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Research article

On generalized Hermite polynomials

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Abstract: This article is devoted to establishing new formulas concerning generalized Hermite polynomials (GHPs) that generalize the classical Hermite polynomials. Derivative expressions of these polynomials that involve one parameter are found in terms of other parameter polynomials. Some other important formulas, such as the linearization and connection formulas between these polynomials and some other polynomials, are also given. Most of the coefficients are represented in terms of hypergeometric functions that can be reduced in some specific cases using some standard formulas. Two applications of the developed formulas in this paper are given. The first application is concerned with introducing some weighted definite integrals involving the GHPs. In contrast, the second is concerned with establishing the operational matrix of the integer derivatives of the GHPs.

Keywords: generalized Hermite polynomials; hypergeometric functions; connection; linearization formulas; high-order derivatives
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1. Introduction

Orthogonal polynomials (OPs) are crucial due to their numerous applications. In mathematics, they are the backbone for several contributions regarding approximation theory and numerical analysis. Also, they have many practical applications in the physical sciences; for instance, they arise in electromagnetism and quantum mechanics. Among the advantages of the orthogonal polynomials is that the quadrature rules are based on employing these polynomials; see, for example, [1]. For the uses of OPs in spectral methods, one can refer to [2, 3]. Some other applications can be found in [4–6]. The most-used orthogonal polynomials in the literature are the classical polynomials, which include Hermite, Laguerre, and Jacobi polynomials. Some theoretical contributions regarding these polynomials can be found in [7, 8], while some applications to these polynomials can be found in [9–11].

Hermite polynomials are a family of classical orthogonal polynomials. They have several applications in various branches of the applied sciences. Hermite polynomials have a crucial role in quantum field theory. Furthermore, these polynomials arise in many applications, such as statistical mechanics, signal processing, computer graphics, and probability theory. Some applications of Hermite polynomials can be found in [12, 13]. Many authors were interested in investigating Hermite polynomials and their related polynomials from theoretical and practical perspectives. Hwang and Ryoo [14] derived some identities involving two-variable partially degenerate Hermite polynomials. In [15], some representations of degenerate Hermite polynomials were developed. Two-variable q-Hermite polynomials were introduced in [16]. A study regarding the (p,q)-Hermite Polynomials was presented in [17]. Muhiuddin et al. [18] studied a class of Bernoulli polynomials associated with Lagrange-Hermite polynomials. Artioli et al. [19] studied some families of polynomials, including Hermite polynomials. The authors of [20, 21] studied some other polynomials associated with Hermite polynomials. In [22], a new family of Hermite polynomials was established.

Chihara presented a class of orthogonal polynomials, generalized Hermite polynomials (GHPs), in his essential book [23]. These polynomials generalize the classical Hermite polynomials. They have been the subject of both historical and contemporary research. In [24], spectral analysis is carried out on these polynomials. Chaggara and Koepf [25] provided these polynomials' linearization and connection coefficients. The authors of [26] also provided more findings regarding monic GHPs. Some formulae involving these polynomials will be developed in this study. In particular, these formulas may be helpful in approximation theory and numerical analysis.

Investigating special functions, including orthogonal polynomials and their generalized ones, is of interest from theoretical and practical aspects. Studying these generalized polynomials may be helpful in different disciplines, such as numerical analysis, so investigating them is a target for many authors. The authors of [27] introduced new polynomials that unify the four kinds of Chebyshev polynomials. Some formulas regarding the shifted Jacobi polynomials were developed in [28]. Some other formulas regarding the generalized Bernoulli polynomials were introduced in [29]. A class of generalized polynomials associated with Laguerre and Bernoulli polynomials was introduced in [30]. The authors in [31] established new formulas with some applications for two polynomials that generalize Fibonacci and Lucas polynomials. The authors in [32] investigated some generalized q-Bernoulli polynomials. Another type of generalized polynomials was introduced in [33]. Following a matrix approach, the authors of [34] investigated some Appell polynomials. Other formulas for general polynomial sequences were given in [35]. The theory of special functions extensively uses the mathematical framework known as umbral calculus, which is concerned with manipulating sequences and polynomials. For example, the authors in [36, 37] investigated many sequences of special functions using the umbral calculus.

Numerical analysis highlights the importance of utilizing various polynomial sequences. For example, the authors of [38] developed new formulas for certain Jacobi polynomials and employed them to treat some DEs of even-order. The authors of [39] used the generalized Bessel polynomials for treating some fractional DEs. The authors of [40] used a kind of generalized Chebyshev polynomials to treat some fractional optimal control problems. The authors of [41] numerically treated the fractional Rayleigh-Stokes problem utilizing some orthogonal combinations of Chebyshev polynomials. In [42], convolved Fibonacci polynomials were developed and used to solve the Fitzhugh numerically-Nagumo nonlinear DE. For some other studies of generalized polynomials, one can refer to [43–45].

Many disciplines encounter the celebrated special functions known as hypergeometric functions (HGFs). They arise in combinatorics, number theory, probability, and physics. HGFs can express several vital functions, including famous polynomials. HGFs can solve important problems such as duplication, connection, and linearization. For example, in [46–48], HGFs are used in the linearization coefficients of Jacobi polynomials with different parameters. For some important problems that can be solved via the different HGFs, one can see [49–51].

In this paper, we are concerned with establishing some new formulas related to the GHPs. To be more specific, we can list the current paper's main objectives as follows:

- Establishing new derivative expressions for the GHPs in terms of other GHPs.
- Deducing the connection formulas between two GHPs of different parameters.
- Deriving new product formulas of two different GHPs in terms of other GHPs.
- Deriving other product formulas of GHPs with some celebrated polynomials.
- Expressing the derivatives of GHPs as combinations of different polynomials.
- Presenting some applications to the derived formulas.

The paper's organization is as follows: Section 2 presents an overview of the GHPs. In Section 3, new derivative expressions between two different classes of GHPs are established. From these relations, some connection formulas can be deduced as special cases. New moments formulas of the GHPs are derived in Section 4. Some new linearization formulas involving the GHPs are established in Section 5. Some other derivatives and connection formulas are found in Section 6. Some definite weighted integrals are presented in Section 7. Moreover, this section establishes the operational matrix of the integer derivatives. Finally, some concluding remarks are reported in Section 8.

2. An overview on generalized Hermite polynomials

This section presents some fundamental properties of the GHPs and an overview of some well-known polynomials.

It is well-known that Hermite polynomials are classical orthogonal polynomials regarding the weight function: $w(x) = e^{-x^2}$ on $(-\infty, \infty)$ in the sense that

$$\int_{-\infty}^{\infty} e^{-x^2}\, H_i(x)\, H_j(x)\, dx = \sqrt{\pi}\, 2^i\, i!\, \delta_{i,j}, \tag{2.1}$$

and $\delta_{i,j}$ is the celebrated Kronecker delta function.

Some authors investigated the polynomials of the generalized weight function: $w(x) = |x|^{2\mu}\, e^{-x^2}$. These polynomials, of course, generalize those of the standard Hermite polynomials. In Chihara [23], it was proven that these polynomials can be represented as

$$H_m^{(\mu)}(x) = e^{x^2}\, x^{-2\mu}\, \frac{d^m}{dx^m}\left(e^{-x^2}\, x^{2\mu+m}\, K_m^{(\mu)}\right), \tag{2.2}$$

where

$$K_m^{(\mu)} = \begin{cases} \frac{(-1)^{\frac{m}{2}}}{(\mu+1)_{\frac{m}{2}}} \, {}_1F_1\left(\frac{m}{2}; \frac{m}{2}+\mu+1; x^2\right), & m \text{ even}, \\ \frac{(-1)^{\frac{m-1}{2}}}{(\mu+1)_{\frac{m+1}{2}}} x \, {}_1F_1\left(\frac{m+1}{2}; \frac{m-1}{2}+\mu+2; x^2\right), & m \text{ odd}. \end{cases}$$

The polynomials can be written alternatively in terms of the generalized Laguerre polynomials as follows [52]:

$$H_m^{(\mu)}(x) = \begin{cases} (-1)^{m/2} 2^m \left(\frac{m}{2}\right)! \, L_{\frac{m}{2}}^{\mu-\frac{1}{2}}\left(x^2\right), & m \text{ even}, \\ (-1)^{\frac{m-1}{2}} \, 2^m \left(\frac{m-1}{2}\right)! \, L_{\frac{m-1}{2}}^{\mu+\frac{1}{2}}\left(x^2\right), & m \text{ odd}. \end{cases} \tag{2.3}$$

Moreover, the orthogonality relation of $H_m^{(\mu)}(x)$ is given by

$$\int_{-\infty}^{\infty} |x|^{2\mu} \, e^{-x^2} H_j^{(\mu)}(x) \, H_k^{(\mu)}(x) \, dx = \begin{cases} 0, & k \neq j, \\ 4^k \, \left(\frac{k}{2}\right)! \, \Gamma\left(\frac{k+1}{2}+\mu\right), & k = j, k \text{ even}, \\ 4^k \left(\frac{k-1}{2}\right)! \, \Gamma\left(1+\frac{k}{2}+\mu\right), & k = j, k \text{ odd}. \end{cases} \tag{2.4}$$

The following two lemmas are of basic importance. The first gives the GHPs' analytic forms, and the second provides their inversion formulas. These formulas will be the fundamental basis for deriving new formulas for the GHPs.

Lemma 2.1. *For a non-negative integer j, the following are the two analytic forms of the GHPs:*

$$H_{2j}^{(\mu)}(x) = 2^{2j} \sum_{r=0}^{j} \frac{(-1)^r \left(\mu+j-r+\frac{1}{2}\right)_r (j-r+1)_r}{r!} \, x^{2j-2r}, \tag{2.5}$$

$$H_{2j+1}^{(\mu)}(x) = 2^{2j+1} \sum_{r=0}^{j} \frac{(-1)^r \left(\mu+j-r+\frac{3}{2}\right)_r (j-r+1)_r}{r!} \, x^{2j-2r+1}. \tag{2.6}$$

Proof. The above two formulas are direct consequences of the representation (2.3) along with the analytic form of the generalized Laguerre polynomial given by [53]

$$L_n^{(\mu)}(x) = \frac{\Gamma(n+\alpha+1)}{n!} \sum_{k=0}^{n} \frac{(-1)^{n-k}\binom{n}{k}}{\Gamma(n+\alpha+1-k)} \, x^{n-k}. \tag{2.7}$$

Lemma 2.1 is now proved. □

Lemma 2.2. *For a non-negative integer j, the following are the two inversion formulas of the GHPs:*

$$x^{2j} = \sum_{r=0}^{j} \frac{2^{2r-2j}(1+j-r)_r \left(\frac{1}{2}+j-r+\mu\right)_r}{r!} \, H_{2j-2r}^{(\mu)}(x), \tag{2.8}$$

$$x^{2j+1} = \sum_{r=0}^{j} \frac{2^{2r-2j-1} \left(\mu+j-r+\frac{3}{2}\right)_r (j-r+1)_r}{r!} \, H_{2j-2r+1}^{(\mu)}(x). \tag{2.9}$$

Proof. The proofs of (2.8) and (2.9) are similar. Now, we prove (2.8). If we assume the identity:

$$x^{2j} = \sum_{r=0}^{j} F_{r,j}\, H_{2j-2r}^{(\mu)}(x), \tag{2.10}$$

then we have to compute the coefficients $F_{r,j}$. For this purpose, multiply both sides of (2.10) by $w(x)\, H_{2m}^{(\mu)}(x)$, and integrate from to $-\infty$ to ∞ to get

$$\sum_{r=0}^{j} F_{r,j} \int_{-\infty}^{\infty} w(x) H_{2m}^{(\mu)}(x)\, H_{2j-2r}^{(\mu)}(x)\, dx = \int_{-\infty}^{\infty} w(x)\, x^{2j}\, H_{2m}^{(\mu)}(x)\, dx. \tag{2.11}$$

In virtue of the orthogonality relation (2.4) together with the power form representation (2.5), it is not difficult to express the coefficients $F_{r,j}$ explicitly in the form

$$F_{r,j} = \frac{1}{h_{2j-2r}} \sum_{\ell=0}^{j-r} A_{\ell,j-r} \int_{-\infty}^{\infty} w(x)\, x^{4j-2r-2\ell}\, dx, \tag{2.12}$$

where

$$A_{r,j} = \frac{(-1)^r\, 2^{2j} \left(\mu + j - r + \frac{1}{2}\right)_r (j-r+1)_r}{r!},$$

The integral on the right-hand side of (2.12) can be computed in terms of the Gamma function as

$$\int_{-\infty}^{\infty} w(x)\, x^{4j-2r-2\ell}\, dx = \Gamma\left(\frac{1}{2} + 2j - \ell - r + \mu\right). \tag{2.13}$$

Therefore, the coefficients $F_{r,j}$ reduce to the form

$$F_{r,j} = 4^{-j+r} \sum_{\ell=0}^{j-r} \frac{(-1)^{\ell}\Gamma\left(\frac{1}{2} + 2j - \ell - r + \mu\right)}{\ell!\,(j-\ell-r)!\,\Gamma\left(\frac{1}{2} + j - \ell - r + \mu\right)}.$$

In hypergeometric form, we can write

$$\sum_{\ell=0}^{j-r} \frac{(-1)^{\ell}\Gamma\left(\frac{1}{2} + 2j - \ell - r + \mu\right)}{\ell!\,(j-\ell-r)!\,\Gamma\left(\frac{1}{2} + j - \ell - r + \mu\right)} = \frac{\Gamma\left(\frac{1}{2} + 2j - r + \mu\right)}{(j-r)!\,\Gamma\left(\frac{1}{2} + j - r + \mu\right)} \times {}_2F_1\left(\left.\begin{matrix} -j+r,\ \frac{1}{2} - j + r - \mu \\ \frac{1}{2} - 2j + r - \mu \end{matrix}\right| 1\right).$$

The last ${}_2F_1(1)$ can be summed by the Chu-Vandemonde identity [53], and thus, we get

$$F_{r,j} = \frac{2^{2r-2j}(1+j-r)_r \left(\frac{1}{2} + j - r + \mu\right)_r}{r!}.$$

Formula (2.8) is now proved. Formula (2.9) can be similarly proved. □

2.1. An overview on some classes of polynomials

In this part, we give specific properties of some classes of polynomials that will be important in what follows.

The ultraspherical polynomials are the orthogonal normalized Gegenbauer polynomials defined as

$$U_j^{(\lambda)}(x) = \frac{j!\, C_j^{(\lambda)}(x)}{(2\lambda)_j},$$

where $C_j^{(\lambda)}(x)$ are the well-known Gegenbauer polynomials.
These polynomials can be expressed in the form

$$U_j^{(\lambda)}(x) = \frac{j!\,\Gamma(2\lambda+1)}{\Gamma(\lambda+1)\,\Gamma(j+2\lambda)} \sum_{r=0}^{\lfloor \frac{j}{2} \rfloor} \frac{(-1)^r\, 2^{j-2r-1}\,\Gamma(j-r+\lambda)}{(j-2r)!\, r!}\, x^{j-2r}, \tag{2.14}$$

while their inversion formula is

$$x^j = \frac{\Gamma(\lambda+1)\, j!}{\Gamma(2\lambda+1)\, 2^{j-1}} \sum_{r=0}^{\lfloor \frac{j}{2} \rfloor} \frac{(j-2r+\lambda)\,\Gamma(j-2r+2\lambda)}{(j-2r)!\, r!\,\Gamma(1+j-r+\lambda)}\, U_{j-2r}^{(\lambda)}(x). \tag{2.15}$$

In addition, these polynomials are orthogonal on $[-1, 1]$ in the sense that

$$\int_{-1}^{1} (1-x^2)^{\lambda-\frac{1}{2}}\, U_j^{(\lambda)}(x)\, U_k^{(\lambda)}(x) = \begin{cases} \dfrac{k!\,\Gamma(\frac{1}{2})\,\Gamma(\lambda+\frac{1}{2})}{(2\,\lambda)_k\,\Gamma(\lambda)\,(\lambda+k)}. & j = k, \\ 0, & j \neq k. \end{cases} \tag{2.16}$$

Among the important families of polynomials are the two classes of generalized Fibonacci and generalized Lucas polynomials. $F_j^{a,b}(x)$ and $L_j^{c,d}(x)$ can be generated using the following recursive formulas [47]:

$$F_j^{a,b}(x) = a\,x\,F_{j-1}^{a,b}(x) + b\,F_{j-2}^{a,b}(x), \quad F_0^{a,b}(x) = 1,\ F_1^{a,b}(x) = a\,x, \quad j \geq 2. \tag{2.17}$$

$$L_j^{c,d}(x) = c\,x\,L_{j-1}^{c,d}(x) + d\,L_{j-2}^{c,d}(x), \quad L_0^{c,d}(x) = 2,\ L_1^{c,d}(x) = c\,x, \quad j \geq 2. \tag{2.18}$$

Also, among the celebrated classes of polynomials are the four classes of Chebyshev polynomials that can be generated with the following unified recursive formula:

$$\phi_j(x) = 2\,x\,\phi_{j-1}(x) - \phi_{j-2}(x), \quad j \geq 2, \tag{2.19}$$

with the following initials:

$$T_0(x) = 1,\ T_1(x) = x, \quad U_0(x) = 1,\ U_1(x) = 2x,$$
$$V_0(x) = 1,\ V_1(x) = 2x - 1, \quad W_0(x) = 1,\ W_1(x) = 2x + 1.$$

These polynomials have the following unified moment formula:

$$x^m\,\phi_j(x) = \frac{1}{2^m} \sum_{s=0}^{m} \binom{m}{s} \phi_{j+m-2s}(x). \tag{2.20}$$

3. New derivative expressions of the GHPs

This section is interested in developing new derivative expressions for the GHPs. In fact, the general derivative for the GHPs of a certain parameter will be given in terms of other GHPs. From these formulas, the following expressions will be deduced:

- The expressions of the derivatives of the GHPs in terms of the original polynomials themselves.
- The expressions of the derivatives of Hermite polynomials in terms of the GHPs.
- The derivative expressions of the GHPs in terms of the standard Hermite polynomials.
- The connection formulas between the standard Hermite polynomials and their generalized ones.

In the following theorem, we give an expression for $D^q H_j^{(\mu)}(x)$ in terms of $H_j^{(\lambda)}(x)$.

Theorem 3.1. *Let j and q be non-negative integers with $j \geq q$. The following formula applies:*

$$D^q H_j^{(\mu)}(x) = \sum_{\ell=0}^{\left\lfloor \frac{j-q}{2} \right\rfloor} B_{\ell,j,q}\, H_{j-q-2\ell}^{(\lambda)}(x), \tag{3.1}$$

where

$$B_{\ell,j,q} = \frac{\sqrt{\pi}\, j!\, 2^{-j+2(\ell+q)}}{\ell!} \times \begin{cases} \dfrac{\Gamma\left(\frac{1}{2}(1+j-q)+\lambda\right)}{\left(\frac{j-q}{2}-\ell\right)!\ \Gamma\left(\frac{1}{2}(1+j-q)\right)\ \Gamma\left(\frac{1}{2}(1+j-2\ell-q)+\lambda\right)} \times \\ {}_3F_2\left(\left. \begin{array}{c} -\ell, \frac{1}{2}-\frac{j}{2}+\frac{q}{2}, \frac{1}{2}-\frac{j}{2}-\mu \\ \frac{1}{2}-\frac{j}{2}, \frac{1}{2}-\frac{j}{2}+\frac{q}{2}-\lambda \end{array} \right| 1 \right), & q \text{ and } j \text{ even}, \\ \dfrac{\Gamma\left(\frac{1}{2}(2+j-q)+\lambda\right)}{\left(\frac{1}{2}(j-q-1)-\ell\right)!\ \Gamma\left(\frac{1}{2}(2+j-q)\right)\ \Gamma\left(\frac{1}{2}(2+j-2\ell-q)+\lambda\right)} \times \\ {}_3F_2\left(\left. \begin{array}{c} -\ell, -\frac{j}{2}+\frac{q}{2}, -\frac{j}{2}-\mu \\ -\frac{j}{2}, -\frac{j}{2}+\frac{q}{2}-\lambda \end{array} \right| 1 \right), & q \text{ even},\ j \text{ odd}, \\ \dfrac{\Gamma\left(\frac{1}{2}(2+j-q)+\lambda\right)}{\left(\frac{1}{2}(j-q-1)-\ell\right)!\ \Gamma\left(\frac{1}{2}(2+j-q)\right)\ \Gamma\left(\frac{1}{2}(2+j-2\ell-q)+\lambda\right)} \times \\ {}_3F_2\left(\left. \begin{array}{c} -\ell, -\frac{j}{2}+\frac{q}{2}, \frac{1}{2}-\frac{j}{2}-\mu \\ \frac{1}{2}-\frac{j}{2}, -\frac{j}{2}+\frac{q}{2}-\lambda \end{array} \right| 1 \right), & q \text{ odd},\ j \text{ even}, \\ \dfrac{\Gamma\left(\frac{1}{2}(1+j-q+2\lambda)\right)}{\left(\frac{j-q}{2}-\ell\right)!\ \Gamma\left(\frac{1}{2}(1+j-q)\right)\ \Gamma\left(\frac{1}{2}(1+j-2\ell-q+2\lambda)\right)} \times \\ {}_3F_2\left(\left. \begin{array}{c} -\ell, \frac{1}{2}-\frac{j}{2}+\frac{q}{2}, -\frac{j}{2}-\mu \\ -\frac{j}{2}, \frac{1}{2}-\frac{j}{2}+\frac{q}{2}-\lambda \end{array} \right| 1 \right), & q \text{ and } j \text{ odd}. \end{cases} \tag{3.2}$$

Proof. To prove formula (3.1), it is required to prove the following four formulas:

$$\begin{aligned} D^{2q} H_{2j}^{(\mu)}(x) =& \frac{\sqrt{\pi}\,(2j)!\,\Gamma\left(\frac{1}{2}+j-q+\lambda\right)}{\Gamma\left(\frac{1}{2}+j-q\right)} \sum_{\ell=0}^{j-q} \frac{4^{-j+\ell+2q}}{\ell!\,(j-\ell-q)!\,\Gamma\left(\frac{1}{2}+j-\ell-q+\lambda\right)} \times \\ & {}_3F_2\left(\left.\begin{matrix} -\ell, \frac{1}{2}-j+q, \frac{1}{2}-j-\mu \\ \frac{1}{2}-j, \frac{1}{2}-j+q-\lambda \end{matrix}\right| 1\right) H_{2j-2\ell-2q}^{(\lambda)}(x), \end{aligned} \tag{3.3}$$

$$\begin{aligned} D^{2q} H_{2j+1}^{(\mu)}(x) =& \frac{\sqrt{\pi}\,(2j+1)!}{\Gamma\left(\frac{3}{2}+j-q\right)} \sum_{\ell=0}^{j-q} \frac{2^{-1-2j+2\ell+4q}\Gamma\left(\frac{3}{2}+j-q+\lambda\right)}{\ell!\,(j-\ell-q)!\,\Gamma\left(\frac{3}{2}+j-\ell-q+\lambda\right)} \times \\ & {}_3F_2\left(\left.\begin{matrix} -\ell, -\frac{1}{2}-j+q, -\frac{1}{2}-j-\mu \\ -\frac{1}{2}-j, -\frac{1}{2}-j+q-\lambda \end{matrix}\right| 1\right) H_{2j-2\ell-2q+1}^{(\lambda)}(x), \end{aligned} \tag{3.4}$$

$$\begin{aligned} D^{2q+1} H_{2j}^{(\mu)}(x) =& \frac{\sqrt{\pi}\,(2j)!}{\Gamma\left(j-q+\frac{1}{2}\right)} \sum_{\ell=0}^{j-q} \frac{2^{2-2j+2\ell+4q}\Gamma\left(j-q+\lambda+\frac{1}{2}\right)}{\ell!\,(j-\ell-q-1)!\,\Gamma\left(j-\ell-q+\lambda+\frac{1}{2}\right)} \times \\ & {}_3F_2\left(\left.\begin{matrix} -\ell, \frac{1}{2}-j+q, \frac{1}{2}-j-\mu \\ \frac{1}{2}-j, \frac{1}{2}-j+q-\lambda \end{matrix}\right| 1\right) H_{2j-2\ell-2q-1}^{(\lambda)}(x), \end{aligned} \tag{3.5}$$

$$\begin{aligned} D^{2q+1} H_{2j+1}^{(\mu)}(x) =& \frac{\sqrt{\pi}(2j+1)!}{\Gamma\left(\frac{1}{2}+j-q\right)} \sum_{\ell=0}^{j-q} \frac{2^{1-2j+2\ell+4q}\Gamma\left(\frac{1}{2}+j-q+\lambda\right)}{\ell!\,(j-\ell-q)!\,\Gamma\left(\frac{1}{2}+j-\ell-q+\lambda\right)} \times \\ & {}_3F_2\left(\left.\begin{matrix} -\ell, \frac{1}{2}-j+q, -\frac{1}{2}-j-\mu \\ -\frac{1}{2}-j, \frac{1}{2}-j+q-\lambda \end{matrix}\right| 1\right) H_{2j-2\ell-2q}^{(\lambda)}(x). \end{aligned} \tag{3.6}$$

To prove (3.3), we make use of (2.5) to obtain the following formula:

$$D^{2q} H_{2j}^{(\mu)}(x) = = 2^{2j} \sum_{r=0}^{j-q} \frac{(-1)^r (1+2j-2q-2r)_{2q} (1+j-r)_r \left(\frac{1}{2}+j-r+\mu\right)_r}{r!} x^{2j-2r-2q}. \tag{3.7}$$

Inserting the inversion formula (2.8) into (3.7) yields the following formula:

$$\begin{aligned} D^{2q} H_{2j}^{(\mu)}(x) =& 2^{2j} \sum_{r=0}^{j-q} \frac{(-1)^r (1+2j-2q-2r)_{2q} (1+j-r)_r \left(\frac{1}{2}+j-r+\mu\right)_r}{r!} \times \\ & \sum_{\ell=0}^{j-r-q} \frac{4^{-j+\ell+q+r} (1+j-\ell-q-r)_\ell \left(\frac{1}{2}+j-\ell-q-r+\lambda\right)_\ell}{\ell!} H_{2j-2\ell-2q-2r}^{(\lambda)}(x). \end{aligned} \tag{3.8}$$

The last formula can be transformed into the following formula:

$$\begin{aligned} D^{2q} H_{2j}^{(\mu)}(x) =& \sum_{\ell=0}^{j-q} 2^{2(\ell+q)} \sum_{p=0}^{\ell} \frac{(-1)^p (1+j-p)_p (1+2j-2p-2q)_{2q} (1+j-\ell-q)_{\ell-p}}{(\ell-p)!\,p!} \times \\ & \left(\frac{1}{2}+j-\ell-q+\lambda\right)_{\ell-p} \left(\frac{1}{2}+j-p+\mu\right)_p H_{2j-2\ell-2q}^{(\lambda)}(x), \end{aligned} \tag{3.9}$$

and accordingly, the following formula can be obtained:

$$D^{2q} H_{2j}^{(\mu)}(x) = \frac{\sqrt{\pi}\,(2j)!\,\Gamma\left(\frac{1}{2}+j-q+\lambda\right)}{\Gamma\left(\frac{1}{2}+j-q\right)} \sum_{\ell=0}^{j-q} \frac{4^{-j+\ell+2q}}{\ell!\,(j-\ell-q)!\,\Gamma\left(\frac{1}{2}+j-\ell-q+\lambda\right)} \times \\ {}_3F_2\left(\begin{matrix} -\ell, \frac{1}{2}-j+q, \frac{1}{2}-j-\mu \\ \frac{1}{2}-j, \frac{1}{2}-j+q-\lambda \end{matrix} \middle| 1\right) H_{2j-2\ell-2q}^{(\lambda)}(x).$$

This proves (3.3). The other formulas can be similarly proved. ◻

Remark 3.1. *Several important formulas can be deduced from formula* (3.1). *The following corollaries exhibit these results.*

Corollary 3.1. *In terms of* $H_j^{(\mu)}(x)$, $D^q H_j^{(\mu)}(x)$ *can be written as follows:*

$$D^q H_j^{(\mu)}(x) = \sum_{\ell=0}^{\left\lfloor \frac{j-q}{2} \right\rfloor} G_{\ell,j,q}\, H_{j-q-2\ell}^{(\mu)}(x), \quad j \geq q, \tag{3.10}$$

where

$$G_{\ell,j,q} = \frac{\sqrt{\pi}\, j!\, 2^{-j+2(\ell+q)}}{\ell!} \times \begin{cases} \dfrac{\Gamma\left(\frac{1}{2}(1+j-q)+\mu\right)}{\left(\frac{1}{2}(j-2\ell-q)\right)!\,\Gamma\left(\frac{1}{2}(1+j-q)\right)\Gamma\left(\frac{1}{2}(1+j-2\ell-q)+\mu\right)} \times \\ {}_3F_2\left(\begin{matrix} -\ell, \frac{1}{2}-\frac{j}{2}+\frac{q}{2}, \frac{1}{2}-\frac{j}{2}-\mu \\ \frac{1}{2}-\frac{j}{2}, \frac{1}{2}-\frac{j}{2}+\frac{q}{2}-\mu \end{matrix} \middle| 1\right), & q \text{ and } j \text{ even}, \\ \dfrac{\Gamma\left(\frac{1}{2}(2+j-q)+\mu\right)}{\Gamma\left(\frac{1}{2}(2+j-q)\right)\left(\frac{1}{2}(j-2\ell-q-1)\right)!\,\Gamma\left(\frac{1}{2}(2+j-2\ell-q)+\mu\right)} \times \\ {}_3F_2\left(\begin{matrix} -\ell, -\frac{j}{2}+\frac{q}{2}, -\frac{j}{2}-\mu \\ -\frac{j}{2}, -\frac{j}{2}+\frac{q}{2}-\mu \end{matrix} \middle| 1\right), & q \text{ even}, \ j \text{ odd}, \\ \dfrac{\Gamma\left(\frac{1}{2}(2+j-q)+\mu\right)}{\Gamma\left(\frac{1}{2}(2+j-q)\right)\left(\frac{1}{2}(j-2\ell-q-1)\right)!\,\Gamma\left(\frac{1}{2}(2+j-2\ell-q)+\mu\right)} \times \\ {}_3F_2\left(\begin{matrix} -\ell, -\frac{j}{2}+\frac{q}{2}, \frac{1}{2}-\frac{j}{2}-\mu \\ \frac{1}{2}-\frac{j}{2}, -\frac{j}{2}+\frac{q}{2}-\mu \end{matrix} \middle| 1\right), & q \text{ odd}, \ j \text{ even}, \\ \dfrac{\Gamma\left(\frac{1}{2}(1+j-q)+\mu\right)}{\left(\frac{1}{2}(j-2\ell-q)\right)!\,\Gamma\left(\frac{1}{2}(1+j-q)\right)\Gamma\left(\frac{1}{2}(1+j-2\ell-q)+\mu\right)} \times \\ {}_3F_2\left(\begin{matrix} -\ell, \frac{1}{2}-\frac{j}{2}+\frac{q}{2}, -\frac{j}{2}-\mu \\ -\frac{j}{2}, \frac{1}{2}-\frac{j}{2}+\frac{q}{2}-\mu \end{matrix} \middle| 1\right), & q \text{ and } j \text{ odd}. \end{cases} \tag{3.11}$$

Proof. Formula (3.10) can be easily obtained from (3.1) only by setting $\lambda = \mu$. ◻

Corollary 3.2. *The q-th derivative of $H_j(x)$ is given by*

$$D^q H_j(x) = \frac{2^q\, j!}{(j-q)!}\, H_{j-q}(x), \quad j \geq q. \tag{3.12}$$

Proof. The substitution by $\mu = 0$ in (3.10) reduces it into the following form:

$$D^q H_j(x) = 2^q\, j! \sum_{\ell=0}^{\left\lfloor \frac{j-q}{2} \right\rfloor} \frac{{}_1F_0(-\ell;\,;1)}{\ell!\,(j-2\ell-q)!}\, H_{j-q-2\ell}(x), \tag{3.13}$$

which immediately gives

$$D^q H_j(x) = \frac{2^q\, j!}{(j-q)!}\, H_{j-q}(x). \tag{3.14}$$

This proves (3.12). □

Corollary 3.3. *Consider two non-negative integers j and q such that $j \geq q$. The following formulas hold:*

$$D^{2q} H_{2j}^{(\mu)}(x) = 4^q (2j)! \sum_{\ell=0}^{j-q} \frac{(-1)^\ell (\mu)_\ell}{\ell!\,(2j-2\ell-2q)!\,\left(\frac{1}{2}+j-\ell\right)_\ell} H_{2j-2\ell-2q}(x), \tag{3.15}$$

$$D^{2q} H_{2j+1}^{(\mu)}(x) = (2j+1)!\, 2^{2q} \sum_{\ell=0}^{j-q} \frac{(-1)^\ell (\mu)_\ell}{\ell!\,(2j-2\ell-2q+1)!\,\left(\frac{3}{2}+j-\ell\right)_\ell} H_{2j-2\ell-2q+1}(x), \tag{3.16}$$

$$D^{2q+1} H_{2j}^{(\mu)}(x) = (2j)!\, 2^{1+2q} \sum_{\ell=0}^{j-q} \frac{(-1)^\ell (\mu)_\ell}{\ell!\,(2j-2\ell-2q-1)!\,\left(\frac{1}{2}+j-\ell\right)_\ell} H_{2j-2\ell-2q-1}(x), \tag{3.17}$$

$$D^{2q+1} H_{2j+1}^{(\mu)}(x) = 2^{1+2q} (2j+1)! \sum_{\ell=0}^{j-q} \frac{(-1)^\ell (\mu)_\ell}{\ell!\,(2j-2\ell-2q)!\,\left(\frac{3}{2}+j-\ell\right)_\ell} H_{2j-2\ell-2q}(x). \tag{3.18}$$

Proof. To prove (3.15), we set $\lambda = 0$ in (3.3) to obtain the following relation:

$$D^{2q} H_{2j}^{(\mu)}(x) = 4^q (2j)! \sum_{\ell=0}^{j-q} \frac{1}{\ell!\,(2j-2\ell-2q)!}\, {}_2F_1\left(\begin{matrix} -\ell, \frac{1}{2}-j-\mu \\ \frac{1}{2}-j \end{matrix} \,\middle|\, 1 \right) H_{2j-2\ell-2q}(x).$$

The Chu-Vandermonde identity leads to the following reduced formula:

$$D^{2q} H_{2j}^{(\mu)}(x) = 4^q (2j)! \sum_{\ell=0}^{j-q} \frac{(-1)^\ell (\mu)_\ell}{\ell!\,(2j-2\ell-2q)!\,\left(\frac{1}{2}+j-\ell\right)_\ell} H_{2j-2\ell-2q}(x).$$

Other reduction formulas can be obtained similarly by applying the well-known Chu-Vandermonde identity. □

Corollary 3.4. *Consider two non-negative integers j and q such that $j \geq q$. The following formulas hold:*

$$D^{2q} H_{2j}(x) = \frac{\sqrt{\pi}(2j)!}{\Gamma\left(\frac{1}{2}+j-q\right)} \sum_{\ell=0}^{j-q} \frac{4^{-j+\ell+2q}(1-\ell+\lambda)_\ell}{\ell!\,(j-\ell-q)!}\, H_{2j-2\ell-2q}^{(\lambda)}(x),$$

$$D^{2q} H_{2j+1}(x) = \frac{\sqrt{\pi}(2j+1)!}{\Gamma\left(\frac{3}{2}+j-q\right)} \sum_{\ell=0}^{j-q} \frac{2^{-1-2j+2\ell+4q}(1-\ell+\lambda)_\ell}{\ell!\,(j-\ell-q)!} H^{(\lambda)}_{2j-2\ell-2q+1}(x),$$

$$D^{2q+1} H_{2j}(x) = \frac{\sqrt{\pi}\,(2j)!}{\Gamma\left(\frac{1}{2}+j-q\right)} \sum_{\ell=0}^{j-q} \frac{4^{1-j+\ell+2q}(1-\ell+\lambda)_\ell}{\ell!\,(j-\ell-q-1)!} H^{(\lambda)}_{2j-2\ell-2q-1}(x),$$

$$D^{2q+1} H_{2j+1}(x) = \frac{\sqrt{\pi}(2j+1)!}{\Gamma\left(\frac{1}{2}+j-q\right)} \sum_{\ell=0}^{j-q} \frac{2^{1-2j+2\ell+4q}(1-\ell+\lambda)_\ell}{\ell!\,(j-\ell-q)!} H^{(\lambda)}_{2j-2\ell-2q}(x).$$

Proof. Similar to the proof of Corollary 3.3. □

Corollary 3.5. *The following four connection formulas between Hermite and generalized Hermite polynomials are valid:*

$$H^{(\mu)}_{2j}(x) = j! \sum_{\ell=0}^{j} \frac{(-4)^\ell(\mu)_\ell}{\ell!\,(j-\ell)!} H_{2j-2\ell}(x),$$

$$H^{(\mu)}_{2j+1}(x) = j! \sum_{\ell=0}^{j} \frac{(-4)^\ell(\mu)_\ell}{\ell!\,(j-\ell)!} H_{2j-2\ell+1}(x),$$

$$H_{2j}(x) = j! \sum_{\ell=0}^{j} \frac{4^\ell(\lambda-\ell+1)_\ell}{\ell!\,(j-\ell)!} H^{(\mu)}_{2j-2\ell}(x),$$

$$H_{2j+1}(x) = j! \sum_{\ell=0}^{j} \frac{4^\ell(\lambda-\ell+1)_\ell}{\ell!\,(j-\ell)!} H^{(\mu)}_{2j-2\ell+1}(x).$$

Proof. Simply set $q = 0$ respectively in formulas (3.15)–(3.18). □

4. Moments formulas of the GHPs

This section focuses on deriving the moment formulas of the GHPs in terms of other GHPs. More precisely, we will determine the coefficients $G_{p,j,m}$ in the following equation:

$$x^m H^{(\mu)}_j(x) = \sum_{p=0}^{\left\lfloor \frac{j+m}{2} \right\rfloor} G_{p,j,m} H^{(\lambda)}_{j+m-2p}(x).$$

Theorem 4.1. *For all non-negative integers m and j, the following moment formula holds:*

$$x^m H^{(\mu)}_j(x) = \sum_{p=0}^{\left\lfloor \frac{j+m}{2} \right\rfloor} G_{p,j,m} H^{(\lambda)}_{j+m-2p}(x), \tag{4.1}$$

where the coefficients $G_{p,j,m}$ are given as follows:

$$G_{p,j,m}=\frac{2^{2p-m}}{m!}\begin{cases}\dfrac{\frac{j+m}{2}!\,\Gamma\left(\frac{1}{2}(1+j+m)+\lambda\right)}{\frac{1}{2}(j+m-2p)!\,\Gamma\left(\frac{1}{2}(1+j+m-2p)+\lambda\right)}\times\\ {}_3F_2\left(\begin{matrix}-\frac{j}{2}-\frac{m}{2},\frac{1}{2}-\frac{j}{2}-\frac{m}{2}-\lambda\\ -\frac{j}{2},\frac{1}{2}-\frac{j}{2}+\frac{q}{2}-\lambda\end{matrix}\middle|1\right), & m\ even,\ j\ even,\\ \dfrac{\frac{j+m}{2}!\Gamma\left(\frac{1}{2}(1+j+m)+\lambda\right)}{\left(\frac{1}{2}(j+m-2p)\right)!\Gamma\left(\frac{1}{2}(1+j+m-2p)+\lambda\right)}\times\\ {}_3F_2\left(\begin{matrix}\frac{1}{2}-\frac{j}{2},-p,-\frac{j}{2}-\mu\\ -\frac{j}{2}-\frac{m}{2},\frac{1}{2}-\frac{j}{2}-\frac{m}{2}-\lambda\end{matrix}\middle|1\right), & m\ odd,\ j\ odd,\\ \dfrac{\left(\frac{1}{2}(j+m-1)\right)!\,\Gamma\left(\frac{1}{2}(2+j+m)+\lambda\right)}{\left(\frac{1}{2}(j+m-2p-1)\right)!\,\Gamma\left(\frac{1}{2}(2+j+m-2p)+\lambda\right)}\times\\ {}_3F_2\left(\begin{matrix}\frac{1}{2}-\frac{j}{2},-p,-\frac{j}{2}-\mu\\ \frac{1}{2}-\frac{j}{2}-\frac{m}{2},-\frac{j}{2}-\frac{m}{2}-\lambda\end{matrix}\middle|1\right), & m\ even,\ j\ odd,\\ \dfrac{\left(\frac{1}{2}(j+m-1)\right)!\,\Gamma\left(\frac{1}{2}(2+j+m)+\lambda\right)}{\left(\frac{1}{2}(j+m-2p-1)\right)!\,\Gamma\left(\frac{1}{2}(2+j+m-2p)+\lambda\right)}\times\\ {}_3F_2\left(\begin{matrix}-\frac{j}{2},-p,\frac{1}{2}-\frac{j}{2}-\mu\\ \frac{1}{2}-\frac{j}{2}-\frac{m}{2},-\frac{j}{2}-\frac{m}{2}-\lambda\end{matrix}\middle|1\right), & m\ odd,\ j\ even.\end{cases}\tag{4.2}$$

Proof. Formula (4.1) can be split into the following four formulas:

$$\begin{aligned}x^{2m}\,H_{2j}^{(\mu)}(x)=&(j+m)!\,\Gamma\left(\frac{1}{2}+j+m+\lambda\right)\sum_{p=0}^{j+m}\frac{1}{2^{2m-2p}p!(j+m-p)!\,\Gamma\left(\frac{1}{2}+j+m-p+\lambda\right)}\times\\&{}_3F_2\left(\begin{matrix}-j,-p,\frac{1}{2}-j-\mu\\ -j-m,\frac{1}{2}-j-m-\lambda\end{matrix}\middle|1\right)H_{2j+2m-2p}^{(\lambda)}(x),\end{aligned}\tag{4.3}$$

$$\begin{aligned}x^{2m}\,H_{2j+1}^{(\mu)}(x)=&(j+m)!\,\Gamma\left(\frac{3}{2}+j+m+\lambda\right)\sum_{p=0}^{j+m}\frac{1}{2^{2m-2p}p!(j+m-p)!\,\Gamma\left(\frac{3}{2}+j+m-p+\lambda\right)}\times\\&{}_3F_2\left(\begin{matrix}-j,-p,-\frac{1}{2}-j-\mu\\ -j-m,-\frac{1}{2}-j-m-\lambda\end{matrix}\middle|1\right)H_{2j+2m-2p+1}^{(\lambda)}(x),\end{aligned}\tag{4.4}$$

$$\begin{aligned}x^{2m+1}\,H_{2j}^{(\mu)}(x)=&(j+m)!\,\Gamma\left(\frac{3}{2}+j+m+\lambda\right)\sum_{p=0}^{j+m}\frac{1}{2^{2m-2p+1}p!(j+m-p)!\,\Gamma\left(\frac{3}{2}+j+m-p+\lambda\right)}\times\\&{}_3F_2\left(\begin{matrix}-j,-p,\frac{1}{2}-j-\mu\\ -j-m,-\frac{1}{2}-j-m-\lambda\end{matrix}\middle|1\right)H_{2j+2m-2p+1}^{(\lambda)}(x),\end{aligned}\tag{4.5}$$

$$
\begin{aligned}
x^{2m+1}\, H^{(\mu)}_{2j+1}(x) =&(j+m+1)!\,\Gamma\left(\tfrac{3}{2}+j+m+\lambda\right)\times \\
&\sum_{p=0}^{j+m+1} \frac{1}{2^{2m-2p+1}\,p!(j+m-p+1)!\,\Gamma\left(\tfrac{3}{2}+j+m-p+\lambda\right)}\times \\
&{}_3F_2\left(\begin{matrix} -j,-p,-\frac{1}{2}-j-\mu \\ -1-j-m,-\frac{1}{2}-j-m-\lambda \end{matrix}\,\middle|\,1\right) H^{(\lambda)}_{2j+2m-2p+2}(x).
\end{aligned}
\tag{4.6}
$$

The above formulas can be proved using the power form representations (2.5) and (2.6), and their inversion formulas (2.8) and (2.9). We will show (4.6). We can demonstrate the other formulas similarly.
Formula (2.5) leads to

$$
x^{2m+1}\, H^{(\mu)}_{2j}(x) = 2^{2j} \sum_{r=0}^{j} \frac{(-1)^r \left(\mu+j-r+\frac{1}{2}\right)_r (j-r+1)_r}{r!}\, x^{2j-2r+2m+1}. \tag{4.7}
$$

The inversion formula (2.9) converts the last formula to the following one:

$$
\begin{aligned}
x^{2m+1}\, H^{(\mu)}_{2j}(x) =&2^{2j} \sum_{r=0}^{j} \frac{(-1)^r \left(\mu+j-r+\frac{1}{2}\right)_r (j-r+1)_r}{r!}\times \\
&\sum_{\ell=0}^{j-r+m} \frac{2^{-1+2\ell-2(j+m-r)}(1+j-\ell+m-r)_\ell \left(\frac{3}{2}+j-\ell+m-r+\lambda\right)_\ell}{\ell!}\, H^{(\lambda)}_{1-2\ell+2j+2m-2r}(x).
\end{aligned}
\tag{4.8}
$$

After some algebraic computation, the last formula can be turned into (4.6). □

5. Some linearization formulas of generalized Hermite polynomials

This section is devoted to deriving some linearization formulas involving the GHPs. We will give a product formula for two different GHPs in terms of other GHPs. In addition, GHP products with some well-known polynomials will be presented.

Theorem 5.1. *The following linearization formula for three different parameters generalized Hermite polynomials holds:*

$$
H^{(\alpha)}_i(x)\, H^{(\mu)}_j(x) = \sum_{p=0}^{i+j} G_{p,i,j}\, H^{(\lambda)}_{i+j-2p}(x), \tag{5.1}
$$

where the coefficients $G_{p,i,j}$ are given as follows:

$$G_{p,i,j}=\begin{cases}\dfrac{4^{p}}{\Gamma\left(\frac{1}{2}(1+i+j-2p)+\lambda\right)}\displaystyle\sum_{\ell=0}^{p}\frac{(-1)^{\ell}\left(\frac{1}{2}(i+j-2\ell)\right)!\,\Gamma\left(\frac{1}{2}(1+i+j-2\ell)+\lambda\right)}{\ell!\,\left(\frac{1}{2}(i+j-2p)\right)!\,(p-\ell)!}\times\\ \left(1+\frac{i}{2}-\ell\right)_{\ell}\left(\frac{1}{2}(1+i-2\ell)+\alpha\right)_{\ell}\ {}_{3}F_{2}\left(\begin{matrix}-\frac{j}{2},\ell-p,\frac{1}{2}-\frac{j}{2}-\mu\\ -\frac{i}{2}-\frac{j}{2}+\ell,\frac{1}{2}-\frac{i}{2}-\frac{j}{2}+\ell-\lambda\end{matrix}\middle|1\right), & i\ \text{even},\ j\ \text{even},\\ \dfrac{4^{p}}{\Gamma\left(\frac{1}{2}(2+i+j-2p)+\lambda\right)}\displaystyle\sum_{\ell=0}^{p}\frac{(-1)^{\ell}\Gamma\left(\frac{1}{2}(1+i+j-2\ell)\right)\Gamma\left(\frac{1}{2}(2+i+j-2\ell)+\lambda\right)}{\ell!\,(p-\ell)!\,\Gamma\left(\frac{1}{2}(1+i+j-2p)\right)}\times\\ \left(1+\frac{i}{2}-\ell\right)_{\ell}\left(\frac{1}{2}(1+i-2\ell)+\alpha\right)_{\ell}\ {}_{3}F_{2}\left(\begin{matrix}\frac{1}{2}-\frac{j}{2},\ell-p,-\frac{j}{2}-\mu\\ \frac{1}{2}-\frac{i}{2}-\frac{j}{2}+\ell,-\frac{i}{2}-\frac{j}{2}+\ell-\lambda\end{matrix}\middle|1\right), & i\ \text{odd},\ j\ \text{even},\\ \dfrac{4^{p}}{\Gamma\left(\frac{1}{2}(1+i+j-2p)+\lambda\right)}\displaystyle\sum_{\ell=0}^{p}\frac{(-1)^{\ell}\left(\frac{1}{2}(i+j-2\ell)\right)!\,\Gamma\left(\frac{1}{2}(1+i+j-2\ell)+\lambda\right)}{\ell!\,\left(\frac{1}{2}(i+j-2p)\right)!\,(p-\ell)!}\times\\ \left(1+\frac{i}{2}-\ell\right)_{\ell}\left(\frac{1}{2}(1+i-2\ell)+\alpha\right)_{\ell}\ {}_{3}F_{2}\left(\begin{matrix}-\frac{j}{2},\ell-p,\frac{1}{2}-\frac{j}{2}-\mu\\ -\frac{i}{2}-\frac{j}{2}+\ell,-\frac{i}{2}-\frac{j}{2}+\ell-\lambda\end{matrix}\middle|1\right), & i\ \text{even},\ j\ \text{odd},\\ \dfrac{4^{p}}{\Gamma\left(\frac{1}{2}(1+i+j-2p)+\lambda\right)}\displaystyle\sum_{\ell=0}^{p}\frac{(-1)^{\ell}\Gamma\left(\frac{1}{2}(2+i+j-2\ell)\right)\Gamma\left(\frac{1}{2}(1+i+j-2\ell)+\lambda\right)}{\ell!\,(p-\ell)!\,\Gamma\left(\frac{1}{2}(2+i+j-2p)\right)}\times\\ \left(\frac{1}{2}(1+i-2\ell)\right)_{\ell}\left(1+\frac{i}{2}-\ell+\alpha\right)_{\ell}\ {}_{3}F_{2}\left(\begin{matrix}\frac{1}{2}-\frac{j}{2},\ell-p,-\frac{j}{2}-\mu\\ -\frac{i}{2}-\frac{j}{2}+\ell,\frac{1}{2}-\frac{i}{2}-\frac{j}{2}+\ell-\lambda\end{matrix}\middle|1\right), & i\ \text{odd},\ j\ \text{odd}.\end{cases}\tag{5.2}$$

Proof. The linearization formula (5.1) can be split into the following linearization formulas:

$$\begin{aligned}H_{2i}^{(\alpha)}(x)\,H_{2j}^{(\mu)}(x)=\sum_{p=0}^{i+j}&\frac{4^{p}}{\Gamma\left(\frac{1}{2}+i+j-p+\lambda\right)}\times\\ &\sum_{\ell=0}^{p}\frac{(-1)^{\ell}(i+j-\ell)!\,\Gamma\left(\frac{1}{2}+i+j-\ell+\lambda\right)(1+i-\ell)_{\ell}\left(\frac{1}{2}+i-\ell+\alpha\right)_{\ell}}{\ell!\,(i+j-p)!\,(p-\ell)!}\times\\ &{}_{3}F_{2}\left(\begin{matrix}-j,\ell-p,\frac{1}{2}-j-\mu\\ -i-j+\ell,\frac{1}{2}-i-j+\ell-\lambda\end{matrix}\middle|1\right)H_{2j+2i-2p}^{(\lambda)}(x),\end{aligned}\tag{5.3}$$

$$\begin{aligned}H_{2i}^{(\alpha)}(x)\,H_{2j+1}^{(\mu)}(x)=\sum_{p=0}^{i+j}&\frac{4^{p}}{\Gamma\left(\frac{3}{2}+i+j-p+\lambda\right)}\times\\ &\sum_{\ell=0}^{p}\frac{(-1)^{\ell}(i+j-\ell)!\,\Gamma\left(\frac{3}{2}+i+j-\ell+\lambda\right)(1+i-\ell)_{\ell}\left(\frac{1}{2}+i-\ell+\alpha\right)_{\ell}}{\ell!\,(i+j-p)!\,(p-\ell)!}\times\\ &{}_{3}F_{2}\left(\begin{matrix}-j,\ell-p,-\frac{1}{2}-j-\mu\\ -i-j+\ell,-\frac{1}{2}-i-j+\ell-\lambda\end{matrix}\middle|1\right)H_{2j+2i-2p+1}^{(\lambda)}(x),\end{aligned}\tag{5.4}$$

$$H_{2i+1}^{(\alpha)}(x)\,H_{2j}^{(\mu)}(x)=\sum_{p=0}^{i+j}\frac{4^p}{\Gamma\left(\frac{3}{2}+i+j-p+\lambda\right)}\times$$
$$\sum_{\ell=0}^{p}\frac{(-1)^{\ell}(i+j-\ell)!\,\Gamma\left(\frac{3}{2}+i+j-\ell+\lambda\right)(1+i-\ell)_{\ell}\left(\frac{3}{2}+i-\ell+\alpha\right)_{\ell}}{\ell!\,(i+j-p)!\,(p-\ell)!}\times \tag{5.5}$$
$${}_3F_2\left(\left.\begin{array}{c}-j,\ell-p,\frac{1}{2}-j-\mu\\ -i-j+\ell,-\frac{1}{2}-i-j+\ell-\lambda\end{array}\right|1\right)H_{2j+2i-2p+1}^{(\lambda)}(x),$$

$$H_{2i+1}^{(\alpha)}(x)\,H_{2j+1}^{(\mu)}(x)=\sum_{p=0}^{i+j+1}\frac{4^p}{\Gamma\left(\frac{3}{2}+i+j-p+\lambda\right)}\times$$
$$\sum_{\ell=0}^{p}\frac{(-1)^{\ell}(i+j-\ell+1)!\,\Gamma\left(\frac{3}{2}+i+j-\ell+\lambda\right)(1+i-\ell)_{\ell}\left(\frac{3}{2}+i-\ell+\alpha\right)_{\ell}}{\ell!\,(p-\ell)!\,(i+j-p+1)!}\times \tag{5.6}$$
$${}_3F_2\left(\left.\begin{array}{c}-j,\ell-p,-\frac{1}{2}-j-\mu\\ -1-i-j+\ell,-\frac{1}{2}-i-j+\ell-\lambda\end{array}\right|1\right)H_{2j+2i-2p+2}^{(\lambda)}(x).$$

The proofs of the four formulas are similar. We are going to prove (5.4). Starting from formula (2.5), we can write

$$H_{2i}^{(\alpha)}(x)\,H_{2j+1}^{(\mu)}(x)=2^{2i}\sum_{r=0}^{i}\frac{(-1)^r(1+i-r)_r\left(\frac{1}{2}+i-r+\alpha\right)_r}{r!}x^{2i-2r}H_{2j+1}^{(\mu)}.$$

Based on the moment formula (4.4), the following formula can be obtained:

$$H_{2i}^{(\alpha)}(x)\,H_{2j+1}^{(\mu)}(x)=4^i\sum_{r=0}^{i}\frac{(-1)^r(i+j-r)!\,\Gamma\left(\frac{3}{2}+i+j-r+\lambda\right)(1+i-r)_r\left(\frac{1}{2}+i-r+\alpha\right)_r}{r!}\times$$
$$\sum_{p=0}^{j+i-r}\frac{2^{2p-2(i-r)}}{p!(i+j-p-r)!\,\Gamma\left(\frac{3}{2}+i+j-p-r+\lambda\right)}\times \tag{5.7}$$
$${}_3F_2\left(\left.\begin{array}{c}-j,-p,-\frac{1}{2}-j-\mu\\ -i-j+r,-\frac{1}{2}-i-j+r-\lambda\end{array}\right|1\right)H_{2j+2i-2r-2p+1}^{(\lambda)}(x),$$

which can be transformed—after performing some algebraic computations—into (5.4). □

In the next part, we will offer product formulas for the GHPs and some polynomials.

Theorem 5.2. *Consider i and j to be two non-negative integers, and let $\phi_j(x)$ denote any of the four kinds of Chebyshev polynomials that are generated by* (2.19). *The following linearization formula holds:*

$$H_i^{(\mu)}(x)\,\phi_j(x)=\sum_{p=0}^{i}\gamma_{p,i}\,\phi_{j+i-2p}(x), \tag{5.8}$$

where the coefficients $\gamma_{p,i}$ are given by the following formula:

$$\gamma_{p,i} = \frac{i!}{(i-p)!\,p!} \times \begin{cases} {}_3F_1\left(\begin{matrix} -p, -i+p, \frac{1}{2}-\frac{i}{2}-\mu \\ \frac{1}{2}-\frac{i}{2} \end{matrix} \middle| -1 \right), & i\ even, \\ {}_3F_1\left(\begin{matrix} (-p, -i+p, -\frac{i}{2}-\mu \\ -\frac{i}{2} \end{matrix} \middle| -1 \right), & i\ odd. \end{cases}$$

Proof. We are going to show that the following two linearization formulas hold:

$$H_{2i}^{(\mu)}(x)\,\phi_j(x) = (2i)! \sum_{p=0}^{2i} \frac{1}{p!(2i-p)!}\, {}_3F_1\left(\begin{matrix} -p, -2i+p, \frac{1}{2}-i-\mu \\ -i+\frac{1}{2} \end{matrix} \middle| -1 \right) \phi_{j+2i-2p}(x), \tag{5.9}$$

$$H_{2i+1}^{(\mu)}(x)\,\phi_j(x) = (2i+1)! \sum_{p=0}^{2i+1} \frac{1}{p!(2i-p+1)!}\, {}_3F_1\left(\begin{matrix} -p, -2i+p-1, -\frac{1}{2}-i-\mu \\ -i-\frac{1}{2} \end{matrix} \middle| -1 \right) \phi_{j+2i-2p+1}(x). \tag{5.10}$$

To prove relation (5.9), we make use of the power form representation of $H_{2i}(x)$ along with the moment formula of $\phi_j(x)$ given in (2.20) to write:

$$H_{2i}^{(\mu)}(x)\,\phi_j(x) = \sum_{r=0}^{i} \frac{(-1)^r 2^{2r} (1+i-r)_r \left(\frac{1}{2}+i-r+\mu\right)_r}{r!} \sum_{s=0}^{2i-2r} \binom{2i-2r}{s} \phi_{j+2i-2r-2s}(x), \tag{5.11}$$

that can be rewritten again in the form

$$H_{2i}^{(\mu)}(x)\,\phi_j(x) = \sum_{p=0}^{2i} \sum_{\ell=0}^{p} \frac{(-4)^\ell \binom{2i-2\ell}{p-\ell} (1+i-\ell)_\ell \left(\frac{1}{2}+i-\ell+\mu\right)_\ell}{\ell!} \phi_{j+2i-2p}(x).$$

Based on the identity:

$$\sum_{\ell=0}^{p} \frac{(-4)^\ell \binom{2i-2\ell}{p-\ell} (1+i-\ell)_\ell \left(\frac{1}{2}+i-\ell+\mu\right)_\ell}{\ell!} = \frac{(2i)!}{p!(2i-p)!}\, {}_3F_1\left(\begin{matrix} -p, -2i+p, \frac{1}{2}-i-\mu \\ -i+\frac{1}{2} \end{matrix} \middle| -1 \right),$$

the following linearization formula can be obtained:

$$H_{2i}^{(\mu)}(x)\,\phi_j(x) = (2i)! \sum_{p=0}^{2i} \frac{1}{p!(2i-p)!}\, {}_3F_1\left(\begin{matrix} -p, -2i+p, \frac{1}{2}-i-\mu \\ -i+\frac{1}{2} \end{matrix} \middle| -1 \right) \phi_{j+2i-2p}(x).$$

Formula (5.10) can be similarly proved. □

Theorem 5.3. *Consider i and j to be two non-negative integers, and let $F_j^{a,b}(x)$ denote the generalized Fibonacci polynomials that are generated by* (2.17). *The following linearization formula holds:*

$$H_i^{(\mu)}(x)\,F_j^{a,b}(x) = \sum_{p=0}^{i} G_{p,i}\, F_{j+i-2p}^{a,b}(x), \tag{5.12}$$

where $G_{p,i}$ is given as

$$G_{p,i} = \frac{2^i\, a^{-i}\,(-b)^p i!}{(i-p)!\,p!} \times \begin{cases} {}_3F_1\left(\begin{matrix} -p, -i+p, \frac{1}{2}-\frac{i}{2}-\mu \\ \frac{1}{2}-\frac{i}{2} \end{matrix} \middle| \frac{a^2}{4\,b} \right), & i \text{ even}, \\ {}_3F_1\left(\begin{matrix} -p, -i+p, -\frac{i}{2}-\mu \\ -\frac{i}{2} \end{matrix} \middle| \frac{a^2}{4\,b} \right), & i \text{ odd}. \end{cases}$$

Proof. Similar to the proof of Theorem 5.2. □

Theorem 5.4. *Consider i and j to be two non-negative integers, and let $L_j^{c,d}(x)$ denote the generalized Lucas polynomials that are generated by* (2.18). *The following linearization formula holds:*

$$H_i^{(\mu)}(x)\, L_j^{c,d}(x) = \sum_{p=0}^{i} G_{p,i}\, L_{j+i-2p}^{c,d}(x), \tag{5.13}$$

where $G_{p,i}$ is given as

$$G_{p,i} = \frac{2^i\, c^{-i}\,(-d)^p i!}{(i-p)!\,p!} \times \begin{cases} {}_3F_1\left(\begin{matrix} -p, -i+p, \frac{1}{2}-\frac{i}{2}-\mu \\ \frac{1}{2}-\frac{i}{2} \end{matrix} \middle| \frac{c^2}{4\,d} \right), & i \text{ even}, \\ {}_3F_1\left(\begin{matrix} -p, -i+p, -\frac{i}{2}-\mu \\ -\frac{i}{2} \end{matrix} \middle| \frac{c^2}{4\,d} \right), & i \text{ odd}. \end{cases}$$

Proof. Similar to the proof of Theorem 5.2. □

6. Some other derivatives and connection formulas

This section is devoted to presenting the derivative formulas of the GHPs in terms of different polynomials. Some inverse formulas are also found. The derivation of these formulas is based on the analytic and inversion formulas of these polynomials.

Theorem 6.1. *In terms of the ultraspherical polynomials $U_j^{(\alpha)}(x)$, one has the following expression of $D^q H_j^{(\mu)}(x)$ for $j \geq q \geq 1$:*

$$D^q H_j^{(\mu)}(x) = \sum_{p=0}^{\left\lfloor \frac{j-q}{2} \right\rfloor} A_{p,j,q}\, U_{j-q-2p}^{(\alpha)}(x), \tag{6.1}$$

with $A_{p,j,q}$ given as

$$\begin{aligned} A_{p,j,q} =& \frac{2^{1+q-2\alpha}\sqrt{\pi}(j-2p-q+\alpha)j!\Gamma(j-2p-q+2\alpha)}{p!(j-2p-q)!\,\Gamma\left(\frac{1}{2}+\alpha\right)\Gamma(1+j-p-q+\alpha)} \times \\ & \begin{cases} {}_3F_1\left(\begin{matrix} -p, -j+p+q-\alpha, \frac{1}{2}-\frac{j}{2}-\mu \\ \frac{1}{2}-\frac{i}{2} \end{matrix} \middle| -1 \right), & j \text{ even}, \\ {}_3F_1\left(\begin{matrix} -p, -j+p+q-\alpha, -\frac{j}{2}-\mu \\ -\frac{j}{2} \end{matrix} \middle| -1 \right), & j \text{ odd}. \end{cases} \end{aligned} \tag{6.2}$$

Proof. Based on formula (2.5), we can write

$$D^q H_{2j}^{(\mu)}(x) = 2^{2j} \sum_{r=0}^{j-\lfloor \frac{q}{2} \rfloor} \frac{(-1)^r (1+2j-q-2r)_q (1-r+j)_r \left(\frac{1}{2}-r+\mu+j\right)_r}{r!} x^{2j-2r-q}, \tag{6.3}$$

which can be transformed into the following formula by applying the inversion formula (2.15)

$$\begin{aligned} D^q H_{2j}^{(\mu)}(x) =& 4^j \sum_{r=0}^{j-\lfloor \frac{q}{2} \rfloor} \frac{(-1)^r 2^{1-2j+q+2r} (1+2j-q-2r)_q (1-r+j)_r \left(\frac{1}{2}-r+\mu+j\right)_r}{r!} \times \\ & \sum_{t=0}^{j-\lfloor \frac{q}{2} \rfloor - r} \frac{(2j-q-2(r+t)+\alpha)(2j-q-2r)!\,\Gamma(2j-q-2(r+t-\alpha))\Gamma(1+\alpha)}{t!(2j-q-2(r+t))!\,\Gamma(1+2j-q-2r-t+\alpha)\Gamma(1+2\alpha)} U_{2j-2r-q-2t}^{(\alpha)}(x). \end{aligned} \tag{6.4}$$

Some lengthy algebraic computations turn (6.4) into the following formula:

$$\begin{aligned} D^q H_{2j}^{(\mu)}(x) =& \frac{\Gamma(1+\alpha)\Gamma\left(\frac{1}{2}+j+\mu\right)}{\Gamma(1+2\alpha)} \sum_{p=0}^{j-\lfloor \frac{q}{2} \rfloor} \frac{(-2j+2p+q-\alpha)\Gamma(2j-2p-q+2\alpha)}{(2j-2p-q)!} \times \\ & \sum_{r=0}^{p} \frac{(-1)^{1+r} 2^{1+q+2r} (1+j-r)_r (2j-2r)!}{(p-r)!\, r! \Gamma(1+2j-p-q-r+\alpha)\Gamma\left(\frac{1}{2}+j-r+\mu\right)} U_{j-q-2p}^{(\alpha)}(x). \end{aligned} \tag{6.5}$$

In hypergeometric form, the last formula can be written as

$$\begin{aligned} D^q H_{2j}^{(\mu)}(x) =& \frac{2^{1+q-2\alpha} \sqrt{\pi}(2j)!}{\Gamma\left(\frac{1}{2}+\alpha\right)} \sum_{p=0}^{j-\frac{q}{2}} \frac{(2j-2p-q+\alpha)\Gamma(2j-2p-q+2\alpha)}{p!(2j-2p-q)!\,\Gamma(1+2j-p-q+\alpha)} \times \\ & {}_3F_1\left(\left. \begin{matrix} -p, \frac{1}{2}-j-\mu, -2j+p+q-\alpha \\ \frac{1}{2}-j \end{matrix} \right| -1 \right) U_{j-q-2p}^{(\alpha)}(x). \end{aligned} \tag{6.6}$$

Similarly, we can find an explicit expression for $D^q H_{2j+1}^{(\mu)}(x)$ in terms of $U_j^{(\alpha)}(x)$. This expression is

$$\begin{aligned} D^q H_{2j+1}^{(\mu)}(x) =& \frac{2^{1+q-2\alpha} \sqrt{\pi}(2j+1)!}{\Gamma\left(\frac{1}{2}+\alpha\right)} \sum_{p=0}^{j-\frac{q}{2}+1} \frac{(1+2j-2p-q+\alpha)\Gamma(1+2j-2p-q+2\alpha)}{p!(2j-2p-q+1)!\,\Gamma(2+2j-p-q+\alpha)} \times \\ & {}_3F_1\left(\left. \begin{matrix} -p, -\frac{1}{2}-j-\mu, -1-2j+p+q-\alpha \\ -\frac{1}{2}-j \end{matrix} \right| -1 \right) U_{j-q-2p+1}^{(\alpha)}(x). \end{aligned} \tag{6.7}$$

Merging the two expressions in (6.6) and (6.7) gives the expression in (6.1). □

Theorem 6.2. *In terms of $F_j^{a,b}(x)$, one has the following expression of $D^q H_j^{(\mu)}(x)$ for $j \geq q \geq 1$:*

$$D^q H_j^{(\mu)}(x) = \sum_{p=0}^{\lfloor \frac{j-q}{2} \rfloor} M_{p,j,q}\, F_{j-q-2p}^{a,b}(x), \tag{6.8}$$

with $M_{p,j,q}$ given as

$$M_{p,j,q} = \frac{(-1)^p\, 2^j\, a^{-j+q}\, b^p\, j!\,(1+j-2p-q)}{p!\,(j-p-q+1)!} \times$$
$$\begin{cases} {}_3F_1\left(\begin{matrix} -p, -1-j+p+q, \frac{1}{2}-\frac{j}{2}-\mu \\ \frac{1}{2}-\frac{j}{2} \end{matrix} \middle| \frac{a^2}{4b}\right), & j \text{ even}, \\ {}_3F_1\left(\begin{matrix} -p, -1-j+p+q, -\frac{j}{2}-\mu \\ -\frac{j}{2} \end{matrix} \middle| \frac{a^2}{4b}\right), & j \text{ odd}. \end{cases}$$

Proof. Similar to the proof of Theorem 6.1. □

Theorem 6.3. *In terms of $L_j^{c,d}(x)$, one has the following expression of $D^q H_j^{(\mu)}(x)$ for $j \geq q \geq 1$:*

$$D^q H_j^{(\mu)}(x) = \sum_{p=0}^{\left\lfloor \frac{j-q}{2} \right\rfloor} \bar{M}_{p,j,q}\, L_{j-q-2p}^{c,d}(x), \tag{6.9}$$

with $\bar{M}_{p,j,q}$ given as

$$\bar{M}_{p,j,q} = \frac{(-1)^p 2^j c^{-j+q} d^p \eta_{j-2p-q}\, j!}{p!(j-p-q)!} \times$$
$$\begin{cases} {}_3F_1\left(\begin{matrix} -p, -j+p+q, \frac{1}{2}-\frac{j}{2}-\mu \\ \frac{1}{2}-\frac{j}{2} \end{matrix} \middle| \frac{c^2}{4d}\right), & j \text{ even}, \\ {}_3F_1\left(\begin{matrix} -p, -j+p+q, -\frac{j}{2}-\mu \\ -\frac{j}{2} \end{matrix} \middle| \frac{c^2}{4d}\right), & j \text{ odd}. \end{cases}$$

Proof. Similar to the proof of Theorem 6.1. □

Remark 6.1. *The inversion formulas to the derivative formulas stated in Theorems 6.1–6.3 can also be derived using similar procedures. The following theorem exhibits the derivatives of the ultraspherical polynomials in terms of the GHPs.*

Theorem 6.4. *In terms of $H_j^{(\alpha)}(x)$, one has the following expression of $D^q U_j^{(\alpha)}(x)$ for $j \geq q \geq 1$:*

$$D^q U_j^{(\alpha)}(x) = \sum_{p=0}^{\left\lfloor \frac{j-q}{2} \right\rfloor} Z_{p,j,q}\, H_{j-q-2p}^{(\alpha)}(x), \tag{6.10}$$

with $Z_{p,j,q}$ given as

$$Z_{p,j,q} = \frac{j!2^{-1-j+2p+2q+2\alpha}\Gamma\left(\frac{1}{2}+\alpha\right)\Gamma(j+\alpha)}{\Gamma(j+2\alpha)p!}\times$$

$$\begin{cases} \frac{\Gamma\left(\frac{1}{2}(1+j-q)+\mu\right)}{\left(\frac{1}{2}(j-2p-q)\right)!\,\Gamma\left(\frac{1}{2}(1+j-q)\right)\Gamma\left(\frac{1}{2}(1+j-2p-q)+\mu\right)}\ {}_2F_2\left(\begin{matrix} -p, \frac{1}{2}-\frac{j}{2}+\frac{q}{2} \\ 1-j-\alpha, \frac{1}{2}-\frac{j}{2}+\frac{q}{2}-\mu \end{matrix}\middle| -1\right), & j \text{ even}, q \text{ even}, \\ \frac{\Gamma\left(\frac{1}{2}(2+j-q)+\mu\right)}{\Gamma\left(\frac{1}{2}(1+j-2p-q)\right)\Gamma\left(\frac{1}{2}(2+j-q)\right)\Gamma\left(\frac{1}{2}(2+j-2p-q)+\mu\right)}\ {}_2F_2\left(\begin{matrix} -p, -\frac{j}{2}+\frac{q}{2} \\ 1-j-\alpha, -\frac{j}{2}+\frac{q}{2}-\mu \end{matrix}\middle| -1\right), & j \text{ even}, q \text{ odd}, \\ \frac{\Gamma\left(\frac{1}{2}(2+j-q)+\mu\right)}{\Gamma\left(\frac{1}{2}(1+j-2p-q)\right)\Gamma\left(\frac{1}{2}(2+j-q)\right)\Gamma\left(\frac{1}{2}(2+j-2p-q)+\mu\right)}\ {}_2F_2\left(\begin{matrix} -p, -\frac{j}{2}+\frac{q}{2} \\ 1-j-\alpha, -\frac{j}{2}+\frac{q}{2}-\mu \end{matrix}\middle| -1\right), & j \text{ odd}, q \text{ even}, \\ \frac{\Gamma\left(\frac{1}{2}(1+j-q)+\mu\right)}{\Gamma\left(\frac{1}{2}(2+j-2p-q)\right)\Gamma\left(\frac{1}{2}(1+j-q)\right)\Gamma\left(\frac{1}{2}(1+j-2p-q)+\mu\right)}\ {}_2F_2\left(\begin{matrix} -p, \frac{1}{2}-\frac{j}{2}+\frac{q}{2} \\ 1-j-\alpha, \frac{1}{2}-\frac{j}{2}+\frac{q}{2}-\mu \end{matrix}\middle| -1\right), & j \text{ odd}, q \text{ odd}. \end{cases}$$

Proof. We employ the analytic form of the ultraspherical polynomials (2.14) and the GHPs' inversion formulas (2.8) and (2.9) to obtain (6.10). □

7. Applications to some of the derived formulas

This section presents two applications based on the formulas developed in the previous sections. The first concerns obtaining closed formulas for certain weighted definite integrals involving the GHPs. The second concerns deriving the operational matrix of the GHPs' derivatives.

7.1. Some new definite and weighted definite integrals

This section will display new weighted definite integrals based on moments, linearization, and derivatives formulas.

Corollary 7.1. *For all positive integers i, m, j with $(j+m+i)$ even, the following identity holds:*

$$\int_{-\infty}^{\infty} x^m\, |x|^{2\lambda}\, e^{-x^2} H_j^{(\mu)}(x)\, H_j^{(\lambda)}(x)\, dx = G_{\frac{j+m-i}{2},j,m}\, h_i, \tag{7.1}$$

where

$$G_{p,j,m} = \frac{2^{-m+2p}\left(\left\lfloor\frac{j+m}{2}\right\rfloor!\,\Gamma\left(\frac{1}{2}+\left\lceil\frac{j+m}{2}\right\rceil+\lambda\right)\right)}{p!\left(\left\lfloor\frac{1}{2}(j+m-2p)\right\rfloor!\,\Gamma\left(\frac{1}{2}+\left\lceil\frac{j+m}{2}\right\rceil+\lambda-p\right)\right)}\times \\ {}_3F_2\left(\begin{matrix} -\left\lfloor\frac{j}{2}\right\rfloor, -p, \frac{1}{2}-\left\lceil\frac{j}{2}\right\rceil-\mu \\ \left\lfloor\frac{1}{2}(1-j-m)\right\rfloor, \frac{1}{2}-\left\lceil\frac{j+m}{2}\right\rceil-\lambda \end{matrix}\middle| 1\right), \tag{7.2}$$

where $\lfloor z \rfloor$ and $\lceil z \rceil$ represent, respectively, the floor and ceiling functions, and h_i is given by:

$$h_i = 4^i \begin{cases} \left(\frac{i}{2}\right)!\,\Gamma\left(\frac{i+1}{2}+\mu\right), & i=j,\ i \text{ even}, \\ \left(\frac{i-1}{2}\right)!\,\Gamma\left(1+\frac{i}{2}+\mu\right), & i=j,\ i \text{ odd}. \end{cases} \tag{7.3}$$

Proof. The moment formula of $H_j^{(\mu)}(x)$ can be written as:

$$x^m H_j^{(\mu)}(x) = \sum_{p=0}^{\left\lfloor \frac{j+m}{2} \right\rfloor} G_{p,j,m} H_{j+m-2p}^{(\lambda)}(x), \tag{7.4}$$

and $G_{p,j,m}$ are given by (7.2). The moment formula (4.1) along with the orthogonality relation (2.4) leads to (7.1). □

Corollary 7.2. *For all non-negative integers i, j, s with $j \geq i$, one has the following integral formulas:*

$$\int_{-1}^{1} \frac{H_i^{(\mu)}(x)\, T_j(x)\, T_s(x)}{\sqrt{1-x^2}}\, dx = \frac{\pi}{2\, c_s}\, \eta_{i,s}, \tag{7.5}$$

$$\int_{-1}^{1} \sqrt{1-x^2}\, H_i^{(\mu)}(x)\, U_j(x)\, U_s(x)\, dx = \frac{\pi}{2}\, \eta_{i,s}, \tag{7.6}$$

$$\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}\, H_i^{(\mu)}(x)\, V_j(x)\, V_s(x)\, dx = \pi\, \eta_{i,s}, \tag{7.7}$$

$$\int_{-1}^{1} \sqrt{\frac{1-x}{1+x}}\, H_i^{(\mu)}(x)\, W_j(x)\, W_s(x)\, dx = \pi\, \eta_{i,s}, \tag{7.8}$$

where

$$c_s = \begin{cases} \frac{1}{2}, & s = 0, \\ 1, & s \geq 1, \end{cases}$$

and $\eta_{i,s}$ is given by

$$\eta_{i,s} = i! \begin{cases} \dfrac{1}{\left(\frac{1}{2}(i+j-s)\right)!\, \left(\frac{1}{2}(i-j+s)\right)!}\ {}_3F_1\left(\left. \begin{matrix} -\frac{i}{2}+\frac{j}{2}-\frac{s}{2}, -\frac{i}{2}-\frac{j}{2}+\frac{s}{2}, \frac{1}{2}-\frac{i}{2}-\mu \\ \frac{1}{2}-\frac{i}{2} \end{matrix} \right| -1 \right), & i \text{ even}, \\ \dfrac{1}{\Gamma\left(\frac{1}{2}(i+j-s+2)\right)\Gamma\left(\frac{1}{2}(i-j+s+2)\right)}\ {}_3F_1\left(\left. \begin{matrix} -\frac{i}{2}+\frac{j}{2}-\frac{s}{2}, -\frac{i}{2}-\frac{j}{2}+\frac{s}{2}, -\frac{i}{2}-\mu \\ -\frac{i}{2} \end{matrix} \right| -1 \right), & i \text{ odd}. \end{cases}$$

Proof. The four integrals (7.5)–(7.8) are direct consequences of the unified linearization formula (5.8) along with the orthogonality relations of the four kinds of Chebyshev polynomials [54]. □

Corollary 7.3. *Let j, k, q be positive integers, with $j \geq q$. The following integral formula holds:*

$$\int_{-1}^{1} \left(1-x^2\right)^{\alpha-\frac{1}{2}}\, D^q H_j^{(\mu)}(x)\, U_k^{(\alpha)}(x)\, dx = \begin{cases} A_{\frac{j-k-q}{2}}\, h_k, & (j-k-q) \text{ even}, \\ 0, & \text{otherwise}, \end{cases} \tag{7.9}$$

where $A_{p,j,q}$ are defined in (6.2)*, and h_k is given by*

$$h_k = \frac{k!\, \Gamma\left(\frac{1}{2}\right)\, \Gamma\left(\alpha+\frac{1}{2}\right)}{(2\alpha)_k\, \Gamma(\alpha)\, (\alpha+k)}.$$

Proof. Starting from formula (6.1), and multiplying both terms by $(1-x^2)^{\alpha-\frac{1}{2}}\, U_k^{(\alpha)}(x)$, we can write

$$\int_{-1}^{1}\left(1-x^2\right)^{\alpha-\frac{1}{2}}\, D^q H_j^{(\mu)}(x)\, U_k^{(\alpha)}(x)\,dx = \sum_{p=0}^{\left\lfloor\frac{j-q}{2}\right\rfloor} A_{p,j,q}\int_{-1}^{1}\left(1-x^2\right)^{\alpha-\frac{1}{2}}\, U_k^{(\alpha)}(x)\, U_{j-q-2p}^{(\alpha)}(x)\,dx. \tag{7.10}$$

The application of the orthogonality relation of $U_k^{(\alpha)}(x)$ in (2.16) leads to formula (7.9). □

7.2. Introducing the operational matrix of integer derivatives of the GHPs

It is well-known that the operational matrices of derivatives of different polynomials are important tools in numerical analysis. Operational matrices are fundamental keys for solving many DEs, particularly non-linear ones. Many authors were interested in establishing these matrices for various polynomials and utilizing them in various applications. For example, the operational matrices of derivatives of some combinations of Legendre polynomials were established and utilized in [55] to solve initial value problems.

In the following part, the operational matrix of integer derivatives of the GHPs will be established. First, the following lemma is needed.

Lemma 7.1. *The first-order derivative of the GHPs may be expressed explicitly in the form*

$$DH_j^{(\mu)}(x) = \begin{cases} 2\,j\,H_{j-1}^{(\mu)}(x), & j \text{ even},\\[2ex] 2\,j\,H_{j-1}^{(\mu)}(x) + \mu \displaystyle\sum_{m=0}^{j-3} \frac{(2)^{j-m+1}\,(-1)^{\frac{j-m+1}{2}}\,\left(\frac{j-1}{2}\right)!}{\Gamma\left(1+\frac{m}{2}\right)}\,H_m^{(\mu)}(x), & j \text{ odd}. \end{cases} \tag{7.11}$$

Proof. Formula (7.11) can be split into the following two formulas:

$$DH_{2j}^{(\mu)}(x) = 4\,j\,H_{2j-1}^{(\mu)}(x), \tag{7.12}$$

$$DH_{2j+1}^{(\mu)}(x) = 2\,(2j+1)H_{2j}^{(\mu)}(x) + \mu\sum_{\ell=1}^{j} 2^{2\ell+2}(-1)^{\ell+1}(j-\ell+1)_\ell\, H_{2j-2\,\ell}^{(\mu)}(x). \tag{7.13}$$

To prove (7.12), we set $q=0$ and $\lambda=\mu$ in (3.5) to get

$$DH_{2j}^{(\mu)}(x) = j!\,\Gamma\left(\frac{1}{2}+j+\mu\right)\sum_{\ell=0}^{j-1}\frac{2^{2\ell+2}\,{}_1F_0(-\ell;;1)}{(j-\ell-1)!\,\ell!\,\Gamma\left(\frac{1}{2}+j-\ell+\mu\right)} H_{2j-2\ell-1}^{(\mu)}(x). \tag{7.14}$$

If we note the simple identity

$${}_1F_0(-\ell;;1) = \begin{cases} 1, & \ell = 0,\\ 0, & \ell > 0, \end{cases}$$

then, it is easy to show the following identity:

$$DH_{2j}^{(\mu)}(x) = 4\,j\,H_{2j-1}^{(\mu)}(x). \tag{7.15}$$

Now, to prove formula (7.13), we set $q = 0$ and $\lambda = \mu$ in (3.6) to get the following formula:

$$D H^{(\mu)}_{2j+1}(x) = \frac{\sqrt{\pi}(2j+1)!\,\Gamma\left(\frac{1}{2}+j+\mu\right)}{\Gamma\left(\frac{1}{2}+j\right)} \sum_{\ell=0}^{j} \frac{2^{1-2j+2\ell}}{\ell!(j-\ell)!\,\Gamma\left(\frac{1}{2}+j-\ell+\mu\right)} \times \\ {}_3F_2\left(\left. \begin{matrix} \frac{1}{2}-j, -\ell, -\frac{1}{2}-j-\mu \\ -\frac{1}{2}-j, \frac{1}{2}-j-\mu \end{matrix} \right| 1 \right) H^{(\mu)}_{2j-2L}(x). \tag{7.16}$$

Now, to find a closed form for the ${}_3F_2(1)$ that appears in the last formula, we set

$$M_{\ell,\mu} = {}_3F_2\left(\left. \begin{matrix} \frac{1}{2}-j, -\ell, -\frac{1}{2}-j-\mu \\ -\frac{1}{2}-j, \frac{1}{2}-j-\mu \end{matrix} \right| 1 \right),$$

and utilize the Zeilberger's algorithm [56] to show that $M_{\ell,\mu}$ has the following closed form:

$$M_{\ell,\mu} = \begin{cases} 1, & \ell = 0, \\ \dfrac{(-1)^{\ell+1}\, 2^{2\ell+2}\, \mu\, \ell!}{2^{2\ell+1}\,(2j+1)\,\left(\mu+j-\ell+\frac{1}{2}\right)_{\ell}}, & \ell \geq 1. \end{cases} \tag{7.17}$$

Inserting the reduced formula (7.17) into (7.16) yields the following formula:

$$D H^{(\mu)}_{2j+1}(x) = 2\,(2j+1) H^{(\mu)}_{2j}(x) + \mu \sum_{\ell=1}^{j} 2^{2\ell+2}(-1)^{\ell+1}(j-\ell+1)_{\ell}\, H^{(\mu)}_{2j-2\,\ell}(x).$$

This completes the proof of Corollary 7.1. □

Corollary 7.4. *Based on formula* (7.11)*, the first derivative of* $H^{(\mu)}_{j}(x)$ *can be written in matrix form as*

$$\frac{d\,\boldsymbol{\mathcal{H}}^{(\mu)}(x)}{d\,x} = \mathcal{S}\,\boldsymbol{\mathcal{H}}^{(\mu)}(x), \tag{7.18}$$

where $\boldsymbol{\mathcal{H}}^{(\mu)}(x) = [\mathcal{H}^{(\mu)}_0(x), \mathcal{H}^{(\mu)}_1(x), \ldots, \mathcal{H}^{(\mu)}_N(x)]^T$*, where* $\mathcal{S} = (s_{m,j})$ *is the operational matrix of derivative whose order is* $(N+1)\times(N+1)$*, and its elements can be expressed in the form*

$$s_{m,j} = \begin{cases} 2j, & \text{if } m = j-1, \\ \dfrac{\mu 2^{1+j-m}(-1)^{\frac{1}{2}(j-m+1)}\left(\frac{j-1}{2}\right)!}{\left(\frac{m}{2}\right)!}, & \text{if } j > m,\ j \text{ odd, and } m \text{ even}, \\ 0, & \text{otherwise}. \end{cases}$$

For example, for $N = 7$*, the matrix* S *is given by*

$$S = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 16\mu & 0 & 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 \\ -128\mu & 0 & 32\mu & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 \\ 1536\mu & 0 & -384\mu & 0 & 48\mu & 0 & 14 & 0 \end{pmatrix}_{8\times 8}.$$

8. Conclusions

This paper extends the theoretical framework beyond the classical Hermite polynomials by establishing essential new formulas related to the GHPs. We first introduced some elementary formulas for these polynomials, then derived a series of new formulas related to these polynomials. These formulas include the derivative expressions of the GHPs, which are expressed in terms of various polynomials. In addition, some product formulas for these polynomials, along with some celebrated polynomials, were also developed. Some applications to the derived formulas were also deduced based on applying some of the introduced formulas. Some new definite and weighted definite integrals were developed. In addition, a new operational matrix of the GHPs was established. We expect it will be useful in numerically treating various differential equations. As far as we know, most of the formulas in this paper are new. In addition, we aim to employ these polynomials in numerical analysis soon.

Author contributions

Waleed Mohamed Abd-Elhameed: Conceptualization, methodology, validation, formal analysis, investigation, project administration, supervision, writing-original draft, writing-review & editing; Omar Mazen Alqubori: Methodology, validation, investigation, funding acquisition. All authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

The authors declare that they have no competing interest.

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