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Research article

Fixed points of generalized φ-concave-convex operators with mixed monotonicity and applications

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Abstract: In this paper, we introduced a new concept of generalized φ-concave-convex operator and proved the existence and uniqueness of fixed points of such operators with mixed monotonicity. As consequences, several new fixed point results about mixed monotone operators with some concavity and convexity were gained. In addition, the main results were applied to nonlinear integral equations on unbounded regions. The research findings generalized and developed recent relevant results in the literature.

Keywords: generalized φ-concave-convex operator; mixed monotone operator; fixed point
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1. Introduction

It is well known that seeking the positive solutions to nonlinear equations is of great importance in nonlinear analysis. In order to meet this goal we are used to utilizing suitable fixed point methods as well as monotone iteration techniques (see e.g., [1, 2]). The concept of monotone operator together with cone and partial order was first introduced by Krasnoselskii [1] and in this book the existence of positive fixed points was investigated. Later on, cone theory and monotone iteration techniques were set up and well-developed (see e.g., [2–8]). The theory about monotone operators has been investigated over six decades and has been applied to various different fields, such as different equations and dynamical systems [9–12], fixed point theory [13–17], control systems [18], theory of Li groups [19] and biomathematics [20]. However, in several applications [21, 22] the operators involved are not monotone but have a class of mixed monotone property. To deal with such situations, the authors in [23] gave the concept of mixed monotone operators and investigated their existence of coupled fixed points. Since mixed monotone operators play a crucial role in the studying of nonlinear analysis, nonlinear differential equations and integral equations, such operators have not only important theoretical meaning (see e.g., [24]) but also wide applications in non-mathematics fields, such as engineering and nuclear physics [3, 4, 25, 26]. Besides, by virtue of the fact that embedding a dynamical system, whose generator has both increasing and decreasing monotonicity property into a larger symmetric monotone dynamical system, mixed monotone operators have significant applications in mathematical biology, chemistry, neural networks and others [27–32]. In order to solve the fixed point problem, two common methods are usually utilized in the study of the fixed point problems for mixed monotone operators. One is to require that the mixed monotone operators should satisfy some compactness or continuity (see e.g., [23,33–35]); the other is to assume the operators discussed exhibit certain concavity or convexity (see e.g., [36–44]). For recent two decades, a number of authors were interested in studying the mixed monotone operators with some concavity and convexity in the setting of ordered real Banach spaces. In [36, 37, 45], the scholars presented the mixed monotone operators that meet the following concave-convex properties:

(H_1) $\mathfrak{A}(t\sigma, t^{-1}\varsigma) \geq t^{\alpha}\mathfrak{A}(\sigma, \varsigma)$;

(H_2) $\mathfrak{A}(t\sigma, t^{-1}\varsigma) \geq t(1 + r)\mathfrak{A}(\sigma, \varsigma)$.

Z. Liang etc. [39] investigated this problem and extended (H_1) to the following condition:

(H_3) $\mathfrak{A}(t\tau, t^{-1}\upsilon) \geq t^{\alpha(t)}\mathfrak{A}(\tau, \upsilon)$.

Later on, Wu [41] continued to study the problems and extended (H_2) and (H_3) respectively to the following

(H_4) $\mathfrak{A}(t\tau, t^{-1}\upsilon) \geq t^{\alpha(t,\tau,\upsilon)}$;

(H_5) $\mathfrak{A}(t\tau, t^{-1}\upsilon) \geq t(1 + \eta(t, \tau, \upsilon))\mathfrak{A}(\tau, \upsilon)$,

and introduced the concepts of $t - \alpha(t, \tau, \upsilon)$ and $t - \theta(t, \tau, \upsilon)$ mixed monotone model operator for the mixed monotone operators satisfying (H_4) and (H_5), respectively.

In addition, Xu and Jia [42] introduced the concept of ϕ concave-(-ψ) convex operator and investigated some mixed monotone operators with certain concavity and convexity in a general way. However, we have not found any general method to cope with such operators with one of the concave-convex properties. In this paper, we introduce the concept of generalized φ-concave-convex operators to solve this problem. The advantage of doing so is that such generalized φ-concave-convex operators can unify a large number of operators satisfying the conditions from (H_1) to (H_5) above and others, and so we can investigate the existence and uniqueness as well as the convergence of the iterated sequences for such operators under weaker conditions. As a result, some new fixed point results on mixed monotone operators with certain concavity and convexity are obtained and some relevant results are improved or extended in the literature.

2. Preliminaries

In this section, we begin by briefly reviewing some basic concepts, symbols and known facts in the theory of cone and partial order, which can be found in Refs. [3, 4, 23, 36–42, 46].

Let the real Banach space K be partially ordered by a cone M of K, i.e., $\sigma \leq \varsigma$ (alternatively denoted by $\varsigma \geq \sigma$) if and only if $\varsigma - \sigma \in M$. We denote by θ the null element of K. Note that a nonempty closed subset M of K is called a cone if it is convex and satisfies

(i) $\forall \sigma \in M, \lambda \geq 0 \Rightarrow \lambda\sigma \in M$;

(ii) $\forall \sigma, -\sigma \in M \Rightarrow \sigma = \theta$.

Denote by int$M$ the interior of M. A cone M is called solid if int$M \neq \emptyset$, i.e., int$M$ is nonempty. M

is called normal if there is a positive constant N such that $\theta \leq \sigma \leq \varsigma$ implies $\|\sigma\| \leq N\|\varsigma\|$. The smallest N satisfying the condition above is called the normal constant of M. For convenience, we will keep using these symbols throughout the rest of the content.

For any $e > \theta$, that is, $e \geq \theta$ and $e \neq \theta$, we define

$$M_e = \{\sigma | \sigma \in K \text{ and there exist } \lambda, \mu > 0 \text{ such that } \lambda e \leq \sigma \leq \mu e\}.$$

Let $U \subset K$. If for any $\sigma \in U, \lambda > 0$ it follows that $\lambda\sigma \in U$, then U is called a wedge in K.

Let $\tau_0, \upsilon_0 \in K$ with $\tau_0 \leq \upsilon_0$. Write

$$[\tau_0, \upsilon_0] = \{\sigma \in K | \tau_0 \leq \sigma \leq \upsilon_0\},$$

where $[\tau_0, \upsilon_0]$ is said to be an ordering interval in K.

Let $U \subset K$. We call an operator $\mathfrak{A} : U \times U \to K$ mixed monotone, if $\forall \sigma_1, \sigma_2, \varsigma_1, \varsigma_2 \in U$, $\sigma_1 \leq \sigma_2$ and $\varsigma_1 \geq \varsigma_2$ imply $\mathfrak{A}(\sigma_1, \varsigma_1) \leq \mathfrak{A}(\sigma_2, \varsigma_2)$. If an element $\sigma^* \in U$ satisfies $\mathfrak{A}(\sigma^*, \sigma^*) = \sigma^*$, then σ^* is said to be a fixed point of $\mathfrak{A}$. An operator $\mathfrak{A} : U \subset K \to K$ is called convex if for all $\sigma, \varsigma \in U$ and each $t \in [0, 1]$, we have

$$\mathfrak{A}(t\sigma + (1-t)\varsigma) \leq t\mathfrak{A}\sigma + (1-t)\mathfrak{A}\varsigma;$$

$\mathfrak{A}$ is called concave if $-\mathfrak{A}$ is convex.

Assume $U = M$ or $U = \text{int}M$ and $0 \leq \alpha < 1$. An operator $\mathfrak{A} : U \to U$ is named α-concave ($(-\alpha)$-convex) if it satisfies

$$\mathfrak{A}(t\sigma) \geq t^\alpha \mathfrak{A}\sigma \ (\mathfrak{A}(t\sigma) \leq t^{-\alpha}\mathfrak{A}\sigma), \forall t \in (0,1), \forall \sigma \in U.$$

Let $\mathfrak{A} : M \to M$ be an operator and $e > \theta$. Suppose that

(i) $\mathfrak{A}e \in M_e$;

(ii) there exists a real number $\eta = \eta(t, \sigma) > 0$ such that

$$\mathfrak{A}(t\sigma) \geq t(1+\eta)\mathfrak{A}\sigma, \forall t \in (0,1), \forall \sigma \in M_e,$$

then $\mathfrak{A}$ is called a generalized e-concave operator, and $\eta = \eta(t, \sigma)$ is called its characteristic function.

Similarly, in the above-mentioned definition, if the condition (ii) is replaced by the following

(ii') $\mathfrak{A}(t\sigma) \leq \frac{1}{t(1+\eta)}\mathfrak{A}\sigma$, $\forall t \in (0,1)$, $\forall \sigma \in M_e$,

then $\mathfrak{A}$ is called a generalized e-convex operator, and $\eta = \eta(t, \sigma)$ is called its characteristic function.

Definition 2.1. ([40, 41]) If the operator $\mathfrak{A} : M_e \times M_e \to K$ is mixed monotone, and satisfies the condition (a) (or (b)) of Lemma 2.1, then $\mathfrak{A}$ is called a $t - \alpha(t)$ (or $t - \eta(t)$) mixed monotone model operator.

Definition 2.2. ([42]) An operator $\mathfrak{A} : U \times U \to K$ is said to be ϕ concave $-(-\psi)$ convex, if there are two functions $\phi : (0,1) \times U \to (0, \infty)$ and $\psi : (0,1] \times U \to (0, \infty)$ such that $(t, \sigma) \in (0,1] \times U$ implies $t < \phi(t, \sigma)\psi(t, \sigma) \leq 1$, and also $\mathfrak{A}$ satisfies the following two conditions:

(H_1) $\mathfrak{A}(t\sigma, \varsigma) \geq \phi(t, \sigma)\mathfrak{A}(\sigma, \varsigma), \forall t \in (0,1), \forall (\sigma, \varsigma) \in U \times U$;

(H_2) $\mathfrak{A}(\sigma, t\varsigma) \leq \frac{1}{\psi(t,\varsigma)}\mathfrak{A}(\sigma, \varsigma), \forall t \in (0,1), \forall (\sigma, \varsigma) \in U \times U$.

Lemma 2.1. ([39]) Let $e > \theta$ and $\mathfrak{A} : M_e \times M_e \to K$ be an operator. Then the following two statements are equivalent:

(a) For all $0 < t < 1$ and $\tau, \upsilon \in M_e$, there exists $0 < \alpha = \alpha(t) < 1$ such that $\mathfrak{A}(t\tau, t^{-1}\upsilon) \geq t^{\alpha(t)}\mathfrak{A}(\tau, \upsilon)$.

(b) For all $0 < t < 1$ and $\tau, \upsilon \in M_e$, there exists $\eta = \eta(t) > 0$ such that $\mathfrak{A}(t\tau, t^{-1}\upsilon) \geq t[1+\eta(t)]\mathfrak{A}(\tau, \upsilon)$, where $t[1+\eta(t)] < 1$.

Definition 2.3. Let U be a wedge of K. An operator $\mathfrak{A} : U \times U \to K$ is said to be generalized φ-concave-convex, if there exists a function $\varphi : (0,1) \times U \times U \to (0, \infty)$ such that

$$\mathfrak{A}(t\sigma, \varsigma) \geq \varphi(t, \sigma, \varsigma)\mathfrak{A}(\sigma, t\varsigma), \forall t \in (0,1), \forall \sigma, \varsigma \in U.$$

Remark 2.1. The definition of generalized φ-concave-convex operator above is different from that discussed in [41, Theorem 3.1], because in [41, Theorem 3.1], the discussed operator $\mathfrak{A}$ is defined on $M \times M$, while in Definition 2.3, we need not require the operator $\mathfrak{A}$ should be only defined on $M \times M$; we may define $\mathfrak{A}$ on $U \times U$, where U may be any wedge of K in a general way. So the concept of generalized φ-concave-convex operator is a generalization of the operator discussed in [41, Theorem 3.1].

Remark 2.2. The concept of generalized φ-concave-convex mixed monotone operator is a generalization of a number of operators such as $t - \alpha(t)$ (or $t - \eta(t)$) mixed monotone model operator, ϕ concave-$(-\psi)$ convex mixed monotone operator.

For example, if $\mathfrak{A}$ is ϕ concave-$(-\psi)$ convex then we have

$$\mathfrak{A}(t\sigma, \varsigma) \geq \phi(t, \sigma)\mathfrak{A}(\sigma, \varsigma), \forall t \in (0,1), \forall \sigma, \varsigma \in U$$

$$\mathfrak{A}(\sigma, t\varsigma) \leq \frac{1}{\psi(t, \varsigma)}\mathfrak{A}(\sigma, \varsigma), \forall t \in (0,1), \forall \sigma, \varsigma \in U.$$

So it follows that

$$\begin{aligned}\mathfrak{A}(t\sigma, \varsigma) &\geq \phi(t, \sigma)\psi(t, \varsigma)\mathfrak{A}(\sigma, t\varsigma)\\ &= \varphi(t, \sigma, \varsigma)\mathfrak{A}(\sigma, t\varsigma), \forall t \in (0,1), \forall \sigma, \varsigma \in U,\end{aligned}$$

where $\varphi(t, \sigma, \varsigma) = \phi(t, \sigma)\psi(t, \varsigma)$. Thus, $\mathfrak{A}$ is generalized φ-concave-convex.

3. Fixed point theorems of generalized φ-concave-convex operators

In this paper, we always assume M is a normal cone of a real Banach space K. In this section, we will explore the existence and uniqueness of the fixed points for generalized φ-concave-convex mixed monotone operators.

Theorem 3.1. Let U be a wedge of K, $\tau_0, \upsilon_0 \in U$ with $\tau_0 \leq \upsilon_0$ and $\mathfrak{A} : U \times U \to K$ be a generalized φ-concave-convex mixed monotone operator. Suppose that

(i) $\tau_0 \leq \mathfrak{A}(\tau_0, \upsilon_0), \mathfrak{A}(\upsilon_0, \tau_0) \leq \upsilon_0$;

(ii) there is a real number r_0 such that $\tau_0 \geq r_0\upsilon_0$;

(iii) $t < \varphi(t, \sigma, \varsigma) \leq 1, \forall t \in (0,1), \forall \sigma, \varsigma \in U$;

(iv) there exist elements $w_0, z_0 \in [\tau_0, \upsilon_0]$ such that

$$\varphi(t, \sigma, \varsigma) \geq \varphi(t, w_0, z_0), \forall t \in (0,1), \forall \sigma, \varsigma \in [\tau_0, \upsilon_0].$$

Then $\mathfrak{A}$ admits the unique fixed point σ^* in $[\tau_0, \upsilon_0]$, and for any initial value $(\sigma_0, \varsigma_0) \in [\tau_0, \upsilon_0] \times [\tau_0, \upsilon_0]$, the iterated sequences

$$\sigma_n = \mathfrak{A}(\sigma_{n-1}, \varsigma_{n-1}), \varsigma_n = \mathfrak{A}(\varsigma_{n-1}, \sigma_{n-1}), n = 1, 2, \cdots, \tag{3.1}$$

always converge to σ^*. Namely, $\|\sigma_n - \sigma^*\| \to 0$, and $\|\varsigma_n - \sigma^*\| \to 0$ as $n \to \infty$.

Proof. Let us first show the existence of the fixed point and the convergence of the iterated sequences. Set

$$\tau_n = \mathfrak{A}(\tau_{n-1}, \upsilon_{n-1}), \upsilon_n = \mathfrak{A}(\upsilon_{n-1}, \tau_{n-1}), n = 1, 2, \cdots. \tag{3.2}$$

Since $\mathfrak{A}$ is mixed monotone, by hypothesis (i) we have

$$\tau_0 \leq \tau_1 \leq \tau_2 \leq \cdots \leq \tau_n \leq \cdots \leq \upsilon_n \leq \cdots \leq \upsilon_2 \leq \upsilon_1 \leq \upsilon_0.$$

Clearly, $0 < r_0 \leq 1$ since $\tau_0 \geq r_0 \upsilon_0$ from (*ii*). Now we assume that $0 < r_0 < 1$ (otherwise, if r_0=1, then $\tau_0 = \upsilon_0$, which implies the $\tau_0 = \upsilon_0$ is the unique fixed point of $\mathfrak{A}$ in $[\tau_0, \upsilon_0]$).

Set

$$t_1 = \sup\{t > 0 | \tau_1 \geq t\upsilon_1\},$$

then we have $0 < r_0 \leq t_1 \leq 1$. In fact, since $\mathfrak{A}$ is a generalized φ-concave-convex mixed monotone operator, it follows from (*ii*) that

$$\begin{aligned} \tau_1 = \mathfrak{A}(\tau_0, \upsilon_0) &\geq \mathfrak{A}(r_0\upsilon_0, \upsilon_0) \geq \varphi(r_0, \upsilon_0, \upsilon_0)\mathfrak{A}(\upsilon_0, r_0\upsilon_0) \\ &\geq \varphi(r_0, \upsilon_0, \upsilon_0)\mathfrak{A}(\upsilon_0, \tau_0) = \varphi(r_0, \upsilon_0, \upsilon_0)\upsilon_1, \end{aligned}$$

which implies that $t_1 \geq \varphi(r_0, \upsilon_0, \upsilon_0) > r_0$, so $0 < r_0 \leq t_1 \leq 1$. In general, we put

$$t_n = \sup\{t > 0 | \tau_n \geq t\upsilon_n\}, n = 1, 2, \cdots. \tag{3.3}$$

Then it is easy to see that $0 \leq t_n \leq 1$ and

$$\tau_n \geq t_n \upsilon_n, n = 1, 2, \cdots. \tag{3.4}$$

By induction, we can prove that

$$0 < t_1 < t_2 < \cdots < t_n < t_{n+1} < \cdots \leq 1. \tag{3.5}$$

In fact, if $0 < t_n < 1$, then by (3.4) and the fact that $\mathfrak{A}$ is generalized φ-concave-convex mixed monotone, we get

$$\begin{aligned} \tau_{n+1} = \mathfrak{A}(\tau_n, \upsilon_n) &\geq \mathfrak{A}(t_n\upsilon_n, \upsilon_n) \\ &\geq \varphi(t_n, \upsilon_n, \upsilon_n)\mathfrak{A}(\upsilon_n, t_n\upsilon_n) \\ &\geq \varphi(t_n, \upsilon_n, \upsilon_n)\mathfrak{A}(\upsilon_n, \tau_n) = \varphi(t_n, \upsilon_n, \upsilon_n)\upsilon_{n+1}. \end{aligned} \tag{3.6}$$

From (3.3), we obtain

$$t_{n+1} = \sup\{t > 0 | \tau_{n+1} \geq t\upsilon_{n+1}\}, n = 1, 2, \cdots. \tag{3.7}$$

From (3.6), (3.7) and the hypothesis (ii), we get

$$t_{n+1} \geq \varphi(t_n, \upsilon_n, \upsilon_n) > t_n, n = 1, 2, \cdots .$$

So, $\{t_n\}$ is nondecreasing and (3.5) holds. Hence $\lim_{n\to\infty} t_n = t^*$ exists and $0 < t^* \leq 1$. We now show $t^* = 1$. Otherwise if $0 < t^* < 1$, then by (3.4) and the fact that $\mathfrak{A}$ is generalized φ-concave-convex mixed monotone, we see

$$\begin{aligned}
\tau_{n+1} &= \mathfrak{A}(\tau_n, \upsilon_n) \geq \mathfrak{A}(t_n\upsilon_n, t_n^{-1}\tau_n) = \mathfrak{A}(\frac{t_n}{t^*} \cdot t^*\upsilon_n, t_n^{-1}\tau_n) \\
&\geq \varphi(\frac{t_n}{t^*}, t^*\upsilon_n, t_n^{-1}\tau_n)\mathfrak{A}(t^*\upsilon_n, \frac{t_n}{t^*} \cdot t_n^{-1}\tau_n) \\
&= \varphi(\frac{t_n}{t^*}, t^*\upsilon_n, t_n^{-1}\tau_n)\mathfrak{A}(t^*\upsilon_n, \frac{1}{t^*}\tau_n) \\
&\geq \varphi(\frac{t_n}{t^*}, t^*\upsilon_n, t_n^{-1}\tau_n)\varphi(t^*, \upsilon_n, \frac{1}{t^*}\tau_n)\mathfrak{A}(\upsilon_n, t^* \cdot \frac{1}{t^*}\tau_n) \\
&\geq \varphi(\frac{t_n}{t^*}, w_0, z_0)\varphi(t^*, w_0, z_0)\mathfrak{A}(\upsilon_n, \tau_n) \\
&> \frac{t_n}{t^*} \cdot \varphi(t^*, w_0, z_0)\upsilon_{n+1}.
\end{aligned} \tag{3.8}$$

It follows from (3.7) and (3.8) that

$$t_{n+1} \geq \frac{t_n}{t^*} \cdot \varphi(t^*, w_0, z_0). \tag{3.9}$$

Letting $n \to \infty$ in (3.9) we get

$$t^* \geq \frac{t^*}{t^*} \cdot \varphi(t^*, w_0, z_0) > t^*,$$

which leads to a contradiction. Thus $t^* = 1$. For any $n, p \geq 1$, we get

$$\theta \leq \upsilon_n - \tau_n \leq \upsilon_n - t_n\upsilon_n = (1 - t_n)\upsilon_n \leq (1 - t_n)\upsilon_0$$

and

$$\theta \leq \tau_{n+p} - \tau_n \leq \upsilon_n - \tau_n, \theta \leq \upsilon_n - \upsilon_{n+p} \leq \upsilon_n - \tau_n.$$

So on account of the normality of the cone M we get $\|\upsilon_n - \tau_n\| \to 0 (n \to \infty)$ and hence $\{\upsilon_n\}$ and $\{\tau_n\}$ are both Cauchy. So, by the fact that K is complete, there exist τ^*, υ^* in $[\tau_0, \upsilon_0]$ such that $\|\tau_n - \tau^*\| \to 0$, $\|\upsilon_n - \upsilon^*\| \to 0$ $(n \to \infty)$, and $\upsilon^* = \tau^*$. Write $\sigma^* = \tau^* = \upsilon^*$, by the standard method (see [3, 36, 44]) we easily get $\|\sigma_n - \sigma^*\| \to 0$, $\|\varsigma_n - \sigma^*\| \to 0 (n \to \infty)$ and the operator $\mathfrak{A}$ has a unique fixed point σ^* in $[\tau_0, \upsilon_0]$. Therefore, the conclusions of Theorem 3.1 hold.

Remark 3.1. In Theorem 3.1, if the condition (iv) is substituted by

(iv') $\varphi(t, \sigma, \varsigma)$ is monotone in σ and ς, respectively,

then the conclusions still hold.

Theorem 3.2. Let M be solid and $\mathfrak{A} : M \times M \to M$ be a mixed monotone operator. Suppose that there exists a function $\varphi : (0, 1) \times M \times M \to (0, \infty)$ such that

(i) $\forall (t, \sigma, \varsigma) \in (0.1) \times M \times M$ implies that

$$\mathfrak{A}(t\sigma, \varsigma) \geq \varphi(t, \sigma, \varsigma)\mathfrak{A}(\sigma, t\varsigma);$$

(ii) for all $(t, \sigma, \sigma) \in (0, 1) \times M \times M$, $t < \varphi(t, \sigma, \sigma) \leq 1$, and $\varphi(t, \sigma, \varsigma)$ is nonincreasing (or alternatively, nondecreasing) in σ and ς, then $\mathfrak{A}$ admits a unique fixed point σ^* in int$M$ if and only if for some $\tau_0, \upsilon_0 \in$ int$M$ with $\tau_0 \leq \upsilon_0$, it holds that

$$\tau_0 \leq \mathfrak{A}(\tau_0, \upsilon_0), \mathfrak{A}(\upsilon_0, \tau_0) \leq \upsilon_0. \tag{3.10}$$

Proof. Necessity. Suppose σ^* is the unique fixed point of $\mathfrak{A}$ in int$M$. Set $\tau_0 = \upsilon_0 = \sigma^*$, then it follows from $\mathfrak{A}(\sigma^*, \sigma^*) = \sigma^*$ that $\tau_0 \leq \mathfrak{A}(\tau_0, \upsilon_0)$ and $\mathfrak{A}(\upsilon_0, \tau_0) \leq \upsilon_0$.

Sufficiency. Since $\tau_0, \upsilon_0 \in$ int$M$, there exists a real number $r_0 > 0$ such that $\tau_0 \geq r_0 \upsilon_0$. Set

$$\tau_n = \mathfrak{A}(\tau_{n-1}, \upsilon_{n-1}), \upsilon_n = \mathfrak{A}(\upsilon_{n-1}, \tau_{n-1}), n = 1, 2, \cdots. \tag{3.11}$$

Then by (3.10), (3.11) and the mixed monotonicity of $\mathfrak{A}$, we have

$$\tau_0 \leq \tau_1 \leq \tau_2 \leq \cdots \leq \tau_n \leq \upsilon_n \leq \cdots \leq \upsilon_2 \leq \upsilon_1 \leq \upsilon_0.$$

Without loss of generality, suppose $\varphi(t, \sigma, \varsigma)$ is nonincreasing in σ and ς, respectively, then for all $t \in (0, 1), \sigma, \varsigma \in [\tau_0, \upsilon_0]$ we get $\varphi(t, \sigma, \varsigma) \geq \varphi(t, w_0, z_0)$, where $w_0 = z_0 = \upsilon_0$. Thus all the conditions of Theorem 3.1 are satisfied. Thus, the conclusions hold from Theorem 3.1.

Remark 3.2. Compared to Theorem 3.1 in [41], Theorem 3.2 deletes the following continuity condition:

"(H) $\varphi(t, \sigma, \sigma)$ is continuous from left in σ",

in which the proof of Theorem 3.1 in [41] strongly depends, while the conclusions concerning fixed point of the operator discussed still hold.

Similar to Theorem 3.2, we have the following four theorems by means of Theorem 3.1.

Theorem 3.3. Let $\mathfrak{A} : M \times M \to M$ be a mixed monotone operator. Assume that

(i) there exist $\tau_0, \upsilon_0 \in M$ with $\tau_0 \leq \upsilon_0$ and a real number r_0 such that $\tau_0 \geq r_0 \upsilon_0$ and

$$\tau_0 \leq \mathfrak{A}(\tau_0, \upsilon_0), \mathfrak{A}(\upsilon_0, \tau_0) \leq \upsilon_0;$$

(ii) there exists a function $\varphi : (0, 1) \times M \times M \to (0, \infty)$ with $t < \varphi(t, \sigma, \varsigma) \leq 1$ satisfying

$$\mathfrak{A}(t\sigma, \varsigma) \geq \varphi(t, \sigma, \varsigma)\mathfrak{A}(\sigma, t\varsigma), \forall t \in (0, 1), \forall \sigma, \varsigma \in M;$$

(iii) $\varphi = \varphi(t, \sigma, \varsigma)$ is monotone (i.e., nondecreasing or nonincreasing) in σ and ς, respectively.

Then $\mathfrak{A}$ admits a unique fixed point σ^* in $[\tau_0, \upsilon_0]$. Moreover, for any initial $\sigma_0, \varsigma_0 \in [\tau_0, \upsilon_0]$, the iterated sequences

$$\sigma_n = \mathfrak{A}(\sigma_{n-1}, \varsigma_{n-1}), \varsigma_n = \mathfrak{A}(\varsigma_{n-1}, \sigma_{n-1}), n = 1, 2, \cdots$$

always converge to σ^*. Namely, $\|\sigma_n - \sigma^*\| \to 0$, and $\|\varsigma_n - \sigma^*\| \to 0$ as $n \to \infty$.

Theorem 3.4. Let $\mathfrak{A}$ be the same as in Theorem 3.3. Suppose that

(i) there exist $\tau_0, \upsilon_0 \in M$ with $\tau_0 \leq \upsilon_0$ and a real number r_0 such that $\tau_0 \geq r_0 \upsilon_0$ and

$$\tau_0 \leq \mathfrak{A}(\tau_0, \upsilon_0), \mathfrak{A}(\upsilon_0, \tau_0) \leq \upsilon_0;$$

(ii) there exists a function $\varphi : (0, 1) \times M \to (0, \infty)$ with $t < \varphi(t, \sigma) \leq 1$ satisfying

$$\mathfrak{A}(t\sigma, \varsigma) \geq \varphi(t, \sigma)\mathfrak{A}(\sigma, t\varsigma), \forall t \in (0, 1), \forall \sigma, \varsigma \in M;$$

(iii) $\varphi = \varphi(t, \sigma)$ is monotone (i.e., nondecreasing or nonincreasing) in σ.
Then the conclusions of Theorem 3.3 also hold.

Theorem 3.5. Let $\mathfrak{A}$ be the same as in Theorem 3.3. Suppose that

(i) there exist $\tau_0, \upsilon_0 \in M$ with $\tau_0 < \upsilon_0$ and a real number r_0 such that $\tau_0 \geq r_0\upsilon_0$ and

$$\tau_0 \leq \mathfrak{A}(\tau_0, \upsilon_0), \mathfrak{A}(\upsilon_0, \tau_0) \leq \upsilon_0;$$

(ii) there exists a function $\varphi : (0, 1) \times M \to (0, \infty)$ with $t < \varphi(t, \varsigma) \leq 1$ satisfying

$$\mathfrak{A}(t\sigma, \varsigma) \geq \varphi(t, \varsigma)\mathfrak{A}(\sigma, t\varsigma), \forall t \in (0, 1), \forall \sigma, \varsigma \in M;$$

(iii) $\varphi = \varphi(t, \varsigma)$ is monotone (i.e., nondecreasing or nonincreasing) in ς.
Then the conclusions of Theorem 3.3 also hold.

Theorem 3.6. Let $\mathfrak{A}$ be the same as in Theorem 3.3. Suppose that

(i) there exist $\tau_0, \upsilon_0 \in M$ with $\tau_0 < \upsilon_0$ and a real number r_0 such that $\tau_0 \geq r_0\upsilon_0$ and

$$\tau_0 \leq \mathfrak{A}(\tau_0, \upsilon_0), \mathfrak{A}(\upsilon_0, \tau_0) \leq \upsilon_0;$$

(ii) there exists a function $\varphi : (0, 1) \to (0, +\infty)$ with $t < \varphi(t) \leq 1$ satisfying

$$\mathfrak{A}(t\sigma, \varsigma) \geq \varphi(t)\mathfrak{A}(\sigma, t\varsigma), \forall t \in (0, 1), \forall \sigma, \varsigma \in M.$$

Then the conclusions of Theorem 3.3 also hold.

Lemma 3.1. Let $\mathfrak{A} : M \times M \to M$ be a generalized φ-concave-convex operator with $\varphi = \varphi(t)$ satisfying $t < \varphi(t) < 1$ for all $t \in (0, 1)$. Suppose $\mathfrak{A} : M \times M \to M$ is mixed monotone and there exists $e > \theta$ such that $\mathfrak{A}(e, e) \in M_e$. Then $\mathfrak{A} : M_e \times M_e \to M_e$; and there exist $\tau_0, \upsilon_0 \in M_e$ and $r_0 \in (0, 1)$ such that $r\upsilon_0 \leq \tau_0 < \upsilon_0$, and $\tau_0 \leq \mathfrak{A}(\tau_0, \upsilon_0), \mathfrak{A}(\upsilon_0, \tau_0) \leq \upsilon_0$.

Proof. Since $\mathfrak{A} : M \times M \to M$ is generalized φ-concave-convex with $\varphi = \varphi(t)$, we have

$$\mathfrak{A}(t\tau, \upsilon) \geq \varphi(t)\mathfrak{A}(\tau, t\upsilon) \tag{3.12}$$

and

$$\mathfrak{A}(\tau, t\upsilon) \leq \frac{1}{\varphi(t)}\mathfrak{A}(t\tau, \upsilon) \tag{3.13}$$

for all $t \in (0, 1)$ and $\tau, \upsilon \in M$. So, for any $\sigma, \varsigma \in M_e$, taking $\tau = t^{-1}\sigma, \upsilon = \varsigma$ in (3.12) we have

$$\mathfrak{A}(t \cdot t^{-1}\sigma, \varsigma) \geq \varphi(t)\mathfrak{A}(t^{-1}\sigma, t\varsigma).$$

Hence, we get

$$\mathfrak{A}(t^{-1}\sigma, t\varsigma) \leq \frac{1}{\varphi(t)}\mathfrak{A}(\sigma, \varsigma) \tag{3.14}$$

and

$$\mathfrak{A}(\sigma, \varsigma) \geq \varphi(t)\mathfrak{A}(t^{-1}\sigma, t\varsigma). \tag{3.15}$$

For any $\sigma, \varsigma \in M_e$, there exist $\lambda_1, \lambda_2 \in (0,1)$ such that

$$\lambda_1 e \leq \sigma \leq \lambda_1^{-1} e, \lambda_2 e \leq \varsigma \leq \lambda_2^{-1} e.$$

Set $\lambda = \min\{\lambda_1, \lambda_2\}$. Then $\lambda \in (0,1)$. So by (3.14) and the fact that $\mathfrak{A}$ is generalized φ-concave-convex mixed monotone, we obtain

$$\mathfrak{A}(\sigma, \varsigma) \leq \mathfrak{A}(\lambda_1^{-1} e, \lambda_2 e) \leq \mathfrak{A}(\lambda^{-1} e, \lambda e) \leq \frac{1}{\varphi(\lambda)} \mathfrak{A}(e, e) \tag{3.16}$$

and

$$\mathfrak{A}(\sigma, \varsigma) \geq \mathfrak{A}(\lambda_1 e, \lambda_2^{-1} e) \geq \mathfrak{A}(\lambda e, \lambda^{-1} e) \geq \varphi(\lambda) \mathfrak{A}(e, e). \tag{3.17}$$

From (3.16) and (3.17) we get

$$\varphi(\lambda) \mathfrak{A}(e, e) \leq \mathfrak{A}(\sigma, \varsigma) \leq \frac{1}{\varphi(\lambda)} \mathfrak{A}(e, e),$$

which implies that $\mathfrak{A}(\sigma, \varsigma) \in M_e$ since $\mathfrak{A}(e, e) \in M_e$. Thus $\mathfrak{A} : M_e \times M_e \to M_e$.

Take a sufficiently small number $0 < t_0 < 1$ such that

$$t_0 e \leq \mathfrak{A}(e, e) \leq \frac{1}{t_0} e. \tag{3.18}$$

Since $t_0 < \varphi(t_0) \leq 1$, choose a sufficient large natural number k such that

$$\left(\frac{\varphi(t_0)}{t_0} \right)^k \geq \frac{1}{t_0}, \tag{3.19}$$

i.e.,

$$(\varphi(t_0))^k t_0 \geq t_0^k. \tag{3.20}$$

Take $\tau_0 = t_0^k e, \upsilon_0 = t_0^{-k} e$. It is easy to see that $\tau_0, \upsilon_0 \in M_e$ and $\tau_0 = t_0^{2k} \upsilon_0 < \upsilon_0$. Choose any $r_0 \in (0, t_0^{2k})$, then $0 < r_0 < 1$ and $\tau_0 \geq r_0 \upsilon_0$. Since $\mathfrak{A}$ is mixed monotone, we see $\mathfrak{A}(\tau_0, \upsilon_0) \leq \mathfrak{A}(\upsilon_0, \tau_0)$. Moreover, by (3.18), (3.19) and the generalized φ-concave-convex property of $\mathfrak{A}$, we get

$$\begin{aligned} \mathfrak{A}(\tau_0, \upsilon_0) &= \mathfrak{A}(t_0^k e, t_0^{-k} e) = \mathfrak{A}\left(t_0 \cdot t_0^{k-1} e, t_0^{-1} \cdot t_0^{-(k-1)} e\right) \\ &\geq \varphi(t_0) \mathfrak{A}\left(t_0^{k-1} e, t_0 \cdot t_0^{-1} \cdot t_0^{-(k-1)} e\right) \\ &= \varphi(t_0) \mathfrak{A}\left(t_0^{k-1} e, t_0^{-(k-1)} e\right) \\ &= \varphi(t_0) \mathfrak{A}\left(t_0, t_0^{k-2} e, t_0^{-1} \cdot t_0^{-(k-2)} e\right) \\ &\geq \varphi(t_0) \varphi(t_0) \mathfrak{A}\left(t_0^{k-2} e, t_0^{-(k-2)} e\right) \geq \cdots \\ &\geq (\varphi(t_0))^k \mathfrak{A}(e, e) \geq (\varphi(t_0))^k t_0 e \geq t_0^k e = \tau_0. \end{aligned}$$

Similarly, we get

$$\mathfrak{A}(\upsilon_0, \tau_0) = \mathfrak{A}(t_0^{-k} e, t_0^k e) = \mathfrak{A}\left(t_0^{-1} \cdot t_0^{-(k-1)} e, t_0 \cdot t_0^{k-1} e\right)$$

$$
\begin{aligned}
&\le \frac{1}{\varphi(t_0)}\mathfrak{A}\left(t_0^{-(k-1)}e, t_0^{k-1}e\right)\\
&= \frac{1}{\varphi(t_0)}\mathfrak{A}\left(t_0^{-1}\cdot t_0^{-(k-2)}e, t_0\cdot t_0^{k-2}e\right)\\
&\le \frac{1}{\varphi(t_0)}\cdot\frac{1}{\varphi(t_0)}\mathfrak{A}\left(t_0^{-(k-2)}e, t_0^{k-2}e\right)\le\cdots\\
&\le \frac{1}{(\varphi(t_0))^k}\mathfrak{A}(e,e)\le\frac{1}{t_0(\varphi(t_0))^k}e.
\end{aligned}
$$

Hence, by (3.20), we have

$$\mathfrak{A}(\upsilon_0,\tau_0)\le\frac{1}{t_0(\varphi(t_0))^k}e\le\frac{1}{t_0^k}e=\upsilon_0.$$

Therefore we obtain $\tau_0\le\mathfrak{A}(\tau_0,\upsilon_0)$, and $\mathfrak{A}(\upsilon_0,\tau_0)\le\upsilon_0$.

Theorem 3.7. Let $e>\theta$ and $\mathfrak{A}: M_e\times M_e\to M_e$ be a generalized φ-concave-convex mixed monotone with $\varphi(t,\sigma,\varsigma)=\varphi(t)$ and $t<\varphi(t)\le 1$ for all $t\in(0,1)$ and $\sigma,\varsigma\in M_e$, namely,

$$\mathfrak{A}(t\sigma,\varsigma)\ge\varphi(t)\mathfrak{A}(\sigma,t\varsigma),\forall t\in(0,1),\forall\sigma,\varsigma\in M_e.$$

Then the operator $\mathfrak{A}$ admits a unique fixed point σ^* in M_e. Moreover, for any initial $(\sigma_0,\varsigma_0)\in M_e\times M_e$, the iterated sequences

$$\sigma_n=\mathfrak{A}(\sigma_{n-1},\varsigma_{n-1}),\varsigma_n=\mathfrak{A}(\varsigma_{n-1},\sigma_{n-1}),n=1,2,\cdots$$

always converge to σ^*. Namely, $\|\sigma_n-\sigma^*\|\to 0$, and $\|\varsigma_n-\sigma^*\|\to 0$ as $n\to\infty$.

Proof. The conclusions of Theorem 3.7 follow from Lemma 3.1 and Theorem 3.1.

Remark 3.3. Since generalized φ-concave-convex operators unify and extend a number of nonlinear operators with certain concavity and convexity, such as ϕ concave-(-ψ) convex operators, here we state that Theorems 3.1–3.7 have a typical advantage over the related results in the existing literature. In fact, in [42, Theorem 2.1], the ϕ concave-(-ψ) convex operator is assumed to satisfy the following limit inequality

"(M) $\lim_{s\to t^-}\phi(s,w_0)\psi(s,w_0)>t$, $\forall t\in(0,1)$",

which is a condition related to the continuity because if the function $\phi(\sigma,\varsigma)\psi(\sigma,\varsigma)$ is continuous from left in σ, then the condition (M) holds. However, Theorems 3.1–3.7 need not require the operators discussed should satisfy such limit inequality as the condition (M). In short, Theorems 3.1–3.7 need not require the generalized φ-concave-convex operators satisfy any kind of continuity condition, so they will derive a number of new fixed point results of the nonlinear operators with certain concavity and convexity under weaker conditions (see the subsequent Sections 4 and 5).

Remark 3.4. Different from Theorems 2.1–2.6 and Corollaries 2.1 and 2.2 in [42], Theorem 3.7 need not require the generalized φ-concave-convex mixed monotone operator should satisfy a pair of coupled upper and lower solutions, which will deduce a number of new fixed point results of mixed monotone operators with certain concavity and convexity without assumption of coupled upper and lower solutions.

4. Fixed point theorems of ϕ concave-(-ψ) convex operators

In this section, we will use the main results concerning generalized φ-concave-convex operators to deduce a number of new fixed point theorems of ϕ concave-(-ψ) convex mixed monotone operators.

Theorem 4.1. Let $\mathfrak{A} : M \times M \to M$ be a ϕ concave-(-ψ) convex operator with $\phi = \phi(t, \sigma)$ and $\psi = \psi(t)$, namely, $\forall (t, \sigma, \varsigma) \in (0, 1) \times M \times M$ implies

$$\mathfrak{A}(t\sigma, \varsigma) \geq \phi(t, \sigma)\mathfrak{A}(\sigma, \varsigma) \tag{4.1}$$

and

$$\mathfrak{A}(\sigma, t\varsigma) \leq \frac{1}{\psi(t)}\mathfrak{A}(\sigma, \varsigma), \tag{4.2}$$

where

$$t < \phi(t, \sigma)\psi(t) \leq 1, \textit{ for all } t \in (0, 1) \textit{ and } \sigma \in M.$$

Suppose that $\mathfrak{A}$ is mixed monotone and satisfies

(i) there exist $\tau_0, \upsilon_0 \in M$ and a real number $r_0 > 0$ such that $\tau_0 \geq r_0\upsilon_0$ and

$$\tau_0 \leq \mathfrak{A}(\tau_0, \upsilon_0), \mathfrak{A}(\upsilon_0, \tau_0) \leq \upsilon_0; \tag{4.3}$$

(ii) there exists an element $w_0 \in [\tau_0, \upsilon_0]$ such that

$$\phi(t, \sigma) \geq \phi(t, w_0), \forall (t, \sigma) \in (0, 1) \times [\tau_0, \upsilon_0].$$

Then $\mathfrak{A}$ admits a unique fixed point σ^* in $[\tau_0, \upsilon_0]$. Moreover, for any initial $\sigma_0, \varsigma_0 \in [\tau_0, \upsilon_0]$, the iterated sequences

$$\sigma_n = \mathfrak{A}(\sigma_{n-1}, \varsigma_{n-1}), \varsigma_n = \mathfrak{A}(\varsigma_{n-1}, \sigma_{n-1}), n = 1, 2, \cdots$$

always converge to σ^*. Namely, $\|\sigma_n - \sigma^*\| \to 0$, and $\|\varsigma_n - \sigma^*\| \to 0$ as $n \to \infty$.

Proof. Since $\mathfrak{A}$ is ϕ concave-(-ψ) convex, it follows from (4.1) and (4.2) that

$$\begin{aligned}\mathfrak{A}(t\sigma, \varsigma) &\geq \phi(t, \sigma)\mathfrak{A}(\sigma, \varsigma)\\ &\geq \phi(t, \sigma)\psi(t)\mathfrak{A}(\sigma, t\varsigma),\end{aligned} \tag{4.4}$$

where $t \in (0, 1)$ and $\sigma, \varsigma \in M$.

Set $\varphi(t, \sigma, \varsigma) = \phi(t, \sigma)\psi(t)$. Then by (4.4), we see $\mathfrak{A} : M \times M \to M$ is a generalized φ-concave-convex operator. Obviously the operator $\mathfrak{A}$ satisfies all the conditions of Theorem 3.1, so the conclusions of Theorem 4.1 follow from Theorem 3.1.

Similarly, we have the following two theorems.

Theorem 4.2. Let $\mathfrak{A} : M \times M \to M$ be a ϕ concave-(-ψ) convex operator with $\phi = \phi(t)$ and $\psi = \psi(t, \varsigma)$, namely, $\forall (t, \sigma, \varsigma) \in (0, 1) \times M \times M$ implies

$$\mathfrak{A}(t\sigma, \varsigma) \geq \phi(t)\mathfrak{A}(\sigma, \varsigma) \tag{4.5}$$

and

$$\mathfrak{A}(\sigma, t\varsigma) \leq \frac{1}{\psi(t, \varsigma)}\mathfrak{A}(\sigma, \varsigma), \tag{4.6}$$

where $t < \phi(t)\psi(t, \varsigma) \leq 1$ for all $t \in (0, 1)$ and $\varsigma \in M$. Suppose that $\mathfrak{A}$ is mixed monotone and satisfies

(i) there exist $\tau_0, \upsilon_0 \in M$ and a real number $r_0 > 0$ such that $\tau_0 \geq r_0\upsilon_0$ and

$$\tau_0 \leq \mathfrak{A}(\tau_0, \upsilon_0), \mathfrak{A}(\upsilon_0, \tau_0) \leq \upsilon_0;$$

(ii) there is a point $w_0 \in [\tau_0, \upsilon_0]$ such that

$$\psi(t, \varsigma) \geq \psi(t, w_0), \forall (t, \varsigma) \in (0, 1) \times [\tau_0, \upsilon_0].$$

Then the conclusions of Theorem 4.1 also hold.

Proof. Since $\mathfrak{A}$ is ϕ concave-(-ψ) convex, it follows from (4.5) and (4.6) that

$$\begin{aligned}\mathfrak{A}(t\sigma, \varsigma) &\geq \phi(t)\mathfrak{A}(\sigma, \varsigma)\\ &\geq \phi(t)\psi(t, \varsigma)\mathfrak{A}(\sigma, t\varsigma),\end{aligned} \tag{4.7}$$

where $t \in (0, 1)$ and $\sigma, \varsigma \in M$.

Set $\varphi(t, \sigma, \varsigma) = \phi(t)\psi(t, \varsigma)$. Then by (4.7), we see $\mathfrak{A} : M \times M \to M$ is a generalized φ-concave-convex operator. Obviously the operator $\mathfrak{A}$ satisfies all the conditions of Theorem 3.1, so the conclusions of Theorem 4.2 follow from Theorem 3.1.

Similarly, we have the following two theorems.

Remark 4.1. If the condition (ii) in Theorem 4.2 is substituted by

(ii') $\psi(\sigma, \varsigma)$ is monotone (nonincreasing or nondecreasing) in ς,

then the conclusions still hold.

Theorem 4.3. Let $\mathfrak{A} : M \times M \to M$ be a ϕ concave-(-ψ) convex operator with $\phi = \phi(t)$ and $\psi = \psi(t)$, namely, $\forall (t, \sigma, \varsigma) \in (0, 1) \times M \times M$ implies

$$\mathfrak{A}(t\sigma, \varsigma) \geq \phi(t)\mathfrak{A}(\sigma, \varsigma)$$

and

$$\mathfrak{A}(\sigma, t\varsigma) \leq \frac{1}{\psi(t)}\mathfrak{A}(\sigma, \varsigma),$$

where $t < \phi(t)\psi(t) \leq 1$ for all $t \in (0, 1)$. Suppose that $\mathfrak{A}$ is mixed monotone and satisfies.

(C) there exist $\tau_0, \upsilon_0 \in M$ and a real number $r_0 > 0$ such that $\tau_0 \geq r_0\upsilon_0$ and

$$\tau_0 \leq \mathfrak{A}(\tau_0, \upsilon_0), \mathfrak{A}(\upsilon_0, \tau_0) \leq \upsilon_0.$$

Then the conclusions of Theorem 4.1 also hold.

Theorem 4.4. Let $e > \theta$ and $\mathfrak{A} : M_e \times M_e \to M_e$ be ϕ concave-(-ψ) convex mixed monotone with $\phi = \phi(t)$ and $\psi = \psi(t)$, namely, for all $t \in (0, 1)$ and $\sigma, \varsigma \in M_e$, it holds that

$$\mathfrak{A}(t\sigma, \varsigma) \geq \phi(t)\mathfrak{A}(\sigma, \varsigma), \mathfrak{A}(\sigma, t\varsigma) \leq \frac{1}{\psi(t)}\mathfrak{A}(\sigma, \varsigma). \tag{4.8}$$

Then $\mathfrak{A}$ admits a unique fixed point σ^* in M_e. Moreover, for any initial $\sigma_0, \varsigma_0 \in M_e$, the iterated sequences

$$\sigma_n = \mathfrak{A}(\sigma_{n-1}, \varsigma_{n-1}), \varsigma_n = \mathfrak{A}(\varsigma_{n-1}, \sigma_{n-1}), n = 1, 2, \cdots$$

always converge to σ^*. Namely, $\|\sigma_n - \sigma^*\| \to 0$, and $\|\varsigma_n - \sigma^*\| \to 0$ as $n \to \infty$.
Proof. Let $\varphi(t) = \phi(t)\psi(t)$. Then for all $\sigma \in (0, 1), \sigma, \varsigma \in M_e$, it follows from (4.8) that

$$\mathfrak{A}(t\sigma, \varsigma) \geq \phi(t)\psi(t)\mathfrak{A}(\sigma, \varsigma) = \varphi(t)\mathfrak{A}(\sigma, t\varsigma),$$

which implies that $\mathfrak{A} : M_e \times M_e \to M_e$ is generalized φ-concave-convex with $\varphi = \phi(t)\psi(t)$. So, the operator $\mathfrak{A}$ satisfies all the conditions of Theorem 3.7. Hence, the conclusions of Theorem 4.4 follow from Theorem 3.7.

Remark 4.2. Compared to Theorem 2.1 in [42], Theorems 4.1–4.3 need not require the ϕ concave-(-ψ) convex operators $\mathfrak{A}$ should satisfy the following condition:

"(H) $\lim_{s\to t^-} \phi(s, w_0)\psi(s, w_0) > t$, $\forall t \in (0, 1)$",

upon which the crucial condition the proof of [42, Theorem 2.1] depends strongly, while the conclusions concerning the ϕ concave-(-ψ)convex operator $\mathfrak{A}$ still hold. So Theorems 4.1–4.3 improve [42, Theorem 2.1] to a certain extent.

We now use the fixed point results about ϕ concave-(-ψ) convex operators obtained above to deduce some new fixed point theorems of mixed monotone operators with certain concavity and convexity.

Remark 4.3. Theorem 4.4 is a new fixed point result of ϕ concave-(-ψ) convex operators and has potential applications to nonlinear equations. This is due to the fact that Theorem 2.1 in [42] is a main fixed point result regarding ϕ concave-(-ψ) convex operators in the existent literature and it can not deduce Theorem 4.4 above since Theorem 4.4 deletes the limit inequality condition

"(H) $\lim_{s\to t^-} \phi(s, w_0)\psi(s, w_0) > t$, $\forall t \in (0, 1)$",

which appears in [42, Theorem 2.1] as a crucial condition for the proof of the existence of the fixed point of the operator. Besides, Theorem 4.4 need not require us to seek a surplus pair of coupled upper and lower solutions. Such advantage will bring about some practical convenience to nonlinear differential equations as well as integral equations.

Corollary 4.1. Let M be solid and $\mathfrak{A} : M \times M \to M$ be a mixed monotone operator. Suppose that

(i) there exist $\tau_0, \upsilon_0 \in \mathrm{int}M$ with $\tau_0 \leq \upsilon_0$, such that

$$\tau_0 \leq \mathfrak{A}(\tau_0, \upsilon_0), \mathfrak{A}(\upsilon_0, \tau_0) \leq \upsilon_0; \tag{4.9}$$

(ii) for fixed ς, $\mathfrak{A}(\cdot, \varsigma) : M \to M$ is concave; for fixed σ, $\mathfrak{A}(\sigma, \cdot) : M \to M$ is generalized e-convex, i.e., there is a function $\eta = \eta(t, \varsigma)$ satisfying

$$\mathfrak{A}(\sigma, t\varsigma) \leq [t(1 + \eta(t, \varsigma))]^{-1}\mathfrak{A}(\sigma, \varsigma), \forall t \in (0, 1), \forall \sigma, \varsigma \in M;$$

(iii) $\eta(t, \varsigma)$ is monotone (i.e., nonincreasing or nondecreasing) in ς, and there exists $\varepsilon > 0$, such that

$$\mathfrak{A}(\theta, \upsilon_0) \geq \varepsilon\mathfrak{A}(\upsilon_0, \tau_0) \tag{4.10}$$

and

$$[\varepsilon + (1 - \varepsilon)t]^{-1} - 1 < \eta(t, \varsigma) < [\varepsilon t + (1 - \varepsilon)t^2]^{-1} - 1. \tag{4.11}$$

Then the conclusions of Theorem 4.2 hold.
Proof. Let $\tau_n = \mathfrak{A}(\tau_{n-1}, \upsilon_{n-1}), \upsilon_n = \mathfrak{A}(\upsilon_{n-1}, \tau_{n-1}), n = 1, 2, ...$ Then by (4.9) and the fact that $\mathfrak{A}$ is mixed monotone, we get that

$$\tau_0 \leq \tau_1 \leq \tau_2 \leq \cdots \leq \tau_n \leq \upsilon_n \leq \cdots \leq \upsilon_2 \leq \upsilon_1 \leq \upsilon_0.$$

Since $\mathfrak{A}$ is mixed monotone, by (4.10) we see $\tau_1 \geq \varepsilon\upsilon_1$ and $\tau_1 \leq \mathfrak{A}(\tau_1, \upsilon_1), \mathfrak{A}(\upsilon_1, \tau_1) \leq \upsilon_1$. Thus $0 < \varepsilon \leq 1$. Now we begin to show $\mathfrak{A} : M \times M \to M$ is ϕ concave-(-ψ) convex. In fact, for all $t \in (0, 1)$ and $\sigma, \varsigma \in M$, we have

$$\begin{aligned}
\mathfrak{A}(t\sigma, \varsigma) &= \mathfrak{A}(t\sigma + (1-t)\theta, \varsigma) \\
&\geq t\mathfrak{A}(\sigma, \varsigma) + (1-t)\mathfrak{A}(\theta, \varsigma) \\
&\geq t\mathfrak{A}(\sigma, \varsigma) + (1-t)\mathfrak{A}(\theta, \upsilon_0) \\
&\geq t\mathfrak{A}(\sigma, \varsigma) + \varepsilon(1-t)\mathfrak{A}(\upsilon_0, \tau_0) \\
&\geq t\mathfrak{A}(\sigma, \varsigma) + \varepsilon(1-t)\mathfrak{A}(\sigma, \varsigma) = \phi(t)\mathfrak{A}(\sigma, \varsigma), \\
\mathfrak{A}(\sigma, t\varsigma) &\leq \frac{1}{t(1+\eta(t, \varsigma))}\mathfrak{A}(\sigma, \varsigma) = \frac{1}{\psi(t, \varsigma)}\mathfrak{A}(\sigma, \varsigma),
\end{aligned}$$

where

$$\phi = \phi(t) = t + \varepsilon(1-t), \psi = \psi(t, \varsigma) = t(1 + \eta(t, \varsigma)). \tag{4.12}$$

By (4.11) and (4.12) we see

$$t < \phi(t)\psi(t, \varsigma) \leq 1, \forall t \in (0, 1), \forall \varsigma \in M.$$

Hence, $\mathfrak{A} : M \times M \to M$ is ϕ concave-(-ψ) convex and all the conditions of Theorem 4.2 are satisfied. Therefore, the conclusions of Corollary 4.1 follows from Theorem 4.2 and Remark 4.1.

Remark 4.4. Compared with Corollary 3.3 in [41], Corollary 4.1 deletes the following continuity condition

"(CC) $\eta(t, \varsigma)$ is continuous from left in t",

which is listed in the assumption (*iii*) in [41, Corollary 3.3], and the conclusions concerning the fixed point of the operator discussed still hold. So Corollary 4.1 improves [41, Corollary 3.3].

Corollary 4.2. Let M be solid and $\mathfrak{A}$: int$M$ × int$M$ → int$M$ be a mixed monotone operator. Suppose that $\mathfrak{A}$ satisfies the following condition:

$(C_{\alpha_1-\alpha_2})$ for fixed $\varsigma, \mathfrak{A}(\cdot, \varsigma)$: int$M$ → int$M$ is α_1-concave; for fixed $\sigma, \mathfrak{A}(\sigma, \cdot)$: int$M$ → int$M$ is (-α_2)-convex, where $0 \leq \alpha_1 + \alpha_2 < 1$.

Then $\mathfrak{A}$ admits a unique fixed point σ^* in int$M$. Moreover, for any initial $(\sigma_0, \varsigma_0) \in$ int$M$ × int$M$, the iterated sequences

$$\sigma_n = \mathfrak{A}(\sigma_{n-1}, \varsigma_{n-1}), \varsigma_n = \mathfrak{A}(\varsigma_{n-1}, \sigma_{n-1}), n = 1, 2, \cdots$$

always converge to σ^*. Namely, $\|\sigma_n - \sigma^*\| \to 0$, and $\|\varsigma_n - \sigma^*\| \to 0$ as $n \to \infty$.

Proof. By the condition $(C_{\alpha_1-\alpha_2})$ we have for all $\sigma \in (0, 1), \sigma, \varsigma \in$ int$M$, it holds that

$$\mathfrak{A}(t\sigma, \varsigma) \geq t^{\alpha_1}\mathfrak{A}(\sigma, \varsigma) = \phi(t)\mathfrak{A}(\sigma, \varsigma),$$

and

$$\mathfrak{A}(\sigma, t\varsigma) \leq \frac{1}{t^{\alpha_2}}\mathfrak{A}(\sigma, \varsigma) = \frac{1}{\psi(t)}\mathfrak{A}(\sigma, \varsigma),$$

where $\phi(t) = t^{\alpha_1}, \psi(t) = t^{\alpha_2}$. It is easy to see that $t < \phi(t)\psi(t) = t^{\alpha_1+\alpha_2} < 1$. Since $0 < \alpha_1 + \alpha_2 < 1$ for all $t \in (0, 1)$. So $\mathfrak{A}$: int$M$ × int$M$ → int$M$ is ϕ concave-(-ψ) convex.

Therefore, the conclusions of Corollary 4.2 follow from Theorem 4.4.

Corollary 4.3. Let $K, M, \mathfrak{A}$ be the same as that in Corollary 4.2. Suppose that $\mathfrak{A}$ satisfies the following condition:

$(C_{\alpha-\alpha})$ for fixed $\varsigma : \mathfrak{A}(\cdot, \varsigma) : \mathrm{int}M \to \mathrm{int}M$ is α-concave; for fixed $\sigma, \mathfrak{A}(\sigma, \cdot) : \mathrm{int}M \to \mathrm{int}M$ is (-α)-convex, where $0 \le \alpha < \frac{1}{2}$.

Then the conclusions of Corollary 4.2 also hold.

Proof. Set $\alpha_1 = \alpha_2 = \alpha$. Then the proof is complete by Corollary 4.2.

Remark 4.5. Compared with Corollaries 2.1 and 2.2 in [42], Corollaries 4.2 and 4.3 remove the following redundant assumption of coupled upper and lower solution condition

"(*ii*) there exist elements $\tau_0, \upsilon_0 \in \mathrm{int}M$ with $\tau_0 \le \upsilon_0$ such that $\tau_0 \le \mathfrak{A}(\tau_0, \upsilon_0), \mathfrak{A}(\upsilon_0, \tau_0) \le \upsilon_0$",
which appears in Corollaries 2.1 and 2.2 in [42], while the conclusions still hold. So Corollaries 4.2 and 4.3 improve Corollaries 2.1 and 2.2 in [42], respectively.

5. Fixed point theorems of $t - \eta(t)$ $(t - \alpha(t))$ mixed monotone model operators

In this section, we will use the fixed point results on generalized φ-concave-convex operators obtained in Section 3 to deduce new fixed point theorems for $t - \eta(t)$ $(t - \alpha(t))$ mixed monotone model operators.

Theorem 5.1. Let $e > \theta$ and $\mathfrak{A} : M_e \times M_e \to M_e$ be a mixed monotone operator. Assume that $\mathfrak{A}$ is a $t - \eta(t)$ mixed monotone model operator, i.e., for all $t \in (0, 1)$ and $\tau, \upsilon \in M_e$, there exists a function $\eta = \eta(t) > 0$ such that

$$\mathfrak{A}(t\tau, t^{-1}\upsilon) \ge t(1 + \eta(t))\mathfrak{A}(\tau, \upsilon). \tag{5.1}$$

Then $\mathfrak{A}$ admits a unique fixed point σ^* in M_e. Moreover, for any initial $\sigma_0, \varsigma_0 \in M_e$, the iterated sequences

$$\sigma_n = \mathfrak{A}(\sigma_{n-1}, \varsigma_{n-1}), \varsigma_n = \mathfrak{A}(\varsigma_{n-1}, \sigma_{n-1}), n = 1, 2, \cdots$$

always converge to σ^*. Namely, $\|\sigma_n - \sigma^*\| \to 0$, and $\|\varsigma_n - \sigma^*\| \to 0$ as $n \to \infty$.

Proof. According to Theorem 3.7, it suffices to check that $\mathfrak{A} : M_e \times M_e \to M_e$ is generalized φ-concave-convex with $\varphi = \varphi(t)$. In fact, for any $t \in (0, 1), \sigma, \varsigma \in M_e$, by (5.1) we have

$$\mathfrak{A}(t\sigma, \varsigma) = \mathfrak{A}(t\sigma, t^{-1} \cdot t\varsigma) \ge t(1 + \eta(t))\mathfrak{A}(\sigma, t\varsigma) = \varphi(t)\mathfrak{A}(\sigma, t\varsigma),$$

where $\varphi(t) = t(1 + \eta(t))$, which means that $\mathfrak{A} : M_e \times M_e \to M_e$ is generalized φ-concave-convex with $\varphi = \varphi(t)$. So the proof is complete by Theorem 3.7.

Remark 5.1. Compared with Theorem 2.1 in [39], Theorem 5.1 removes the surplus assumption of coupled upper and lower solution condition as following: "$\exists \tau_0, \upsilon_0 \in M_e$ with $\tau_0 \le \upsilon_0, \tau_0 \le \mathfrak{A}(\tau_0, \upsilon_0)$ and $\mathfrak{A}(\upsilon_0, \tau_0) \le \upsilon_0$", which appears as an important condition in the proof of [39, Theorem 2.1], while the conclusions still hold. So Theorem 5.1 improves [39, Theorem 2.1].

Next, we will discuss $t - \alpha(t)$ mixed monotone model operators.

Theorem 5.2. ([40]) Let $e > \theta$ and $\mathfrak{A} : M_e \times M_e \to M_e$ be a mixed monotone operator. Suppose that $\mathfrak{A}$ is a $t - \alpha(t)$ mixed monotone model operator, i.e., for all $t \in (0, 1)$ and $\tau, \upsilon \in M_e$, there exists a function $\alpha = \alpha(t)$ with $0 < \alpha(t) < 1$ such that

$$\mathfrak{A}(t\tau, t^{-1}\upsilon) \ge t^{\alpha(t)}\mathfrak{A}(\tau, \upsilon).$$

Then the conclusions of Theorem 5.1 also hold.

Proof. By Lemma 2.1, we easily see that the conclusions of Theorem 5.2 follow from Theorem 5.1.

Remark 5.2. Theorem 5.2 is just the same as Theorem 2.1 in [40], which is one of the main results of [40]. From the proof of Theorem 5.2 we assert that Theorem 5.2 is a special case of Theorem 3.1, so Theorem 3.1 is a generation of [40, Theorem 2.1].

Remark 5.3. If we take $\alpha(t) = \alpha$ which is a constant with $0 < \alpha < 1$ in Theorem 5.2, then Theorem 5.2 is reduced to Theorem 1 in [36]. So Theorem 3.1 is also a generalization of [36, Theorem 1].

6. Applications

In this section, we give two examples to show the fixed point results obtained in previous sections can be applied to nonlinear integral equations on unbounded regions.

Example 6.1. Consider the following nonlinear integral equation

$$\sigma(t) = (\mathfrak{A}\sigma)(t) = \int_{\mathbb{R}^N} K(t,s)[\sigma^{\frac{1}{2}}(s) + \sigma^{-\frac{1}{3}}(s)]ds. \tag{6.1}$$

Conclusion 6.1. Assume that $K : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^1$ is a nonnegative and continuous function. Then Eq (6.1) has a unique positive solution $\sigma^*(t)$. Moreover, constructing successively the sequences $\sigma_n(t)$ and $\varsigma_n(t)$ $(n = 1, 2, \cdots)$ with

$$\sigma_n(t) = \int_{\mathbb{R}^N} K(t,s)[\sigma_{n-1}^{\frac{1}{2}}(s) + \varsigma_{n-1}^{-\frac{1}{3}}(s)]ds$$

and

$$\varsigma_n(t) = \int_{\mathbb{R}^N} K(t,s)[\varsigma_{n-1}^{\frac{1}{2}}(s) + \sigma_{n-1}^{-\frac{1}{3}}(s)]ds$$

for any positive bounded continuous functions σ_0 and ς_0, we have $\sup_{t\in\mathbb{R}^N} |\sigma_n(t) - \sigma^*(t)| \to 0$, and $\sup_{t\in\mathbb{R}^N} |\varsigma_n(t) - \sigma^*(t)| \to 0$ as $n \to \infty$.

Proof. Let $K = C_B(\mathbb{R}^N)$ denote the set of all bounded continuous functions in $\mathbb{R}^N$. Define the norm $\|\sigma\| = \sup_{t\in\mathbb{R}^N} |\sigma(t)|$, then K is a real Banach space. Note the set $M = C_B^+(\mathbb{R}^N)$ of nonnegative functions in $C_B(\mathbb{R}^N)$ is a normal and solid cone in $C_B(\mathbb{R}^N)$. Obviously, Eq (6.1) can be written as $\sigma = \mathfrak{A}(\sigma, \sigma)$, where

$$\mathfrak{A}(\sigma, \varsigma) = \mathfrak{A}_1(\sigma) + \mathfrak{A}_2(\varsigma),$$

$$\mathfrak{A}_1(\sigma) = \int_{\mathbb{R}^N} K(t,s)\sigma^{\frac{1}{2}}(s)dx, \mathfrak{A}_2(\varsigma) = \int_{\mathbb{R}^N} K(t,s)\varsigma^{-\frac{1}{3}}(s)ds.$$

According to Corollary 4.2, it suffices to check that $\mathfrak{A}$ is an α_1-concave-(-α_2)-convex mixed monotone operator where $\alpha_1 = \frac{1}{2}, \alpha_2 = \frac{1}{3}$. In fact, it is easy to verify that $\mathfrak{A}$ is mixed monotone and for fixed $\varsigma, \mathfrak{A}(\cdot, \varsigma) : \mathrm{int}M \to \mathrm{int}M$ is α_1-concave; for fixed $\sigma, \mathfrak{A}(\sigma, \cdot) : \mathrm{int}M \to \mathrm{int}M$ is (-α_2)-convex, where $0 < \alpha_1 + \alpha_2 = \frac{1}{2} + \frac{1}{3} = \frac{5}{6} < 1$. Therefore, we assert Conclusion 6.1 holds by Corollary 4.2.

Remark 6.1. Compared with Example 3.1 in [42], Example 6.1 does not require us to seek another surplus pair of coupled upper and lower solutions τ_0 and υ_0 satisfying:

$$\tau_0 \le \upsilon_0, \tau_0 \le \mathfrak{A}(\tau_0, \upsilon_0), \mathfrak{A}(\upsilon_0, \tau_0) \le \upsilon_0,$$

which appears in [42, Example 3.1] as one of the crucial prerequisites to show the existence of the solution for the integral equation. So Example 6.1 is more workable than [42, Example 3.1]. Similar to Example 6.1, the following is another example to show the application of main results to nonlinear integral equations.

Example 6.2. Consider the following nonlinear integral equation:

$$\sigma(t) = (\mathfrak{A}\sigma)(t) = \int_{\mathbb{R}^N} K(t,s)\left[\sqrt{\sigma(s)} + \frac{1}{\sqrt[4]{\sigma(s)}}\right] ds. \tag{6.2}$$

Conclusion 6.2. Assume that $K : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^1$ is a nonnegative and continuous function. Then Eq (6.2) has a unique positive solution $\sigma^*(t)$. Moreover, constructing successively the sequences $\sigma_n(t)$ and $\varsigma_n(t)$ $(n = 1, 2, \cdots)$ with

$$\sigma_n(t) = \int_{\mathbb{R}^N} K(t,s)\left[\sigma_{n-1}^{\frac{1}{2}}(s) + \varsigma_{n-1}^{-\frac{1}{4}}(s)\right] ds$$

and

$$\varsigma_n(t) = \int_{\mathbb{R}^N} K(t,s)\left[\varsigma_{n-1}^{\frac{1}{2}}(s) + \sigma_{n-1}^{-\frac{1}{4}}(s)\right] ds$$

for any positive bounded continuous functions σ_0 and ς_0, we have $\sup_{t\in\mathbb{R}^N} |\sigma_n(t) - \sigma^*(t)| \to 0$, and $\sup_{t\in\mathbb{R}^N} |\varsigma_n(t) - \sigma^*(t)| \to 0$, as $n \to \infty$.

Remark 6.2. Compared to Example 4.2 in [41], Example 6.2, like Example 6.1, does not require us to check some certain coupled upper and lower solution τ_0 and υ_0 would exist. As a kind of convenience, Example 6.2 deletes the following condition:

$$\text{“}\frac{1}{110} \le \int_{\mathbb{R}^N} K(t,s)ds \le \frac{1}{1+\sqrt{10}}\text{”}$$

which is for the existence of the coupled upper and lower solutions. In addition, in Example 6.2, the initial values σ_0 and ς_0 for the iterated sequences $\{\sigma_n(t)\}$ and $\{\varsigma_n(t)\}$ may be chosen in a wider scope int$M$, namely, we may choose any two positive continuous bounded functions as the values of initial σ_0 and ς_0, while in [41, Example 4.2], the initial σ_0 and ς_0 can be chosen only in the interval $[\tau_0, \upsilon_0]$. So Example 6.2 is more workable than [41, Example 4.2].

7. Conclusions

In this paper, we introduce the notion of generalized ϕ-concave-convex operators. By means of the theory of cone and partial order as well as the monotone iteration techniques, we investigate such kind of operators satisfying mixed monotonicity property and obtain the existence and uniqueness of the fixed points as well as the convergence of the iterated sequence. The main novelty is that the so-called generalized φ-concave-convex operators can unify and extend a number of operators with certain concavity and convexity; and the main results improve and generalize many related results in the existing literature. Further, we delete the redundant conditions and thus make the application examples more practicable. While the study's conclusions are enlightening, the paper has a research limitation in the application. In the application section, it is found that some of the main results are applied to only two nonlinear integral equations on unbounded regions. However, the theory of mixed monotone operators has many other applications in nonlinear equations as well as nonlinear dynamics. Thus, future research could focus on the applications of the obtained new fixed point results of mixed monotone operators to boundary-value problems for nonlinear different equations, nonlinear delay integral equations, population dynamics and chemical reaction networks.

Author contributions

Shaoyuan Xu: Conceptualization, methodology, validation, formal analysis, resources, writing-original draft preparation, writing-review and editing, visualization, supervision, project administration, funding acquisition; Yan Han: Conceptualization, validation, formal analysis, resources, writing-original draft preparation, writing-review and editing, visualization, project administration, funding acquisition; Li Fan: Software, writing-original draft preparation, writing-review and editing, visualization. All authors have read and agreed to the published version of the manuscript.

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Conflicts of interest

The authors declare that they have no conflicts of interest.

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