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Commentary

Unavoidable corrections for $\theta\beta$-ideal approximation spaces

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Abstract: The short article in hand introduces some amendments for the relationships and claims presented in [16] with the investigation of their correct forms. To elucidate those failures and to support the results obtained herein, we provide an illustrative example. We also elucidate that the rough set models proposed by [11] and [16] are incomparable. Moreover, we demonstrate that the observations, given in the application section of [16], contradict the computations of lower and upper approximations, boundary regions, and accuracy measures as well as violate some well-known properties of Pawlak approximation space.

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1. Introduction

Rough set theory, introduced by Pawlak [17], is a powerful mathematical tool for effectively transacting with imprecise and uncertain information. A key advantage of rough set theory is its ability to represent data using granular computing inspired by an equivalence relation. As we know, the granular computing represented by equivalence classes in the original model of Pawlak has been updated using some neighborhood systems inspired by relations weaker than equivalence relation or arbitrary relations; for more details about these neighborhood systems, we refer the readers to [6,14,22] and references mentioned therein. This development assists in canceling a strict condition of an equivalence relation and expanding the scope of its applications in diverse disciplines.

The interconnection of topological and rough set theory was first put forward by Wiweger [21], who explored the topological aspects of rough sets. This led to a fusion of rough set theory and topological structures becoming a central focus of numerous studies [19, 25]. Then, some techniques to institute a topology using rough neighborhoods were proposed to represent rough approximation operators and analyze information systems; see [1, 13]. One of the important topological tools to reduce the vagueness of knowledge is nearly open sets, so many authors applied to describe rough set models, such as $\delta\beta$-open sets [2], somewhat open sets [3], and somewhere dense sets [4]. Quite recently, this interaction has been developed to involve generalizations of topology, such as supra topology [5], infra topology [7], minimal structures [9], and bitopology [18]. It is worth noting that several manuscripts investigated rough set models from different views as [8, 20].

In 2013, Kandi et al. [12] integrated the abstract principle so-called ideal $\mathcal{I}$ with rough neighborhoods to provide a new framework of rough sets paradigms called ideal approximation spaces. As illustrated in published literature like [10, 15, 24], this framework proves its capability in terms of enlarging the domain of confirmed knowledge and thereby maximizing the value of accuracy.

In 2022, Nawar et al. [16] proposed two new rough set models, the first one generated by one of nearly open sets, namely, $\theta\beta_\sigma$-open sets, and the second generated by ideals and $\mathcal{I}$-$\theta\beta_\sigma$-open sets. However, we note that they provided some incorrect results and relationships that cannot be overlooked and require correction, particularly those that compare the superiority of their approach over the one introduced by Hosny [11]. In this regard, we construct a counterexample to show that Theorem 4.1, Corollary 4.1, and items (2) and (4) of Corollary 4.2 displayed in [16] are false. Moreover, we prove that their approach and Hosny's approach [11] are incomparable. Ultimately, we evidence that three observations on page 2494 of [11] about the given application are incorrect, in general.

2. Preliminaries

Here, we recall some definitions and results that are required to understand this work.

Remember that a relation λ on a nonempty set $\mathbb{X}$ is a subset of $\mathbb{X}\times\mathbb{X}$. We write $a\lambda b$ when $(a, b) \in \lambda$.

Definition 2.1. *[17] Let λ be an equivalence relation on $\mathbb{X}$. The lower approximation and upper approximation of $Z \subseteq \mathbb{X}$ are, respectively, given by:*

$$\underline{\lambda}(Z) = \cup\{V \in \mathbb{X}/\lambda \mid V \subseteq Z\}.$$

$$\overline{\lambda}(Z) = \cup\{V \in \mathbb{X}/\lambda \mid V \cap Z \neq \emptyset\},$$

where $\mathbb{X}/\lambda$ denotes the family of equivalence classes induced by λ.

The triple $(\mathbb{X}, \underline{\lambda}, \overline{\lambda})$ is called Pawlak rough set models; it is known as the original (standard) model. The core features of this model are enumerated in the subsequent proposition.

Proposition 2.2. *[17] Consider an equivalence relation λ defined on $\mathbb{X}$. For sets V, W, the next characteristics hold:*

$(L1)$ $\underline{\lambda}(V) \subseteq V$	$(U1)$ $S \subseteq \overline{\lambda}(V)$
$(L2)$ $\underline{\lambda}(\emptyset) = \emptyset$	$(U2)$ $\overline{\lambda}(\emptyset) = \emptyset$
$(L3)$ $\underline{\lambda}(\mathbb{X}) = \mathbb{X}$	$(U3)$ $\overline{\lambda}(\mathbb{X}) = \mathbb{X}$

$(L4)$ $If\, V \subseteq W,\ then\ \underline{\lambda}(V) \subseteq \underline{\lambda}(W)$ $(U4)$ $If\, V \subseteq W,\ then\ \overline{\lambda}(V) \subseteq \overline{\lambda}(W)$

$(L5)$ $\underline{\lambda}(V \cap W) = \underline{\lambda}(V) \cap \underline{\lambda}(W)$ $(U5)$ $\overline{\lambda}(V \cap W) \subseteq \overline{\lambda}(V) \cap \overline{\lambda}(W)$

$(L6)$ $\underline{\lambda}(S) \cup \underline{\lambda}(W) \subseteq \underline{\lambda}(V \cup W)$ $(U6)$ $\overline{\lambda}(V \cup W) = \overline{\lambda}(V) \cup \overline{\lambda}(W)$

$(L7)$ $\underline{\lambda}(V^c) = (\overline{\lambda}(V))^c$ $(U7)$ $\overline{\lambda}(V^c) = (\underline{\lambda}(V))^c$

$(L8)$ $\underline{\lambda}(\underline{\lambda}(V)) = \underline{\lambda}(V)$ $(U8)$ $\overline{\lambda}(\overline{\lambda}(V)) = \overline{\lambda}(V)$

$(L9)$ $\underline{\lambda}((\underline{\lambda}(V))^c) = (\overline{\lambda}(V))^c$ $(U9)$ $\overline{\lambda}((\overline{\lambda}(V))^c) = (\underline{\lambda}(V))^c$

$(L10)$ $\underline{\lambda}(W) = W, \forall W \in \mathbb{X}/\lambda$ $(U10)$ $\overline{\lambda}(W) = W, \forall W \in \mathbb{X}/\lambda$

In many positions, the equivalence relations are not attainable. Consequently, the classical approach has been extended by employing weaker relations than full equivalence. This led to the proposal of different types of neighborhoods as granular computing alternatives for the equivalence classes.

Definition 2.3. *[1, 22, 23] Consider an arbitrary relation λ on $\mathbb{X}$. If $\sigma \in \{r, \langle r\rangle, l, \langle l\rangle, i, \langle i\rangle, u, \langle u\rangle\}$, then the σ-neighborhoods of $a \in \mathbb{X}$, symbolized by $\mathcal{N}_\sigma(a)$, are identified as:*

(*i*) $\mathcal{N}_r(a) = \{b \in \mathbb{X} : a\ \lambda\ b\}$.

(*ii*) $\mathcal{N}_l(a) = \{b \in \mathbb{X} : b\ \lambda\ a\}$.

(*iii*)

$$\mathcal{N}_{\langle r\rangle}(a) = \begin{cases} \bigcap\limits_{a \in \mathcal{N}_r(b)} \mathcal{N}_r(b) & : \ \exists\, \mathcal{N}_r(b) \text{ involving } a \\ \emptyset & : \ Elsewise \end{cases}$$

(*iv*)

$$\mathcal{N}_{\langle l\rangle}(a) = \begin{cases} \bigcap\limits_{a \in \mathcal{N}_l(b)} \mathcal{N}_l(b) & : \ \exists\, \mathcal{N}_l(b) \text{ involving } a \\ \emptyset & : \ Elsewise \end{cases}$$

(*v*) $\mathcal{N}_i(a) = \mathcal{N}_r(a) \bigcap \mathcal{N}_l(a)$.

(*vi*) $\mathcal{N}_u(a) = \mathcal{N}_r(a) \bigcup \mathcal{N}_l(a)$.

(*vii*) $\mathcal{N}_{\langle i\rangle}(a) = \mathcal{N}_{\langle r\rangle}(a) \bigcap \mathcal{N}_{\langle l\rangle}(a)$.

(*viii*) $\mathcal{N}_{\langle u\rangle}(a) = \mathcal{N}_{\langle r\rangle}(a) \bigcup \mathcal{N}_{\langle l\rangle}(a)$.

Henceforward, unless otherwise specified, we will consider σ to belong to the set $\{r, \langle r\rangle, l, \langle l\rangle, i, \langle i\rangle, u, \langle u\rangle\}$.

Remark 2.4. *The authors of [16] incorrectly mentioned the definitions of $\mathcal{N}_{\langle r\rangle}(a)$ and $\mathcal{N}_{\langle l\rangle}(a)$. They overlooked the cases that do not exist $\mathcal{N}_r(b)$ containing a and $\mathcal{N}_l(b)$ containing a, which leads to incorrect computations for some cases, especially when the given binary relation is not serial or inverse serial. Therefore, we should consider these cases when we define $\mathcal{N}_{\langle r\rangle}(a)$ and $\mathcal{N}_{\langle l\rangle}(a)$ as given in (iii) and (iv) of Definition 2.3.*

Definition 2.5. *[1] Consider a relation λ on $\mathbb{X}$ and let ζ_σ denote a mapping from $\mathbb{X}$ to $2^{\mathbb{X}}$, associating each member $a \in \mathbb{X}$ with its σ-neighborhood in $2^{\mathbb{X}}$. Consequently, the triple $(\mathbb{X}, \lambda, \zeta_\sigma)$ is termed a σ-neighborhood space, abbreviated as σ-NS.*

Theorem 2.6. *[1] It may generate a topology ϑ_σ on X using $\mathcal{N}_\sigma$-neighborhoods by the next formula*

$$\vartheta_\sigma = \{V \subseteq X : \mathcal{N}_\sigma(a) \subseteq V \text{ for each } a \in V\}$$

The main concepts of rough set paradigms inspired by a topology given in Theorem 2.6 are mentioned in the next two definitions.

Definition 2.7. *[1] Let $(\mathbb{X}, \lambda, \zeta_\sigma)$ be a σ-NS and ϑ_σ be a topology described in Theorem 2.6. For each σ, the σ-lower and σ-upper, σ-boundary, and σ-accuracy of a subset Z of $\mathbb{X}$ are, respectively, defined by the following formulas.*

(i) $\underline{\mathcal{H}}_\sigma(Z) = \cup\{V \in \vartheta_\sigma : V \subseteq Z\} = int_\sigma(Z)$, *where* $int_\sigma(Z)$ *is the interior points of Z in* $(\mathbb{X}, \vartheta_\sigma)$.

(ii) $\overline{\mathcal{H}}_\sigma(Z) = \cap\{W : Z \subseteq W \text{ and } \in W^c \in \vartheta_\sigma\} = cl_\sigma(Z)$, *where* $cl_\sigma(Z)$ *is the closure points of Z in* $(\mathbb{X}, \vartheta_\sigma)$.

(iii) $\mathcal{B}_\sigma(Z) = \overline{\mathcal{H}}_\sigma(Z) \setminus \underline{\mathcal{H}}_\sigma(Z)$

(iv) $\mathcal{A}_\sigma(Z) = \frac{|\underline{\mathcal{H}}_\sigma(Z)|}{|\overline{\mathcal{H}}_\sigma(Z)|}$, *where Z is a nonempty set.*

Definition 2.8. *[1] A subset Z of $(\mathbb{X}, \lambda, \zeta_\sigma)$ is called an σ-exact (resp., σ-rough) set if $\underline{\mathcal{H}}_\sigma(Z) = \overline{\mathcal{H}}_\sigma(Z)$ (resp., $\underline{\mathcal{H}}_\sigma(Z) \neq \overline{\mathcal{H}}_\sigma(Z)$)*

Definition 2.9. *A nonempty subclass $\mathcal{I}$ of $2^{\mathbb{X}}$ is called an ideal on $\mathbb{X}$ provided that the next conditions are satisfied.*

(i) *The union of any two members in $\mathcal{I}$ is a member of $\mathcal{I}$.*

(ii) *If $V \in \mathcal{I}$, then any subset of V is a member of $\mathcal{I}$.*

Theorem 2.10. *[11, 12] It may generate a topology $\vartheta_\sigma^\mathcal{I}$ on X using $\mathcal{N}_\sigma$-neighborhoods and an ideal $\mathcal{I}$ by the next formula*

$$\vartheta_\sigma^\mathcal{I} = \{V \subseteq X : \mathcal{N}_\sigma(a) \setminus V \in \mathcal{I} \text{ for each } a \in V\}$$

The main concepts of rough set paradigms inspired by a topology given in Theorem 2.10 are mentioned in the next two definitions.

Definition 2.11. *[11] Let $(\mathbb{X}, \lambda, \zeta_\sigma)$ be a σ-NS, $\mathcal{I}$ be an ideal on $\mathbb{X}$, and $\vartheta_\sigma^\mathcal{I}$ be a topology described in Theorem 2.10. For each σ, the $\mathcal{I}\sigma$-lower and $\mathcal{I}\sigma$-upper, $\mathcal{I}\sigma$-boundary, and $\mathcal{I}\sigma$-accuracy of a subset Z of $\mathbb{X}$ are, respectively, defined by the following formulas.*

(i) $\underline{\mathcal{H}}_\sigma^\mathcal{I}(Z) = \cup\{V \in \vartheta_\sigma^\mathcal{I} : V \subseteq Z\} = int_\sigma^\mathcal{I}(Z)$, *where* $int_\sigma^\mathcal{I}(Z)$ *is the interior points of Z in* $(\mathbb{X}, \vartheta_\sigma^\mathcal{I})$.

(ii) $\overline{\mathcal{H}}_\sigma^\mathcal{I}(Z) = \cap\{W : Z \subseteq W \text{ and } \in W^c \in \vartheta_\sigma^\mathcal{I}\} = cl_\sigma^\mathcal{I}(Z)$, *where* $cl_\sigma^\mathcal{I}(Z)$ *is the closure points of Z in* $(\mathbb{X}, \vartheta_\sigma^\mathcal{I})$.

(iii) $\mathcal{B}_\sigma^\mathcal{I}(Z) = \overline{\mathcal{H}}_\sigma^\mathcal{I}(Z) \setminus \underline{\mathcal{H}}_\sigma^\mathcal{I}(Z)$

(iv) $\mathcal{A}_\sigma^\mathcal{I}(Z) = \frac{|\underline{\mathcal{H}}_\sigma^\mathcal{I}(Z)|}{|\overline{\mathcal{H}}_\sigma^\mathcal{I}(Z)|}$, *where Z is a nonempty set.*

Definition 2.12. *[11] Let $(\mathbb{X}, \lambda, \zeta_\sigma)$ be a σ-NS and $\mathcal{I}$ be an ideal on $\mathbb{X}$. A subset Z of $\mathbb{X}$ is called an $\mathcal{I}\sigma$-exact (resp., $\mathcal{I}\sigma$-rough) set if $\underline{\mathcal{H}}_\sigma^\mathcal{I}(Z) = \overline{\mathcal{H}}_\sigma^\mathcal{I}(Z)$ (resp., $\underline{\mathcal{H}}_\sigma^\mathcal{I}(Z) \neq \overline{\mathcal{H}}_\sigma^\mathcal{I}(Z)$).*

3. Main results

This section rectifies some invalid claims and relationships presented in [16]. An elucidative example is provided to support the amendments that were made.

To begin, we recall the definition of $\mathcal{I}$-$\theta\beta_\sigma$-open sets as introduced in [16].

Definition 3.1. *[16] Let $(\mathbb{X}, \lambda, \zeta_\sigma)$ be a σ-NS and $\mathcal{I}$ be an ideal on $\mathbb{X}$. Then, a subset of V of $\mathbb{X}$ is called $\mathcal{I}$-$\theta\beta_\sigma$-open if $V \subseteq cl_\sigma(int_\sigma(cl_\sigma^{\star\theta}(V)))$ such that*

$$cl_\sigma^{\star\theta}(V) = V \cup V_\sigma^{\star\theta} \text{ where } V_\sigma^{\star\theta} = \{a \in \mathbb{X} : V \cap cl_\sigma(G) \notin \mathcal{I} \text{ for every } G \in \vartheta_\sigma \text{ containing } a\}.$$

The complement of an $\mathcal{I}$-$\theta\beta_\sigma$-open set is named $\mathcal{I}$-$\theta\beta_\sigma$-closed. The classes of $\mathcal{I}$-$\theta\beta_\sigma$-open subsets and $\mathcal{I}$-$\theta\beta_\sigma$-closed subsets are respectively symbolized by $\mathcal{I}$-$\theta\beta_\sigma O(\mathbb{X})$ and $\mathcal{I}$-$\theta\beta_\sigma C(\mathbb{X})$.

Then, they introduced the ideas of lower and upper approximations, boundary regions, and accuracy measures in relation to the classes of $\mathcal{I}$-$\theta\beta_\sigma O(\mathbb{X})$ and $\mathcal{I}$-$\theta\beta_\sigma C(\mathbb{X})$ as follows.

Definition 3.2. *[16] Let $(\mathbb{X}, \lambda, \zeta_\sigma)$ be a σ-NS and $\mathcal{I}$ be an ideal on $\mathbb{X}$. For each σ, the $\mathcal{I}$-$\theta\beta_\sigma$-lower and $\mathcal{I}$-$\theta\beta_\sigma$-upper, $\mathcal{I}$-$\theta\beta_\sigma$-boundary, and $\mathcal{I}$-$\theta\beta_\sigma$-accuracy of a subset Z of $\mathbb{X}$ are, respectively, defined by the following formulas.*

(i) $\underline{\mathcal{H}}_\sigma^{\mathcal{I}-\theta\beta}(Z) = \cup\{V \in \mathcal{I}\text{-}\theta\beta_\sigma O(\mathbb{X}) : V \subseteq Z\} = int_\sigma^{\mathcal{I}-\theta\beta}(Z)$, *where $int_\sigma^{\mathcal{I}-\theta\beta}(Z)$ is the interior points of Z in $(\mathbb{X}, \mathcal{I}\text{-}\theta\beta_\sigma O(\mathbb{X}))$.*

(ii) $\overline{\mathcal{H}}_\sigma^{\mathcal{I}-\theta\beta}(Z) = \cap\{W \in \mathcal{I}\text{-}\theta\beta_\sigma C(\mathbb{X}) : Z \subseteq W\} = cl_\sigma^{\mathcal{I}-\theta\beta}(Z)$, *where $cl_\sigma^{\mathcal{I}-\theta\beta}(Z)$ is the closure points of Z in $(\mathbb{X}, \mathcal{I}\text{-}\theta\beta_\sigma O(\mathbb{X}))$.*

(iii) $\mathcal{B}_\sigma^{\mathcal{I}-\theta\beta}(H) = \overline{\mathcal{H}}_\sigma^{\mathcal{I}-\theta\beta_\sigma}(Z) \setminus \underline{\mathcal{H}}_\sigma^{\mathcal{I}-\theta\beta_\sigma}(Z)$

(iv) $\mathcal{A}_\sigma^{\mathcal{I}-\theta\beta}(Z) = \frac{|\underline{\mathcal{H}}_\sigma^{\mathcal{I}-\theta\beta_\sigma}(Z)|}{|\overline{\mathcal{H}}_\sigma^{\mathcal{I}-\theta\beta_\sigma}(Z)|}$, *where Z is a nonempty set.*

Remark 3.3. *One can prove that the structure $(\mathbb{X}, \mathcal{I}\text{-}\theta\beta_\sigma O(\mathbb{X}))$ is closed under arbitrary unions. In contrast, the intersection of two $\mathcal{I}$-$\theta\beta_\sigma$-open sets fails to be an $\mathcal{I}$-$\theta\beta_\sigma$-open set, in general. Therefore, $(\mathbb{X}, \mathcal{I}\text{-}\theta\beta_\sigma O(\mathbb{X}))$ forms a supra topology on $\mathbb{X}$.*

Definition 3.4. *[16] Let $(\mathbb{X}, \lambda, \zeta_\sigma)$ be a σ-NS and $\mathcal{I}$ be an ideal on $\mathbb{X}$. A subset Z of $\mathbb{X}$ is called an $\mathcal{I}$-$\theta\beta_\sigma$-exact (resp., $\mathcal{I}$-$\theta\beta_\sigma$-rough) set if $\underline{\mathcal{H}}_\sigma^{\mathcal{I}-\theta\beta}(Z) = \overline{\mathcal{H}}_\sigma^{\mathcal{I}-\theta\beta}(Z)$ (resp., $\underline{\mathcal{H}}_\sigma^{\mathcal{I}-\theta\beta}(Z) \neq \overline{\mathcal{H}}_\sigma^{\mathcal{I}-\theta\beta}(Z)$)*

The authors of [16] claimed the following theorem and two corollaries; they were presented in [16] in the following order: Theorem 4.1, Corollary 4.1, and items (2) and (4) of Corollary 4.2.

Theorem 3.5. *[16] Let $(\mathbb{X}, \lambda, \zeta_\sigma)$ be a σ-NS and $\mathcal{I}$ be an ideal on $\mathbb{X}$. Then, we have the following properties for any subset Z of $\mathbb{X}$.*

(i) $\underline{\mathcal{H}}_\sigma(Z) \subseteq \underline{\mathcal{H}}_\sigma^{\mathcal{I}}(Z) \subseteq \underline{\mathcal{H}}_\sigma^{\mathcal{I}-\theta\beta}(Z)$.

(ii) $\overline{\mathcal{H}}_\sigma^{\mathcal{I}-\theta\beta}(Z) \subseteq \overline{\mathcal{H}}_\sigma^{\mathcal{I}}(Z) \subseteq \overline{\mathcal{H}}_\sigma(Z)$.

Corollary 3.6. *[16] Let $(\mathbb{X}, \lambda, \zeta_\sigma)$ be a σ-NS and $\mathcal{I}$ be an ideal on $\mathbb{X}$. Then, we have the following properties for any subset Z of $\mathbb{X}$.*

(i) $\mathcal{B}_{\sigma}^{I-\theta\beta}(Z) \subseteq \mathcal{B}_{\sigma}^{I}(Z) \subseteq \mathcal{B}_{\sigma}(Z)$.

(ii) $\mathcal{A}_{\sigma}(Z) \leq \mathcal{A}_{\sigma}^{I}(Z) \leq \mathcal{A}_{\sigma}^{I-\theta\beta}(Z)$.

Corollary 3.7. *[16] Let $(\mathbb{X}, \lambda, \zeta_\sigma)$ be a σ-NS and I be an ideal on $\mathbb{X}$. Then,*

(i) *Every I-σ-exact set is I-$\theta\beta_\sigma$-exact.*

(ii) *Every I-$\theta\beta_\sigma$-rough set is I-σ-rough.*

We give the subsequent counterexample to show that Theorem 3.5 is incorrect in general.

Example 3.8. *Let $\lambda = \{(a,a),(b,b),(c,c),(a,b),(b,a),(x,a),(x,b),(x,c)\}$ be a binary relation on $\mathbb{X} = \{a,b,c,x\}$ and $I = \{\emptyset,\{a\},\{b\},\{a,b\}\}$ be an ideal on $\mathbb{X}$. Then, the r-neighborhoods of elements of $\mathbb{X}$ are $\mathcal{N}_r(a) = \mathcal{N}_r(b) = \{a,b\}$, $\mathcal{N}_r(c) = \{c\}$, and $\mathcal{N}_r(x) = \{a,b,c\}$. Accordingly, we compute the following classes:*

(i) $\vartheta_r = \{\emptyset, \mathbb{X}, \{c\}, \{a,b\}, \{a,b,c\}\}$,

(ii) $\vartheta_r^{I} = \vartheta_r \cup \{\{a\},\{b\},\{a,c\},\{b,c\},\{c,x\},\{a,c,x\},\{b,c,x\}\}$, *and*

(iii) I-$\theta\beta_r O(\mathbb{X}) = \vartheta_r \cup \{\{x\},\{a,x\},\{b,x\},\{c,x\},\{a,b,x\},\{a,c,x\},\{b,c,x\}\}$.

Now, we calculate the lower and upper approximations inspired by the classes of ϑ_r^{I} and I-$\theta\beta_r O(\mathbb{X})$ in Table 1.

Table 1. Lower and upper approximations in relation to the methods of Hosny [11] and Nawar et al. [16]

Methods	Hosny method ϑ_r^{I} [11]		Nawar et al. method I-$\theta\beta_r C(\mathbb{X})$ [16]	
$Z \subseteq \mathbb{X}$	$\underline{\mathcal{H}}_r^{I}(Z)$	$\overline{\mathcal{H}}_r^{I}(Z)$	$\underline{\mathcal{H}}_r^{I-\theta\beta}(Z)$	$\overline{\mathcal{H}}_r^{I-\theta\beta}(Z)$
$\{a\}$	$\{a\}$	$\{a\}$	$\emptyset$	$\{a\}$
$\{b\}$	$\{b\}$	$\{b\}$	$\emptyset$	$\{b\}$
$\{c\}$	$\{c\}$	$\{c,x\}$	$\{c\}$	$\{c\}$
$\{x\}$	$\emptyset$	$\{x\}$	$\{x\}$	$\{x\}$
$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$
$\{a,c\}$	$\{a,c\}$	$\{a,c,x\}$	$\{c\}$	$\{a,c\}$
$\{a,x\}$	$\{a\}$	$\{a,x\}$	$\{a,x\}$	$\{a,b,x\}$
$\{b,c\}$	$\{b,c\}$	$\{b,c,x\}$	$\{c\}$	$\{b,c\}$
$\{b,x\}$	$\{b\}$	$\{b,x\}$	$\{b,x\}$	$\{a,b,x\}$
$\{c,x\}$	$\{c,x\}$	$\{c,x\}$	$\{c,x\}$	$\{c,x\}$
$\{a,b,c\}$	$\{a,b,c\}$	$\mathbb{X}$	$\{a,b,c\}$	$\{a,b,c\}$
$\{a,b,x\}$	$\{a,b\}$	$\{a,b,x\}$	$\{a,b,x\}$	$\{a,b,x\}$
$\{a,c,x\}$	$\{a,c,x\}$	$\{a,c,x\}$	$\{a,c,x\}$	$\mathbb{X}$
$\{b,c,x\}$	$\{b,c,x\}$	$\{b,c,x\}$	$\{b,c,x\}$	$\mathbb{X}$

On the one hand, one can see from Table 1 that $\underline{\mathcal{H}}_r^{I-\theta\beta}(\{a,c\}) = \{c\} \subseteq \underline{\mathcal{H}}_r^{I}(\{a,c\}) = \{a,c\}$ and $\overline{\mathcal{H}}_r^{I}(\{a,c,x\}) = \{a,c,x\} \subseteq \overline{\mathcal{H}}_r^{I-\theta\beta}(\{a,c,x\}) = \mathbb{X}$. Therefore, there exist subsets V, W such that $\underline{\mathcal{H}}_r^{I}(V) \nsubseteq$

$\underline{\mathcal{H}}_r^{I-\theta\beta}(V)$ *and* $\overline{\mathcal{H}}_r^{I-\theta\beta}(W) \nsubseteq \overline{\mathcal{H}}_r^{I}(W)$*, which revokes the claims of Theorem 3.5. On the other hand,* $\underline{\mathcal{H}}_r^{I}(\{b,x\}) = \{b\} \subseteq \underline{\mathcal{H}}_r^{I-\theta\beta}(\{b,x\}) = \{b,x\}$ *and* $\overline{\mathcal{H}}_r^{I-\theta\beta}(\{a,c\}) = \{a,c\} \subseteq \overline{\mathcal{H}}_r^{I}(\{a,c\}) = \{a,c,x\}$.

This implies that the rough approximation operators generated by the methods introduced in [11] and [16] are independent of each other. Hence, the claims given in Theorem 3.5 are false.

Accordingly, one can note that Corollary 4.2 of [16] (mentioned here as Corollary 3.7) is also wrong. To confirm this matter, take a subset $V = \{a\}$*. By Table 1, we find that* $\underline{\mathcal{H}}_r^{I}(V) = \overline{\mathcal{H}}_r^{I}(V) = V$*, so* V *is* $\mathcal{I}$*-r-exact. However,* $\underline{\mathcal{H}}_r^{I-\theta\beta}(V) = \emptyset \neq \underline{\mathcal{H}}_r^{I-\theta\beta}(V) = \{a\}$*, so* V *is not* $\mathcal{I}$*-*$\theta\beta_r$*-exact. Equivalently,* V *is* $\mathcal{I}$*-*$\theta\beta_r$*-rough but not* $\mathcal{I}$*-r-rough.*

Consequently, Corollary 4.1 of [16] (mentioned here as Corollary 3.6) is false. To illustrate this conclusion, we provide Table 2 which is based on the computations of Table 1.

Table 2. Boundary region and accuracy in relation to the methods of Hosny [11] and Nawar et al. [16]

Methods	Hosny method ϑ_r^I [11]		Nawar et al. method $\mathcal{I}$-$\theta\beta_r C(\mathbb{X})$ [16]	
$Z \subseteq \mathbb{X}$	$\mathcal{B}_r^I(Z)$	$\mathcal{A}_r^I(Z)$	$\mathcal{B}_r^{I-\theta\beta}(Z)$	$\mathcal{A}_r^{I-\theta\beta}(Z)$
$\{a\}$	$\emptyset$	1	$\{a\}$	0
$\{b\}$	$\emptyset$	1	$\{b\}$	0
$\{c\}$	$\{x\}$	$\frac{1}{2}$	$\emptyset$	1
$\{x\}$	$\{x\}$	0	$\emptyset$	1
$\{a,b\}$	$\emptyset$	1	$\emptyset$	1
$\{a,c\}$	$\{x\}$	$\frac{2}{3}$	$\{c\}$	$\frac{1}{2}$
$\{a,x\}$	$\{x\}$	$\frac{1}{2}$	$\{b\}$	$\frac{2}{3}$
$\{b,c\}$	$\{x\}$	$\frac{2}{3}$	$\{b\}$	$\frac{1}{2}$
$\{b,x\}$	$\{x\}$	$\frac{1}{2}$	$\{a\}$	$\frac{2}{3}$
$\{c,x\}$	$\emptyset$	1	$\emptyset$	1
$\{a,b,c\}$	$\{x\}$	$\frac{3}{4}$	$\emptyset$	1
$\{a,b,x\}$	$\{x\}$	$\frac{2}{3}$	$\emptyset$	1
$\{a,c,x\}$	$\emptyset$	1	$\{b\}$	$\frac{3}{4}$
$\{b,c,x\}$	$\emptyset$	1	$\{a\}$	$\frac{3}{4}$

On the one hand, one can see from Table 2 that $\mathcal{B}_r^I(\{a\}) = \emptyset \subseteq \mathcal{B}_r^{I-\theta\beta}(\{a\}) = \{a\}$ and $\mathcal{A}_r^{I-\theta\beta}(\{a\}) = 0 < \mathcal{A}_r^I(\{a\}) = 1$. Therefore, there exist subsets V, W such that $\mathcal{B}_r^{I-\theta\beta}(V) \nsubseteq \mathcal{B}_r^I(V)$ and $\mathcal{A}_r^I(W) \nless \mathcal{A}_r^{I-\theta\beta}(W)$, which revokes the claims of Corollary 3.6. On the other hand, $\mathcal{B}_r^{I-\theta\beta}(\{c\}) = \emptyset \subseteq \mathcal{B}_r^I(\{c\}) = \{x\}$ and $\mathcal{A}_r^I(\{c\}) = 0 < \mathcal{A}_r^{I-\theta\beta}(\{c\}) = \frac{1}{2}$.

This implies that the boundary regions and accuracy induced by the methods introduced in [11] and [16] are independent of each other. Hence, the claims given in Corollary 3.6 need not be true, in general.

Now, we put forward the correct relationships between the notions presented in Theorem 3.5 and Corollaries 3.6 and 3.7 in the following remark.

Remark 3.9. *Let* $(\mathbb{X}, \lambda, \zeta_\sigma)$ *be a* σ*-NS and* $\mathcal{I}$ *be an ideal on* $\mathbb{X}$*. Then, there is no relationship between the following concepts inspired by the approaches of [11] and [16].*

(i) *The lower approximations* $\underline{\mathcal{H}}_{\sigma}^{\mathcal{I}}$ *and* $\underline{\mathcal{H}}_{\sigma}^{\mathcal{I}-\theta\beta}$.

(ii) *The upper approximations* $\overline{\mathcal{H}}_{\sigma}^{\mathcal{I}}$ *and* $\overline{\mathcal{H}}_{\sigma}^{\mathcal{I}-\theta\beta}$.

(iii) *The boundary regions* $\mathcal{B}_{\sigma}^{\mathcal{I}}$ *and* $\mathcal{B}_{\sigma}^{\mathcal{I}-\theta\beta}$.

(iv) *The accuracy measures* $\mathcal{A}_{\sigma}^{\mathcal{I}}$ *and* $\mathcal{A}_{\sigma}^{\mathcal{I}-\theta\beta}$.

(v) $\mathcal{I}$*-*σ*-exact and* $\mathcal{I}$*-*$\theta\beta_{\sigma}$*-exact (*$\mathcal{I}$*-*σ*-rough and* $\mathcal{I}$*-*$\theta\beta_{\sigma}$*-rough) sets.*

In Figure 1, three observations were mentioned on page 2494 of [16] concerning the suggested application.

Observation: From the previous comparisons in Table 5, we note the following:

1) There are several approaches to approximate the sets. The finest of these approaches is that there are assumed by using $\boldsymbol{I-\theta\beta_j}$-approximations of the current methods in Definition 4.2 for creating the rough approximations because the boundary regions in these cases are minimized (or removed) by maximizing the lower approximation and minimizing the upper approximation. Moreover, the accuracy degree, in these cases, is more accurate than the other types. For example, all proper subsets are rough in Abd El-Monsef's approaches. But there are many $I-\theta\beta_j$- exact sets in the current methods.
2) The suggested method in Definition 4.2 is more accurate and stronger than M. Hosny's methods. Therefore, the suggested methodologies will be useful in decision-making for extracting the information and help in eliminating the ambiguity of the data in real-life problems.
3) The significance of the suggested approximations is not only that it is decreasing or deleting the boundary regions, but also, it's satisfying all characteristics of Pawlak's model without any restrictions as shown in Proposition 4.1.

Figure 1. Observations given on page 2494 of [16].

In what follows, we show that these observations are incorrect or inaccurate.

(i) The first and second items are not true since the approach of [16] is not the finest one. It is not stronger than the approach proposed by Hosny [11] as we illustrated in the aforementioned discussion. The appropriate description is that the methods of [11] and [16] are incomparable.

(ii) The third item is inaccurate since the approach of [16] does not preserve all properties of the standard model of Pawlak (we mentioned these properties in Proposition 2.7) since it can be noted that the properties $L5$ and $U6$ are not satisfied. To confirm this point, take subsets $V = \{a, b\}$, $W = \{a, x\}$, $Y = \{a\}$, and $Z = \{x\}$. Then, $\underline{\mathcal{H}}_{\sigma}^{\mathcal{I}-\theta\beta}(V) = V$ and $\underline{\mathcal{H}}_{\sigma}^{\mathcal{I}-\theta\beta}(W) = W$, whereas $\underline{\mathcal{H}}_{\sigma}^{\mathcal{I}-\theta\beta}(V \cap W) = \emptyset$ is a proper subset of $\underline{\mathcal{H}}_{\sigma}^{\mathcal{I}-\theta\beta}(V) \cap \underline{\mathcal{H}}_{\sigma}^{\mathcal{I}-\theta\beta}(W)$. Also, $\overline{\mathcal{H}}_{\sigma}^{\mathcal{I}-\theta\beta}(Y \cup Z) = \{a, b, x\}$, whereas $\overline{\mathcal{H}}_{\sigma}^{\mathcal{I}-\theta\beta}(Y) \cup \overline{\mathcal{H}}_{\sigma}^{\mathcal{I}-\theta\beta}(Z) = \{a, x\}$ is a proper subset of $\overline{\mathcal{H}}_{\sigma}^{\mathcal{I}-\theta\beta}(Y \cup Z)$. This means that rough set models provided in [16] violate some properties of the standard model of Pawlak, which disproves the third observation of Figure 1.

4. Conclusions

In this note, we have showed invalid results and relationships introduced in [16]. With the help of an illustrative example, we have demonstrated that Theorem 4.1, Corollary 4.1, and items (2) and (4) of Corollary 4.2 given in [16] are incorrect. Also, we have concluded that the rough set paradigms proposed by Hosny [11] and Nawar et al. [16] are independent of each other; that is, they are incomparable. Then, we pointed out the concrete relationships between these rough set models. Finally, we have elucidated that three observations on page 2494 of [16] about the given application are false, as well as emphasized that there is no preponderance for Nawar et al.'s approach [16] over Hosny's approach [11] and vice versa in terms of improving the approximation operators and reducing the size of uncertainty.

Author contributions

Tareq M. Al-shami: Conceptualization, Methodology, Validation, Formal analysis, Investigation, Writing-original draft, Writing-review and editing, Supervision. Mohammed M. Ali Al-Shamiri: Validation, Formal analysis, Investigation, Funding acquisition. Murad Arar: Validation, Formal analysis, Investigation, Funding acquisition. All authors have read and agreed to the published version of the manuscript.

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Conflict of interest

The authors declare that they have no competing interests.

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