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Research article

Hopf bifurcation and Turing pattern of a diffusive Rosenzweig-MacArthur model with fear factor

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Abstract: In this paper, the dynamic behavior of a diffusive Rosenzweig-MacArthur (R-M) predator-prey model with hyperbolic tangent functional response and fear effect was investigated. For the local system, we gave a detailed classification of equilibria and performed bifurcation analysis. It was shown by numerical simulation that both the capture rate and fear factor have a stabilizing effect. Furthermore, the existence of limit cycles was discussed when the prey was in low fear or carrying capacity was sufficiently large. For the reaction-diffusion system, we considered the local stability of a unique positive equilibrium, Turing instability of both positive equilibrium and homogeneous periodic orbits under weak fear effect or strong carrying capacity, the direction of Hopf bifurcation and the stability of bifurcating periodic solutions under small fear cost and large diffusion coefficients, as well as the existence of positive nonconstant steady states. However, in the absence of fear effect, Turing instability of both positive equilibrium and homogeneous periodic orbits did not occur. Meanwhile, numerical examples were given to illustrate the corresponding analytic results.

Keywords: Rosenzweig-MacArthur model; hyperbolic tangent functional response; fear effect; Hopf bifurcation; Turing instability; steady state
Mathematics Subject Classification: 37G15, 35B32, 35B36, 35K57, 92D25

1. Introduction

The predator-prey system and its complex dynamics are perhaps the most extensively studied topics in mathematical biology and population dynamics nowadays. Biologists strive to illustrate some biological phenomena through experiments carried out in laboratories or in the wild. However, it is often not entirely successful, and we also need to make mathematical progress on the analytic aspects of the system. In terms of the 'paradox of enrichment' phenomenon [1–4], environmental conditions and properties of the community such as web-like structure, shift to inedible prey, and inducible defences have been put forward by many scholars as interpretations for why communities might fail to destabilize in the presence of enrichment [5,6]. The standard Rosenzweig-MacArthur (R-M) predator-prey model [7] is possibly the simplest formulation of a trophic community able to cause realistic dynamic behavior [8]. Rosenzweig and MacArthur in [7] used a graphical representation to predict conditions for stability of the predator-prey interaction. In 2005, Fussmann and Blasius [9] first suggested that it is highly sensitive to the properties of the mathematical model. They discussed the generalized form of the standard Rosenzweig-MacArthur predator-prey model [7], which reads

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = g(u) - \Phi(u)v, \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = (\Phi(u) - m)v, \end{cases} \tag{1.1}$$

where $u(t)$ and $v(t)$ are the densities of prey and predator, respectively. Prey and predator respectively grow logistically at the rate $g(u) = ru(1 - u/K)$ and according to the nonlinear functional response $\Phi(u)$. $m > 0$ denotes the per capita mortality rate, $r > 0$ is called the prey intrinsic growth rate, and $K > 0$ stands for the carrying capacity.

Fussmann and Blasius [9] obtained a surprising result that the extent of destabilization caused by enrichment is highly sensitive to the mathematical nature of the response function.

Seo and Wolkowicz [10] paid attention to the general Rosenzweig-MacArthur predator-prey model with three different response functions: Holling type II, Ivlev, and hyperbolic tangent as mentioned in [11–13], respectively, which take the forms:

$$\begin{aligned} \text{Holling type II} : \Phi_H(u) &= \frac{a_H u}{1 + b_H u}, \\ \text{Ivlev} : \Phi_I(u) &= a_I(1 - \mathrm{e}^{-b_I u}), \\ \text{Hyperbolic tangent} : \Phi_T(u) &= a_T \tanh(b_T u). \end{aligned}$$

Parameters $a_H,\ b_H,\ a_I,\ b_I,\ a_T,\ b_T$ are all positive constants.

Seo and Wolkowicz [10] reconsidered the effects of the above three response functions on the dynamic behavior of model (1.1) applying the bifurcation theory. First, they discussed the locally and globally asymptotic stabilities of equilibria. Second, the dynamics sensitivities of model (1.1) for three different response functions are analyzed. It is shown that there exist supercritical Hopf bifurcations in cases of Holling type II and Ivlev functional response. Meanwhile, in case of hyperbolic tangent functional response, the system also exhibits a subcritical Hopf bifurcation, co-dimension-two Bautin (or generalized Hopf) bifurcation, transcritical bifurcation, and saddle-node bifurcation of periodic orbits inducing two coexisting limit cycles. Finally, Seo and Wolkowicz illustrated the potential to destabilize the dynamics of system (1.1) as more complex in the hyperbolic tangent functional response case.

As for the R-M predator-prey system, little attention has been given to rigorously proving the effects of the hyperbolic tangent response function on the dynamic behavior. Therefore, we focus on the following R-M predator-prey model with hyperbolic tangent response function:

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = ru\left(1 - \frac{u}{K}\right) - a\tanh(bu)v, \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = (a\tanh(bu) - m)v, \end{cases} \tag{1.2}$$

where a is the predator's conversion factor, and b is the efficiency of predator for capturing prey. Other parameters have the same significances as introduced in (1.1).

In recent years, the prey's antipredator behavior has been studied extensively from experimental and mathematical points of view; one can refer to [14–21] and the references therein. In particular, Wang et al. [15] first formulated the fear factor $\frac{1}{1+hv}$, where the parameter h reflects the level of the fear reducing intrinsic growth rate of prey. Inspired by [15], we try to study whether the system exhibits new dynamic behavior when the cost of fear is incorporated into model (1.2). Therefore, we can modify system (1.2) by multiplying the reproduction term ru of prey by a factor $\frac{1}{1+hv}$ as follows:

$$\begin{cases} \dfrac{du}{dt} = \frac{ru}{1+hv}\left(1 - \frac{u}{K}\right) - a\tanh(bu)v, \\ \dfrac{dv}{dt} = (a\tanh(bu) - m)v. \end{cases} \tag{1.3}$$

However, the dynamic behavior of model (1.3) is similar to that of model (1.2) (can be checked by hand; details omitted), which means that model (1.3) will not produce new biological phenomena (there will only be a co-dimension-two Bautin bifurcation, will not be other bifurcations, e.g., Bogdanov-Takens (B-T) bifurcation). Biologically, a high level of fear will reduce the foraging activities of prey due to the energy and time limitations. This antipredator behavior indirectly reduces the capture efficiency of the predator. To incorporate the fear into the low capture rate of the predator as in [22,23], one can modify system (1.2) by multiplying the capture term bu of the predator by a factor $\frac{1}{1+kv}$, where k represents the level of the fear reducing capture rate of the predator.

Based on these reasons above, in this paper we discuss the following modified R-M predator-prey model with both hyperbolic tangent functional response and fear factor:

$$\begin{cases} \dfrac{du}{dt} = ru\left(1 - \frac{u}{K}\right) - a\tanh\left(\frac{bu}{1+kv}\right)v \triangleq f_1(u,v), \\ \dfrac{dv}{dt} = (a\tanh\left(\frac{bu}{1+kv}\right) - m)v \triangleq f_2(u,v). \end{cases} \tag{1.4}$$

Model (1.4) can in principle, be used to model numerous types of interactions, such as hares and foxes, sardines and sharks, mice and owls, and so on.

Since environment, climate, food supplies, and other factors could lead to the densities of predator and prey being spatially inhomogeneous within a fixed bounded domain at any given time, and each species naturally diffuses to areas of smaller population concentration, we study the following reaction-diffusion model associated with (1.4)

$$\begin{cases} \frac{\partial u}{\partial t} - d_1\Delta u = ru\left(1 - \frac{u}{K}\right) - a\tanh\left(\frac{bu}{1+kv}\right)v, & x \in \Omega,\ t > 0, \\ \frac{\partial v}{\partial t} - d_2\Delta v = (a\tanh\left(\frac{bu}{1+kv}\right) - m)v, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \upsilon} = \frac{\partial v}{\partial \upsilon} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x) \geq, \not\equiv 0,\ v(x,0) = v_0(x) \geq, \not\equiv 0, & x \in \Omega, \end{cases} \tag{1.5}$$

where $\Omega \subset \mathbb{R}^N (N \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, and υ is the outward unit normal vector on $\partial\Omega$. $d_1 > 0$ and $d_2 > 0$ are called diffusion coefficients. The homogeneous Neumann boundary conditions indicate that system (1.5) is self-contained with zero population flux across $\partial\Omega$. $u_0(x)$ and $v_0(x)$ are nonnegative smooth functions.

The asymptotic behavior of nonnegative solutions of (1.5) is closely connected with its nonnegative steady states, and we discuss the following steady state problem associated with (1.5):

$$\begin{cases} -d_1\Delta u = ru\left(1-\frac{u}{K}\right) - a\tanh\left(\frac{bu}{1+kv}\right)v, & x\in\Omega, \\ -d_2\Delta v = (a\tanh\left(\frac{bu}{1+kv}\right) - m)v, & x\in\Omega, \\ \frac{\partial u}{\partial \upsilon} = \frac{\partial v}{\partial \upsilon} = 0, & x\in\partial\Omega. \end{cases} \tag{1.6}$$

Note that some other cases of predator-prey R-D equations for which Hopf bifurcations and Turing instability have been established (see [24–28] and references therein). Nevertheless, the diffusive R-M predator-prey model with both hyperbolic tangent functional response and fear factor remains unstudied.

The first purpose of this paper is to investigate the influence of fear factor on the stability of system (1.4). It is shown that strong fear effect can stabilize the system (1.4). Our second purpose is to perform bifurcation analysis for system (1.4). It is shown by numerical simulation that both the capture rate of predator and the fear cost could stabilize the system. Moreover, system (1.4) will generate a limit cycle when the prey is in a state of low fear or carrying capacity is large enough, and will always undergo a transcritical bifurcation regardless of whether the prey is in fear or not. Our third purpose is to discuss and compare the dynamic behavior of Reaction-Diffusion (R-D) model (1.5) with and without fear factor. Case I. $k = 0$. There only exist supercritical Hopf bifurcations and the bifurcating periodic solutions are orbitally asymptotically stable. Case II. $k > 0$. Theoretical analysis and numerical simulation demonstrate that Turing instability of both positive equilibrium and homogeneous periodic orbits occurs under weak fear effect or strong carrying capacity, which induces spatial inhomogeneous patterns. Moreover, if the fear cost is sufficiently small and the diffusion coefficients d_1 and d_2 are large enough, system (1.5) undergoes new Hopf bifurcations which produce temporal inhomogeneous patterns. For completeness, the existence and nonexistence results of positive nonconstant steady states of (1.5) are proved.

Briefly speaking, a strong fear effect can stabilize the system (1.4), while a weak fear effect causes the system (1.4) to generate limit cycles. In system (1.5), a small fear factor gives rise to spatial and temporal inhomogeneous patterns.

The rest of this paper is organized as follows: In Section 2, we first give a detailed classification of the equilibria for Ordinary Differential Equation (ODE) model (1.4). Then, the existence, direction, and stability of Hopf bifurcation for (1.4) are studied by the Poincaré-Andronov-Hopf bifurcation theorem. Next, we use the Poincaré-Bendixson theorem to prove the existence of limit cycles and apply the Sotomayor's theorem to demonstrate that (1.4) undergoes a transcritical bifurcation. Finally, numerical simulations are provided to discuss the dynamics differences between model (1.2) and (1.4). In Section 3, we take into accuont the Turing instability of both positive equilibrium and homogeneous periodic orbits for reaction-diffusion system (1.5), and the direction of Hopf bifurcation and the stability of bifurcating periodic solutions of (1.5) are considered in one space dimension. Meanwhile, the above analytic results are illustrated numerically. In Section 4, we investigate the existence of positive nonconstant solutions of (1.6) according to the fixed point index theory. A brief discussion is presented in Section 5.

2. Stability and bifurcation analysis of ODE model (1.4)

It is easy to verify that model (1.4) has a predator-prey extinction equilibrium $E_0 = (0, 0)$, a predator-extinction equilibrium $E_K = (K, 0)$, and a unique positive equilibrium $\bar{E} = (\bar{u}, \bar{v})$ if, and only if,

$$(\mathbf{H_0}) \quad \operatorname{arctanh}\left(\frac{m}{a}\right) < b\bar{u} < bK, \quad a > m,$$

where

$$\bar{u} = \frac{K\left(rk \operatorname{arctanh}\left(\frac{m}{a}\right) - bm\right) + \sqrt{\triangle}}{2rk \operatorname{arctanh}\left(\frac{m}{a}\right)},$$

$$\bar{v} = \frac{b\bar{u} - \operatorname{arctanh}\left(\frac{m}{a}\right)}{k \operatorname{arctanh}\left(\frac{m}{a}\right)} = \frac{bK\left(rk \operatorname{arctanh}\left(\frac{m}{a}\right) - bm\right) - 2rk \operatorname{arctanh}^2\left(\frac{m}{a}\right) + b\sqrt{\triangle}}{2rk^2 \operatorname{arctanh}^2\left(\frac{m}{a}\right)}$$

with

$$\triangle := K^2\left(rk \operatorname{arctanh}\left(\frac{m}{a}\right) - bm\right)^2 + 4mrkK \operatorname{arctanh}^2\left(\frac{m}{a}\right). \tag{2.1}$$

2.1. Local stability analysis

For model (1.4), the Jacobian matrix at (u, v) can be written as

$$J(u,v) = \begin{pmatrix} r\left(1 - \frac{2u}{K}\right) - \frac{abv}{(1+kv)\cosh^2\left(\frac{bu}{1+kv}\right)} & \frac{abkuv}{(1+kv)^2\cosh^2\left(\frac{bu}{1+kv}\right)} - a\tanh\left(\frac{bu}{1+kv}\right) \\ \frac{abv}{(1+kv)\cosh^2\left(\frac{bu}{1+kv}\right)} & a\tanh\left(\frac{bu}{1+kv}\right) - \frac{abkuv}{(1+kv)^2\cosh^2\left(\frac{bu}{1+kv}\right)} - m \end{pmatrix}.$$

In what follows, we give the classification of the equilibria for model (1.4). Obviously, E_0 is a saddle point. The Jacobian matrix at E_K is

$$J(K,0) = \begin{pmatrix} -r & -a\tanh(bK) \\ 0 & a\tanh(bK) - m \end{pmatrix}.$$

It has two eigenvalues given by $\lambda_1 = -r < 0$ and $\lambda_2 = a\tanh(bK) - m$. Hence, E_K is a saddle point if $a\tanh(bK) > m$ and E_K is a stable node if $a\tanh(bK) < m$.

The Jacobian matrix at $\bar{E}$ is as follows:

$$J(\bar{u},\bar{v}) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \tag{2.2}$$

where

$$\begin{aligned} a_{11} &= \frac{K(a^2-m^2)\left(\operatorname{arctanh}\left(\frac{m}{a}\right) - b\bar{u}\right) + ark\bar{u}(K - 2\bar{u})}{akK\bar{u}}, \\ a_{12} &= \frac{(a^2-m^2)\left(b\bar{u} - \operatorname{arctanh}\left(\frac{m}{a}\right)\right)\operatorname{arctanh}\left(\frac{m}{a}\right)}{ab\bar{u}} - m < 0, \\ a_{21} &= \frac{(a^2-m^2)\left(b\bar{u} - \operatorname{arctanh}\left(\frac{m}{a}\right)\right)}{ak\bar{u}} > 0, \\ a_{22} &= \frac{(a^2-m^2)\left(\operatorname{arctanh}\left(\frac{m}{a}\right) - b\bar{u}\right)\operatorname{arctanh}\left(\frac{m}{a}\right)}{ab\bar{u}} < 0. \end{aligned} \tag{2.3}$$

The corresponding characteristic equation of (2.2) is

$$P(\lambda) = \lambda^2 - \Theta_1\lambda + \Theta_2$$

with

$$\Theta_1 := a_{11} + a_{22} = \frac{K(a^2-m^2)\left(\operatorname{arctanh}\left(\frac{m}{a}\right)-b\bar{u}\right)\left(b+k\operatorname{arctanh}\left(\frac{m}{a}\right)\right)+abrk\bar{u}(K-2\bar{u})}{abkK\bar{u}},$$

$$\Theta_2 := a_{11}a_{22} - a_{12}a_{21} = \frac{(a^2-m^2)\left(b\bar{u}-\operatorname{arctanh}\left(\frac{m}{a}\right)\right)\sqrt{\triangle}}{abkK\bar{u}} > 0.$$

Lemma 2.1. *(1)* $\Theta_1 < 0$ *if, and only if,*

$$(\mathbf{H_1})\quad K(a^2-m^2)\left(b\bar{u}-\operatorname{arctanh}\left(\frac{m}{a}\right)\right)\left(b+k\operatorname{arctanh}\left(\frac{m}{a}\right)\right) > abrk\bar{u}(K-2\bar{u}).$$

(2) $\Theta_1 > 0$ *if, and only if,*

$$(\mathbf{H_2})\quad K(a^2-m^2)\left(b\bar{u}-\operatorname{arctanh}\left(\frac{m}{a}\right)\right)\left(b+k\operatorname{arctanh}\left(\frac{m}{a}\right)\right) < abrk\bar{u}(K-2\bar{u}).$$

(3) $a_{11} < 0$ *if, and only if,*

$$(\mathbf{H_3})\quad K(a^2-m^2)\left(b\bar{u}-\operatorname{arctanh}\left(\frac{m}{a}\right)\right) > ark\bar{u}(K-2\bar{u}).$$

(4) $a_{11} > 0$ *if, and only if,*

$$(\mathbf{H_4})\quad K(a^2-m^2)\left(b\bar{u}-\operatorname{arctanh}\left(\frac{m}{a}\right)\right) < ark\bar{u}(K-2\bar{u}).$$

Remark 2.1. *Denote*

$$k_1 = \frac{b\left[am-(a^2-m^2)\operatorname{arctanh}\left(\frac{m}{a}\right)\right]}{(a^2-m^2)\operatorname{arctanh}\left(\frac{m}{a}\right)} > 0,$$

$$k_2 = \frac{b\left[2am-(a^2-m^2)\operatorname{arctanh}\left(\frac{m}{a}\right)\right]}{\operatorname{arctanh}\left(\frac{m}{a}\right)\left[(a^2-m^2)\operatorname{arctanh}\left(\frac{m}{a}\right)+ar\right]} > 0$$

and

$$s^* = (a^2-m^2)\operatorname{arctanh}\left(\frac{m}{a}\right)\left(b+k\operatorname{arctanh}\left(\frac{m}{a}\right)\right) > 0. \tag{2.4}$$

If one of the following conditions holds

$$(\mathbf{H_{K_1}})\quad k < \min\{k_1, k_2\},$$

$$(\mathbf{H_{K_2}})\quad k > \max\{k_1, k_2\},$$

then

$$\Theta_1 < 0 \;\Leftrightarrow\; 0 < K < \frac{\operatorname{arctanh}\left(\frac{m}{a}\right)(s^*-2abm)^2}{b(s^*-abm)\left(s^*-2abm+ark\operatorname{arctanh}\left(\frac{m}{a}\right)\right)} \triangleq K_0,$$

$$\Theta_1 > 0 \;\Leftrightarrow\; K > K_0.$$

Thus, we have the following conclusion.

Theorem 2.1. *Assume that* $(\mathbf{H_0})$ *and* $(\mathbf{H_{K_1}})$ *hold, or* $(\mathbf{H_0})$ *and* $(\mathbf{H_{K_2}})$ *hold. Then, the positive equilibrium* $\bar{E}$ *is a locally asymptotically stable node or focus if* $(\mathbf{H_1})$ *(i.e.,* $K < K_0$*) holds, whereas it is an unstable node or focus if* $(\mathbf{H_2})$ *(i.e.,* $K > K_0$*) holds.*

The impact of fear factor k on the stability of $\bar{E}$ is investigated in the following example.

Example 2.1. *Choose* $a = 0.85$*,* $b = 3$*,* $m = 0.6$*,* $r = 4$*, and* $K = 2$*. Clearly,* $(\mathbf{H_0})$ *holds, and* $(\mathbf{H_2})$ *is satisfied if* $k < 0.2734847026$*.* $(\mathbf{H_1})$ *will be satisfied as k increases from* 0.2734847026*. It means that* $\bar{E}$ *changes from unstable to locally asymptotically stable as k increases. In the biological sense, the higher level of fear drives prey to exhibit antipredation behaviors including habitat switch, reducing foraging time, raising vigilance, and so on, the more likely prey and predator are to coexist.*

2.2. Bifurcation analysis

In this subsection, we prove the existence of Hopf bifurcation, limit cycle and transcritical bifurcation for system (1.4). The direction of Hopf bifurcation and the stability of bifurcating periodic solution of (1.4) are discussed.

2.2.1. Hopf bifurcation and limit cycle

It is difficult to obtain a unique positive bifurcation parameter value and verify the transversality condition by taking fear factor k as the bifurcation parameter. Hence, we choose K as the bifurcation parameter to look for the condition for Hopf bifurcation of model (1.4) occurring at $\bar{E}$.

Clearly, $\Theta_1 = 0$ if, and only if, $K = K_0$ when one of $(\mathbf{H_{K_1}})$ and $(\mathbf{H_{K_2}})$ holds.

Let $\lambda(K) = \alpha(K) \pm i\beta(K)$ be a pair of complex roots of $P(\lambda) = 0$ when K is near K_0. Then

$$\alpha(K) = \frac{\Theta_1}{2}, \quad \beta(K) = \frac{1}{2}\sqrt{-4a_{12}a_{21} - (a_{11} - a_{22})^2}.$$

Some straightforward calculations imply that

$$\alpha(K_0) = 0,$$

$$\alpha'(K_0) = \frac{r \operatorname{arctanh}\left(\frac{m}{a}\right)(2abm - s^*)\, t^*}{ab\left[K_0\left(rk \operatorname{arctanh}\left(\frac{m}{a}\right) - bm\right) + \sqrt{\triangle_0}\right]^2 \sqrt{\triangle_0}} > 0,$$

where s^* is defined as (2.4) and

$$\begin{aligned}
\triangle_0 &= K_0^2\left(rk \operatorname{arctanh}\left(\frac{m}{a}\right) - bm\right)^2 + 4mrkK_0 \operatorname{arctanh}^2\left(\frac{m}{a}\right),\\
t^* &= \left(rk \operatorname{arctanh}\left(\frac{m}{a}\right) - bm\right)\left[K_0\left(rk \operatorname{arctanh}\left(\frac{m}{a}\right) - bm\right) + \sqrt{\triangle_0}\right] + 2mrk \operatorname{arctanh}^2\left(\frac{m}{a}\right)\\
&> 2K_0\left(rk \operatorname{arctanh}\left(\frac{m}{a}\right) - bm\right)^2 + 2mrk \operatorname{arctanh}^2\left(\frac{m}{a}\right) > 0.
\end{aligned}$$

Indeed, $\alpha'(K_0) > 0$ if $(\mathbf{H_{K_1}})$ holds; and the sign of $\alpha'(K_0)$ depends on the sign of $2abm - s^*$ if $(\mathbf{H_{K_2}})$ holds. It is not hard to prove $2abm - s^* < 0$ is in contradiction with $\operatorname{arctanh}\left(\frac{m}{a}\right) < bK_0$, which means that ($\mathbf{H_0}$) does not hold. Then, $\alpha'(K_0) > 0$ if ($\mathbf{H_{K_2}}$) is satisfied. Therefore, the transversality condition always holds with $\alpha'(K_0) > 0$.

Thus, system (1.4) undergoes a Hopf bifurcation at $\bar{E}$ as K passes through K_0.

A discussion about the detailed property of Hopf bifurcation can be found in Appendix 5 and one can obtain the following result.

Theorem 2.2. *Suppose that* ($\mathbf{H_0}$) *and* ($\mathbf{H_{K_1}}$) *hold, or* ($\mathbf{H_0}$) *and* ($\mathbf{H_{K_2}}$) *hold. Then, system* (1.4) *undergoes a Hopf bifurcation at the positive equilibrium* $\bar{E}$ *when* $K = K_0$.

(i) *The direction of the Hopf bifurcation is subcritical and the bifurcated periodic solutions are unstable if* $a(K_0) > 0$ *(see* (5.5)*).*

(ii) *The direction of the Hopf bifurcation is supercritical and the bifurcated periodic solutions are orbitally asymptotically stable if* $a(K_0) < 0$ *(see* (5.5)*).*

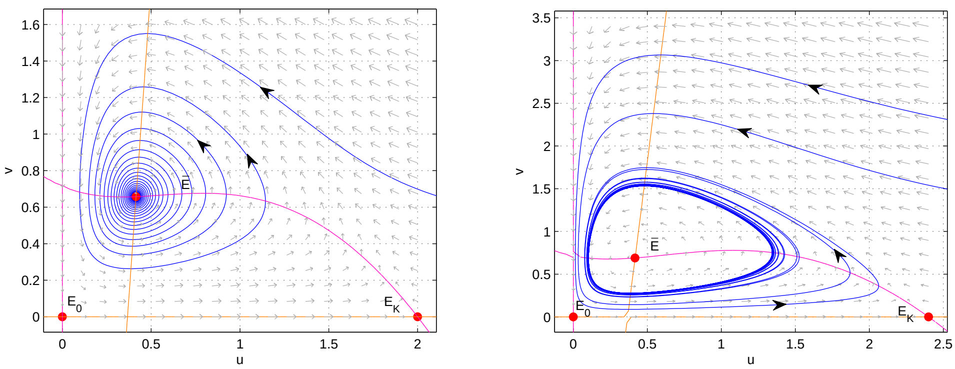
In the following, we illustrate Theorem 2.2 by numerical simulations.

Example 2.2. *Choose two sets of parameters in model* (1.4) *as follows:*

$$a = 3.2,\ m = 2,\ b = 2,\ r = 4,\ k = 0.2, \tag{2.5}$$

$$a = 0.9,\ m = 0.75,\ b = 3,\ r = 2,\ k = 0.6. \tag{2.6}$$

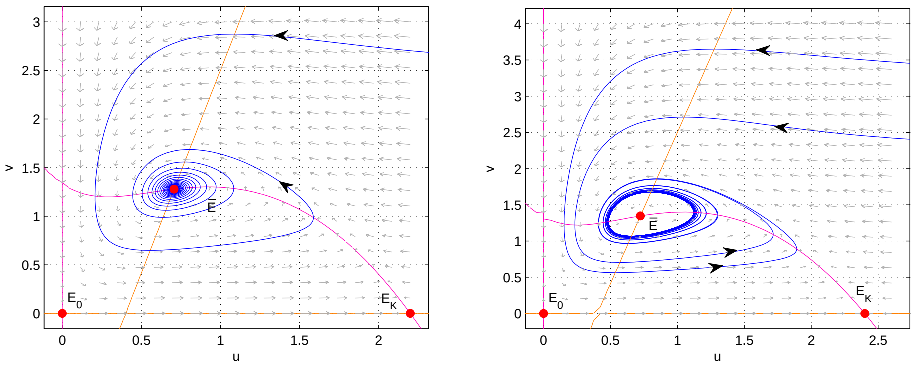
Under (2.5)*,* $K_0 \approx 2.2037$ *and the parameters satisfy* ($\mathbf{H_0}$) *and* ($\mathbf{H_{K_1}}$)*. By computing, we obtain* $a(K_0) \approx 0.0044 > 0$*. It follows from Theorem 2.2 that system* (1.4) *undergoes a subcritical Hopf bifurcation and the bifurcating periodic solutions are unstable. In Figure 1(a), choose* $K = 2 < K_0$*, and by Theorem 2.1,* $\bar{E} \approx (0.4148, 0.6575)$ *is locally asymptotically stable. In Figure 1(b), take* $K = 2.4 > K_0$*, and by Theorem 2.1,* $\bar{E} \approx (0.4171, 0.6893)$ *is unstable.*



(a) $K = 2$ (b) $K = 2.4$

Figure 1. The phase portraits of model (1.4) with (2.5). (a) $\bar{E} \approx (0.4148, 0.6575)$ is locally asymptotically stable; (b) $\bar{E} \approx (0.4171, 0.6893)$ is unstable.

Under (2.6)*,* $K_0 \approx 2.2801$ *and the parameters satisfy* ($\mathbf{H_0}$) *and* ($\mathbf{H_{K_1}}$)*. Calculations yield that* $a(K_0) \approx -0.1990 < 0$*. By Theorem 2.2, system* (1.4) *undergoes a supercritical Hopf bifurcation and the bifurcating periodic solutions are stable. In Figure 2(a), take* $K = 2.2 < K_0$*, and* $\bar{E} \approx (0.6975, 1.2422)$ *is locally asymptotically stable. In Figure 2(b), choose* $K = 2.4 > K_0$*, and* $\bar{E} \approx (0.7226, 1.3468)$ *is unstable. There exists a stable limit cycle that surrounds* $\bar{E}$ *arising from a supercritical Hopf bifurcation.*



(a) $K = 2.2$ (b) $K = 2.4$

Figure 2. The phase portraits of model (1.4) with (2.6). (a) $\bar{E} \approx (0.6975, 1.2422)$ is locally asymptotically stable; (b) $\bar{E} \approx (0.7226, 1.3468)$ is unstable. System (1.4) produces a stable limit cycle that surrounds $\bar{E}$ arising from a supercritical Hopf bifurcation.

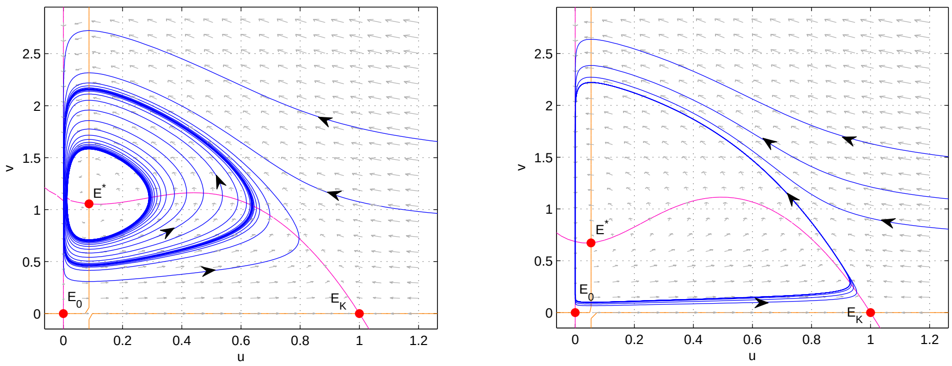
In the absence of fear, system (1.4) degenerates into (1.2). Notice that the cost of fear is incorporated into the capture rate of predator in the modified R-M model (1.4), and we choose b as the bifurcation parameter to compare the dynamics differences between model (1.2) and (1.4). Similar to the above analysis associated with Theorem 2.2, we only give the corresponding numerical results as follows.

Example 2.3. *Choose the following two sets of parameters in model* (1.2)*:*

$$a = 0.9, \ m = 0.3, \ r = 4, \ K = 1, \tag{2.7}$$

$$a = 0.9, \ m = 0.8, \ r = 1, \ K = 2. \tag{2.8}$$

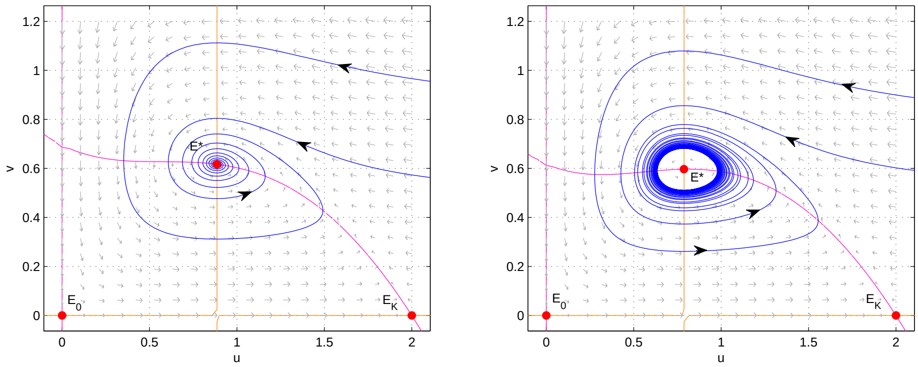
Under (2.7)*,* $b_0 \approx 4.9186$*. System* (1.2) *undergoes a subcritical Hopf bifurcation and the bifurcating periodic solutions are unstable. In Figure 3(a), choose* $b = 4 < b_0$*, and the positive equilibrium* $E^* \approx (0.0866, 1.0552)$ *of system* (1.2) *is locally asymptotically stable. The two limit cycles are generated by saddle-node bifurcation of limit cycles. System* (1.2) *exhibits bistability (a stable equilibrium and a stable limit cycle). In Figure 3(b), take* $b = 6.5 > b_0$*, then* $E^* \approx (0.0533, 0.6730)$ *is unstable.*



(a) $b = 4$ (b) $b = 6.5$

Figure 3. The phase portraits of model (1.2) with (2.7). (a) $E^* \approx (0.0866, 1.0552)$ is locally asymptotically stable. The two limit cycles are generated by saddle-node bifurcation of limit cycles; (b) $E^* \approx (0.0533, 0.6730)$ is unstable.

Under (2.8)*,* $b_0 \approx 1.7726$*. System* (1.2) *undergoes a supercritical Hopf bifurcation and the bifurcating periodic solutions are stable. In Figure 4(a), take* $b = 1.6 < b_0$*, and* $E^* \approx (0.8854, 0.6168)$ *is locally asymptotically stable. In Figure 4(b), choose* $b = 1.8 > b_0$*, and* $E^* \approx (0.7870, 0.5966)$ *is unstable. There is a stable limit cycle that surrounds* E^* *arising from a supercritical Hopf bifurcation.*



(a) $b = 1.6$ (b) $b = 1.8$

Figure 4. The phase portraits of model (1.2) with (2.8). (a) $E^* \approx (0.8854, 0.6168)$ is locally asymptotically stable; (b) $E^* \approx (0.7870, 0.5966)$ is unstable. System (1.2) produces a stable limit cycle that surrounds E^* arising from a supercritical Hopf bifurcation.

Remark 2.2. *(1) Conditions* $(\mathbf{H}_{\mathbf{K}_1})$ *and* $(\mathbf{H}_{\mathbf{K}_2})$ *indicate that system* (1.4) *may exhibit oscillation behaviors under a low or high level of fear.*

(2) It is shown by numerical simulation that system (1.2) *exhibits bistability phenomenon. System* (1.4) *produces a stable limit cycle that surrounds unstable* $\bar{E}$ *arising from a supercritical Hopf bifurcation. This suggests that both the capture rate of predator and the fear factor could stabilize the system.*

Remark 2.3. *There will be a co-dimension-two Bautin (generalized Hopf or degenerate Hopf) bifurcation at positive equilibrium in system* (1.2) *and* (1.4) *when the first Lyapunov coefficient* $\sigma_2 = 0$ *(see* (5.6)*).*

Notice that the above Hopf bifurcation is a small amplitude nonconstant periodic solution around $\bar{E}$, and the bifurcation structure is local. Therefore, we try to prove the existence of limit cycles applying the Poincaré-Bendixson theorem.

Theorem 2.3. *For system* (1.4)*, there exists a limit cycle if* $(\mathbf{H_0})$ *and* $(\mathbf{H_2})$ *hold.*

Proof. By Theorem 2.1, $\bar{E} = (\bar{u}, \bar{v})$ is unstable under conditions $(\mathbf{H_0})$ and $(\mathbf{H_2})$. Furthermore, $E_K = (K, 0)$ is a saddle point. Note that

$$K > \bar{u} > \frac{1}{b}\text{arctanh}\left(\frac{m}{a}\right).$$

First, set $L_1 = u - K$, and one has

$$\frac{\mathrm{d}L_1}{\mathrm{d}t}\Big|_{L_1=0} = \frac{\mathrm{d}u}{\mathrm{d}t}\Big|_{u=K} = -a\tanh\left(\frac{bK}{1+kv}\right)v < 0, \quad \forall v \in (0, \infty).$$

Next, let $L_2 = u + v - \varsigma$ with $\varsigma > 0$ to be specified later. It is not hard to verify that

$$\frac{\mathrm{d}L_2}{\mathrm{d}t}\Big|_{L_2=0} = \left(\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{\mathrm{d}v}{\mathrm{d}t}\right)\Big|_{v=\varsigma-u} = ru\left(1 - \frac{u}{K}\right) - m(\varsigma - u) < 0$$

for sufficiently large $\varsigma > 0$ and $0 < u < K$.

Finally, set $L_3 = \bar{u} + v - \varsigma$, and we obtain

$$\frac{\mathrm{d}L_3}{\mathrm{d}t}\Big|_{L_3=0} = \frac{\mathrm{d}v}{\mathrm{d}t}\Big|_{v=\varsigma-\bar{u}} = \left(a\tanh\left(\frac{bu}{1+k(\varsigma-\bar{u})}\right) - m\right)(\varsigma - \bar{u}) < 0$$

for $0 < u < \frac{1+k(\varsigma-\bar{u})}{b}\text{arctanh}\left(\frac{m}{a}\right)$.

By the Poincaré-Bendixson theorem [29], there exists a limit cycle in system (1.4) if $(\mathbf{H_0})$ and $(\mathbf{H_2})$ hold. □

Remark 2.4. *Similar to Theorem 2.3, system* (1.2) *possesses a limit cycle when* E^* *is unstable.*

2.2.2. Transcritical bifurcation

Since the transcritical bifurcation occurs at a non-hyperbolic equilibrium of system, we need to consider the axial equilibrium E_K. Notice that E_K could coincide with $\bar{E}$ when $a\tanh(bK) = m$, and the transversality condition for transcritical bifurcation of system (1.4) ($k \geq 0$) is verified by Sotomayor's theorem as follows.

Theorem 2.4. *The system* (1.4) *($k \geq 0$) experiences a transcritical bifurcation when the parameters satisfy*

$$b \equiv b_{TC} = \frac{1}{K}\text{arctanh}\left(\frac{m}{a}\right).$$

Proof. The Jacobian matrix at E_K

$$J_{E_K} = \begin{pmatrix} -r & -m \\ 0 & 0 \end{pmatrix}$$

has a zero eigenvalue. Let S and V be the two eigenvectors corresponding to the zero eigenvalue of J_{E_K} and $J^T_{E_K}$, respectively, and

$$S = \begin{pmatrix} S_1 \\ S_2 \end{pmatrix} = \begin{pmatrix} -\frac{m}{r} \\ 1 \end{pmatrix}, \quad V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

One can obtain

$$f_b(E_K;\ b_{TC}) = \begin{pmatrix} -auv(1-\tanh^2(bu)) \\ auv(1-\tanh^2(bu)) \end{pmatrix}_{(E_K;\ b_{TC})} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$Df_b(E_K;\ b_{TC})S = \begin{pmatrix} 0 & -au(1-\tanh^2(bu)) \\ 0 & au(1-\tanh^2(bu)) \end{pmatrix} \begin{pmatrix} -\frac{m}{r} \\ 1 \end{pmatrix}_{(E_K;\ b_{TC})} = \begin{pmatrix} \frac{K(m^2-a^2)}{a} \\ \frac{K(a^2-m^2)}{a} \end{pmatrix},$$

$$\begin{aligned} D^2 f(E_K;\ b_{TC})(S,S) &= \begin{pmatrix} \frac{\partial^2 f_1}{\partial u^2} S_1 S_1 + 2\frac{\partial^2 f_1}{\partial u \partial v} S_1 S_2 + \frac{\partial^2 f_1}{\partial v^2} S_2 S_2 \\ \frac{\partial^2 f_2}{\partial u^2} S_1 S_1 + 2\frac{\partial^2 f_2}{\partial u \partial v} S_1 S_2 + \frac{\partial^2 f_2}{\partial v^2} S_2 S_2 \end{pmatrix}_{(E_K;\ b_{TC})} \\ &= \begin{pmatrix} -\frac{2r}{K} S_1^2 - 2ab(1-\tanh^2(bu))S_1 S_2 \\ 2ab(1-\tanh^2(bu))S_1 S_2 \end{pmatrix}_{(E_K;\ b_{TC})} \\ &= \begin{pmatrix} \frac{2rm(a^2-m^2)\operatorname{arctanh}\left(\frac{m}{a}\right) - 2arm^2}{ar^2 K} \\ \frac{2m(m^2-a^2)\operatorname{arctanh}\left(\frac{m}{a}\right)}{arK} \end{pmatrix}. \end{aligned}$$

Hence, S and V satisfy the transversality conditions

$$\begin{aligned} V^\top f_b(E_K;\ b_{TC}) &= \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0, \\ V^\top [Df_b(E_K;\ b_{TC})S] &= \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{K(m^2-a^2)}{a} \\ \frac{K(a^2-m^2)}{a} \end{pmatrix} = \frac{K(a^2-m^2)}{a} \neq 0, \\ V^\top [D^2 f(E_K;\ b_{TC})(S,S)] &= \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{2rm(a^2-m^2)\operatorname{arctanh}\left(\frac{m}{a}\right) - 2arm^2}{ar^2 K} \\ \frac{2m(m^2-a^2)\operatorname{arctanh}\left(\frac{m}{a}\right)}{arK} \end{pmatrix} \neq 0. \end{aligned}$$

According to Sotomayor's theorem [30], system (1.4) ($k \geq 0$) undergoes a transcritical bifurcation at E_K. □

Remark 2.5. *For the R-M predator-prey model* (1.1) *with Holling type* II *or Ivlev functional response, a similar bifurcation analysis indicates that there only exist supercritical Hopf bifurcations. However, in case of hyperbolic tangent functional response ($k \geq 0$), it can also cause to occur a subcritical Hopf bifurcation or a transcritical bifurcation.*

Remark 2.6. *Note that the number of positive equilibria does not split from one to two when the parameter crosses critical value, the stability will not change, and, thus, saddle-node bifurcation cannot occur in system* (1.4). *On the other hand, the occurrence of transcritical bifurcation also implies that system* (1.4) *will not give rise to a saddle-node bifurcation. The linearized matrix of system* (1.4) *at the positive equilibrium has no zero eigenvalue with multiplicity 2. Hence, B-T bifurcation does not occur in* (1.4).

3. Turing instability and Hopf bifurcation of R-D model (1.5)

In this section, we study the Turing instability of both positive equilibrium and homogeneous periodic orbits for (1.5). Moreover, the direction of Hopf bifurcation and the stability of bifurcating periodic solutions to (1.5) are discussed in one space dimension.

3.1. Turing instability of positive equilibrium

In 1952, Turing demonstrated that a system of coupled reaction-diffusion equations is applicable to depict patterns and forms within biological systems. Turing's theory reveals that the interaction between chemical reaction and diffusion might render the stable equilibrium of the local system unstable for the diffusive system, and give rise to the spontaneous formation of a spatially periodic stationary structure. This type of instability is named *Turing instability* or *diffusion-driven instability*.

Let $0 = \mu_1 < \mu_2 < \mu_3 < \cdots$ be the eigenvalues of the operator $-\Delta$ on Ω with the homogeneous Neumann boundary conditions and $W(\mu_n)$ be the eigen-space corresponding to μ_n in $H^1(\Omega)$. Let $\mathbf{X}$ be the closure of $(C^1(\bar{\Omega}))^2$ in $(H^1(\Omega))^2$, $\{\phi_{ns} : s = 1, 2, \cdots, \dim W(\mu_n)\}$ be an orthonormal basis of $W(\mu_n)$, and $\mathbf{X}_{ns} = \{\mathbf{c}\phi_{ns} : \mathbf{c} \in \mathbb{R}^2\}$. Then,

$$\mathbf{X} = \bigoplus_{n=1}^{+\infty} \mathbf{X}_n, \quad \mathbf{X}_n = \bigoplus_{s=1}^{\dim W(\mu_n)} \mathbf{X}_{ns}.$$

Theorem 3.1. *Assume that* $(\mathbf{H_0})$ *and* $(\mathbf{H_3})$ *hold. Then, the unique positive equilibrium* $\bar{E}$ *of model* (1.5) *is locally uniformly asymptotically stable.*

The proof of Theorem 3.1 is given in Appendix 5.

It follows from Theorem 2.1 that the positive equilibrium $\bar{E}$ of ODE model (1.4) is locally asymptotically stable if $(\mathbf{H_1})$ holds. In what follows, we look for the conditions for the Turing instability of the spatially homogeneous equilibrium $\bar{E}$ of R-D model (1.5) under the assumption $(\mathbf{H_1})$ and $(\mathbf{H_4})$.

For convenience, denote

$$\xi(\mu_n) := N_n = d_1 d_2 \mu_n^2 - (a_{11} d_2 + a_{22} d_1)\mu_n + \Theta_2,$$

which is a quadratic polynomial with respect to (w.r.t.) μ_n. If $g_n(\lambda) = 0$ has two real eigenvalues with different signs, then the positive equilibrium $\bar{E}$ of (1.5) is unstable. Note that if

$$H(d_1, d_2) := -a_{11} d_2 - a_{22} d_1 < 0,$$

then $\xi(\mu_n)$ will take its minimum value

$$\min_{\mu_n} \xi(\mu_n) = \Theta_2 - \frac{(a_{11}d_2 + a_{22}d_1)^2}{4d_1d_2}$$

at the critical value $\bar{\mu} = \frac{a_{11}d_2+a_{22}d_1}{2d_1d_2} > 0$.

Define the radio $\gamma = d_2/d_1$ and

$$\Delta(d_1, d_2) = (a_{11}d_2 + a_{22}d_1)^2 - 4d_1d_2\Theta_2 = a_{11}^2 d_2^2 + 2(2a_{12}a_{21} - a_{11}a_{22})d_1d_2 + a_{22}^2 d_1^2.$$

Thus,

$$\begin{aligned} \Delta(d_1, d_2) = 0 \;&\Leftrightarrow\; a_{11}^2\gamma^2 + 2(2a_{12}a_{21} - a_{11}a_{22})\gamma + a_{22}^2 = 0, \\ H(d_1, d_2) = 0 \;&\Leftrightarrow\; \gamma = -\frac{a_{22}}{a_{11}} \equiv \gamma^* > 0. \end{aligned}$$

Moreover,

$$4(2a_{12}a_{21} - a_{11}a_{22})^2 - 4a_{11}^2a_{22}^2 = 16a_{12}a_{21}(a_{12}a_{21} - a_{11}a_{22}) = -16a_{12}a_{21}\Theta_2 > 0.$$

Then $\Delta(d_1, d_2) = 0$ has two positive real roots

$$\begin{aligned} \gamma_1 &= \frac{-(2a_{12}a_{21} - a_{11}a_{22}) + 2\sqrt{a_{12}a_{21}(a_{12}a_{21} - a_{11}a_{22})}}{a_{11}^2}, \\ \gamma_2 &= \frac{-(2a_{12}a_{21} - a_{11}a_{22}) - 2\sqrt{a_{12}a_{21}(a_{12}a_{21} - a_{11}a_{22})}}{a_{11}^2}. \end{aligned}$$

By direct calculations, $\gamma_1 > \gamma^* > \gamma_2 > 0$. Hence, $H(d_1, d_2) < 0$ and $\min_{\mu_n} \xi(\mu_n) < 0$ when $\gamma > \gamma_1$. This implies that diffusion-driven instability appears.

Similarly, $\min_{\mu_n} \xi(\mu_n) > 0$ at $\bar{\mu}$ can be determined when $\gamma_1 > \gamma > \gamma^*$ or $\gamma^* > \gamma > \gamma_2$. In these two cases, all the roots of $g_n(\lambda) = 0$ have negative real parts, and $\bar{E}$ is locally asymptotically stable.

Analyzing the distribution of the roots of $g_n(\lambda) = 0$, one can obtain the following result.

Theorem 3.2. *Suppose that* $(\mathbf{H_0})$, $(\mathbf{H_1})$, *and* $(\mathbf{H_4})$ *hold (in this case,* $\bar{E}$ *is stable w.r.t. ODE model* (1.4)*). Then there exist unbounded regions*

$$U_1 := \{(d_1, d_2) : d_1 > 0,\ d_2 > 0,\ \gamma_1 d_1 > d_2 > -d_1a_{22}/a_{11}\}, \tag{3.1}$$

$$U_2 := \{(d_1, d_2) : d_1 > 0,\ d_2 > 0,\ -d_1a_{22}/a_{11} > d_2 > \gamma_2 d_1\}, \tag{3.2}$$

$$U_3 := \{(d_1, d_2) : d_1 > 0,\ d_2 > 0,\ d_2 > \gamma_1 d_1\} \tag{3.3}$$

such that $\bar{E}$ *is locally asymptotically stable w.r.t. R-D model* (1.5) *in* U_1 *or* U_2*, and unstable w.r.t. R-D model* (1.5) *in* U_3*, which means that Turing instability occurs.*

Now, we provide an example to explain Theorem 3.2 and investigate Turing instability of R-D model (1.5).

Example 3.1. *Fix a set of parameters in model* (1.5) *as follows:*

$$a = 3.2,\ b = 4,\ m = 2.4,\ r = 3,\ k = 2,\ K = 3,\ d_1 = 0.01. \tag{3.4}$$

The parameters in (3.4) *satisfy* $(\mathbf{H_0})$ *and* $\Theta_1 \approx -0.1026 < 0$, *i.e.,* $(\mathbf{H_1})$ *holds. Then, the unique positive equilibrium* $\bar{E} \approx (0.4943, 0.5161)$ *in model* (1.4) *is locally asymptotically stable. Now we choose* $d_2 = 0.1$, *then* $\gamma_1 \approx 25.5802$, $\gamma_1 d_1 - d_2 \approx 0.1558 > 0$, *and* $d_2 + d_1 a_{22}/a_{11} \approx 0.0883 > 0$, *i.e., parameters in* U_1. $\bar{E}$ *of model* (1.5) *is also locally asymptotically stable by Theorem 3.2 (see Figure 5).*

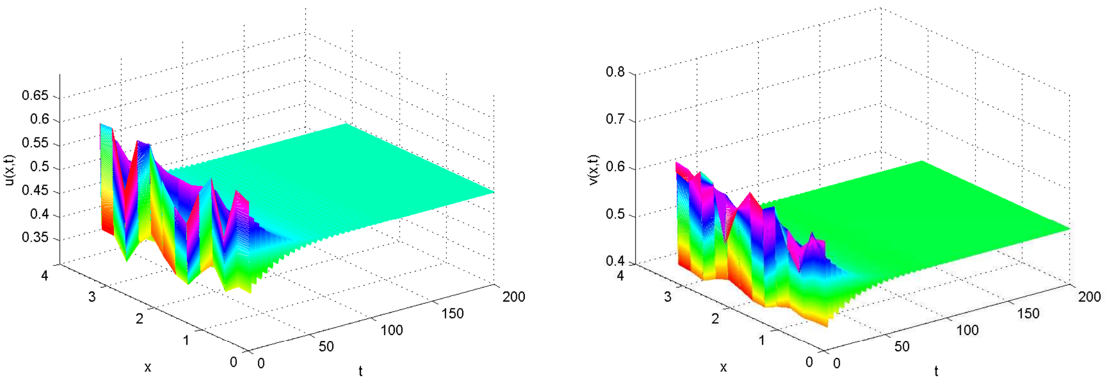


Figure 5. Stable behavior of model (1.5) with parameters in (3.4) and $d_2 = 0.1$.

Notice that if we increase d_2 *to* 0.3, *then* $\gamma_1 \approx 25.5802$ *and* $d_2 - \gamma_1 d_1 \approx 0.0442 > 0$, *i.e., parameters in* U_3. *As a consequence,* $\bar{E} \approx (0.4943, 0.5161)$ *is stable w.r.t. ODE model* (1.4), *and unstable w.r.t. R-D model* (1.5) *by Theorem 3.2. It indicates that Turing instability occurs in* (1.5) *(see Figure 6).*

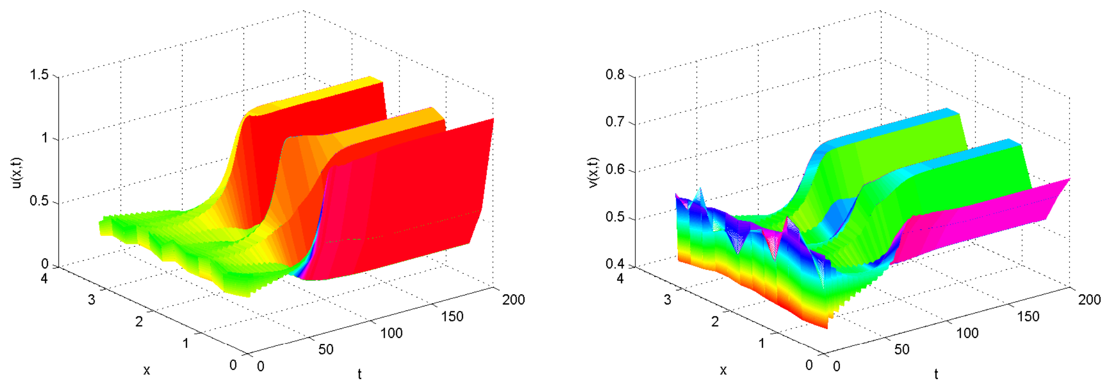


Figure 6. Turing instability of model (1.5) with parameters in (3.4) and $d_2 = 0.3$.

Remark 3.1. *In the absence of fear, positive equilibrium* E^* *of ODE system* (1.2) *and positive equilibrium* $\bar{E}$ *of diffusive system* (1.5) *with* $k = 0$ *are always stable, which means that the Turing instability of both positive equilibrium and homogeneous periodic orbits does not occur in* (1.5) *with* $k = 0$. *Compared with Theorem 3.2, the fear effect induces Turing instability, which implies that the cost of fear could create spatial inhomogeneous patterns.*

3.2. The existence and stability of Hopf bifurcation

This subsection is devoted to determining the direction of Hopf bifurcation and stability of the bifurcating periodic solutions for system (1.5) by the normal form theory and center manifold theorem. For the sake of convenience, we consider the following system in $\Omega = (0, \ell\pi),\ \ell \in \mathbb{R}^+$:

$$\begin{cases} u_t - d_1 u_{xx} = ru\left(1 - \frac{u}{K}\right) - a\tanh\left(\frac{bu}{1+kv}\right)v, & x \in (0, \ell\pi),\ t > 0, \\ v_t - d_2 v_{xx} = \left(a\tanh\left(\frac{bu}{1+kv}\right) - m\right)v, & x \in (0, \ell\pi),\ t > 0, \\ u_x(0,t) = u_x(\ell\pi, t) = v_x(0,t) = v_x(\ell\pi, t) = 0, & t > 0, \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in (0, \ell\pi). \end{cases} \tag{3.5}$$

It is well-known that the eigenvalue problem

$$-\varphi'' = \mu\varphi, \quad x \in (0, \ell\pi), \quad \varphi'(0) = \varphi'(\ell\pi) = 0$$

has eigenvalues $\mu_n = \frac{n^2}{\ell^2}$ $(n = 0, 1, 2, \cdots)$ with corresponding eigenfunctions $\varphi_n(x) = \cos\frac{n}{\ell}x$.

As in [31] (see also [32, 33]), we shall derive our results in 3 steps as follows.

Step 1. Linearization analysis.

For system (3.5), we take the perturbation $u = \tilde{u} + \bar{u}$, $v = \tilde{v} + \bar{v}$, and still denote $(\tilde{u}, \tilde{v})$ by (u, v). Then the problem (3.5) is transformed into

$$\begin{cases} u_t - d_1 u_{xx} = r(u + \bar{u})\left(1 - \frac{u+\bar{u}}{K}\right) - a\tanh\left(\frac{b(u+\bar{u})}{1+k(v+\bar{v})}\right)(v + \bar{v}), & x \in (0, \ell\pi),\ t > 0, \\ v_t - d_2 v_{xx} = \left(a\tanh\left(\frac{b(u+\bar{u})}{1+k(v+\bar{v})}\right) - m\right)(v + \bar{v}), & x \in (0, \ell\pi),\ t > 0, \\ u_x(0,t) = u_x(\ell\pi, t) = v_x(0,t) = v_x(\ell\pi, t) = 0, & t > 0, \\ u(x,0) = u_0(x) - \bar{u},\ v(x,0) = v_0(x) - \bar{v}, & x \in (0, \ell\pi). \end{cases} \tag{3.6}$$

The linearized operator of system (3.6) evaluated at $(0, 0)$ is

$$L(K) := \begin{pmatrix} d_1\frac{\mathrm{d}^2}{\mathrm{d}x^2} + A(K) & B(K) \\ C(K) & d_2\frac{\mathrm{d}^2}{\mathrm{d}x^2} + D(K) \end{pmatrix}$$

and

$$L_n(K) := \begin{pmatrix} A(K) - \frac{d_1 n^2}{\ell^2} & B(K) \\ C(K) & D(K) - \frac{d_2 n^2}{\ell^2} \end{pmatrix},$$

where

$$\begin{aligned} A(K) &= \frac{K(a^2 - m^2)\left(\operatorname{arctanh}\left(\frac{m}{a}\right) - b\bar{u}\right) + ark\bar{u}(K - 2\bar{u})}{akK\bar{u}}, \\ B(K) &= \frac{(a^2 - m^2)\left(b\bar{u} - \operatorname{arctanh}\left(\frac{m}{a}\right)\right)\operatorname{arctanh}\left(\frac{m}{a}\right)}{ab\bar{u}} - m, \\ C(K) &= \frac{(a^2 - m^2)\left(b\bar{u} - \operatorname{arctanh}\left(\frac{m}{a}\right)\right)}{ak\bar{u}}, \end{aligned}$$

$$D(K)=\frac{(a^2-m^2)\left(\operatorname{arctanh}\left(\frac{m}{a}\right)-b\bar{u}\right)\operatorname{arctanh}\left(\frac{m}{a}\right)}{ab\bar{u}}.$$

The characteristic equation of $L_n(K)$ is as follows:

$$\lambda^2-\lambda T_n(K)+D_n(K)=0,\quad n=0,1,2,\cdots,\tag{3.7}$$

where

$$\begin{cases} T_n(K)=A(K)+D(K)-\frac{(d_1+d_2)n^2}{\ell^2},\\ D_n(K)=\frac{d_1d_2n^4}{\ell^4}-\frac{(d_1D(K)+d_2A(K))n^2}{\ell^2}+A(K)D(K)-B(K)C(K),\end{cases}$$

and the eigenvalues are given by

$$\lambda(K)=\frac{T_n(K)\pm\sqrt{T_n^2(K)-4D_n(K)}}{2},\quad n=0,1,2,\cdots.$$

Step 2. Identify possible Hopf bifurcation value and verify transversality conditions.

To look for Hopf bifurcation value K_H, we need to verify that the following necessary and sufficient condition [31] is satisfied:

($\mathbf{H_5}$) There exists $n\geq 0$ such that

$$T_n(K_H)=0,\ \ D_n(K_H)>0\ \text{ and }\ T_j(K_H)\neq 0,\ \ D_j(K_H)\neq 0\ \text{ for }\ j\neq n,$$

as well as for the unique pair of complex eigenvalues near the imaginary axis $\alpha(K)\pm i\beta(K)$,

$$\alpha'(K_H)\neq 0.\tag{3.8}$$

Let $\lambda(K)=\alpha(K)\pm i\beta(K)$ be the roots of (3.7). Clearly, $\alpha(K)=T_n(K)/2$. Notice that

$$\begin{aligned}T_n(K)=&\frac{1}{abk\operatorname{arctanh}\left(\frac{m}{a}\right)\left[K\left(rk\operatorname{arctanh}\left(\frac{m}{a}\right)-bm\right)+\sqrt{\triangle}\right]}\left\{bK\left(bm-rk\operatorname{arctanh}\left(\frac{m}{a}\right)\right)\right.\\&\left(s^*-2abm+ark\operatorname{arctanh}\left(\frac{m}{a}\right)\right)+2rks^*\operatorname{arctanh}^2\left(\frac{m}{a}\right)+b\sqrt{\triangle}(2abm-s^*)\\&\left.-abrk\operatorname{arctanh}\left(\frac{m}{a}\right)\left(4m\operatorname{arctanh}\left(\frac{m}{a}\right)+\sqrt{\triangle}\right)\right\}-\frac{(d_1+d_2)n^2}{\ell^2}\end{aligned}$$

where $\triangle$ and s^* are defined as (2.1) and (2.4), respectively.

Assume that ($\mathbf{H_{K_1}}$) or ($\mathbf{H_{K_2}}$) holds. Then,

$$T_n'(K)=\frac{r\operatorname{arctanh}\left(\frac{m}{a}\right)(2abm-s^*)\,\widetilde{t}}{ab\left[K\left(rk\operatorname{arctanh}\left(\frac{m}{a}\right)-bm\right)+\sqrt{\triangle}\right]^2\sqrt{\triangle}}>0$$

with

$$\begin{aligned}\widetilde{t}=&\left(rk\operatorname{arctanh}\left(\frac{m}{a}\right)-bm\right)\left[K\left(rk\operatorname{arctanh}\left(\frac{m}{a}\right)-bm\right)+\sqrt{\triangle}\right]+2mrk\operatorname{arctanh}^2\left(\frac{m}{a}\right)\\>&\,2K\left(rk\operatorname{arctanh}\left(\frac{m}{a}\right)-bm\right)^2+2mrk\operatorname{arctanh}^2\left(\frac{m}{a}\right)>0.\end{aligned}$$

It indicates that $T_n(K)$ is monotonically increasing w.r.t. K, and

$$\lim_{K\to 0} T_n(K) = \lim_{K\to 0} \frac{2rk \operatorname{arctanh}^2\left(\frac{m}{a}\right)(s^* - 2abm)}{abk \operatorname{arctanh}\left(\frac{m}{a}\right)\left[K\left(rk \operatorname{arctanh}\left(\frac{m}{a}\right) - bm\right) + \sqrt{\triangle}\right]} = -\infty,$$

$$\lim_{K\to +\infty} T_n(K) = \frac{2abm - s^* - ark \operatorname{arctanh}\left(\frac{m}{a}\right)}{ak \operatorname{arctanh}\left(\frac{m}{a}\right)} - \frac{(d_1 + d_2)n^2}{\ell^2} \triangleq \Upsilon.$$

Note that $\Upsilon > 0$ if, and only if, $(\mathbf{H_{K_1}})$ and

$$\frac{(d_1 + d_2)n^2}{\ell^2} < \frac{2abm - s^* - ark \operatorname{arctanh}\left(\frac{m}{a}\right)}{ak \operatorname{arctanh}\left(\frac{m}{a}\right)}. \tag{3.9}$$

Then, $T_n(K_n^H) = 0$ has a unique positive root for each $n \geq 0$. Hence, one obtains

$$T_n(K_n^H) = 0, \ \ \alpha'(K_n^H) > 0 \ \text{ and } \ T_j(K_n^H) \neq 0, \ \text{ for } \ j \neq n.$$

The transversality condition (3.8) is satisfied. We claim that $D_j(K_n^H) \neq 0$ for $n = 0, 1, 2, \cdots$. Indeed, if

$$\frac{n^2}{\ell^2} > \frac{d_1 D(K) + d_2 A(K)}{d_1 d_2}, \tag{3.10}$$

then $D_n(K_n^H) > 0$. Therefore, the condition $(\mathbf{H_5})$ is satisfied, which means that (3.5) undergoes a Hopf bifurcation at $K = K_n^H$. Obviously, $K = K_0^H (= K_0)$ is always the unique value for the Hopf bifurcation of spatially homogeneous periodic solution of (3.5). The above analysis can be summarized as follows.

Theorem 3.3. *Assume* $(\mathbf{H_0})$, $(\mathbf{H_{K_1}})$, (3.9), *and* (3.10) *hold. Then, system* (3.5) *undergoes Hopf bifurcations at* K_n^H, $n = 0, 1, 2, \cdots$.

Step 3. Verify the sign of the first Lyapunov coefficient, which will be defined later.

Recall that $\alpha'(K_0) > 0$, as in [31]. We know that the bifurcating periodic solutions are unstable (resp., stable) and the bifurcation is subcritical (resp., supercritical) if $\mathrm{Re}(c_1(K_0)) > 0$ (resp., < 0).

Under the conditions of Theorem 3.3, it is easy to see that all other eigenvalues of $L(K_0)$ have negative real parts, and for any $n \geq 1$, $L(K_n^H)$ has at least one eigenvalue whose real part is positive. Hence, the periodic solutions bifurcating from $(0, 0, K_n^H)$ are unstable.

We make a further consideration for the bifurcation solution to discuss the direction and stability of the periodic solutions bifurcating from $(0, 0, K_0)$.

As defined in (B.1), L is a linear operator with domain $X_{\mathbb{C}} := X \oplus iX = \{x_1 + ix_2 | x_1, x_2 \in X\}$, where

$$X := \{(u, v) \in H^2(0, \ell\pi) \times H^2(0, \ell\pi) | (u_x, v_x)|_{x=0,\ell\pi} = 0\}$$

denotes a real-valued Sobolev space. Let L^* be the conjugate operator of L. Then

$$L^* E := D\Delta E + J^* E,$$

where $J^* = J^\top$ with the domain $D_{L^*} = X_{\mathbb{C}}$. Set

$$q := \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{\kappa_1}{\kappa_2} + \frac{\beta_0}{\kappa_2} i \end{pmatrix}$$

with $\kappa_1 = a_{11}|_{K=K_0}$, $\kappa_2 = a_{12}|_{K=K_0}$ and

$$q^* := \begin{pmatrix} q_1^* \\ q_2^* \end{pmatrix} = \frac{\kappa_2}{2\pi\beta_0}\begin{pmatrix} \frac{\beta_0}{\kappa_2} + \frac{\kappa_1}{\kappa_2}i \\ i \end{pmatrix}.$$

For any $\pi_1 \in D_{L^*}$, $\pi_2 \in D_L$, it is easy to verify that $\langle L^*\pi_1, \pi_2\rangle = \langle \pi_1, L\pi_2\rangle$, $L(K_0)q = i\beta_0 q$, $L^*(K_0)q^* = -i\beta_0 q^*$, $\langle q^*, q\rangle = 1$, $\langle q^*, \bar{q}\rangle = 0$, where $\langle \pi_1, \pi_2\rangle = \int_0^{\ell\pi} \bar{\pi}_1{}^\top \pi_2 \, \mathrm{d}x$ denotes the inner product in $L^2[(0, \ell\pi)] \times L^2[(0, \ell\pi)]$. On the basis of [31], we decompose $X = X^C \oplus X^S$ with $X^C = \{zq + \bar{z}\bar{q} | z \in \mathbb{C}\}$ and $X^S = \{\psi \in X | \langle q^*, \psi\rangle = 0\}$.

For any $(u, v) \in X$, there exist $z \in \mathbb{C}$ and $\psi = (\psi_1, \psi_2) \in X^S$ such that

$$\begin{pmatrix} u \\ v \end{pmatrix} = zq + \bar{z}\,\bar{q} + \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad z = \langle q^*, (u, v)^\top\rangle.$$

Then

$$\begin{cases} u = z + \bar{z} + \psi_1, \\ v = z\left(-\frac{\kappa_1}{\kappa_2} + \frac{\beta_0}{\kappa_2}i\right) + \bar{z}\left(-\frac{\kappa_1}{\kappa_2} - \frac{\beta_0}{\kappa_2}i\right) + \psi_2. \end{cases}$$

Reduce system (3.5) to the following system in (z, ψ) coordinates

$$\begin{cases} \frac{\mathrm{d}z}{\mathrm{d}t} = i\beta_0 z + \langle q^*, F_0\rangle, \\ \frac{\mathrm{d}\psi}{\mathrm{d}t} = L\psi + H(z, \bar{z}, \psi) \end{cases} \tag{3.11}$$

with

$$H(z, \bar{z}, \psi) = F_0 - \langle q^*, F_0\rangle q - \langle \bar{q}^*, F_0\rangle \bar{q} \quad \text{and} \quad F_0 := F_0(zq + \bar{z}\bar{q} + \psi). \tag{3.12}$$

We write F_0 in the form

$$F_0(E) = \frac{1}{2}Q(E, E) + \frac{1}{6}C(E, E, E) + O(|E|^4), \tag{3.13}$$

where $E = (u, v)$, Q, and C are symmetric multi-linear forms. For simplicity, we denote $Q_{XY} = Q(X, Y)$. Then,

$$F_0 = \frac{1}{2}Q_{qq}z^2 + Q_{q\bar{q}}z\bar{z} + \frac{1}{2}Q_{\bar{q}\bar{q}}\bar{z}^2 + O(|z|^3, |z| \cdot |\psi|, |\psi|^2),$$

$$\langle q^*, F_0\rangle = \frac{1}{2}\langle q^*, Q_{qq}\rangle z^2 + \langle q^*, Q_{q\bar{q}}\rangle z\bar{z} + \frac{1}{2}\langle q^*, Q_{\bar{q}\bar{q}}\rangle \bar{z}^2 + O(|z|^3, |z| \cdot |\psi|, |\psi|^2),$$

$$\langle \bar{q}^*, F_0\rangle = \frac{1}{2}\langle \bar{q}^*, Q_{qq}\rangle z^2 + \langle \bar{q}^*, Q_{q\bar{q}}\rangle z\bar{z} + \frac{1}{2}\langle \bar{q}^*, Q_{\bar{q}\bar{q}}\rangle \bar{z}^2 + O(|z|^3, |z| \cdot |\psi|, |\psi|^2).$$

Hence,

$$H(z, \bar{z}, \psi) = \frac{N_{20}}{2}z^2 + N_{11}z\bar{z} + \frac{N_{02}}{2}\bar{z}^2 + O(|z|^3, |z| \cdot |\psi|, |\psi|^2).$$

By (3.12) and (3.13), we have

$$\begin{cases} N_{20} = Q_{qq} - \langle q^*, Q_{qq}\rangle q - \langle \bar{q}^*, Q_{qq}\rangle \bar{q}, \\ N_{11} = Q_{q\bar{q}} - \langle q^*, Q_{q\bar{q}}\rangle q - \langle \bar{q}^*, Q_{q\bar{q}}\rangle \bar{q}, \\ N_{02} = Q_{\bar{q}\bar{q}} - \langle q^*, Q_{\bar{q}\bar{q}}\rangle q - \langle \bar{q}^*, Q_{\bar{q}\bar{q}}\rangle \bar{q}. \end{cases}$$

Moreover, $N_{20} = N_{11} = N_{02} = (0,0)^\top$ and $H(z,\bar{z},\psi) = O(|z|^3, |z|\cdot|\psi|, |\psi|^2)$. The model (3.11) possesses a center manifold by Appendix A of [31], and then we can write ψ in the following form:

$$\psi = \frac{\psi_{20}}{2}z^2 + \psi_{11}z\bar{z} + \frac{\psi_{02}}{2}\bar{z}^2 + O(|z|^3).$$

Therefore, one has

$$\begin{cases} \psi_{20} = (2i\beta_0 I - L)^{-1}N_{20}, \\ \psi_{11} = (-L)^{-1}N_{11}, \\ \psi_{02} = \bar{\psi}_{20}, \end{cases}$$

which implies that $\psi_{20} = \psi_{02} = \psi_{11} = 0$. For later uses, define

$$\begin{aligned} g_0 &:= f_{uu}q_1^2 + 2f_{uv}q_1q_2 + f_{vv}q_2^2 = 2a_1 + 2a_2q_2 + 2a_3q_2^2, \\ h_0 &:= g_{uu}q_1^2 + 2g_{uv}q_1q_2 + g_{vv}q_2^2 = 2b_1 + 2b_2q_2 + 2b_3q_2^2, \\ i_0 &:= f_{uu}|q_1|^2 + f_{uv}(q_1\bar{q}_2 + \bar{q}_1q_2) + f_{vv}|q_2|^2 = 2a_1 + a_2(q_2+\bar{q}_2) + 2a_3|q_2|^2, \\ j_0 &:= g_{uu}|q_1|^2 + g_{uv}(q_1\bar{q}_2 + \bar{q}_1q_2) + g_{vv}|q_2|^2 = 2b_1 + b_2(q_2+\bar{q}_2) + 2b_3|q_2|^2, \\ l_0 &:= f_{uuu}|q_1|^2q_1 + f_{uuv}(2|q_1|^2q_2 + q_1^2\bar{q}_2) + f_{uvv}(2q_1|q_2|^2 + \bar{q}_1q_2^2) + f_{vvv}|q_2|^2q_2 \\ &\quad = 6a_4 + 2a_5(2q_2+\bar{q}_2) + 2a_6(2|q_2|^2 + q_2^2), \\ m_0 &:= g_{uuu}|q_1|^2q_1 + g_{uuv}(2|q_1|^2q_2 + q_1^2\bar{q}_2) + g_{uvv}(2q_1|q_2|^2 + \bar{q}_1q_2^2) + g_{vvv}|q_2|^2q_2 \\ &\quad = 6b_4 + 2b_5(2q_2+\bar{q}_2) + 2b_6(2|q_2|^2 + q_2^2), \end{aligned}$$

where all the partial derivatives evaluate at the point $(u,v,K) = (0,0,K_0)$. Thus, the reaction-diffusion system restricted to the center manifold in z and $\bar{z}$ coordinates is given by

$$\begin{aligned} \frac{\mathrm{d}z}{\mathrm{d}t} &= i\beta_0 z + \langle q^*, F_0\rangle \\ &= i\beta_0 z + \frac{1}{2}\chi_{20}z^2 + \chi_{11}z\bar{z} + \frac{1}{2}\chi_{02}\bar{z}^2 + \frac{1}{2}\chi_{21}z^2\bar{z} + O(|z|^4) \end{aligned}$$

with

$$\chi_{20} = \langle q^*, (g_0,h_0)^\top\rangle, \quad \chi_{11} = \langle q^*, (i_0,j_0)^\top\rangle, \quad \chi_{21} = \langle q^*, (l_0,m_0)^\top\rangle.$$

Notice that $b_1 = -a_1 - \frac{r}{K}$, $b_2 = -a_2$, $b_3 = -a_3$, $b_4 = -a_4$, $b_5 = -a_5$, and $b_6 = -a_6$. Then, by the straightforward but tedious calculations, we yield

$$\begin{aligned} \chi_{20} &= \frac{\kappa_2}{2\beta_0}\left(\left(\frac{\beta_0}{\kappa_2} - \frac{\kappa_1}{\kappa_2}i\right)g_0 - ih_0\right) \\ &= a_1 - a_2 - \frac{(\kappa_1^2 - 2\kappa_1\kappa_2 + \beta_0^2)a_3}{\kappa_2^2} - \frac{i}{\beta_0\kappa_2^2}\Big[\kappa_2^2(\kappa_1-\kappa_2)a_1 \\ &\quad + \kappa_2(\kappa_1\kappa_2 - \kappa_1^2 - \beta_0^2)a_2 + (\kappa_1^3 - \kappa_1^2\kappa_2 + \beta_0^2\kappa_1 - \beta_0^2\kappa_2)a_3 - \frac{r\kappa_2^3}{K_0}\Big], \end{aligned}$$

$$\chi_{11} = \frac{\kappa_2}{2\beta_0}\left(\left(\frac{\beta_0}{\kappa_2} - \frac{\kappa_1}{\kappa_2}i\right)i_0 - ij_0\right)$$

$$= a_1 - \frac{\kappa_1 a_2}{\kappa_2} + \frac{(\kappa_1^2+\beta_0^2)a_3}{\kappa_2^2} - \frac{i}{\beta_0\kappa_2^2}\Big[\kappa_2^2(\kappa_1-\kappa_2)a_1 + \kappa_1\kappa_2(\kappa_2-\kappa_1)a_2 + (\kappa_1^3 - \kappa_1^2\kappa_2 + \beta_0^2\kappa_1 + \beta_0^2\kappa_2)a_3 - \frac{r\kappa_2^3}{K_0}\Big]$$

and

$$\begin{aligned}\chi_{21} &= \frac{\kappa_2}{2\beta_0}\left(\left(\frac{\beta_0}{\kappa_2} - \frac{\kappa_1}{\kappa_2}i\right)l_0 - im_0\right)\\ &= 3a_4 - \frac{(2\kappa_1+\kappa_2)a_5}{\kappa_2} + \frac{(\kappa_1^2 + 2\kappa_1\kappa_2 + \beta_0^2)a_6}{\kappa_2^2} - \frac{i}{\beta_0\kappa_2^2}\Big[3\kappa_2^2(\kappa_1-\kappa_2)a_4\\ &\quad + \kappa_2(3\kappa_1\kappa_2 - 3\kappa_1^2 - \beta_0^2)a_5 + (3\kappa_1^3 - 3\kappa_1^2\kappa_2 + 3\beta_0^2\kappa_1 - \beta_0^2\kappa_2)a_6\Big].\end{aligned}$$

Thus, we have

$$\begin{aligned}\mathrm{Re}(c_1(K_0)) &= \mathrm{Re}\left\{\frac{i}{2\beta_0}\left(\chi_{20}\chi_{11} - 2|\chi_{11}|^2 - \frac{1}{3}|\chi_{02}|^2\right) + \frac{1}{2}\chi_{21}\right\}\\ &= -\frac{1}{2\beta_0}[\mathrm{Re}(\chi_{20})\mathrm{Im}(\chi_{11}) + \mathrm{Im}(\chi_{20})\mathrm{Re}(\chi_{11})] + \frac{1}{2}\mathrm{Re}(\chi_{21})\\ &= \frac{1}{2K_0\beta_0^2\kappa_2^3}\Big[2K_0\kappa_2^3(\kappa_1-\kappa_2)a_1^2 - K_0\kappa_2^2(3\kappa_1^2 - 2\kappa_1\kappa_2 - \kappa_2^2 + \beta_0^2)a_1a_2 + 2K_0\kappa_1\kappa_2(\kappa_1^2 - \kappa_2^2 + \beta_0^2)a_1a_3\\ &\quad - 2r\kappa_2^4 a_1 + K_0\kappa_1\kappa_2(\kappa_1^2 - \kappa_2^2 + \beta_0^2)a_2^2 - K_0(\kappa_1^4 + 2\kappa_1^3\kappa_2 - 3\kappa_1^2\kappa_2^2 + 2\beta_0^2\kappa_1^2 + 2\beta_0^2\kappa_1\kappa_2 - \beta_0^2\kappa_2^2\\ &\quad + \beta_0^4)a_2a_3 + r\kappa_2^3(\kappa_1+\kappa_2)a_2 + 2K_0(\kappa_1^2+\beta_0^2)(\kappa_1^2 - \kappa_1\kappa_2 + \beta_0^2)a_3^2 - 2r\kappa_1\kappa_2^3a_3 + 3K_0\beta_0^2\kappa_2^3a_4\\ &\quad - K_0\beta_0^2\kappa_2^2(2\kappa_1+\kappa_2)a_5 + K_0\beta_0^2\kappa_2^2(\kappa_1^2 + 2\kappa_1\kappa_2 + \beta_0^2)a_6\Big].\end{aligned}$$

Based on the above analysis, the results can be given as follows.

Theorem 3.4. *Suppose that* $(\mathbf{H_0})$, $(\mathbf{H_{K_1}})$, (3.9), *and* (3.10) *hold. Then, the reaction-diffusion system* (3.5) *undergoes a Hopf bifurcation at* $K = K_0$.

(i) *The direction of Hopf bifurcation is supercritical and the bifurcating periodic solutions are asymptotically stable if* $\mathrm{Re}(c_1(K_0)) < 0$. *Moreover, they are orbitally asymptotically stable in unbounded region* U_1 *or* U_2 *(see* (3.1) *or* (3.2)*), and unstable in unbounded region* U_3 *(see* (3.3)*).*

(ii) *The direction of Hopf bifurcation is subcritical and the bifurcating periodic solutions are unstable if* $\mathrm{Re}(c_1(K_0)) > 0$.

We give the following example to illustrate Theorem 3.4.

Example 3.2. *Choose two sets of coefficients as follows:*

$$a = 0.9,\ b = 2,\ m = 0.8,\ r = 2.4,\ k = 0.6,\ d_1 = 1,\ d_2 = 0.1,\ \ell = 4,\ n = 2, \tag{3.14}$$

$$a = 2,\ b = 1,\ m = 0.7,\ r = 2.4,\ k = 0.2,\ d_1 = 0.1,\ d_2 = 0.01,\ \ell = 3,\ n = 4. \tag{3.15}$$

Under (3.14), $K_0 \approx 12.6419$ *and the parameters satisfy* $(\mathbf{H_0})$, $(\mathbf{H_{K_1}})$, (3.9), *and* (3.10). *By calculations,* $\mathrm{Re}(c_1(K_0)) \approx -0.0314 < 0$. *It follows from Theorem 3.4 that Hopf bifurcation is supercritical and the bifurcating temporal periodic solutions are asymptotically stable. Moreover,*

$\gamma_2 \approx 0.0451$, $-d_1a_{22}/a_{11} - d_2 \approx 0.9 > 0$, *and* $d_2 - \gamma_2 d_1 \approx 0.0549 > 0$, *that is, all parameters lie in* U_2. *The bifurcating periodic orbits of model* (1.5) *are orbitally asymptotically stable (see Figure 7).*

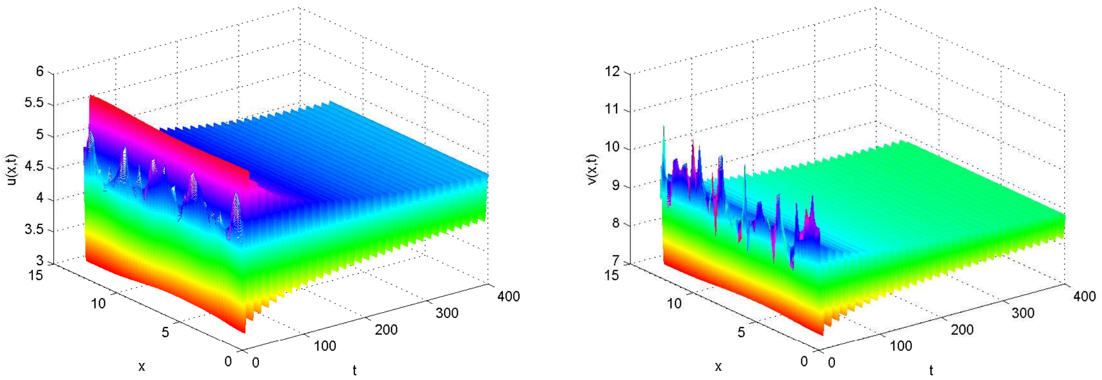


Figure 7. Periodic solutions of model (3.5) with parameters set (3.14), and $K_0 = 12.6419$.

Under (3.15), $K_0 \approx 29.3260$ *and the parameters satisfy* $(\mathbf{H_0})$, $(\mathbf{H_{K_1}})$, (3.9), *and* (3.10). $\mathrm{Re}(c_1(K_0)) \approx 0.3205 > 0$ *implies that Hopf bifurcation is subcritical and the bifurcating temporal periodic solutions are unstable (see Figure 8).*

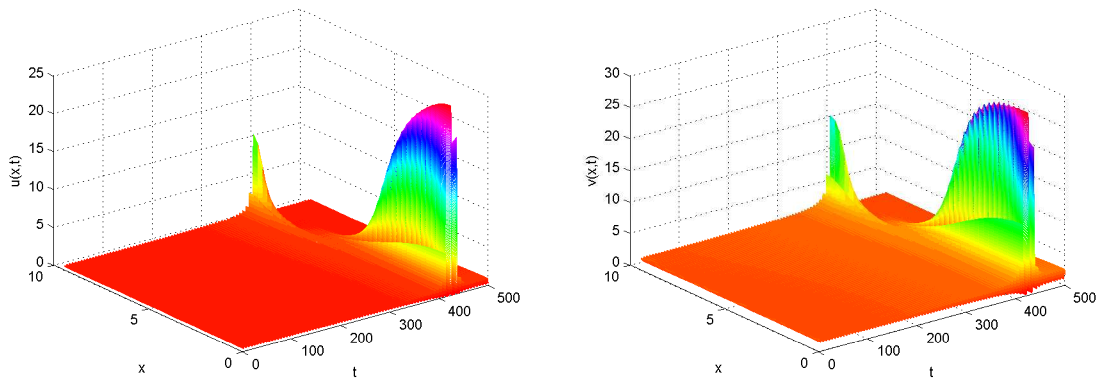


Figure 8. Periodic solutions of model (3.5) with parameters set (3.15), and $K_0 = 29.3260$.

In the absence of fear, we still choose b as the bifurcation parameter. Some direct computations are similar to the above analysis associated with Theorem 3.4, hence we only give the following corresponding numerical results.

Example 3.3. *Take the system coefficients as follows:*

$$a = 0.9,\ m = 0.75,\ r = 6.4,\ K = 3,\ d_1 = 3,\ d_2 = 0.1,\ \ell = 1.2,\ n = 1. \tag{3.16}$$

Similar to the verification in Example 3.2, system (3.5) *(with* $k = 0$*) undergoes a supercritical Hopf bifurcation and the bifurcating temporal periodic solutions are orbitally asymptotically stable (see Figure 9).*

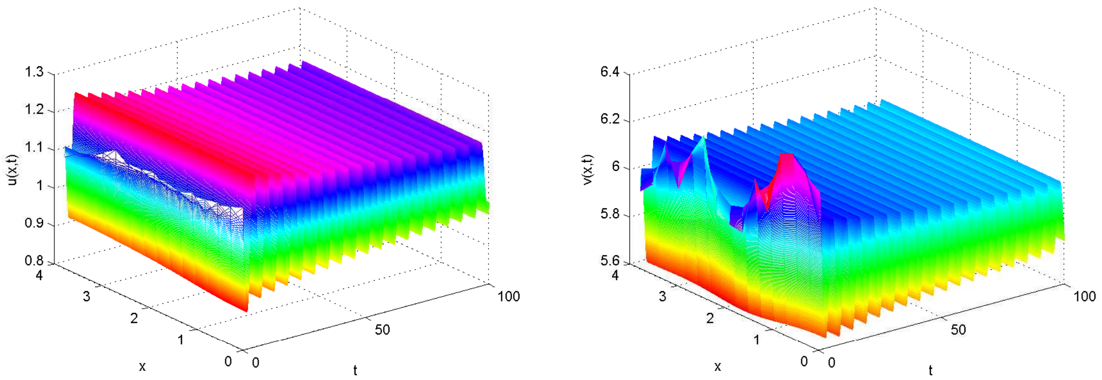


Figure 9. Periodic solutions of model (3.5) with $k = 0$, parameters set (3.16), and $b_0 = 1.1128$.

Remark 3.2. *Denote*

$$\eta_1 = rk^2\ell^2(a^2-m^2)\text{arctanh}^3\left(\frac{m}{a}\right)+bk\ell^2(m+r)(a^2-m^2)\text{arctanh}^2\left(\frac{m}{a}\right) + bm\left[akn^2(d_1+d_2) + b\ell^2(a^2-m^2)\right]\text{arctanh}\left(\frac{m}{a}\right)-2ab^2m^2\ell^2,$$

$$\eta_2 = k\ell^2(a^2-m^2)\text{arctanh}^2\left(\frac{m}{a}\right)+\left[akn^2(d_1+d_2)+b\ell^2(a^2-m^2) + akr\ell^2\right]\text{arctanh}\left(\frac{m}{a}\right)-2abm\ell^2,$$

$$\bar{K} = \left\{K\middle|T_n(K) = 0,\ K > \text{arctanh}\left(\frac{m}{a}\right)/b\right\}.$$

If

$$\frac{\eta_1}{\eta_2\left[abmn^2(d_1+d_2) + r\ell^2(s^*-abm)\right]} > 0$$

with s^* defined as (2.4)*, where* $(\mathbf{H_{K_1}})$ *or* $(\mathbf{H_{K_2}})$ *holds, then system* (3.5) *undergoes new Hopf bifurcations at the Hopf bifurcation point* $\bar{K} \in \left(\text{arctanh}\left(\frac{m}{a}\right)/b, K_0\right)$*. The new Hopf bifurcations produce temporal inhomogeneous patterns.*

Remark 3.3. *There will be a Turing-Hopf bifurcation in system* (1.5)*. The existence and normal form of the Turing-Hopf bifurcation can be analyzed as in [34]. Considering the limited length of this paper, we will not expand the discussion.*

4. Positive nonconstant steady states of (1.5)

In this section, the nonexistence and existence of positive nonconstant steady states for elliptic system (1.6) are investigated.

4.1. A priori estimates

A prior estimates for the positive solution of (1.6) are established by the maximum principle in Lou and Ni [35] as follows.

Theorem 4.1. *Assume that* $\operatorname{arctanh}\left(\frac{m}{a}\right) < bK < 2\operatorname{arctanh}\left(\frac{m}{a}\right)$ *and* $\frac{4a}{rK}\left(bK - \operatorname{arctanh}\left(\frac{m}{a}\right)\right) < k$. *Then any positive solution* (u, v) *of* (1.6) *satisfies*

$$\frac{rkK + \sqrt{\triangle_*}}{2rk} \le u(x) \le K,$$

$$\frac{rk\left(bK - 2\operatorname{arctanh}\left(\frac{m}{a}\right)\right) + b\sqrt{\triangle_*}}{2rk^2\operatorname{arctanh}\left(\frac{m}{a}\right)} \le v(x) \le \frac{bK - \operatorname{arctanh}\left(\frac{m}{a}\right)}{k\operatorname{arctanh}\left(\frac{m}{a}\right)}$$

with

$$\triangle_* := r^2k^2K^2 - 4arkK\left(bK - \operatorname{arctanh}\left(\frac{m}{a}\right)\right) > 0.$$

Proof. Let (u, v) be a given positive solution of (1.6) and

$$u_M = \max_{\bar{\Omega}} u(x), \quad u_L = \min_{\bar{\Omega}} u(x),$$

$$v_M = \max_{\bar{\Omega}} v(x), \quad v_L = \min_{\bar{\Omega}} v(x).$$

Applying the maximum principle in [35] to two equations of (1.6) yields that

$$\begin{cases} ru_M\left(1 - \frac{u_M}{K}\right) - a\tanh\left(\frac{bu_M}{1+kv_M}\right)v_L \ge 0, \\ ru_L\left(1 - \frac{u_L}{K}\right) - a\tanh\left(\frac{bu_L}{1+kv_L}\right)v_M \le 0 \end{cases} \tag{4.1}$$

and

$$\begin{cases} a\tanh\left(\frac{bu_M}{1+kv_M}\right) - m \ge 0, \\ a\tanh\left(\frac{bu_L}{1+kv_L}\right) - m \le 0. \end{cases} \tag{4.2}$$

Solving the first inequality of (4.1) and (4.2), respectively, one obtains

$$u_M \le K, \quad v_M \le \frac{bK - \operatorname{arctanh}\left(\frac{m}{a}\right)}{k\operatorname{arctanh}\left(\frac{m}{a}\right)}.$$

The second inequality of (4.2) entails

$$\frac{bu_L}{1 + kv_L} \le \operatorname{arctanh}\left(\frac{m}{a}\right). \tag{4.3}$$

Further, solve the second inequality of (4.1) to see that

$$ru_L\left(1 - \frac{u_L}{K}\right) - \frac{a}{k}\left(bK - \operatorname{arctanh}\left(\frac{m}{a}\right)\right) \le ru_L\left(1 - \frac{u_L}{K}\right) - \frac{abu_Lv_M}{1 + kv_L} \le 0.$$

Thus,

$$u_L \ge \frac{rkK + \sqrt{\triangle_*}}{2rk}.$$

Moreover, from (4.3) we have

$$v_L \ge \frac{bu_L - \operatorname{arctanh}\left(\frac{m}{a}\right)}{k\operatorname{arctanh}\left(\frac{m}{a}\right)} \ge \frac{rk\left(bK - 2\operatorname{arctanh}\left(\frac{m}{a}\right)\right) + b\sqrt{\triangle_*}}{2rk^2\operatorname{arctanh}\left(\frac{m}{a}\right)}.$$

This completes the proof. □

4.2. Nonexistence of positive nonconstant steady states

One nonexistence result of positive solutions is considered as follows.

Theorem 4.2. *If* arctanh$\left(\frac{m}{a}\right) \geq bK$*, then* (1.6) *has no positive nonconstant solution.*

Proof. On the contrary, suppose that (1.6) has a positive nonconstant solution (u, v) when arctanh$\left(\frac{m}{a}\right) \geq bK$. Integrating the second equation of (1.6) on Ω, one has

$$0 = -d_2 \int_\Omega \Delta v \mathrm{d}x = \int_\Omega \left(a \tanh\left(\frac{bu}{1 + kv}\right) - m\right) v \mathrm{d}x < \int_\Omega (a \tanh(bK) - m) v \mathrm{d}x.$$

Obviously, this is a contradiction with arctanh$\left(\frac{m}{a}\right) \geq bK$. □

Now, we prove another nonexistence result of positive solutions. For notational convenience, one can denote the parameter set consisting of a, b, m, r, k, and K as $M^* = M^*(a, b, m, r, k, K)$.

Theorem 4.3. *There exist positive constants* $d_1^* = d_1^*(M^*, \mu_1)$ *and* $d_2^* = d_2^*(M^*, \mu_1)$ *such that system* (1.6) *has no positive nonconstant solution when* $d_1 \geq d_1^*$ *and* $d_2 \geq d_2^*$.

Proof. Suppose that (u, v) is a positive solution of (1.6) when $d_1 > d_1^*$ and $d_2 > d_2^*$ for some positive constants d_1^* and d_2^*. Denote

$$\hat{u} = \frac{1}{|\Omega|} \int_\Omega u(x) \mathrm{d}x, \quad \hat{v} = \frac{1}{|\Omega|} \int_\Omega v(x) \mathrm{d}x.$$

Multiply the two equations in (1.6) by $u - \hat{u}$ and $v - \hat{v}$, and then integrate the obtained equations by parts, respectively. It follows from Theorem 4.1 that

$$\begin{aligned}
\int_\Omega d_1 |\nabla u|^2 \mathrm{d}x &= \int_\Omega \left[r\left(1 - \frac{u + \hat{u}}{K}\right)(u - \hat{u})^2 - a(u - \hat{u})\left(\tanh\left(\frac{bu}{1 + kv}\right) v - \tanh\left(\frac{b\hat{u}}{1 + k\hat{v}}\right)\hat{v}\right)\right] \mathrm{d}x \\
&< \int_\Omega \left[r(u - \hat{u})^2 - a(u - \hat{u})\left(\tanh\left(\frac{bu}{1 + kv}\right) v - \tanh\left(\frac{bu}{1 + kv}\right)\hat{v} \right.\right. \\
&\quad \left.\left. + \tanh\left(\frac{bu}{1 + kv}\right)\hat{v} - \tanh\left(\frac{bu}{1 + k\hat{v}}\right)\hat{v} + \tanh\left(\frac{bu}{1 + k\hat{v}}\right)\hat{v} - \tanh\left(\frac{b\hat{u}}{1 + k\hat{v}}\right)\hat{v}\right)\right] \mathrm{d}x \\
&< \int_\Omega \Big((r + ab\hat{v})(u - \hat{u})^2 + a(1 + bkK\hat{v})|u - \hat{u}||v - \hat{v}|\Big) \mathrm{d}x \\
&\leq \int_\Omega (r + ab\hat{v})(u - \hat{u})^2 \mathrm{d}x + a(1 + bkK\hat{v})\left(\frac{\mu_1}{4} \int_\Omega (u - \hat{u})^2 \mathrm{d}x + \frac{1}{\mu_1} \int_\Omega (v - \hat{v})^2 \mathrm{d}x\right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\int_\Omega d_2 |\nabla v|^2 \mathrm{d}x &< \int_\Omega \Big(-m(v - \hat{v})^2 + a|v - \hat{v}|(|v - \hat{v}| + bkK\hat{v}|v - \hat{v}| + b\hat{v}|u - \hat{u}|)\Big) \mathrm{d}x \\
&\leq \int_\Omega (a - m + abkK\hat{v})(v - \hat{v})^2 \mathrm{d}x + ab\hat{v}\left(\frac{\mu_1}{4} \int_\Omega (u - \hat{u})^2 \mathrm{d}x + \frac{1}{\mu_1} \int_\Omega (v - \hat{v})^2 \mathrm{d}x\right).
\end{aligned}$$

The above estimates imply that

$$d_1 \int_\Omega |\nabla u|^2 \mathrm{d}x + d_2 \int_\Omega |\nabla v|^2 \mathrm{d}x < \Big(r + ab\hat{v} + \frac{a\mu_1}{4}(1 + b\hat{v} + bkK\hat{v}) \Big) \int_\Omega (u - \hat{u})^2 \mathrm{d}x + \Big(a - m + abkK\hat{v} + \frac{a}{\mu_1}(1 + b\hat{v} + bkK\hat{v}) \Big) \int_\Omega (v - \hat{v})^2 \mathrm{d}x.$$

Applying the Poincaré inequality $\mu_1 \int_\Omega (\omega - \bar{\omega})^2 \mathrm{d}x \le \int_\Omega |\nabla \omega|^2 \mathrm{d}x$ yields

$$d_1\mu_1 \int_\Omega (u - \hat{u})^2 \mathrm{d}x + d_2\mu_1 \int_\Omega (v - \hat{v})^2 \mathrm{d}x \le d_1 \int_\Omega |\nabla u|^2 \mathrm{d}x + d_2 \int_\Omega |\nabla v|^2 \mathrm{d}x.$$

Let

$$d_1^* = \frac{a(1 + b\hat{v} + bkK\hat{v})\mu_1 + 4(r + ab\hat{v})}{4\mu_1},$$
$$d_2^* = \frac{(a - m + abkK\hat{v})\mu_1 + a(1 + b\hat{v} + bkK\hat{v})}{\mu_1^2}.$$

Then

$$d_1 \int_\Omega (u - \hat{u})^2 \mathrm{d}x + d_2 \int_\Omega (v - \hat{v})^2 \mathrm{d}x < d_1^* \int_\Omega (u - \hat{u})^2 \mathrm{d}x + d_2^* \int_\Omega (v - \hat{v})^2 \mathrm{d}x.$$

Obviously, it leads to a contradiction with $d_1 \ge d_1^*$ and $d_2 \ge d_2^*$. □

4.3. Existence of positive nonconstant steady states

The existence of positive nonconstant solutions is studied by applying the fixed point index theory [36, 37]. Theorem 3.1 indicates that there is no positive nonconstant solution to (1.6) near $\bar{E}$ when $(\mathbf{H_3})$ holds. Hence, we always assume that $(\mathbf{H_0})$ and $(\mathbf{H_4})$ are satisfied in the sequel.

The following preliminaries for later use can be found in [38, 39].

Let $\mathbf{E}$ be a real Banach space and $P \subset \mathbf{E}$ be a closed convex set. For any real number $\alpha \ge 0$, P is called a wedge on $\mathbf{E}$ if $\alpha P \subset P$. A wedge P is termed as a cone if $P \cap \{-P\} = \{0\}$. For $\zeta \in P$, denote

$$P_\zeta = \{\hbar \in \mathbf{E} | \zeta + l\hbar \in P,\ l > 0\}, \quad Q_\zeta = \{\hbar \in \bar{P}_\zeta | -\hbar \in \bar{P}_\zeta\}.$$

Then, $\bar{P}_\zeta$ is a wedge and it is convex, and Q_ζ is a closed convex set on $\mathbf{E}$.

Set Y to be a compact linear map on $\mathbf{E}$ which is invariant on $\bar{P}_\zeta$. If there exist $\iota \in (0, 1)$ and $y \in \bar{P}_\zeta \backslash Q_\zeta$ such that $y - \iota Y y \in Q_\zeta$, then Y has property α. Assume P is a wedge on $\mathbf{E}$ and $F : P \to P$ is a compact map with fixed point $\zeta_0 \in P$. The Fréchet derivative of F at ζ_0 is defined as $T = F'(\zeta_0)$. Then, T maps $\bar{P}_{\zeta_0}$ into itself.

Now the fixed point index of compact maps is given as follows.

Lemma 4.1. *(1) Suppose that $I - T$ is invertible in $\mathbf{E}$ and T has property α on $\bar{P}_{\zeta_0}$. Then,* $\mathrm{index}(F, \zeta_0) = 0$.

(2) Suppose that $I - T$ is not invertible in $\mathbf{E}$ but is on $\bar{P}_{\zeta_0} \backslash \{0\}$. Moreover, $I - T : \bar{P}_{\zeta_0} \to \bar{P}_{\zeta_0}$ is not surjective. Then, $\mathrm{index}(F, \zeta_0) = 0$.

As is well-known index(F, ζ_0) means the Leray-Schauder degree $\deg(I - F, U(\zeta_0), 0)$, where $U(\zeta_0)$ is a neighborhood of ζ_0 in P.

If (u, v) is a positive solution of (1.6), it follows from Theorem 4.1 that there exist positive constants D_1 and D_2 such that $D_1 < u, v < D_2$. Denote

$$\begin{aligned}
X &= \{(u, v) \in (C^1(\bar{\Omega}))^2 |\ \partial_\upsilon u = \partial_\upsilon v = 0,\ x \in \partial\Omega\},\\
X^+ &= \{(u, v) \in X | u, v \geq 0,\ x \in \bar{\Omega},\ \partial_\upsilon u = \partial_\upsilon v = 0,\ x \in \partial\Omega\},\\
\mathcal{D} &= \{(u, v) \in X^+ | D_1 < u, v < D_2,\ x \in \bar{\Omega}\},\\
\mathcal{A} &= \{u \in C_0(\bar{\Omega}) | u \geq 0,\ x \in \bar{\Omega},\ \partial_\upsilon u = 0,\ x \in \partial\Omega\},\\
P &= \mathcal{A} \oplus \mathcal{A}.
\end{aligned}$$

Then, $\mathcal{A}$ is a cone in $C_0(\bar{\Omega})$.

We can find a large constant $\Theta > 0$ due to the boundedness of (u, v) such that

$$\max\Big\{\max_{\bar{\Omega}}\{|ru(1 - u/K) - a\tanh(bu/(1 + kv))v|\}, \max_{\bar{\Omega}}\{|(a\tanh(bu/(1 + kv)) - m)v|\}\Big\} < \Theta.$$

To calculate the index(F, E_0) and index(F, E_K), one can let

$$F(u, v) = \begin{pmatrix} (-d_1\Delta + \Theta)^{-1}\Big(ru\left(1 - \frac{u}{K}\right) - a\tanh\left(\frac{bu}{1+kv}\right)v + \Theta u\Big) \\ (-d_2\Delta + \Theta)^{-1}\Big((a\tanh\left(\frac{bu}{1+kv}\right) - m)v + \Theta v\Big) \end{pmatrix},$$

where $(-d_i\Delta + \Theta)^{-1}$, $i = 1, 2$ are positive compact linear operators, and F is the direct sum of positive compact linear operators according to the strong maximum principle. Obviously, (u, v) is the solution of (1.6) if, and only if, (u, v) is the fixed point of F, which is independent of the choice of Θ.

Take into account the system

$$\begin{cases} -d_1\Delta u = ru\left(\vartheta - \frac{u}{K}\right) - a\tanh\left(\frac{bu}{1+kv}\right)v, & x \in \Omega,\\ -d_2\Delta v = \left(a\vartheta\tanh\left(\frac{bu}{1+kv}\right) - m\right)v, & x \in \Omega,\\ \frac{\partial u}{\partial \upsilon} = \frac{\partial v}{\partial \upsilon} = 0, & x \in \partial\Omega, \end{cases} \tag{4.4}$$

with $\vartheta \in [0, 1]$. It can be similarly proved that $(u_\vartheta, v_\vartheta)$ is bounded if $(u_\vartheta, v_\vartheta)$ is a solution of (4.4) and $D_1 < u_\vartheta, v_\vartheta < D_2$.

Denote

$$F_\vartheta(u, v) = \begin{pmatrix} (-d_1\Delta + \Theta)^{-1}\Big(ru\left(\vartheta - \frac{u}{K}\right) - a\tanh\left(\frac{bu}{1+kv}\right)v + \Theta u\Big) \\ (-d_2\Delta + \Theta)^{-1}\Big((a\vartheta\tanh\left(\frac{bu}{1+kv}\right) - m)v + \Theta v\Big) \end{pmatrix}.$$

Clearly, $(u_\vartheta, v_\vartheta)$ is a solution of (4.4) if, and only if, $(u_\vartheta, v_\vartheta)$ is a fixed point of F_ϑ. Moreover, by the boundedness of solutions of (4.4), there is no fixed point of F_ϑ on $\partial\mathcal{D}$. Then, the homotopy invariance of degree shows that index$(F_\vartheta, \mathrm{int}\mathcal{D}, P)$ does not depend on ϑ. Furthermore, if (u, v) is a fixed point of F_ϑ, then

$$\deg(I - F_\vartheta, U(u, v), (0, 0)) = \mathrm{index}(F'_\vartheta(u, v), (0, 0), P) = (-1)^\sigma, \tag{4.5}$$

where $U(u, v)$ is a neighborhood of (u, v), and σ is the sum of algebraic multiplicities of all eigenvalues of $F'_\vartheta(u, v)$ which are greater than 1. Therefore, index$(F'_\vartheta(u, v), (0, 0), P) = 1$ by (4.5) if the spectral radius $R(F'_\vartheta(u, v)) \leq 1$.

Now, we take $\vartheta = 0$. Obviously, (4.4) has a unique nonnegative solution $(0, 0)$. Further, $(0, 0)$ is a unique fixed point of F_ϑ on P. Hence,

$$\text{index}(F_\vartheta, \text{int}\mathcal{D}, P) = \deg(I - F_\vartheta, U(0, 0), (0, 0)).$$

By direct computations, one has

$$F'_\vartheta(0, 0) = \begin{pmatrix} (-d_1\Delta + \Theta)^{-1}(r\vartheta + \Theta) & 0 \\ 0 & (-d_2\Delta + \Theta)^{-1}(\Theta - m) \end{pmatrix}.$$

It is obvious that $F'_\vartheta(0, 0)$ has no eigenvalues which are greater than 1 when $\vartheta = 0$. Thus, $R(F'_\vartheta(u, v)) \leq 1$ and

$$\begin{aligned} \text{index}(F_\vartheta, \text{int}\mathcal{D}, P) &= \deg(I - F_\vartheta, U(0, 0), (0, 0)) \\ &= \text{index}(F'_\vartheta(0, 0), (0, 0), P) \\ &= 1 \end{aligned} \tag{4.6}$$

holds for all $\vartheta \in [0, 1]$. Particularly, take $\vartheta = 1$. For $F_1 := F$, one has

$$\text{index}(F, \text{int}\mathcal{D}, P) = \text{index}(F_1, \text{int}\mathcal{D}, P) = 1.$$

We discuss the indexes of F at E_0, E_K and $\bar{E}$ below. To obtain the index of F at $\bar{E}$, some other preliminaries need to be provided. Let $D = \text{diag}(d_1, d_2)$, $E = (u, v)$, and

$$G(E) = \begin{pmatrix} ru\left(1 - \frac{u}{K}\right) - a\tanh\left(\frac{bu}{1+kv}\right)v \\ \left(a\tanh\left(\frac{bu}{1+kv}\right) - m\right)v \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

with a_{ij} given as (2.3). Then, system (1.6) could be rewritten as

$$-D\Delta E = G(E),\ x \in \Omega; \quad \partial_\nu E = 0,\ x \in \partial\Omega. \tag{4.7}$$

Hence E is a solution of (1.6), if, and only if, E is a solution of (4.7), if, and only if, E is a solution of the problem

$$\Psi(E) := E - (I - \Delta)^{-1}(D^{-1}G(E) + E) = 0,\ x \in \Omega; \quad \partial_\nu E = 0,\ x \in \partial\Omega$$

in X^+. Obviously, the operator Ψ is a compact perturbation of the identity operator I. In view of the definition of the Leray-Schauder degree, we conclude from $\Psi(E) \neq 0$ on $\partial\Omega$ that $\deg(\Psi, \mathcal{D}, 0)$ is well-defined. Furthermore, E is a fixed point of F if, and only if, E is a zero point of Ψ.

In what follows, we look for the index of F at $\bar{E}$ by Ψ. The main result of this subsection is as follows.

Theorem 4.4. *Assume that $I - F'(E_K)$ is invertible on $(C_0(\bar{\Omega}))^2$. If the determinant* $\det(A + D) < 0$*, then the system* (1.6) *has at least a positive nonconstant solution.*

Proof. We first calculate the index of F at E_0. Clearly, $\bar{P}_{E_0} = P$ and $Q_{E_0} = \{(0, 0)\}$. Consider

$$F'(E_0) = \begin{pmatrix} (-d_1\Delta + \Theta)^{-1}(r + \Theta & 0) \\ (-d_2\Delta + \Theta)^{-1}(0 & \Theta - m) \end{pmatrix}.$$

Take $(\tau, \jmath) \in (C_0(\bar{\Omega}))^2$. If $(I - F'(E_0))(\tau, \jmath)^\top = 0$, then

$$\begin{cases} -d_1\Delta\tau = r\tau, \\ -d_2\Delta\jmath = -m\jmath. \end{cases} \tag{4.8}$$

The invertibility of $I - F'(E_0)$ on $(C_0(\bar{\Omega}))^2$ is investigated as follows.

The second equation of (4.8) means that $\jmath \equiv 0$. For the first equation, the following two cases need to be considered.

Case 1. $\frac{r}{d_1} \notin \sigma(-\Delta)$, where $\sigma(-\Delta)$ represents the spectrum set of $-\Delta$ with homogenous Neumann boundary conditions. Thus, $\tau \equiv 0$, which implies that $I - F'(E_0)$ is invertible on $(C_0(\bar{\Omega}))^2$. Set $\imath = \frac{\Theta}{r+\Theta}$. Then, $\imath \in (0, 1)$. Let f_0 be the principal eigenfunction of $-\Delta$. Then, $(f_0, 0) \in \bar{P}_{E_0} \backslash Q_{E_0}$. Hence,

$$(I - \imath F'(E_0))\begin{pmatrix} f_0 \\ 0 \end{pmatrix} = \begin{pmatrix} f_0 - \imath(-d_1\Delta + \Theta)^{-1}(r + \Theta)f_0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in Q_{E_0},$$

which indicates that $F'(E_0)$ has property α on $\bar{P}_{E_0}$. Therefore, index$(F, E_0) = 0$ due to Lemma 4.1.

Case 2. $\frac{r}{d_1} \in \sigma(-\Delta)$. Then, there exist $\tau,\ \jmath \in C_0(\bar{\Omega})$, $\tau \not\equiv 0$, satisfying $(I - F'(E_0))(\tau, \jmath)^\top = 0$. Thus, $I - F'(E_0)$ is not invertible on $(C_0(\bar{\Omega}))^2$. Assume that τ_r is the eigenfunction of $I - F'(E_0)$ corresponding to $\frac{r}{d_1}$. Then, τ_r must change sign in Ω, which means that $I - F'(E_0)$ is invertible on $\bar{P}_{E_0}$. Furthermore, the nodes of τ_r divide Ω into finite subregions. Hence, we can find a positive function $\tilde{\tau}$, $\partial_\upsilon\tilde{\tau}|_{\partial\Omega} = 0$ on Ω that satisfies $\int_\Omega \tilde{\tau}\tau_r\mathrm{d}x \neq 0$. We claim that $I - F'(E_0)$ is not surjective on $\bar{P}_{E_0}$.

In fact, take $\jmath \in \mathcal{A}$ and assume that there exist $\tau_0,\ \jmath_0 \in \mathcal{A}$ such that $(I - F'(E_0))(\tau_0, \jmath_0)^\top = (\tilde{\tau}, \jmath)^\top$. Then,

$$\begin{cases} \tau_0 - (r + \Theta)(-d_1\Delta + \Theta)^{-1}\tau_0 = \tilde{\tau}, \\ \jmath_0 - (\Theta - m)(-d_2\Delta + \Theta)^{-1}\jmath_0 = \jmath. \end{cases} \tag{4.9}$$

Multiply the first equation of (4.9) by τ_r and then integrate over Ω to obtain

$$\int_\Omega (-d_1\Delta - r)\tau_0\tau_r\mathrm{d}x = \int_\Omega (-d_1\Delta + \Theta)\tilde{\tau}\tau_r\mathrm{d}x. \tag{4.10}$$

Clearly, the lefthand side is $\int_\Omega(-d_1\Delta - r)\tau_0\tau_r\mathrm{d}x = 0$ of (4.10), whereas, the righthand side is $(r + \Theta)\int_\Omega \tilde{\tau}\tau_r\mathrm{d}x \neq 0$ of (4.10). This implies that there does not exist $\tau_0,\ \jmath_0 \in \mathcal{A}$ such that $(I - F'(E_0))(\tau_0, \jmath_0)^\top = (\tilde{\tau}, \jmath)^\top$. Consequently, $I - F'(E_0)$ is not surjective on $\bar{P}_{E_0}$. One can still obtain index$(F, E_0) = 0$ due to Lemma 4.1.

Next, we calculate index(F, E_K). For E_K, $\bar{P}_{E_K} = C_0(\bar{\Omega}) \oplus \mathcal{A}$, and $Q_{E_K} = C_0(\bar{\Omega}) \oplus \{0\}$, consider

$$F'(E_K) = \begin{pmatrix} (-d_1\Delta + \Theta)^{-1}(- r + \Theta & - a\tanh(bK)) \\ (-d_2\Delta + \Theta)^{-1}(0 & a\tanh(bK) - m + \Theta) \end{pmatrix}.$$

Take $\imath = \frac{-(r+\delta)+\Theta}{-r+\Theta}$ for some $\delta > 0$, then $\imath \in (0, 1)$ since Θ is large. Moreover, f_0 is still denoted as the principal eigenfunction of $-\Delta$. Then, for $(f_0, 0) \in \bar{P}_{E_K} \backslash Q_{E_K}$, one has

$$\begin{aligned} (I - \imath F'(E_K))\begin{pmatrix} f_0 \\ 0 \end{pmatrix} &= \begin{pmatrix} f_0 - \imath(-d_1\Delta + \Theta)^{-1}(-r + \Theta)f_0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} f_0 - (-d_1\Delta + \Theta)^{-1}\left(-(r + \delta) + \Theta\right) f_0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{r+\delta}{\Theta} f_0 \\ 0 \end{pmatrix} \in Q_{E_K}, \end{aligned}$$

which induces that $F'(E_K)$ has property α on $\bar{P}_{E_K}$. We have already assume that $I - F'(E_K)$ is invertible on $(C_0(\bar{\Omega}))^2$, thus index$(F, E_K) = 0$ by Lemma 4.1.

Then, we will change our ideas about calculating index$(F, \bar{E})$.

Notice that the Fréchet derivative of F at $\bar{E}$ is precisely the matrix A, then

$$I - \Psi'(\bar{E}) = (I - \Delta)^{-1}(D^{-1}A + I).$$

By the definition of fixed point index, if $I - \Psi'(\bar{E})$ is invertible, then the fixed point index of $I - \Psi$ at $\bar{E}$ is denoted by index$(I - \Psi, \bar{E}) = (-1)^{\iota}$, where ι is the number of negative eigenvalues of $I - \Psi'(\bar{E})$. Therefore, we need to investigate the eigenvalues of $I - \Psi'(\bar{E})$.

It is not hard to verify that μ is an eigenvalue of $I - \Psi'(\bar{E})$ if, and only if, μ is an eigenvalue of $\frac{1}{1+\mu_n}(D^{-1}A + I)$. This indicates that $I - \Psi'(\bar{E})$ is invertible if, and only if, the matrix $D^{-1}A + I$ is non-degenerate. Hence we only need to focus on the determinant $D^{-1}A + I$. Since

$$\det(D^{-1}A + I) = \frac{1}{d_1 d_2}\det(A + D),$$

if $\det(A + D) \neq 0$, then $I - \Psi'(\bar{E})$ is invertible and the number of negative eigenvalues of $I - \Psi'(\bar{E})$ is odd if, and only if, $\det(D^{-1}A + I) < 0$. In fact, our assumption implies that

$$\text{index}(I - \Psi, \bar{E}) = -1,$$

which means that index$(F, \bar{E}) = -1$. Recall index$(F, E_0) = $ index$(F, E_K) = 0$. We derive the result and the proof is completed. □

Remark 4.1. *A priori estimates in Subsection 4.1 play an important role in later discussions. However, it is difficult to establish a priori positive lower bounds for the positive solution of* (1.6) *when* $k = 0$. *Then, the homotopy mapping defined later is zero at the boundary and the Leray-Schauder degree has no definition. Hence, we cannot investigate the existence of positive nonconstant solutions applying neither the fixed point index theory nor Leray-Schauder degree theory. It further indicates that the fear effect makes it possible to apply the fixed point index theory or Leray-Schauder degree theory to study spatially nonconstant steady states. On the other hand, this is another reason why model* (1.4) *is discussed instead of* (1.3).

5. Conclusions

In this paper, Hopf bifurcation and Turing pattern of a diffusive R-M predator-prey model with both hyperbolic tangent functional response and fear factor were studied. Mathematical analysis and numerical simulations reveal that fear factor plays a key role in the dynamic behavior of the system.

For the ODE system (1.4), we first gave a detailed classification of equilibria based on stability analysis. It is shown that positive equilibrium $\bar{E}$ becomes locally asymptotically stable as k increases to a critical value k_0 (see Example 2.1). Biologically, strong fear effect could stabilize the system by changing the prey's foraging activity, defense behavior, reproduction capacity, and so on. Then, we proved the existence, direction and stability of Hopf bifurcation for (1.4) by the Poincaré-Andronov-Hopf bifurcation theorem. It is observed by numerical simulation that system (1.2) exhibits the bistability phenomenon. System (1.4) produces a stable limit cycle that surrounds unstable $\bar{E}$ arising from a supercritical Hopf bifurcation. It indicates that both the capture rate of predator and fear factor have a stabilizing effect. Finally, the existence of limit cycles and the transversality condition for transcritical bifurcation of (1.4) were discussed. It is shown that system (1.4) will generate a limit cycle when the prey is in a state of low fear or carrying capacity is large enough, and will always undergo a transcritical bifurcation regardless of whether the prey is in fear or not.

For the diffusive system (1.5), Case I. $k = 0$, the stability of positive equilibrium E^* of (1.2) is exactly the same as for (1.5) with $k = 0$, which means that the Turing instability of both positive equilibrium and homogeneous periodic orbits does not occur in (1.5) with $k = 0$. Case II. $k > 0$, we first demonstrated that positive equilibrium $\bar{E}$ is locally asymptotically stable w.r.t. R-D model (1.5) in unbounded region U_1 or U_2 (see (3.1) or (3.2)), and unstable w.r.t. R-D model (1.5) in unbounded region U_3 (see (3.3)), which implies that Turing instability of positive equilibrium $\bar{E}$ occurs under weak fear effect or strong carrying capacity. Then, the direction of Hopf bifurcation and the stability of bifurcating periodic solutions for system (3.5) were discussed. If the fear cost is sufficiently small and the diffusion coefficients d_1 and d_2 are large enough, system (3.5) undergoes new Hopf bifurcations and exhibits Turing instability of spatially homogeneous periodic orbits. It is shown that fear effect can drive Turing instability and create spatial inhomogeneous patterns. Finally, the existence and nonexistence of positive nonconstant steady states of (1.5) were studied applying the fixed point index theory.

To sum up, the qualitative analysis of the R-M predator-prey model with hyperbolic tangent functional response reveals that different response functions play an important role in determining dynamics of the model. We mainly investigated the impact of fear on the R-M predator-prey model with hyperbolic tangent functional response. It is shown that fear factor could stabilize the system, cause the system to generate limit cycles, and give rise to spatial and temporal inhomogeneous patterns.

Author contributions

Jing Zhang: Formal analysis, writing-original draft, review & editing; Shengmao Fu: Supervision, writing-original draft, review & editing, project administration, methodology. All authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

The authors declare no competing interests.

Appendix A. The proof of Theorem 2.2

In order to understand the detailed property of the Hopf bifurcation, we need a further analysis for the normal form. One can translate the positive equilibrium $\bar{E}$ to the origin by the transformation $\breve{u} = u - \bar{u}$ and $\breve{v} = v - \bar{v}$. For the sake of convenience, we still denote $\breve{u}$ and $\breve{v}$ by u and v, respectively. Thus, the local system (1.4) can be given by

$$\begin{cases} \dfrac{du}{dt} = r(u+\bar{u})\left(1 - \frac{u+\bar{u}}{K}\right) - a\tanh\left(\frac{b(u+\bar{u})}{1+k(v+\bar{v})}\right)(v+\bar{v}), \\ \dfrac{dv}{dt} = \left(a\tanh\left(\frac{b(u+\bar{u})}{1+k(v+\bar{v})}\right) - m\right)(v+\bar{v}). \end{cases} \tag{5.1}$$

Rewrite the system (5.1) as

$$\begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} = J\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f(u,v,K) \\ g(u,v,K) \end{pmatrix}, \tag{5.2}$$

where

$$f(u,v,K) = a_1u^2 + a_2uv + a_3v^2 + a_4u^3 + a_5u^2v + a_6uv^2 + \cdots,$$
$$g(u,v,K) = b_1u^2 + b_2uv + b_3v^2 + b_4u^3 + b_5u^2v + b_6uv^2 + \cdots,$$

and

$$\begin{aligned}
a_1 &= -\frac{r}{K} - \frac{m(a^2-m^2)}{a^2k\bar{u}^2}\text{arctanh}\left(\frac{m}{a}\right)\left(\text{arctanh}\left(\frac{m}{a}\right) - b\bar{u}\right),\\
a_2 &= \frac{a^2-m^2}{a^2b\bar{u}^2}\text{arctanh}^2\left(\frac{m}{a}\right)\left(2m\,\text{arctanh}\left(\frac{m}{a}\right) - 2mb\bar{u} - a\right),\\
a_3 &= \frac{k(a^2-m^2)}{a^2b^2\bar{u}^2}\text{arctanh}^3\left(\frac{m}{a}\right)\left(mb\bar{u} + a - m\,\text{arctanh}\left(\frac{m}{a}\right)\right),\\
a_4 &= \frac{(a^2-m^2)(a^2-3m^2)}{3a^3k\bar{u}^3}\text{arctanh}^2\left(\frac{m}{a}\right)\left(b\bar{u} - \text{arctanh}\left(\frac{m}{a}\right)\right),\\
a_5 &= \frac{a^2-m^2}{a^3b\bar{u}^3}\text{arctanh}^2\left(\frac{m}{a}\right)\left[(a^2-3m^2)\text{arctanh}^2\left(\frac{m}{a}\right) + (b\bar{u}(3m^2-a^2)+2am)\text{arctanh}\left(\frac{m}{a}\right) - amb\bar{u}\right],\\
a_6 &= \frac{k(m^2-a^2)}{a^3b^2\bar{u}^3}\text{arctanh}^3\left(\frac{m}{a}\right)\left[(a^2-3m^2)\text{arctanh}^2\left(\frac{m}{a}\right) + (b\bar{u}(3m^2-a^2)+4am)\text{arctanh}\left(\frac{m}{a}\right) - 2amb\bar{u} - a^2\right],\\
b_1 &= -a_1 - \frac{r}{K},\ b_2 = -a_2,\ b_3 = -a_3,\ b_4 = -a_4,\ b_5 = -a_5,\ b_6 = -a_6.
\end{aligned}$$

Set the matrix

$$\mathcal{P} := \begin{pmatrix} N & 1 \\ M & 0 \end{pmatrix}$$

with $M = -\frac{a_{21}}{\beta}$ and $N = \frac{a_{22}-a_{11}}{2\beta}$. Then,

$$\mathcal{P}^{-1}J\mathcal{P} = \Phi(K) := \begin{pmatrix} \alpha(K) & -\beta(K) \\ \beta(K) & \alpha(K) \end{pmatrix}.$$

Let

$$M_0 := M|_{K=K_0}, \quad N_0 := N|_{K=K_0}, \quad \beta_0 := \beta(K_0).$$

By the transformation $(u, v)^\top = \mathcal{P}(x, y)^\top$, the system (5.2) is transformed into

$$\begin{pmatrix} \frac{\mathrm{d}x}{\mathrm{d}t} \\ \frac{\mathrm{d}y}{\mathrm{d}t} \end{pmatrix} = \Phi(K)\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f^1(x, y, K) \\ g^1(x, y, K) \end{pmatrix}, \tag{5.3}$$

where

$$\begin{aligned} f^1(x, y, K) =& \frac{1}{M} g(Nx + y, Mx, K) \\ =& \left(-\frac{rN^2}{KM} - \frac{N^2}{M}a_1 - Na_2 - Ma_3\right)x^2 + \left(-\frac{2rN}{KM} - \frac{2N}{M}a_1 - a_2\right)xy \\ & - \left(\frac{r}{KM} + \frac{a_1}{M}\right)y^2 - \left(\frac{N^2}{M}a_4 + Na_5 + Ma_6\right)Nx^3 \\ & - \left(\frac{3N^2}{M}a_4 + 2Na_5 + Ma_6\right)x^2y - \left(\frac{3N}{M}a_4 + a_5\right)xy^2 - \frac{a_4}{M}y^3 + \cdots, \end{aligned}$$

$$\begin{aligned} g^1(x, y, K) =& f(Nx + y, Mx, K) - \frac{N}{M} g(Nx + y, Mx, K) \\ =& \left[\frac{rN^3}{KM} + N^2\left(1 + \frac{N}{M}\right)a_1 + N(M + N)a_2 + M(M + N)a_3\right]x^2 \\ & + \left[\frac{2rN^2}{KM} + 2N\left(1 + \frac{N}{M}\right)a_1 + (M + N)a_2\right]xy + \left[\frac{rN}{KM} + \left(1 + \frac{N}{M}\right)a_1\right]y^2 \\ & + \left[N^3\left(1 + \frac{N}{M}\right)a_4 + N^2(M + N)a_5 + MN(M + N)a_6\right]x^3 \\ & + \left[3N^2\left(1 + \frac{N}{M}\right)a_4 + 2N(M + N)a_5 + M(M + N)a_6\right]x^2y \\ & + \left[3N\left(1 + \frac{N}{M}\right)a_4 + (M + N)a_5\right]xy^2 + \left(1 + \frac{N}{M}\right)a_4y^3 + \cdots. \end{aligned}$$

The polar coordinate form of (5.3) is as follows:

$$\begin{aligned} \dot{\rho} &= \alpha(K)\rho + a(K)\rho^3 + \cdots, \\ \dot{\theta} &= \beta(K) + c(K)\rho^2 + \cdots. \end{aligned} \tag{5.4}$$

Then, the Taylor expansion of (5.4) at $K = K_0$ yields

$$\begin{aligned} \dot{\rho} &= \alpha'(K_0)(K - K_0)\rho + a(K_0)\rho^3 + o\left((K - K_0)^2\rho,\ (K - K_0)\rho^3,\ \rho^5\right), \\ \dot{\theta} &= \beta(K_0) + \beta'(K_0)(K - K_0) + c(K_0)\rho^2 + o\left((K - K_0)^2,\ (K - K_0)\rho^2,\ \rho^4\right). \end{aligned}$$

We need to verify the sign of the coefficient $a(K_0)$, which is given by

$$a(K_0) := \frac{1}{16}\left(f^1_{xxx} + f^1_{xyy} + g^1_{xxy} + g^1_{yyy}\right) + \frac{1}{16\beta_0}\left[f^1_{xy}\left(f^1_{xx} + f^1_{yy}\right) - g^1_{xy}\left(g^1_{xx} + g^1_{yy}\right) - f^1_{xx}g^1_{xx} + f^1_{yy}g^1_{yy}\right]$$

to determine the stability of Hopf bifurcation periodic solution, where all partial derivatives are evaluated at bifurcation point $(x, y, K) = (0, 0, K_0)$, and

$$f^1_{xxx}(0, 0, K_0) = -6N_0\left(\frac{N_0^2}{M_0}a_4 + N_0a_5 + M_0a_6\right),$$

$$f^1_{xyy}(0,0,K_0) = -2\left(\frac{3N_0}{M_0}a_4 + a_5\right),$$
$$g^1_{xxy}(0,0,K_0) = 2\left[3N_0^2\left(1+\frac{N_0}{M_0}\right)a_4+2N_0(M_0+N_0)a_5+M_0(M_0+N_0)a_6\right],$$
$$g^1_{yyy}(0,0,K_0) = 6\left(1 + \frac{N_0}{M_0}\right)a_4,$$
$$f^1_{xx}(0,0,K_0) = -2\left(\frac{rN_0^2}{K_0M_0} + \frac{N_0^2}{M_0}a_1 + N_0a_2 + M_0a_3\right),$$
$$f^1_{xy}(0,0,K_0) = -\frac{2rN_0}{K_0M_0} - \frac{2N_0}{M_0}a_1 - a_2,$$
$$f^1_{yy}(0,0,K_0) = -\frac{2}{M_0}\left(\frac{r}{K_0} + a_1\right),$$
$$g^1_{xx}(0,0,K_0) = 2\left[\frac{rN_0^3}{K_0M_0}+N_0^2\left(1+\frac{N_0}{M_0}\right)a_1+N_0(M_0+N_0)a_2+M_0(M_0+N_0)a_3\right],$$
$$g^1_{xy}(0,0,K_0) = \frac{2rN_0^2}{K_0M_0}+2N_0\left(1+\frac{N_0}{M_0}\right)a_1+(M_0+N_0)a_2,$$
$$g^1_{yy}(0,0,K_0) = 2\left[\frac{rN_0}{K_0M_0}+\left(1+\frac{N_0}{M_0}\right)a_1\right].$$

By direct calculations, we have

$$\begin{aligned} a(K_0) = -\frac{1}{8K_0M_0\beta_0}\Big[&2K_0\left(N_0^4+M_0N_0^3+2N_0^2+M_0N_0+1\right)a_1^2 \\ &- K_0\left(N_0^4-2M_0N_0^3-(3M_0^2-2)N_0^2-2M_0N_0-M_0^2+1\right)a_1a_2 + 2K_0M_0N_0\left(M_0^2-N_0^2-1\right)a_1a_3 \\ &+ 2r\left(N_0^4+2N_0^2+1\right)a_1 + K_0M_0N_0\left(M_0^2-N_0^2-1\right)a_2^2 - K_0M_0^2\left(3N_0^2+2M_0N_0-M_0^2+1\right)a_2a_3 \\ &- r\left(N_0^4-M_0N_0^3+2N_0^2-M_0N_0+1\right)a_2 - 2K_0M_0^3(M_0+N_0)a_3^2 - 2rM_0N_0\left(N_0^2+1\right)a_3 \\ &- 3K_0M_0\beta_0\left(N_0^2+1\right)a_4 + K_0M_0\beta_0\left(N_0^2-2M_0N_0+1\right)a_5 + K_0M_0^2\beta_0(2N_0-M_0)a_6\Big]. \end{aligned} \tag{5.5}$$

Denote

$$\sigma_2 = -\frac{a(K_0)}{\alpha'(K_0)}. \tag{5.6}$$

Recall that $\alpha'(K_0) > 0$, and one can obtain the result of Theorem 2.2 by the Poincaré-Andronov-Hopf bifurcation theorem [40].

Appendix B. The proof of Theorem 3.1

Let $D = \text{diag}(d_1, d_2)$, $E = (u, v)$, and $L = D\Delta + J_E(\bar{E})$. Then, the linearized system of (1.5) at $\bar{E}$ is

$$E_t = LE. \tag{B.1}$$

The eigenvalues of the operator L are the eigenvalues of the matrix $-\mu_n D + J_E(\bar{E}),\ \forall n \geq 1$, where

$$J_E(\bar{E}) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

The characteristic equation of $-\mu_n D + J_E(\bar{E})$ is given by

$$g_n(\lambda) \triangleq |\lambda I + \mu_n D - J_E(\bar{E})| = \lambda^2 + M_n\lambda + N_n,$$

where

$$M_n = (d_1 + d_2)\mu_n - \Theta_1, \quad N_n = d_1 d_2 \mu_n^2 - (a_{11}d_2 + a_{22}d_1)\mu_n + \Theta_2.$$

(**H$_3$**) implies that $a_{11} < 0$, then $M_n > 0$ and $N_n > 0$. The roots $\lambda_{n,1}$ and $\lambda_{n,2}$ of $g_n(\lambda) = 0$ all have negative real parts.

We claim that there exists a positive constant $\bar{\delta}$ such that

$$\mathrm{Re}\{\lambda_{n,1}\},\ \mathrm{Re}\{\lambda_{n,2}\} \leq -\bar{\delta}, \quad \forall n \geq 1. \tag{B.2}$$

In fact, let $\lambda = \mu_n \varrho$, then

$$g_n(\lambda) = \mu_n^2 \varrho^2 + M_n \mu_n \varrho + N_n \triangleq \bar{g}_n(\varrho)$$

and

$$\lim_{n\to\infty} \frac{\bar{g}_n(\varrho)}{\mu_n^2} = \varrho^2 + (d_1 + d_2)\varrho + d_1 d_2 \triangleq \bar{g}(\varrho).$$

Notice that $\bar{g}_n(\varrho) = 0$ has two negative roots $-d_1$ and $-d_2$. By continuity, there exists an n_0 such that the two roots $\varrho_{n,1}$ and $\varrho_{n,2}$ of $\bar{g}_n(\varrho) = 0$ satisfy

$$\mathrm{Re}\{\varrho_{n,1}\},\ \mathrm{Re}\{\varrho_{n,2}\} \leq -\frac{\hat{d}}{2},\ \ \hat{d} = \min\{d_1, d_2\},\ \ \forall n \geq n_0.$$

In turn, $\mathrm{Re}\{\lambda_{n,1}\},\ \mathrm{Re}\{\lambda_{n,2}\} \leq -\frac{\hat{d}}{2},\ \forall n \geq n_0$.

Denote

$$\max_{1\leq n\leq n_0} \{\mathrm{Re}\{\lambda_{n,1}\}, \mathrm{Re}\{\lambda_{n,2}\}\} = -\varpi.$$

Then, $\varpi > 0$, and (B.2) holds for $\delta = \min\{\varpi, \frac{\hat{d}}{2}\}$. This implies that all eigenvalues of L lie in $\{\mathrm{Re}\ \lambda \leq -\bar{\delta}\}$. Therefore, $\bar{E}$ is locally, uniformly, asymptotically stable. The proof of Theorem 3.1 is completed.

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