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Research article

About one unsolved problem in asymptotic p-stability of stochastic systems with delay

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Abstract: The problem of asymptotic p-stability for a linear differential equation with delay and stochastic perturbations, described by a set of mutually independent standard Wiener processes and the Poisson measure, is considered. It is shown the solution of this stability problem for some particular cases of the considered stochastic delay differential equation. However, for the general case of the considered equation, the proposed problem remains open and is presented to the attention of potential readers.

Keywords: stochastic perturbations; white noise; Poisson's jumps; asymptotic p-stability; stability in probability; general method of Lyapunov functionals construction
Mathematics Subject Classification: Primary: 34F05, 34K20, 37J25; Secondary: 60G52, 65C30

1. Introduction

The problem of asymptotic p-stability or L^p-boundedness in the theory of stochastic systems is studied in a lot of different works (see, e.g., [1–3, 6, 7, 14–19]). However, it cannot be said that this problem has been studied sufficiently thoroughly. Here some new results are considered, obtained in this direction, as well as one unsolved problem about the rate of fading on the infinity of stochastic perturbations, at which the stability of the zero solution of the equation under consideration is saved. This unsolved problem complements the series of recently published unsolved problems in stability and optimal control theory of stochastic systems (see, e.g., [8–11] and references therein).

Let $\{\Omega, \mathfrak{F}, \mathbf{P}\}$ be a complete probability space, $\{\mathfrak{F}_t\}_{t\geq 0}$ be a nondecreasing family of sub-σ-algebras of $\mathfrak{F}$, i.e., $\mathfrak{F}_s \subset \mathfrak{F}_t$ for $s < t$, $\mathbf{P}\{\cdot\}$ be the probability of an event enclosed in the braces, $\mathbf{E}$ be the mathematical expectation, H_2 be the space of $\mathfrak{F}_0$-adapted stochastic processes $\varphi(s)$, $s \leq 0$, $||\varphi||_0 = \sup_{s\leq 0} |\varphi(s)|$, $||\varphi||_1^p = \sup_{s\leq 0} \mathbf{E}|\varphi(s)|^p$, $p > 0$.

Following Gikhman and Skorokhod [4], let us consider the linear stochastic differential equation with delay

$$\begin{gathered}dx(t) = (Ax(t) + Bx(t-h))dt + \sum_{i=1}^{m} C_i(t)x(t)dw_i(t) + \int G(t,u)x(t)\tilde{\nu}(dt,du), \quad t \geq 0, \\ x(s) = \phi(s) \in H_2, \quad s \in [-h, 0],\end{gathered} \tag{1.1}$$

where $x(t) \in \mathbf{R}^n$, $A, B, C_i(t)$, and $G(t,u)$ are $n \times n$-matrices, $h > 0$, $w_1(t), ..., w_m$ are mutually independent standard Wiener processes, $\tilde{\nu}(t,A) = \nu(t,A) - t\Pi(A)$, $\nu(t,A)$ is the Poisson measure with $\mathbf{E}\nu(t,A) = t\Pi(A)$ [4].

Consider a functional $V(t,\varphi) : [0,\infty) \times H_2 \to \mathbf{R}_+$ that can be represented in the form $V(t,\varphi) = V(t,\varphi(0),\varphi(s))$, $s < 0$, and for $\varphi = x_t$ put [12]

$$\begin{gathered}V_\varphi(t,x) = V(t,\varphi) = V(t,x_t) = V(t,x,x(t+s)), \\ x = \varphi(0) = x(t), \quad s < 0.\end{gathered} \tag{1.2}$$

Note that here and everywhere below $x(t)$ denotes a value of the solution of the Eq (1.1) in the time moment t, x_t denotes a trajectory of the solution $x(s)$ of the Eq (1.1) for $s \leq t$.

Let D be the set of the functionals for which the function $V_\varphi(t,x)$ defined by (1.2) has a continuous derivative with respect to t and two continuous derivatives with respect to x. The generator L of the Eq (1.1) is defined on the functionals from D and has the form [4, 12]

$$\begin{aligned}LV(t,x_t) =& \frac{\partial}{\partial t} V_\varphi(t,x(t)) + \nabla V'_\varphi(t,x(t))(Ax(t) + Bx(t-h)) \\ &+ \frac{1}{2}\sum_{i=1}^{m} x'(t)C'_i(t)\nabla^2 V_\varphi(t,x(t))C_i(t)x(t) \\ &+ \int [V_\varphi(t,x(t) + G(t,u)x(t)) - V_\varphi(t,x(t)) \\ &- \nabla V'_\varphi(t,x(t))G(t,u)x(t)]\Pi(du).\end{aligned} \tag{1.3}$$

Definition 1.1. *[12] The zero solution of the Eq (1.1) is called:*
- p-stable, $p > 0$, if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $\mathbf{E}|x(t,\phi)|^p < \varepsilon$, $t \geq 0$, provided that $\|\phi\|_1^p < \delta$;
- asymptotically p-stable if it is p-stable and $\lim_{t\to\infty} \mathbf{E}|x(t,\phi)|^p = 0$ for each initial function ϕ;
- stable in probability if for any $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ there exists $\delta > 0$, such that the solution $x(t,\phi)$ of the Eq (1.1) satisfies the condition $\mathbf{P}\{\sup_{t\geq 0}|x(t,\phi)| > \varepsilon_1/\mathfrak{F}_0\} < \varepsilon_2$ for any initial function ϕ such that $\mathbf{P}\{\|\phi\|_0 < \delta\} = 1$.

Theorem 1.1. *[12] Let there exist a functional $V(t,\varphi) \in D$, positive numbers c_1, c_2, c_3 and $p \geq 2$, such that the following conditions hold:*

$$\mathbf{E}V(t,x_t) \geq c_1\mathbf{E}|x(t)|^p, \quad \mathbf{E}V(0,\phi) \leq c_2\|\phi\|^p, \quad \mathbf{E}LV(t,x_t) \leq -c_3\mathbf{E}|x(t)|^p, \quad t \geq 0. \tag{1.4}$$

Then the zero solution of the Eq (1.1) is asymptotically p-stable.

Theorem 1.2. *[12] Let there exist a functional $V(t,\varphi) \in D$, positive numbers c_1, c_2, p, such that the following conditions hold:*

$$V(t,x_t) \geq c_1|x(t)|^p, \quad V(0,\phi) \leq c_2\|\phi\|_0^p, \quad LV(t,x_t) \leq 0, \quad t \geq 0. \tag{1.5}$$

Then the zero solution of the Eq (1.1) *is stable in probability.*

Below conditions of asymptotic p-stability for some particular cases of the Eq (1.1) are presented in the hope that the currently unsolved problem of obtaining the best conditions on the rate of fading stochastic perturbations for asymptotic p-stability of the zero solution of the Eq (1.1) in the general case will attract the attention of potential readers.

2. Some particular cases

2.1. The case $p = 2$

Theorem 2.1. *Let there exist positive definite $n \times n$-matrices P, R, and the function $\rho(t)$, such that the following inequalities hold:*

$$\begin{gathered}\sum_{i=1}^{m} C_i'(t)PC_i(t) + \int G'(t,u)PG(t,u)\Pi(du) \leq \rho(t)P,\\ \Phi = \begin{bmatrix} A'P + PA + R & PB \\ B'P & -R \end{bmatrix} < 0, \quad \int_0^\infty \rho(t)dt < \infty.\end{gathered} \tag{2.1}$$

Then the zero solution of the Eq (1.1) *is asymptotically mean square stable.*

The proof of Theorem 2.1 is presented in [11] (in the case $m = 1$), where via the general method of Lyapunov functional construction [5, 12, 13] it is shown that the Lyapunov functional $V(t,x_t) = V_1(t,x(t)) + V_2(t,x_t)$ with

$$V_1(t,x(t)) = \gamma(t)x'(t)Px(t), \quad V_2(t,x_t) = \int_{t-h}^{t} \gamma(s+h)x'(s)Rx(s)ds, \quad \gamma(t) = e^{-\int_0^t \rho(s)ds}, \tag{2.2}$$

satisfies the conditions of Theorem 1.1 with $p = 2$.

2.2. The case $B = 0$, $G(t,u) = 0$

Theorem 2.2. *Let there exists a positive definite $n \times n$-matrix P and the function $\rho(t)$ such that the following inequalities hold:*

$$PA + A'P < 0, \quad \sum_{i=1}^{m} C_i'(t)PC_i(t) \leq \rho(t)P, \quad \int_0^\infty \rho(s)ds < \infty. \tag{2.3}$$

Then the zero solution of the Eq (1.1) *is asymptotically p-stable for $p \geq 2$.*

Proof. Via (1.3) for the function

$$V(t,x) = \gamma(t)(x'Px)^{p/2}, \quad \gamma(t) = e^{-q\int_0^t \rho(s)ds}, \quad q = \frac{1}{2}p(p-1), \tag{2.4}$$

we have

$$
\begin{aligned}
LV(t,x(t)) =& \gamma(t)\Big[-q\rho(t)(x'(t)Px(t))^{p/2}+\frac{p}{2}(x'(t)Px(t))^{p/2-1}2x'(t)PAx(t)\\
&+\frac{p}{2}(p-2)(x'(t)Px(t))^{p/2-2}\sum_{i=1}^{m}(x'(t)PC_i(t)x(t))^2\\
&+\frac{p}{2}(x'(t)Px(t))^{p/2-1}\sum_{i=1}^{m}x'(t)C_i'(t)PC_i(t)x(t)\Big].
\end{aligned}
$$

Via the inequality $(a'b)^2 \le (a'a)(b'b)$ with $a = P^{0.5}x(t)$, $b = P^{0.5}C_i(t)x(t)$, we obtain

$$
(x'(t)PC_i(t)x(t))^2 \le (x'(t)Px(t))(x'(t)C_i'(t)PC_i(t)x(t)).
$$

From here (2.4), (2.3), and $2x'(t)PAx(t) = x'(t)(PA + A'P)x(t)$, it follows that

$$
\begin{aligned}
LV(t,x(t)) \le& \gamma(t)(x'(t)Px(t))^{p/2-1}\Big[-q\rho(t)(x'(t)Px(t))\\
&+\frac{p}{2}x'(t)(PA+A'P)x(t)+q\sum_{i=1}^{m}(x'(t)C_i'(t)PC_i(t)x(t))\Big]\\
\le& \frac{p}{2}\gamma(t)(x'(t)Px(t))^{p/2-1}x'(t)(PA+A'P)x(t)\\
\le& \frac{p}{2}\gamma(\infty)(x'(t)Px(t))^{p/2-1}x'(t)(PA+A'P)x(t)\\
\le& \frac{p}{2}\gamma(\infty)\lambda_{min}|x(t)|^{p-2}x'(t)(PA+A'P)x(t) \le -c|x(t)|^p,
\end{aligned}
\tag{2.5}
$$

where $\lambda_{min} > 0$ is a minimal eigenvalue of the matrix P and $c > 0$.

Via Theorem 1.1, it means that the zero solution of the Eq (1.1) in the case $B = 0$, $G(t,u) = 0$, is asymptotically p-stable. The proof is completed. □

2.3. Scalar case

Consider the Eq (1.1) in the scalar case:

$$
A = -a < 0, \quad B = b, \quad C_i(t) = c_i(t), \quad G(t,u) = g(t,u). \tag{2.6}
$$

Lemma 2.1. *[13] Arbitrary positive numbers $a, b, \alpha, \beta, \gamma$ satisfy the inequality*

$$
a^{\alpha}b^{\beta} \le \frac{\alpha}{\alpha+\beta}a^{\alpha+\beta}\gamma^{\beta}+\frac{\beta}{\alpha+\beta}b^{\alpha+\beta}\gamma^{-\alpha}. \tag{2.7}
$$

Equality is reached for $\gamma = ba^{-1}$.

Theorem 2.3. *If $a > |b|$ and the function*

$$
\rho(t) = \sum_{i=1}^{m} p(2p-1)c_i^2(t) + \int[(1+g(t,u))^{2p}-1-2pg(t,u)]\Pi(du), \quad p \ge 1, \tag{2.8}
$$

satisfies the condition $\int_0^{\infty}\rho(t)dt < \infty$ then the zero solution of the Eq (1.1), (2.6) *is asymptotically 2p-stable.*

Proof. Via the generator (1.3) for the function $V_1(t,x) = \gamma(t)x^{2p}$, where $\gamma(t) = e^{-\int_0^t \rho(s)ds}$ and $\rho(t)$ is defined in (2.8), we have

$$\begin{aligned}
LV_1(t,x(t)) =& \gamma(t)\Big[-\rho(t)x^{2p}(t) + 2px^{2p-1}(t)(-ax(t) + bx(t-h)) \\
&+ \sum_{i=1}^{m} p(2p-1)c_i^2(t)x^{2p}(t) \\
&+ \int [(1+g(t,u))^{2p} - 1 - 2pg(t,u)]\Pi(du)x^{2p}(t)\Big] \\
=& \gamma(t)\Big[2pbx^{2p-1}(t)x(t-h) + \Big(-2pa - \rho(t) + \sum_{i=1}^{m} p(2p-1)c_i^2(t) \\
&+ \int [(1+g(t,u))^{2p} - 1 - 2pg(t,u)]\Pi(du)\Big)x^{2p}(t)\Big] \\
=& \gamma(t)\Big[2pbx^{2p-1}(t)x(t-h) - 2pax^{2p}(t)\Big].
\end{aligned}$$

Using (2.7), we obtain

$$2p|bx^{2p-1}(t)x(t-h)| \le |b|[(2p-1)x^{2p}(t) + x^{2p}(t-h)].$$

So,

$$\begin{aligned}
LV_1(t,x(t)) \le& \gamma(t)\Big[|b|[(2p-1)x^{2p}(t) + x^{2p}(t-h)] - 2pax^{2p}(t)\Big] \\
=& \gamma(t)\Big[[2p(|b|-a) - |b|]x^{2p}(t) + |b|x^{2p}(t-h)\Big].
\end{aligned}$$

Using that $\gamma(t+h) \le \gamma(t)$ and the additional functional $V_2(t,x_t) = |b|\int_{t-h}^{t} \gamma(s+h)x^{2p}(s)ds$ with

$$\begin{aligned}
LV_2(t,x_t) &= |b|\Big[\gamma(t+h)x^{2p}(t) - \gamma(t)x^{2p}(t-h)\Big] \\
&\le \gamma(t)\Big[|b|x^{2p}(t) - |b|x^{2p}(t-h)\Big],
\end{aligned}$$

for the functional $V(t,x_t) = V_1(t,x(t)) + V_2(t,x_t)$, we obtain

$$LV(t,x_t) \le -\gamma(t)2p(a-|b|)x^{2p}(t) \le -cx^{2p}(t), \quad c = 2p(a-|b|)\gamma(\infty) > 0. \tag{2.9}$$

Via Theorem 1.1, it means that the zero solution of the Eq (1.1), (2.6) is asymptotically 2p-stable. The proof is completed. □

Remark 2.1. *Note that in the case of asymptotic mean square stability ($p = 1$), the function* (2.8) *takes the form*

$$\rho(t) = \sum_{i=1}^{m} c_i^2(t) + \int g^2(t,u)\Pi(du).$$

Remark 2.2. *Via Theorem 1.2, from the conditions* (2.5) *and* (2.9)*, it follows that by the condition $\gamma(\infty) = 0$, i.e., $\int_0^\infty \rho(t)dt = \infty$, the zero solution of the Eq* (1.1) *is stable in probability, that is weaker than asymptotic p-stability.*

Remark 2.3. *(About an unsolved problem) Note that the condition* $\int_0^\infty \rho(t)dt < \infty$ *means that the stochastic perturbations in the Eq* (1.1) *fade on the infinity quickly enough. This condition is essentially used in the proofs of Theorems 2.1–2.3. By that, the following question appears: can this condition be relaxed? Above it is shown that under the weaker condition* $\int_0^\infty \rho(t)dt = \infty$ *it is possible to prove only weaker stability in probability. Is it possible under this weaker condition to prove asymptotic p-moment stability—this problem remains unsolved until now. It is clear that this problem requires a proof that is fundamentally different from the proofs of Theorems 2.1–2.3 and is currently an unsolved problem. An interesting result might seem to be the statement that under the condition* $\int_0^\infty \rho(t)dt = \infty$ *asymptotic p-stability is impossible. But a simple example shows that this is not so. Really, it is well known that the zero solution of the equation* $dx(t) = -ax(t)dt + \sigma x(t)dw(t)$ *is asymptotically mean square stable if and only if* $2a > \sigma^2$ *[12], but for this equation* $\rho(t) = \sigma^2$ *and therefore* $\int_0^\infty \rho(t)dt = \infty$.

3. Conclusions

To readers attention an unsolved problem about the acceptable fade rate of stochastic perturbations of the type of white noise and Poisson's jumps for asymptotic p-stability of the solution of a stochastic linear delay differential equation is proposed. It is shown that for some particular cases of the considered equation, the proposed problem can be solved using the general method of Lyapunov functionals construction. Whether to use the method of Lyapunov functionals construction or to find a new way to solve the given unsolved problem is the choice of the potential readers.

Conflict of interest

Dr. Leonid Shaikhet is the Guest Editor of special issue "Problems of Stability and Optimal Control for Stochastic Systems" for AIMS Mathematics. Leonid Shaikhet was not involved in the editorial review and the decision to publish this article. The author declares no conflict of interest.

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