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Research article

On solutions of fractional differential equations for the mechanical oscillations by using the Laplace transform

Changdev P. Jadhav[1], **Tanisha B. Dale**[2], **Vaijanath L. Chinchane**[1], **Asha B. Nale**[3], **Sabri T. M. Thabet**[4,5,6,*], **Imed Kedim**[7] **and Miguel Vivas-Cortez**[8,*]

[1] Department of Mathematics, Deogiri Institute of Engineering and Management Studies Chhatrapati Sambhaji nagar-431005, India

[2] Department of Mathematics, Rajashri Shahu Science, Commerce and Arts Mahavidyalaya, Pathri, India

[3] Department of Mathematics, MGM University, Chhatrapati Sambhaji nagar-431 003, India

[4] Department of Mathematics, Saveetha School of Engineering, Saveetha Institute of Medical and Technical Sciences, Saveetha University, Chennai 602105, Tamil Nadu, India

[5] Department of Mathematics, Radfan University College, University of Lahej, Lahej, Yemen

[6] Department of Mathematics, College of Science, Korea University, 145 Anam-ro, Seongbuk-gu, Seoul 02814, Republic of Korea

[7] Department of Mathematics, College of Science and Humanities in Al-Kharj, Prince Sattam Bin Abdulaziz University, Al-Kharj 11942, Saudi Arabia

[8] Faculty of Exact and Natural Sciences, School of Physical Sciences and Mathematics, Pontifical Catholic University of Ecuador, Sede Quito, Ecuador

* **Correspondence:** Email: th.sabri@yahoo.com, mjvivas@puce.edu.ec.

Abstract: In this article, we employ the Laplace transform (LT) method to study fractional differential equations with the problem of displacement of motion of mass for free oscillations, damped oscillations, damped forced oscillations, and forced oscillations (without damping). These problems are solved by using the Caputo and Atangana-Baleanu (AB) fractional derivatives, which are useful fractional derivative operators consist of a non-singular kernel and are efficient in solving non-local problems. The mathematical modelling for the displacement of motion of mass is presented in fractional form. Moreover, some examples are solved.

Keywords: fractional derivative; Laplace transform; fractional differential equations; oscillations
Mathematics Subject Classification: 26A33, 34A08, 44A10

1. Introduction

Fractional calculus has garnered much attention in the last few years, leading to a concentration on fractional calculus in applied mathematics, applied physics, mathematical biology, and engineering; for example, see these works [3, 9, 10] and references therein. The Riemann-Liouville definition is the most frequently employed [4, 16, 17, 26]. Since fractional differential equations are encountered in many practical disciplines, this research area is now growing. Using the Mittag-Leffler function as well as the generalized sine and cosine functions, the system of fractional differential equations has been analytically solved. The fractional derivative operator used in this solution is of the Jumarie type; see [33]. In [1, 4, 17, 20, 21, 24, 27, 31, 32, 34], numerous mathematicians have dealt with fractional integrals that are barely different, like Hadamard, Caputo, Caputo-Fabrizio, AB, etc. In [36], fractional derivatives models are produced by using fractional calculus in the constitutive relation. This generic form made fractional derivatives models more adaptable and suited to represent the characteristics and behavior of many materials or structures. In [35], the authors used the differential transform method to offer approximate analytical solutions for systems of fractional differential equations. The fractional derivative model is used to analyze significant viscoelastic beam deflection; see [30]. The solution of systems with fractional differential equations, integral equations, and fractional calculus are some of the mathematical applications of the Mittag-Leffler functions; see [24, 27]. Currently, there is a great deal of literature available on their attributes and history because of all this effort, see [6, 7, 15]. Among the evaluations that have been conducted, the monograph by Gorenflo, Kilbas, Mainardi, and Rogosin is particularly noteworthy, see [8, 11, 22, 25]. The LT used to solve linear fractional-order differential equations, see [28]. In [23], authors have studied the exponential function-based solution to a linear fractional differential equation with constant coefficients using the modified LT method. It was explained how this method can convert a fractional problem into an ordinary one by using multiple instances for fractional order. Lin and Lu in [29] presented some problems on oscillations of spring for particular solutions of fractional differential equations using the Riemann-Liouville approach with the Laplace transform.

In [12], authors proposed a novel fractional differential problem to explain the mechanical oscillations of a simple system. Particularly, they analyzed the systems spring-damper and mass-spring. To explain the motion of a linear oscillator using fractional derivatives with singular or non-singular kernels. Zafar et al. have studied the fractional differential equation; see [5]. Our goal is to study the dynamics of a fractional oscillator by replacing the second-order derivative and damping term in the classical equation of a damped oscillator with fractional-order derivatives as defined by Caputo and AB by using the Laplace transform method.

2. Preliminaries

In this section, we discuss several essential properties, definitions, and lemmas of fractional calculus that will be important throughout the paper.

Definition 2.1. *[2] A real function $f(x)$, $x > 0$ is said to be in space C_μ, $\mu \in R$ if there exits a real number $p > \mu$, such that $f(t) = tpf_1(t)$, where $f_1(t) \in C(0, \infty)$ and is said to be in the space C_μ^n if and only if $n \in C_\mu$, $n \in N$.*

Definition 2.2. *[17] The integral operator in the sense of Riemann-Liouville of order $\alpha > 0$ for a* *function* $f \in C_\mu$, $\mu \geq -1$, *is given by,*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \alpha > 0. \tag{2.1}$$

The following are some aspects of the operator I^α, that are required in this case:

For $f \in C_\mu$, $\mu \geq -1, \alpha, \beta \geq 0$ *and* $\gamma \geq -1$,

$$I^\alpha I^\beta f(t) = I^{\alpha+\beta},$$

$$I^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} t^{\alpha+\gamma}.$$

Definition 2.3. *[19] The fractional derivative in the Caputo sense for a function* $f(t)$ *is given by*

$$D^\alpha f(t) = I^{m-\alpha} D^m f(t), \tag{2.2}$$

for each $f \in C_{-1}^m, m-1 < \alpha \leq m, m \in N,$ *and* $t > 0$.

The fractional integral operator of Riemann-Liouville is a linear operation, just like the integer-order integration.

$$I^\alpha \sum_{i=1}^n c_i f_i(t) = \sum_{i=1}^n c_i I^\alpha f_i(t),$$

where $\{c_i\}_{i=1}^n \in R$.

Definition 2.4. *[18] The LT* $F(S)$ *for a function* $f(t)$ $0 < t < \infty$, *is defined as*

$$F(s) = L\{f(t)\} = \int_0^\infty f(t) e^{-st} dt, \tag{2.3}$$

provided the integral converges.

Definition 2.5. *[11] (Generalized Mittag-Leffler function) The generalized Mittag-Leffler functions are defined for* $\alpha, \beta > 0$ *and* $z \in C$,*- as follows:*

$$E_{\alpha,\beta}(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(n\alpha+\beta)}.$$

Where, $\Gamma(.)$ *is gamma function.*

When $\beta = 1$, *it is abbreviated as below:*

$$E_\alpha(z) = E_{\alpha,1}(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(n\alpha+1)}.$$

Definition 2.6. *[1, 13, 14] The AB fractional derivative in the framework of Liouville–Caputo (ABC) is given as follows:*

$$ {}_a^{ABC}D_t^\alpha f(t) = \frac{B(\alpha)}{1-\alpha} \int_a^t f(\theta) E_\alpha \left[\frac{-\alpha(t-\theta)^\alpha}{(1-\alpha)} \right] d\theta, \tag{2.4}$$

such that $B(\alpha)$ *is a normalization function, and* $B(0) = B(1) = 1$.

In addition to being very advantageous when applying the LT to solve various physical problems with initial conditions, this definition will be useful in discussing real-world challenges.

Definition 2.7. *[1, 13, 14] The LT of the AB fractional derivative in the framework of Liouville-Caputo (ABC) of $f(t)$ is given as follows:*

$$L\left[{}_{a}^{ABC}D_t^{\alpha}f(t)\right](s) = \frac{B(\alpha)}{1-\alpha}L\left[\int_a^t f(\theta)E_\alpha\left[\frac{-\alpha(t-\theta)^\alpha}{(1-\alpha)}\right]d\theta\right] \tag{2.5}$$
$$= \frac{B(\alpha)}{1-\alpha}\left[\frac{s^\alpha L[f(t)](s) - s^{\alpha-1}f(0)^\alpha}{s}\right] + \frac{\alpha}{1-\alpha}.$$

Consider $f(t)$ to be a continuous function on $[0,\infty)$, which is of exponential order, that is, for some $c \in R$ and $t > 0$

$$sup\left\{\frac{|f(t)|}{e^{ct}}\right\} < \infty.$$

In this case, the LT exists for all $s > c$.

Herein, several useful properties of the LTs that are required for our study,

$$L\{t^\beta\} = \frac{\Gamma(\beta+1)}{s^{\beta+1}}, \beta > -1,$$

$$L\{f^n(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}\acute{f}(0) - .... - f^{(n-1)}(0),$$

$$L\{t^nf^n(t)\} = (-1)^nF^n(s),$$

$$L\left[\int_0^t f(x)dx\right] = \frac{F(s)}{s},$$

$$L\left[\int_0^t f(t-x)g(x)dx\right] = F(s)G(s).$$

Lemma 2.1. *[28] The LT of the Riemann-Liouville fractional integral operator order $\alpha > 0$ is as follows:*

$$L\{I^\alpha f(t)\} = \frac{F(s)}{s^\alpha}. \tag{2.6}$$

Lemma 2.2. *[28] The LT of the Caputo fractional derivative for $m-1 < \alpha \leq m, m \in N$ is given below*

$$L\{D^\alpha f(t)\} = \frac{s^mF(s) - s^{m-1}f(0) - s^{m-2}\acute{f}(0) - ... - f^{m-1}f(0)}{s^{m-\alpha}}. \tag{2.7}$$

Furthermore, the inverse LT of $F(s)$ is defined as,

$$L^{-1}[F(s)] = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\sigma-iT}^{\sigma+iT} e^{sx}F(s)ds, \tag{2.8}$$

where $F(s)$ is defined for the real part of $s \geq \sigma$, and σ is large enough.

Lemma 2.3. *[28] For $s^\alpha > |a|$, $a \in R$ and $\alpha, \beta > 0$, one has the next inverse LT form:*

$$L^{-1}\left[\frac{s^{\alpha-\beta}}{s^\alpha + a}\right] = t^{\beta-1}E_{\alpha,\beta}(-at^\alpha). \tag{2.9}$$

3. Main results

Herein, we look at the displacement problem for damped oscillations, forced oscillations (without damping), free oscillations, and damped forced oscillations.

3.1. The Caputo fractional derivative case

Propsition 3.1. *Assume a light spring has a fixed end at O and a mass m hanging from it at point A as shown in Figure 1. Consider the elongation $e = AB$ which is caused by the hanging mass in equilibrium position. Due to the elasticity of the material of the spring, it has restoring force per unit stretch of the spring. Then condition for the equilibrium position at point B is $mg = T = ke$.*

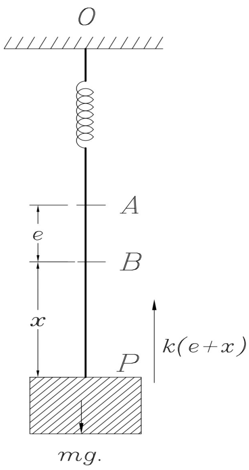

Figure 1. Free oscillations.

At any time t, let the position of mass be at P, where $BP = x$.

Consider the mathematical equation for the displacement of motion of mass m at any time t having a system of single degrees of freedom given by,

$$ {}^{C}D_t^{2\alpha} x(t) + \mu^2 x(t) = 0, \;\; X(0) = x_0 \textit{ and } \acute{X}(0) = x_1. \tag{3.1}$$

Where

$$\mu^2 = \frac{k}{m}.$$

Taking LT of (3.1)*, we obtain,*

$$\frac{s^2 X(s) - sX(0) - \acute{X}(0)}{s^{2-\alpha}} + \mu^2 X(s) = 0.$$

$$s^\alpha X(s) - s^{-(1+\alpha)} x_0 - s^{\alpha-2} x_1 + \mu^2 X(s) = 0.$$

$$\left(s^\alpha + \mu^2\right) X(s) = s^{-(1+\alpha)} x_0 + s^{\alpha-2} x_1.$$

$$X(s) = \frac{s^{-(1+\alpha)} x_0}{s^\alpha + \mu^2} + \frac{s^{(\alpha-2)} x_1}{s^\alpha + \mu^2}. \tag{3.2}$$

Taking the inverse LT of (3.2)*, we obtain,*

$$x(t) = t^0 x_0 E_{\alpha,1}\left(-\mu^2 t^{-\alpha}\right) + x_1 t E_{\alpha,2}\left(\mu^2 t^{\alpha}\right).$$

Example. A 15 *kg* body is suspended from a spring. The spring will extend to 20 *cm* with a 25 *kg* pull. After being drawn down to a location 25 *cm* below the static equilibrium, the body is released. Calculate the displacement of the body from equilibrium at time *t* seconds, taking into account $x(0) = 1$ and $x(\acute{0}) = 2$.

Considering that a 25 *kg* weight lift extends the spring by 0.2 *m*. $25 = T_0 = k \times e \Rightarrow k = 125kg/m$. Also, $15 = T_B = k \times AB \Rightarrow AB = \frac{T_B}{k} = 0.12m$.

It is now necessary to lower the weight to C, where $BC = 25cm = 0.25m$. When the weight is released from C, let it be at P, where $BP = x$, at any time t seconds later. Here, $\mu^2 = \frac{k}{m} = 81.667 \Rightarrow \mu = 9.037$.

Therefore, the displacement of the body at any time t is given by,

$$x(t) = t^0 E_{\alpha,1}\left(81.667 t^{-\alpha}\right) + 2t E_{\alpha,2}\left(81.667 t^{\alpha}\right).$$

3.1.1. Damped oscillations

Here, we study the mechanical system having damped oscillation.

Propsition 3.2. *Considering the damping force proportional to velocity as shown in Figure 2, i.e.,* $r^C D_t^{\alpha} x(t)$*, the equation for the displacement of the motion of the mass m having a system of single degrees of freedom is given by,*

$$^C D_t^{2\alpha} x(t) + 2\lambda^C D_t^{\alpha} \mu^2 x(t) = 0,\ X(0) = x_0 \text{ and } \acute{X}(0) = x_1. \tag{3.3}$$

Where,

$$\frac{r}{m} = 2\lambda \text{ and } \mu^2 = \frac{k}{m}.$$

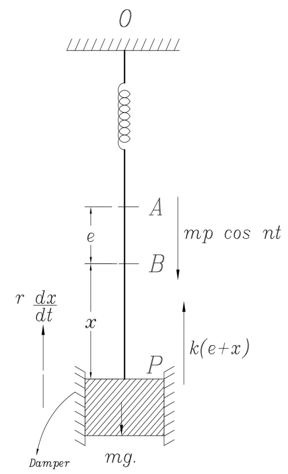

Figure 2. Damped oscillations.

Taking LT of (3.3)*, we have,*

$$s^{\alpha}X(s) - s^{-(1+\alpha)}X(0) - s^{(\alpha-2)}X\acute{(}0) + 2\lambda\left[\frac{sX(s) - X(0)}{s^{1-\alpha}}\right] + \mu^2 X(s) = 0.$$

$$s^{\alpha}X(s) - s^{-(1+\alpha)}x_0 - s^{(\alpha-2)}x_1 + 2\lambda s^{\alpha}X(s) - 2\lambda s^{(\alpha-1)}x_0 + \mu^2 X(s) = 0.$$

$$s^{\alpha}X(s) - s^{-(1+\alpha)}x_0 - s^{(\alpha-2)}x_1 + 2\lambda s^{\alpha}X(s) - 2\lambda s^{(\alpha-1)}x_0 + \mu^2 X(s) = 0.$$

$$\left(s^{\alpha} + 2\lambda s^{\alpha} + \mu^2\right)X(s) - \left(s^{-(1+\alpha)} - 2\lambda s^{(\alpha-1)}\right)x_0 - s^{\alpha-2}x_1 = 0.$$

$$\left(s^{\alpha} + 2\lambda s^{\alpha} + \mu^2\right)X(s) = \left(s^{-(1+\alpha)} - 2\lambda s^{(\alpha-1)}\right)x_0 + s^{\alpha-2}x_1.$$

$$X(s) = \frac{s^{-(1+\alpha)}x_0}{(1+2\lambda)s^{\alpha} + \mu^2} - \frac{2\lambda s^{\alpha-1}x_0}{(1+2\lambda)s^{\alpha} + \mu^2} + \frac{s^{\alpha-2}x_1}{(1+2\lambda)s^{\alpha} + \mu^2}.$$

$$X(s) = \frac{1}{(1+2\lambda)}\frac{\left[s^{-(1+\alpha)}\right]x_0}{\left[s^{\alpha} + \frac{\mu^2}{(1+2\lambda)}\right]} - \frac{2\lambda x_0}{(1+2\lambda)}\frac{s^{\alpha-1}}{\left[s^{\alpha} + \frac{\mu^2}{(1+2\lambda)}\right]} + \frac{1}{(1+2\lambda)}\frac{s^{\alpha-2}}{\left[s^{\alpha} + \frac{\mu^2}{(1+2\lambda)}\right]}x_1. \tag{3.4}$$

Taking the inverse LT of (3.4)*, we obtain,*

$$x(t) = \frac{1}{(1+2\lambda)}x_0 t^0 E_{\alpha,1}\left[-\frac{\mu^2}{1+2\lambda}t^{-\alpha}\right] - \frac{2\lambda x_0 t^0}{(1+2\lambda)}E_{\alpha,1}\left[-\frac{\mu^2 t^{\alpha}}{1+2\lambda}\right] + \frac{x_1 t}{1+2\lambda}E_{\alpha,2}\left[-\frac{\mu^2}{1+2\lambda}t^{\alpha}\right].$$

3.1.2. Forced oscillations (without damping)

Here, we study the mechanical system having forced oscillation without damping.

Propsition 3.3. *Forced oscillation is the motion that results when the spring's support point vibrates in response to an external periodic force as shown in Figure 3. Assuming the external periodic force to be mPcos(nt), the equation for the displacement of motion of mass m having a system of single degrees of freedom is given by,*

$${}^{C}D_t^{2\alpha}x(t) + \mu^2 x(t) = Pcosnt, \;\; X(0) = x_0 \text{ and } \acute{X}(0) = x_1. \tag{3.5}$$

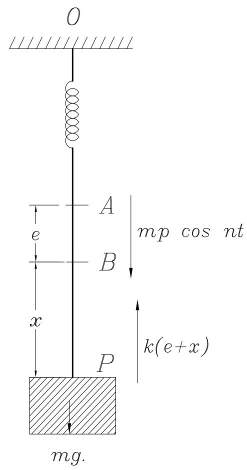

Figure 3. Forced oscillations.

Where,

$$\mu^2 = \frac{k}{m}.$$

Taking LT of (3.5)*, we obtain,*

$$s^{\alpha}X(s) - s^{-(1+\alpha)}X(0) - s^{\alpha-2}\acute{X}(0) + \mu^2 X(s) = P\frac{s}{s^2+n^2}.$$

$$\left(s^{\alpha} + \mu^2\right)X(s) - s^{-(1+\alpha)}x_0 - s^{\alpha-2}x_1 = P\frac{s}{s^2+n^2}.$$

$$\left(s^{\alpha} + \mu^2\right)X(s) = P\frac{s}{s^2+n^2} + s^{-(1+\alpha)}x_0 + s^{\alpha-2}x_1.$$

$$X(s) = P\frac{s}{(s^2+n^2)}\frac{1}{(s^{\alpha}+\mu^2)} + \frac{s^{-(1+\alpha)}x_0}{s^{\alpha}+\mu^2} + \frac{s^{\alpha-2}x_1}{s^{\alpha}+\mu^2}. \tag{3.6}$$

Taking the inverse LT of (3.6)*, we have,*

$$x(t) = P\int_0^t \left[cos\left[n(t-u)\right]t^{\alpha-1}E_{\alpha,\alpha}\left[-\mu^2 t^{\alpha}\right]\right]du + t^0 x_0 E_{\alpha,1}\left[-\mu^2 t^{-\alpha}\right] + x_1 t E_{\alpha,2}\left[-\mu^2 t^{\alpha}\right].$$

3.1.3. Damped forced oscillations

Here, we study the mechanical system having forced oscillation with damping.

Propsition 3.4. *Considering the extra damping force proportional to velocity* $r\frac{d^{\alpha}x}{dt^{\alpha}}$ *as shown in Figure 4, the equation for the displacement of motion for mass m having a system of single degrees of freedom is given by,*

$${}^{C}D_t^{2\alpha}x(t) + 2\lambda{}^{C}D_t^{\alpha}x(t) + \mu^2 x(t) = Pcosnt,\ X(0) = x_0 \ and\ \acute{X}(0) = x_1. \tag{3.7}$$

Where,

$$\frac{r}{m} = 2\lambda \ and\ \mu^2 = \frac{k}{m}.$$

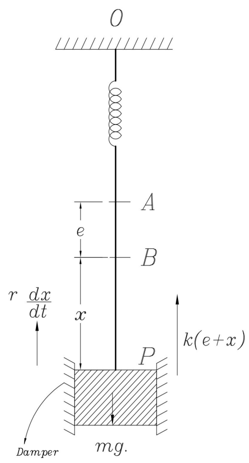

Figure 4. Damped forced oscillations.

Taking LT of (3.7)*, we obtain,*

$$s^{\alpha}X(s) - s^{-(1+\alpha)}X(0) - s^{(\alpha-2)}\acute{X}(0) + 2\lambda\left[\frac{sX(s) - X(0)}{s^{1-\alpha}}\right] + \mu^2 X(s) = P\frac{s}{s^2+n^2}.$$

$$s^{\alpha}X(s) - s^{-(1+\alpha)}x_0 - s^{(\alpha-2)}x_1 + 2\lambda s^{\alpha}X(s) - 2\lambda s^{(\alpha-1)}x_0 + \mu^2 X(s) = P\frac{s}{s^2+n^2}.$$

$$\left(s^{\alpha} + 2\lambda s^{\alpha} + \mu^2\right)X(s) = P\frac{s}{s^2+n^2} + s^{-(1+\alpha)}x_0 + s^{\alpha-2}x_1 + 2\lambda s^{(\alpha-1)}x_0.$$

$$X(s) = P\frac{s}{(s^2+n^2)}\frac{1}{[s^{\alpha}(1+2\lambda)+\mu^2]} + x_0\frac{s^{-(1+\alpha)}}{[s^{\alpha}(1+2\lambda)+\mu^2]}$$
$$+x_1\frac{s^{\alpha-2}}{[s^{\alpha}(1+2\lambda)+\mu^2]} + 2\lambda x_0\frac{s^{\alpha-1}}{[s^{\alpha}(1+2\lambda)+\mu^2]}.$$

$$X(s) = P\frac{s}{(s^2+n^2)}\frac{1}{(1+2\lambda)\left[s^{\alpha}+\frac{\mu^2}{(1+2\lambda)}\right]} + \frac{x_0}{(1+2\lambda)}\frac{s^{-(1+\alpha)}}{\left[s^{\alpha}+\frac{\mu^2}{1+2\lambda}\right]}$$
$$+\frac{x_1}{(1+2\lambda)}\frac{s^{\alpha-2}}{\left[s^{\alpha}+\frac{\mu^2}{(1+2\lambda)}\right]} + \frac{2\lambda.x_0}{(1+2\lambda)}\frac{1}{\left[s^{\alpha}+\frac{\mu^2}{(1+2\lambda)}\right]} \tag{3.8}$$

Taking the inverse LT of (3.8)*, we obtain,*

$$x(t) = P\frac{1}{(1+2\lambda)}\int_0^t \left[cos[n(t-u)]t^{\alpha-1}E_{\alpha,\alpha}\left[-\frac{\mu^2}{(1+2\lambda)t\alpha}\right]\right]du$$
$$+\frac{x_0}{(1+2\lambda)}t^0 x_0 E_{\alpha,1}\left[-\frac{\mu^2}{(1+2\lambda)}t^{-\alpha}\right]$$
$$+\frac{x_1}{(1+2\lambda)}tE_{\alpha,2}\left[-\frac{\mu^2}{(1+2\lambda)}t^{\alpha}\right] + \frac{2\lambda x_0}{(1+2\lambda)}t^0 E_{\alpha,1}\left[-\frac{\mu^2}{(1+2\lambda)}t^{\alpha}\right].$$

3.2. The Atangana-Baleanu fractional derivative case

3.2.1. Free oscillations

Here, we study the mechanical system having free oscillation.

Propsition 3.5. *Considering the free oscillations of a system of single degrees of freedom given in Figure 1, the mathematical equation for the displacement of motion of mass m at any time t is given by,*

$$\frac{1}{\sigma^{2(1-\alpha)}}\,{}_a^{ABC}D_t^{2\alpha}x(t) + \mu^2 x(t) = 0,\ \ X(0) = x_0 \ and\ \acute{X}(0) = x_1. \tag{3.9}$$

Where,

$$\mu^2 = \frac{k}{m}.$$

$${}_a^{ABC}D_t^{2\alpha}x(t) + \omega^2 x(t) = 0,$$

where,

$$\omega^2 = \mu^2\sigma^{2(1-\alpha)}.$$

Taking LT of (3.9)*, we obtain,*

$$\frac{B(\alpha)^2}{(1-\alpha)^2}\left[\frac{s^{2\alpha}X(s)-s^{(2\alpha-1)}X(0)}{\left(s^{\alpha}+\frac{\alpha}{1-\alpha}\right)^2}\right]+\mu^2\omega^2X(s)=0.$$

$$\frac{B(\alpha)^2}{(1-\alpha)^2}\left[\frac{s^{2\alpha}X(s)}{\left(s^{\alpha}+\frac{\alpha}{1-\alpha}\right)^2}\right]-\frac{B(\alpha)^2}{(1-\alpha)^2}\left[\frac{s^{2\alpha-1}x_0}{\left(s^{\alpha}+\frac{\alpha}{1-\alpha}\right)^2}\right]+\mu^2\omega^2X(s)=0.$$

$$\left[\frac{B(\alpha)^2}{(1-\alpha)^2}\frac{s^{2\alpha}}{\left(s^{\alpha}+\frac{\alpha}{1-\alpha}\right)^2}+\mu^2\omega^2\right]X(s)=\frac{B(\alpha)^2}{(1-\alpha)^2}\left[\frac{s^{2\alpha-1}x_0}{\left(s^{\alpha}+\frac{\alpha}{1-\alpha}\right)^2}\right].$$

$$\left[B(\alpha)^2s^{2\alpha}+\mu^2\omega^2(1-\alpha)^2\left(s^{\alpha}+\frac{\alpha}{1-\alpha}\right)^2\right]X(s)=B(\alpha)^2s^{2\alpha-1}x_0.$$

$$X(s)=\frac{B(\alpha)^2s^{2\alpha-1}x_0}{B(\alpha)^2s^{2\alpha}+\mu^2\omega^2(1-\alpha)^2\left(s^{\alpha}+\frac{\alpha}{1-\alpha}\right)^2}.$$

$$X(s)=\frac{B(\alpha)^2s^{2\alpha-1}x_0}{[B(\alpha)+i\omega\mu(1-\alpha)]\,[B(\alpha)-i\omega\mu(1-\alpha)]}\frac{1}{\left[s^{\alpha}+\frac{i\omega\mu}{B(\alpha)+i\omega\mu(1-\alpha)}\right]\left[s^{\alpha}-\frac{i\omega\mu}{B(\alpha)-i\omega\mu(1-\alpha)}\right]}. \tag{3.10}$$

Taking the inverse LT of (3.10)*, we obtain,*

$$x(t)=\frac{B(\alpha)^2x_0}{B(\alpha)^2+\mu^2\omega^2(1-\alpha)^2}\int_0^t(t-\tau)^{\alpha-1}.$$

$$E_{\alpha,-\alpha}\left[\frac{i\omega\mu\alpha}{B(\alpha)+i\omega\mu(1-\alpha)}\right](t-\tau)^{\alpha}\tau^{\alpha}E_{\alpha,\alpha+1}\left[\left(\frac{i\omega\mu\alpha}{B(\alpha)-i\omega\mu(1-\alpha)}\right)t^{\alpha}\right]d\tau.$$

3.2.2. Forced oscillations

Here, we study the mechanical system having forced oscillation.

Propsition 3.6. *Assuming the system of forced oscillations given in Figure 3 has a single degree of freedom and considering the external periodic force to be mPcos*(nt)*, the equation for the displacement of motion of mass m is given by,*

$${}_{a}^{ABC}D_t^{2\alpha}+\mu^2x=Pcos(nt),\ X(0)=x_0 \text{ and } \acute{X}(0)=x_1. \tag{3.11}$$

Where,

$$\mu^2=\frac{k}{m}.$$

Taking LT of (3.11)*, we obtain,*

$$\frac{B(\alpha)^2}{(1-\alpha)^2}\left[\frac{s^{2\alpha}X(s)-s^{2\alpha-1}X(0)}{\left(s^{\alpha}+\frac{\alpha}{(1-\alpha)}\right)^2}\right]+\mu^2X(s)=P\frac{s}{s^2+n^2}.$$

$$\frac{B(\alpha)^2}{(1-\alpha)^2}\left[\frac{s^{2\alpha}}{\left(s^\alpha+\frac{\alpha}{(1-\alpha)}\right)^2}\right]X(s)-\frac{B(\alpha)^2}{(1-\alpha)^2}\left[\frac{s^{2\alpha-1}x_0}{\left(s^\alpha+\frac{\alpha}{(1-\alpha)}\right)^2}\right]$$
$$+\frac{(1-\alpha)^2\left(s^\alpha+\frac{\alpha}{(1-\alpha)}\right)^2\mu^2X(s)}{(1-\alpha)^2\left(s^\alpha+\frac{\alpha}{(1-\alpha)}\right)^2}=P\frac{s}{s^2+n^2}.$$

$$\left[\frac{B(\alpha)^2}{(1-\alpha)^2}\frac{s^{2\alpha}}{\left(s^\alpha+\frac{\alpha}{(1-\alpha)}\right)^2}+\frac{(1-\alpha)^2\left(s^\alpha+\frac{\alpha}{(1-\alpha)}\right)^2}{(1-\alpha)^2\left(s^\alpha+\frac{\alpha}{(1-\alpha)}\right)^2}\right]x(s)=P\frac{s}{s^2+n^2}+\frac{B(\alpha)^2s^{2\alpha-1}x_0}{(1-\alpha)^2\left(s^\alpha+\frac{\alpha}{(1-\alpha)}\right)^2}.$$

$$X(s)=P\frac{s}{(s^2+n^2)}\frac{(1-\alpha)^2\left(s^\alpha+\frac{\alpha}{1-\alpha}\right)^2}{\left[B(\alpha)^2s^{2\alpha}+(1-\alpha)^2\left(s^\alpha+\frac{\alpha}{(1-\alpha)}\right)^2\mu^2\right]}$$
$$+\frac{B(\alpha)^2s^{2\alpha-1}x_0}{\left[B(\alpha)^2s^{2\alpha}+(1-\alpha)^2\left(s^\alpha+\frac{\alpha}{(1-\alpha)}\right)^2\mu^2\right]}.$$

$$X(s)=P\frac{\left[(1-\alpha)^2s^{2\alpha}+2\alpha(1-\alpha)s^\alpha+\alpha^2\right]}{[B(\alpha)^2+(1-\alpha)^2]\left[s^{2\alpha}+\frac{\alpha^2\mu^2}{B(\alpha)^2+(1-\alpha)^2}\right]}\frac{s}{(s^2+n^2)}$$
$$+\frac{B(\alpha)^2x_0s^{2\alpha-1}}{[B(\alpha)^2+(1-\alpha)^2]\left[s^{2\alpha}+\frac{\alpha^2\mu^2}{B(\alpha)^2+(1-\alpha)^2}\right]}.$$

$$\begin{aligned}X(s)=&\frac{P}{[B(\alpha)^2+(1-\alpha)^2]}\frac{(1-\alpha)^2s^{2\alpha}}{\left(s^\alpha+\frac{i\alpha^2\mu^2}{B(\alpha)^2+(1-\alpha)^2}\right)\left(s^\alpha-\frac{i\alpha^2\mu^2}{B(\alpha)^2-(1-\alpha)^2}\right)}\frac{s}{(s^2+n^2)}\\&+\frac{P}{[B(\alpha)^2+(1-\alpha)^2]}\frac{2\alpha s^\alpha(1-\alpha)}{\left(s^\alpha+\frac{i\alpha^2\mu^2}{B(\alpha)^2+(1-\alpha)^2}\right)\left(s^\alpha-\frac{i\alpha^2\mu^2}{B(\alpha)^2-(1-\alpha)^2}\right)}\frac{s}{(s^2+n^2)}\\&+\frac{P}{[B(\alpha)^2+(1-\alpha)^2]}\frac{\alpha^2}{\left(s^\alpha+\frac{i\alpha^2\mu^2}{B(\alpha)^2+(1-\alpha)^2}\right)\left(s^\alpha-\frac{i\alpha^2\mu^2}{B(\alpha)^2-(1-\alpha)^2}\right)}\frac{s}{(s^2+n^2)}\\&+\frac{B(\alpha)^2x_0}{[B(\alpha)^2+(1-\alpha)^2]}\frac{s^{2\alpha-1}}{\left(s^\alpha+\frac{i\alpha^2\mu^2}{B(\alpha)^2+(1-\alpha)^2}\right)\left(s^\alpha-\frac{i\alpha^2\mu^2}{B(\alpha)^2-(1-\alpha)^2}\right)}.\end{aligned}\tag{3.12}$$

Taking the inverse LT of (3.12)*, we have,*

$$\begin{aligned}x(t)=&\frac{P(1-\alpha)^2}{[B(\alpha)^2+(1-\alpha)^2]}\int_0^t\left[\frac{aE_{\alpha,1}(a\tau^\alpha)+bE_{\alpha,-1}(-b\tau^\alpha)}{a+b}\right]\left[\frac{1-sin\left(n^2(t-\tau)\right)}{n^2}\right]\tau^{-2}d\tau\\&+\frac{P2\alpha(1-\alpha)^2}{[B(\alpha)^2+(1-\alpha)^2]}\int_0^t\left[\frac{aE_{\alpha,\alpha}(a\tau^\alpha)+bE_{\alpha,\alpha}(-b\tau^\alpha)}{a+b}\right]\left[cos\left(n^2(t-\tau)\right)\right]\tau d\tau\\&+\frac{P\alpha^2}{[B(\alpha)^2+(1-\alpha)^2]}\int_0^t\left[\frac{aE_{\alpha,2\alpha}(a\tau^\alpha)+bE_{\alpha,2\alpha}(-b\tau^\alpha)}{a+b}\right]\left[cos\left(n^2(t-\tau)\right)\right]\tau^{(2\alpha-1)}d\tau\\&+\frac{B(\alpha)^2x_0}{[B(\alpha)^2+(1-\alpha)^2]}\int_0^t\left[\frac{aE_{\alpha,1}(a\tau^\alpha)+bE_{\alpha,1}(-b\tau^\alpha)}{a+b}\right]d\tau.\end{aligned}$$

3.2.3. Transient vibrations

The transient and steady states are the two states of motion in a system that is periodically excited. In the majority of these situations, the steady state portion endures while the transient portion fades away quickly. On the other hand, the system's reaction is only transient when the excitation is not of a periodic character, such as a shock pulse or transient excitation. Following the period of the excitation, the system vibrates at its inherent frequency and amplitude, which vary according to the kind and intensity of the stimulation. In these situation, the transient vibrations are significant. Rock explosions, gunfires, parcels being loaded or unloaded by dumping them on hard floors, punching operations, fast-moving cars slamming over potholes or road curbs, etc. are some real-world instances of shock-excited transient vibrations. Response to an impulsive input. Let the damped spring of mass m that is exposed to an impulse $\hat{F}\delta(t)$ with a strength of $\hat{F}$. Because the impulse works for such a short period, it has the effect of giving the mass m's initial velocity, which is given by

$$\hat{F} = mdv.$$

The symbol dv represents how the impulse $\hat{F}$ affects the velocity change of mass m. Impulse gives the stationary system a starting speed

$$dv = \frac{\hat{F}}{m}.$$

The mass m at the equilibrium position experiences zero initial displacement due to the very short impulse duration. Therefore, the initial conditions for the mass are specified as,

$$X(0) = 0 \ \ and \ \ X\acute{(0)} = \frac{\hat{F}}{m}.$$

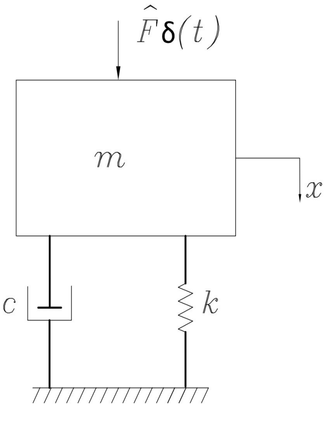

Figure 5. Response to an impulsive input.

Propsition 3.7. *The differential equation for the displacement of motion of mass m as shown in Figure 5, when the forcing function has been taken to be zero since the impulse effectively gives only the initial conditions is given by,*

$$^{C}D_t^{2\alpha}x(t) + 2\xi\omega_n {}^{C}D_t^{\alpha}x(t) + \omega_n^2 x(t) = 0, \tag{3.13}$$

where

$$\frac{c}{m} = 2\xi\omega_n \ and \ \frac{k}{m} = w_n^2.$$

Applying the LT of (3.13)*, we obtain,*

$$s^\alpha X(s) - s^{(1+\alpha)}X(0) - s^{(\alpha-2)}\acute{X}(0) + 2\xi\omega_n\left[\frac{sX(s) - X(0)}{s^{1-\alpha}}\right] + \omega_n^2 X(s) = 0.$$

By applying the initial condition

$$s^\alpha X(s) - s^{(\alpha-2)}\frac{\hat{F}}{m} + 2\xi\omega_n s^\alpha X(s) + \omega_n^2 X(s) = 0.$$

$$\left[s^\alpha + 2\xi\omega_n s^\alpha + \omega_n^2\right]X(s) = s^{(\alpha-2)}\frac{\hat{F}}{m}.$$

$$X(s) = s^{(\alpha-2)}\frac{\hat{F}}{m}\frac{1}{\left[s^\alpha + 2\xi\omega_n s^\alpha + \omega_n^2\right]}.$$

$$X(s) = \frac{\hat{F}}{m}\frac{s^{(\alpha-2)}}{\left[s^\alpha(1 + 2\xi\omega_n) + \omega_n^2\right]}.$$

$$X(s) = \frac{\hat{F}}{m(1 + 2\xi\omega_n)}\frac{s^{(\alpha-2)}}{\left[s^\alpha + \frac{\omega_n^2}{(1+2\xi\omega_n)}\right]}.$$

$$X(s) = \frac{\hat{F}}{m(1 + 2\xi\omega_n)}\frac{s^{(\alpha-2)}}{\left[s^\alpha + \frac{\omega_n^2}{(1+2\xi\omega_n)}\right]}. \tag{3.14}$$

Taking the inverse LT of (3.14)*, we have,*

$$x(t) = \frac{\hat{F}}{m(1 + 2\xi\omega_n)}tE_(\alpha, 2)\left[-\frac{\omega_n}{\sqrt{1 + 2\xi\omega_n}}t^\alpha\right].$$

Response to a step input.

Propsition 3.8. *Let a spring of mass m dashpot system and is subjected to a step force $F_0u(t)$ as shown in Figure 6. Whenever the time is larger than or equal to zero, the magnitude of the force remains constant at F_0. The force is zero for $t < 0$. The differential equation of the motion of the mass at any time t is given by,*

$$^C D_t^{2\alpha} x(t) + 2\xi\omega_n^C D_t^\alpha x(t) + \omega_n^2 x(t) = \frac{F_0}{m}u(t), \ \ X(0 = 0) \ and \ \acute{X}(0) = 0. \tag{3.15}$$

Where,

$$\frac{c}{m} = 2\xi\omega_n \ and \ \frac{k}{m} = w_n^2.$$

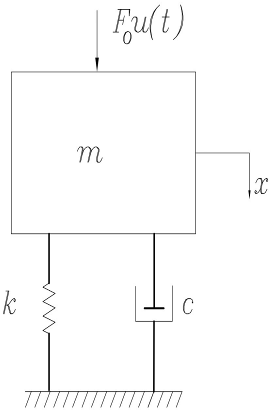

Figure 6. Response to a step input.

Taking the LT of (3.15)*, we obtain,*

$$s^{\alpha}X(s) - s^{(1+\alpha)}X(0) - s^{(\alpha-2)}\acute{X}(0) + 2\xi\omega_n\left[\frac{sX(s) - X(0)}{s^{1-\alpha}}\right] + \omega_n^2 X(s) = \frac{F_0}{ms}.$$

As the second-order system exposed to a finite step cannot possess any initial velocity or displacement. Hence, all initial conditions being assumed as zero in the equation stated above, we obtain,

$$s^{\alpha}X(s) + 2\xi\omega_n s^{\alpha}X(s) + \omega_n^2 X(s) = \frac{F_0}{ms}.$$

$$\left[s^{\alpha} + 2\xi\omega_n s^{\alpha} + \omega_n^2\right]X(s) = \frac{F_0}{ms}.$$

$$X(s) = \frac{F_0}{ms}\frac{1}{\left[s^{\alpha} + 2\xi\omega_n s^{\alpha} + \omega_n^2\right]}.$$

$$X(s) = \frac{F_0}{m}\frac{1}{(1+2\xi\omega_n)}\frac{1}{s}\frac{1}{\left[s^{\alpha} + \frac{\omega_n^2}{(1+2\xi\omega_n)}\right]}. \tag{3.16}$$

Taking the inverse LT of (3.16)*, we have,*

$$x(t) = \frac{F_0}{m}\frac{1}{(1+2\xi\omega_n)}\int_0^t t^{\alpha-1}E_{(\alpha,\alpha)}\left[-\frac{\omega_n}{\sqrt{(1+2\xi\omega_n)}}t^{\alpha}\right]dt.$$

4. Conclusions

In this research work, the result is obtained for the displacement of motion of mass m by solving the fractional differential equations considering the system of single degrees of freedom of free oscillations, forced oscillations, damped oscillations, and damped forced oscillations. The fractional differential equations are considered for the problems of oscillations of spring using the Caputo and AB fractional derivatives. Moreover, the obtained results are applied for the system of a single degree of freedom of the transient vibrations in which the initial conditions are taken in the form of impulse function, which has very much importance in geography. On the other hand, the result is applied to the system, which is subjected to a step force.

The solution of fractional differential equations formed for the different types of oscillations of spring is obtained in the form of the Mittag-Leffler function $E_\alpha(Z)$ and the generalized Mittag-Leffler function $E_{\alpha,\beta}(z)$. It has been found that the Laplace transform method is a powerful tool in applied mathematics and engineering. It will allow us to transform fractional differential equations into algebraic equations, and then by solving these algebraic equations, the unknown function by using the inverse Laplace transform can be obtained.

Author's contributions

Changdev P. Jadhav and Tanisha B. Dale: Conceptualization, methodology and writing-original draft; Vaijanath L. Chinchane and Asha B. Nale: Formal analysis, methodology and investigation; Sabri T. M. Thabet, Imed Kedim and Miguel Vivas-Cortez: Methodology, investigation review and editing. All authors have read and agreed to the published version of the article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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