 10. MLEs and BEs of different entropy measures under the GT-II HCS.

Entropy		MLE	SE	LINEX		GE	
				$q = (4)$	$q = (-4)$	$q = (4)$	$q = (-4)$
E_1		0.87621	0.80134	0.81222	0.79066	0.81134	0.78419
E_2	Case 1	0.43147	0.49738	0.50214	0.49264	0.50444	0.48498
E_3		0.64721	0.74759	0.75809	0.7372	0.75794	0.72962
E_4		0.73657	0.85118	0.86477	0.83793	0.8629	0.83111
E_1		0.8699	0.74041	0.75034	0.73028	0.75034	0.72235
E_2	Case 2	0.32401	0.43615	0.4409	0.43133	0.44414	0.42118
E_3		0.48601	0.65422	0.66488	0.64335	0.66621	0.63178
E_4		0.55312	0.74455	0.75835	0.73046	0.7582	0.71901
E_1		0.77377	0.61321	0.6271	0.5993	0.62968	0.58181
E_2	Case 3	0.32039	0.37454	0.38353	0.36519	0.39153	0.3149
E_3		0.48059	0.56181	0.58189	0.54052	0.5873	0.47235
E_4		0.54695	0.63938	0.6653	0.61165	0.66839	0.53757

5. Conclusions

Entropy is a useful metric for measuring information uncertainty. Likewise, in the fields of survival analysis and reliability engineering, the Weibull distribution is a crucial lifetime model. Thus this work examines the maximum likelihood and Bayesian estimators of Shannon entropy, ζ-entropy, Arimoto entropy, and Havrda and Charvat entropy for the Weibull distribution using the GT-II HCS. The Weibull distribution's entropy expressions are established in Section 2. In classical estimation, the ML estimators of parameters are first derived, next the ML entropy estimators may be acquired through the invariance property, and then the ACIs are calculated in terms of their average length and CPs. In Bayesian estimation, the SLOF and the ASLOF are selected. Nevertheless, computing the forms of Bayesian estimators and HPD is challenging due to their complexity. This issue is resolved by applying the MCMC techniques, specifically employing the M-H algorithm. The numerical results lead to the following conclusions:

- The MLEs and BEs for different entropy measurements show a decreasing trend in their MSEs as the termination time increases in most scenarios.
- Bayesian estimates under different LOFs outperform the MLEs for all entropy measurements in the majority of cases, indicating superior performance.
- Bayesian estimates of entropy measurements using LINEX and GE LOFs at $q = -4$ exhibit a high level of uncertainty, suggesting potential challenges in estimating entropy under these conditions.
- The CPs of the BEs for the entropy measures are smaller than that corresponding to the MLEs and the average length of the intervals decreases when the sample size increases.
- The results obtained from the analysis of real data examples align with those from the simulated data, indicating the reliability and validity of the findings across different data sets.

One of the study's drawbacks is that it only considers the Weibull distribution when utilizing both classical and Bayesian estimation approaches under the GT-II HCS. Furthermore, simulation studies are conducted using large sample sizes. Future research could explore alternative probability distributions beyond the Weibull distribution and investigate the performance of different loss functions in entropy estimation. Additionally, alternative methods such as the Lindley and Tierney-Kadane approximation methods could be considered for calculating entropy measures. Furthermore, conducting simulation studies with both small and large sample sizes would provide a comprehensive understanding of the behavior of entropy estimation methods across different data scenarios.

Author contributions

Amal S. Hassan, Najwan Alsadat and Baria A. Helmy: Conceptualization, Original draft, Methodology, Formal analysis, Writing; Oluwafemi Samson Balogun: Conceptualization, Original draft, Methodology, Formal analysis. All authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

The authors declare no conflicts of interest.

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