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Research article

Novel linguistic q-rung orthopair fuzzy Aczel-Alsina aggregation operators for group decision-making with applications

Ghous Ali[1,*], **Kholood Alsager**[2,*] **and Asad Ali**[1]

[1] Department of Mathematics, Division of Science & Technology, University of Education, Lahore, Pakistan

[2] Department of Mathematics, College of Science, Qasim University, Buraydah, Saudi Arabia

* **Correspondence:** Email: ghous.ali@ue.edu.pk, ksakr@qu.edu.sa.

Abstract: In this article, we presented two novel approaches for group decision-making (GDM) that were derived from the initiated linguistic q-rung orthopair fuzzy Aczel-Alsina weighted arithmetic (Lq-ROFAAWA) aggregation operator (AgOp) using linguistic q-rung orthopair fuzzy numbers (Lq-ROFNs). To introduce these GDM techniques, we first defined new operational laws for Lq-ROFNs based on Aczel-Alsina t-norm and t-conorm. The developed scalar multiplication and addition operations of Lq-ROFNs addressed the limitations of operations when $q = 1$. The first proposed GDM methodology assumed that both experts' weights and attribute weights were fully known, while the second technique assumed that both sets of weights were entirely unknown. We also discussed properties of Lq-ROFNs under the Lq-ROFAAWA operators, such as idempotency, boundedness, and monotonicity. Furthermore, we solved problems related to environmental and economic issues, such as ranking countries by air pollution, selecting the best company for bank investments, and choosing the best electric vehicle design. Finally, we validated the proposed GDM approaches using three validity tests and performed a sensitivity analysis to compare them with preexisting models.

Keywords: linguistic q-rung orthopair fuzzy set; aggregation operator; Aczel-Alsina t-norm; air pollution; sensitivity analysis; decision-making
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1. Introduction

Nowadays, aggregation operators (AgOps) are playing a significant role as powerful mathematical tools for combining various inputs into an output, considering the uncertainty associated with these input values. When investigating environmental and economic issues, AgOps and their fuzzy extensions can help in finding solutions by integrating multiple vital factors and stakeholders' preferences. The revolutionized concept of fuzzy set theory was launched by Zadeh [1] in 1965 as a modification of crisp set theory. The fuzzy set theory has emerged as an effective mathematical tool in decision science, catalyzing improvements in various fields by enhancing our capacity to model, scrutinize, and control complicated systems involving uncertainty [2]. To date, numerous important studies have been presented about collective decision-making using the combination of fuzzy set and rough set [3]. For instance, Wang et al. [4] presented a hybrid decision-making model that addresses uncertainty and relationships across criteria by combining rough fuzzy judgments with Choquet-like integrals. Additionally, Wang and Zhang [5] proposed new methods of analysis for intuitionistic fuzzy covering-based rough sets utilizing the intuitionistic fuzzy granular matrix. Due to the lack of a non-membership part in fuzzy set theory, Atanassov [6] extended the study of fuzzy sets by adding a non-membership function called intuitionistic fuzzy set (IFS), which can tackle non-membership degrees. Later, Yager [7] extended the idea of IFSs and presented a new concept of Pythagorean fuzzy sets that enlarge the space of input membership and non-membership values, but the sum of their square is restricted by unity. Additionally, Yager [8] lifted this study to its peak by presenting the idea of generalized or q-rung orthopair fuzzy sets (q-ROFSs) in which the sum of q^{th} power of the membership and non-membership values is bounded by 1.

The fuzzy sets, IFSs, Pythagorean fuzzy sets, and q-ROFSs are quantitative mathematical tools. Besides, for the evaluation of alternatives in qualitative format, Zadeh [9] introduced the idea of linguistic variables (LVs). For instance, to precisely estimate the structure of any house, the terms extremely good, very good, good, fair, poor, etc. are used rather than numerical values. Following the concept of LVs by Zadeh, Chen et al. [10] introduced the idea of linguistic IFSs for dealing with linguistic IF numbers. To date, several significant studies have been reported for group decision-making based on linguistic IF numbers and other uncertainty theories. The process of collective decision-making in which a group of experts estimates the rank of given alternatives to choose the most suitable alternative from a group of options is called group decision-making (GDM), which is very significant in several fields [11]. For example, Akram et al. [12] introduced a GDM method by the fusion of 2-tuple linguistic Fermatean fuzzy sets with traditional MULTIMOORA (Multiple Objective Optimization on the basis of Ratio Analysis plus Full Multiplicative Form) technique, and explored its application in urban quality management. Seikh and Mandal [13] presented an interval-valued Fermatean fuzzy Dombi AgOps-based GDM method using the concepts of the PROMETHEE II technique under SWARA (Step-wise Weight Assessment Ratio Analysis) method. Liu et al. [14] integrated the evidence theory of Dempster-Shafer with linguistic IF numbers for GDM (see also, Yuan et al. [15]). Gou et al. [16] improved the classical ORESTE (Organization, Rangement Et Synthese De Donnes Relationnelles) method with linguistic preference orderings for GDM in medical domain. Cheng et al. [17] presented a model for large-scale GDM. Seikh and Mandal [18, 19] introduced q-ROF Archimedean and Frank AgOps and explored their applications in selecting the best site for a software house and the best way for industrialization, respectively. Garg and Kumar [20] proposed an AgOps-based GDM method using a possibility degree measure of linguistic IF numbers. Besides, Garg and Kumar [21] introduced another AgOps-based GDM approach under the set-pair investigation of linguistic IF numbers (see also, Garg et al. [22]). Kumar and Chen [23] proposed an improvement in the linguistic interval-valued IF averaging AgOps and explored its application to GDM. Moreover, Kumar and Chen [24] suggested an advanced multiple-criteria GDM method using linguistic IF averaging AgOps for handling linguistic IF information. Liu et al. [25] introduced an emergency GDM method based on linguistic IF sets. Gou et al. [26] developed an improved VIKOR method based on the probabilistic double hierarchy linguistic term set, and explored its applications in smart healthcare domain. Tang and Meng [27] proposed linguistic IF Hamacher AgOps and studied their application in GDM. Kumar and Chen [28] introduced a GDM approach based on Yager's operation for the aggregation of linguistic intuitionistic fuzzy information. Verma and Agarwal [29] developed an AgOps-based generalized GDM approach under a linguistic interval-valued Pythagorean fuzzy environment. Fahmi et al. [30] introduced a novel GDM method based on cubic linguistic hesitant fuzzy AgOps. Kumar and Chen [31] suggested novel q-ROF Yager prioritized weighted arithmetic AgOPs with their application in GDM. Cheng et al. [32] proposed a large-scale GDM model that integrates risk attitudes by addressing the uncertainties in group decision environments.

After the successful production and implementation of various linguistic IF set-based AgOps, a number of scholars were attracted to the study of linguistic q-ROFS-based AgOps [33]. For instance, Lin et al. [34] introduced a powerful generalization of linguistic IF sets called linguistic q-ROFSs (Lq-ROFSs) and their related properties. Liu et al. [35] proposed a novel GDM method using power Bonferroni mean-based AgOps for the aggregation of Lq-ROFNs. Additionally, they [36] introduced another GDM approach based on linguistic q-ROF power Muirhead mean AgOps using the entropy weight strategy. Ranjan et al. [37] proposed novel Archimedean operations-based AgOps using probabilistic linguistic q-ROFSs for GDM (see also Wang et al. [38]; Ali et al. [39]). Deb et al. [40] developed linguistic q-ROF prioritized AgOps under Hamacher operations and explored their applications to multiple attribute GDM. Liu et al. [41] established a new GDM model based on linguistic q-ROF generalized point-weighted AgOps. Akram et al. [42] developed a novel framework under Einstein's operations for linguistic q-ROF numbers and solved certain daily life applications in the mobile industry. Jana et al. [43] launched a new linguistic q-ROF Choquet integral method and discussed the evaluation criteria of sustainable strategies for urban parcel delivery. Naz et al. [44] presented a GDM model by incorporating power Muirhead mean operations for the aggregation of 2-tuple linguistic q-ROF information.

In the beginning of the 1980s, Aczel-Alsina t-norm (TN) and t-conorm (TCN) were first proposed by Aczel and Alsina [45], which are updated versions of the algebraic TN and TCN, respectively. Inspection of the literature reveals that more accurate decision-making is performed by the Aczel-Alsina's TN and TCN, in comparison to other TNs and TCNs. To justify this, Farahbod and Efekhari [46] compared various TN operators for classification issues. Aczel-Alsina operators are superior to the other operators due to the least error. Consequently, researchers have concentrated on Aczel-Alsina TN and TCN-based AgOps for each of the theories mentioned above. For example, Mahmood et al. [47] discussed the use of Aczel-Alsina TN and TCN-based AgOps in the bipolar complex fuzzy environment and studied their application in choosing the optimal operating system (see also, Mahmood and Ali [48]). Garg et al. [49] presented Choquet integral-based Aczel-Alsina AgOps using interval-valued IFSs, and studied their application in decision-making. Liu et al. [50] introduced prioritized AgOps using Aczel-Alsina TN and TCN operations for aggregating complex IF information. Recently, Rehman et al. [51] initiated m-polar fuzzy Aczel-Alsina AgOps and investigated their applications to identify suitable sites for wind power and desalination plants. Ali et al. [52] established a novel decision-making method under Yager's TN and TCN for the aggregation of m-polar fuzzy information. Additionally, Ali et al. [53] presented a GDM strategy using Aczel-Alsina's operations and IF soft information. For more contributions related Aczel-Alsina's TN and TCN, the readers are referred to [54, 55]. It can be easily observed that Lq-ROF information is not aggregated using Aczel-Alsina's operations to date. Motivated by all these concerns, in this study, we propose two novel GDM approaches based on linguistic q-ROF sets with Aczel-Alsina TN and TCN that are omitted in the literature.

The Following are our major motivations for this study:

1) In group decision-making (or GDM), Lq-ROFS-based models play a crucial role, as they allow for linguistic assessments, enabling better management of the ambiguity and uncertainty inherent in the information.

2) Compared to linguistic IFS theory, Lq-ROFSs provide greater flexibility in modeling uncertainty and reluctance in decision-making. They improve the correctness of decisions made in complicated situations by enabling wider representation of both membership and non-membership degrees.

3) To achieve a more flexible and nuanced consensus, the Aczel-Alsina TN and TCN-based AgOps are widely used in GDM [46, 48], which effectively balances a range of inputs by adjusting the influence of extreme values. The accuracy of collective decisions is increased by its parametric structure, which enables effortless changes between challenging and comfortable aggregation.

4) The aggregation of Lq-ROFNs is unattended using the exceptional aggregation features of Aczel-Alsina TN and TCN. Motivated by these settings, two novel GDM methods are initiated by combining the aggregation features of Aczel-Alsina TN and TCN with Lq-ROFSs.

Moreover, we investigate some limitations in Kumar and Chen's [24] scalar multiplication and addition operations of linguistic IF sets in some specific situations. Then, we suggest new addition and scalar multiplication operations of Lq-ROFNs based on Aczel Alsina's TN and TCN [45]. When $q = 1$, the suggested addition and multiplication operations of Lq-ROFNs can overcome the deficiencies of Kumar and Chen's [24] multiplication and addition operations. For a more detailed review of AgOps and GDM approaches, the readers are referred to [56].

The significant contributions of this study are given below:

1) A new kind of averaging weighted AgOps based on Aczel Alsina's TN and TCN operations is initiated for the aggregation of Lq-ROFNs.

2) Some basic properties of suggested AgOps are investigated.

3) Next, two novel GDM methodologies based on the suggested Aczel Alsina's operations-based AgOps are developed to solve decision-making problems. The first GDM approach considers known weights in the aggregation process, while the second GDM method deals with unknown weights of experts and attributes.

4) To validate the applicability scope of our suggested GDM method, five numerical examples are solved using our suggested algorithms for each methodology.

5) Finally, to verify the authenticity and reliability of our proposed approach, a brief comparison is depicted between the suggested and existing approaches, and between the different values of q.

The remaining sections are structured as follows: In Section 2, we provide fundamental notions, including the linguistic term set, linguistic q-ROFS, Aczel-Alsina TN, and TCN. In Section 3, we develop some basic operations for linguistic q-ROFSs based on Aczel-Alsina TN and TCN. Subsequently, based on these operations, the linguistic q-ROF Aczel-Alsina weighted averaging (Lq-ROFAAWA) operators, along with their basic operations and properties, are presented in this section. In Section 4, we propose two GDM approaches considering both known and unknown weights for the aggregation of linguistic q-ROF numbers based on the initiated (Lq-ROFAAWA) operators and successfully implementing them in different practical situations. Section 5 validates our introduced approaches using existing tests and also compares them with certain preexisting methods. Finally, in Section 6, we provide concluding arguments and future directions.

2. Preliminaries

In this section, we review some fundamental notions related to a linguistic term set, q-ROFSs, and Aczel-Alsina TN and TCN. Further, we describe shortcomings in the multiplication and addition operations presented by Kumar and Chen [28].

Definition 2.1. [10, 21] Suppose $\mathcal{L} = \{\mathfrak{a}_0, \mathfrak{a}_1, \ldots, \mathfrak{a}_\mathfrak{h}\}$ is a linguistic term set (L$^\mathrm{T}$S) having an odd cardinal number, here $\mathfrak{h}$ is an even positive integer, and $\mathfrak{a}_t$ is a reference value for the corresponding linguistic variable.

The following characteristics hold for a L$^\mathrm{T}$S $\mathcal{L} = \{\mathfrak{a}_0, \mathfrak{a}_1, ..., \mathfrak{a}_\mathfrak{h}\}$ where $\mathfrak{h}$ is an even positive integer [10, 21, 58]:

1) $\neg(\mathfrak{a}_\mathfrak{k}) = \mathfrak{a}_{\mathfrak{h}-\mathfrak{k}}$,

2) $\mathfrak{a}_\mathfrak{k} \leq \mathfrak{a}_\mathfrak{t} \Leftrightarrow \mathfrak{k} \leq \mathfrak{t}$,

3) $\min(\mathfrak{a}_\mathfrak{k}, \mathfrak{a}_\mathfrak{t}) = \mathfrak{a}_\mathfrak{t} \Leftrightarrow \mathfrak{a}_\mathfrak{k} \geq \mathfrak{a}_\mathfrak{t}$,

4) $\max(\mathfrak{a}_\mathfrak{k}, \mathfrak{a}_\mathfrak{t}) = \mathfrak{a}_\mathfrak{k} \Leftrightarrow \mathfrak{a}_\mathfrak{k} \geq \mathfrak{a}_\mathfrak{t}$.

Now, we define a continuous L$^\mathrm{T}$S (CL$^\mathrm{T}$S) $\mathcal{L}_{[0,\mathfrak{h}]}$ as below [59]:

$$\mathcal{L}_{[0,\mathfrak{h}]} = \{\mathfrak{a}_\mathfrak{z} \mid \mathfrak{a}_0 \leq \mathfrak{a}_\mathfrak{z} \leq \mathfrak{a}_\mathfrak{h}\}, \tag{2.1}$$

where $\mathfrak{h}$ is an even positive integer.

Definition 2.2. [10] Suppose $\mathcal{L}_{[0,\mathfrak{h}]} = \{\mathfrak{a}_\mathfrak{z} \mid \mathfrak{a}_0 \leq \mathfrak{a}_\mathfrak{z} \leq \mathfrak{a}_\mathfrak{h}\}$ is a CL$^\mathrm{T}$S, where $\mathfrak{h}$ is an even positive integer and suppose $\mathcal{U} = \{\mathfrak{u}_1, \mathfrak{u}_2, ..., \mathfrak{u}_n\}$ is a finite universal set. Then, the mathematical expression of linguistic intuitionistic fuzzy set (LIFS) $\mathcal{M}$ in $\mathcal{U}$ is provided by

$$\mathcal{M} = \{\langle \mathfrak{u}, \mathfrak{a}_{\xi(\mathfrak{u})}, \mathfrak{a}_{\eta(\mathfrak{u})}\rangle \mid \mathfrak{u} \in \mathcal{U}\}, \tag{2.2}$$

where $\mathfrak{a}_{\xi(\mathfrak{u})}$ and $\mathfrak{a}_{\eta(\mathfrak{u})}$ represent the MD and NMD of object $\mathfrak{u} \in \mathcal{M}$, respectively. Note that $\mathfrak{a}_{\xi(\mathfrak{u})}, \mathfrak{a}_{\eta(\mathfrak{u})} \in \mathcal{L}_{[0,\mathfrak{h}]}$ and clearly the MD $\mathfrak{a}_{\xi(\mathfrak{u})}$ and NMD $\mathfrak{a}_{\eta(\mathfrak{u})}$ satisfy the condition $0 \leq \xi(\mathfrak{u}) + \eta(\mathfrak{u}) \leq \mathfrak{h}$.

The pair $\langle \mathfrak{a}_{\xi(\mathfrak{u})}, \mathfrak{a}_{\eta(\mathfrak{u})}\rangle$ in LIFS $\mathcal{M} = \{\langle \mathfrak{u}, \mathfrak{a}_{\xi(\mathfrak{u})}, \mathfrak{a}_{\eta(\mathfrak{u})} \mid \mathfrak{u} \in \mathcal{U}\}$ denotes a LIFN. Let $\Gamma_{[0,\mathfrak{h}]}$ represent a collection of LIFNs provided in the CL$^\mathrm{T}$S $\mathcal{L}_{[0,\mathfrak{h}]} = \{\mathfrak{a}_\mathfrak{z} \mid \mathfrak{a}_0 \leq \mathfrak{a}_\mathfrak{z} \leq \mathfrak{a}_\mathfrak{h}\}$, where $\mathfrak{h}$ is an even positive integer.

Definition 2.3. [35] Suppose $\mathcal{L}_{[0,\mathfrak{h}]} = \{\mathfrak{a}_{\mathfrak{z}} \mid \mathfrak{a}_0 \le \mathfrak{a}_{\mathfrak{z}} \le \mathfrak{a}_{\mathfrak{h}}\}$ is a CL^{T}S, where $\mathfrak{h}$ is an even positive integer and suppose $\mathcal{U} = \{\mathfrak{u}_1, \mathfrak{u}_2, ..., \mathfrak{u}_n\}$ is a finite universal set. Then, the mathematical expression of linguistic q-rung orthopair fuzzy set (Lq-ROFS) $\mathcal{Q}$ on $\mathcal{U}$ is given as:

$$\mathcal{Q} = \{\langle \mathfrak{u}, \mathfrak{a}_{\xi(\mathfrak{u})}, \mathfrak{a}_{\eta(\mathfrak{u})}\rangle \mid \mathfrak{u} \in \mathcal{U}\}, \tag{2.3}$$

where $\mathfrak{a}_{\xi(\mathfrak{u})}$ and $\mathfrak{a}_{\eta(\mathfrak{u})}$ are respectively denoted the MD and NMD of element $\mathfrak{u}$ belong to the Lq-ROFS $\mathcal{Q}$, respectively. Here $\mathfrak{a}_{\xi(\mathfrak{u})}, \mathfrak{a}_{\eta(\mathfrak{u})} \in \mathcal{L}_{[0,\mathfrak{h}]}$ that satisfy the conditions $0 \le \xi(\mathfrak{u}) \le \mathfrak{h}$, $0 \le \eta(\mathfrak{u}) \le \mathfrak{h}$ and $0 \le (\xi(\mathfrak{u}))^q + (\eta(\mathfrak{u}))^q \le \mathfrak{h}^q$ where $q \ge 1$.

The pair $\langle \mathfrak{a}_{\xi(\mathfrak{u})}, \mathfrak{a}_{\eta(\mathfrak{u})}\rangle$ in Lq-ROFS $\mathcal{Q} = \{\langle \mathfrak{u}, \mathfrak{a}_{\xi(\mathfrak{u})}, \mathfrak{a}_{\eta(\mathfrak{u})}\rangle \mid \mathfrak{u} \in \mathcal{U}\}$ denotes a linguistic q-ROF number (Lq-ROFN). Let $\Gamma_{[0,\mathfrak{h}]}$ represent a set of Lq-ROFNs provided on the CL^{T}S $\mathcal{L}_{[0,\mathfrak{h}]} = \{\mathfrak{a}_{\mathfrak{z}} \mid \mathfrak{a}_0 \le \mathfrak{a}_{\mathfrak{z}} \le \mathfrak{a}_{\mathfrak{h}}\}$ where $\mathfrak{h}$ is an even positive integer. Let $a_{\pi}(\mathfrak{u}) = \mathfrak{a}_{(\mathfrak{h}^q - \xi^q - \eta^q)^{1/q}}$, then the term $\mathfrak{a}_{\pi}(\mathfrak{u})$ is the hesitancy degree (HD) of the element $\mathfrak{u}$ being a member of the Lq-ROFS $\mathcal{Q}$.

In the following, we recall the notions of score and accuracy functions for Lq-ROFNs.

Definition 2.4. [35] Given a CL^{T}S $\mathcal{L}_{[0,\mathfrak{h}]} = \{\mathfrak{a}_{\mathfrak{z}} \mid \mathfrak{a}_0 \le \mathfrak{a}_{\mathfrak{z}} \le \mathfrak{a}_{\mathfrak{h}}\}$ and an Lq-ROFN $\sigma = (\mathfrak{a}_{\xi}, \mathfrak{a}_{\eta})$ and $(\mathfrak{a}_{\xi}, \mathfrak{a}_{\eta}) \in \mathcal{L}$, then the value of score function of Lq-ROFN $\langle \mathfrak{a}_{\xi_{\mathfrak{u}}}, \mathfrak{a}_{\eta_{\mathfrak{u}}}\rangle$ is defined as:

$$\mathcal{D} = \mathfrak{a}_{\left((\mathfrak{h}^q + \eta^q - \xi^q)/2\right)^{1/q}},$$

and the value of accuracy function is defined as:

$$\mathcal{J} = \mathfrak{a}_{(\xi^q + \eta^q)^{1/q}}.$$

Then, the comparison of any two Lq-ROFNs σ_1 and σ_2 is defined as follows [35]:

1) If $\mathcal{D}(\sigma_1) \ge \mathcal{D}(\sigma_2)$, then $\sigma_1 \ge \sigma_2$, in other words the Lq-ROFN σ_1 is considered to be superior to the Lq-ROFN σ_2.
2) If $\mathcal{D}(\sigma_1) = \mathcal{D}(\sigma_2)$, then their exact ranking is computed using the accuracy function as below:
   (a) If $\mathcal{J}(\sigma_1) > \mathcal{J}(\sigma_2)$, then $\sigma_1 \ge \sigma_2$,
   (b) If $\mathcal{J}(\sigma_1) = \mathcal{J}(\sigma_2)$, then $\sigma_1 = \sigma_2$.

Definition 2.5. [45] For $\lambda \in [0, \infty]$ and $\sigma_1, \sigma_2 \in [0, 1]$, the Aczel-Alsina TN ($\mathcal{AA} - TN$) $\mathcal{T}_A^{\lambda}(\sigma_1, \sigma_2)$ and Aczel-Alsina TCN ($\mathcal{AA} - TCN$) $\mathcal{S}_A^{\lambda}(\sigma_1, \sigma_2)$ are respectively given as:

$$\mathcal{T}_A^{\lambda}(\sigma_1, \sigma_2) = \begin{cases} \mathcal{T}_{dra}(\sigma_1, \sigma_2), & \text{if } \lambda = 0, \\ \min(\sigma_1, \sigma_2), & \text{if } \lambda = \infty, \\ 1 - \exp^{-((-\log \sigma_1)^{\lambda} + (-\log \sigma_2)^{\lambda})^{\frac{1}{\lambda}}}, & \text{otherwise.} \end{cases}$$

$$\mathcal{S}_A^{\lambda}(\sigma_1, \sigma_2) = \begin{cases} \mathcal{S}_{dra}(\sigma_1, \sigma_2), & \text{if } \lambda = 0, \\ \max(\sigma_1, \sigma_2), & \text{if } \lambda = \infty, \\ 1 - \exp^{-((-\log(1-\sigma_1))^{\lambda} + (-\log(1-\sigma_2))^{\lambda})^{\frac{1}{\lambda}}}, & \text{otherwise.} \end{cases}$$

Definition 2.6. [24] Suppose $\sigma_1 = \langle \mathfrak{a}_{\xi_1}, \mathfrak{a}_{\eta_1} \rangle, \sigma_2 = \langle \mathfrak{a}_{\xi_2}, \mathfrak{a}_{\eta_2} \rangle$ and $\sigma = \langle \mathfrak{a}_{\xi}, \mathfrak{a}_{\eta} \rangle$ be Lq-ROFNs, where $\sigma_1, \sigma_2, \sigma \in \Gamma_{[0,\mathfrak{h}]}$ and $\mathfrak{h}$ is an even positive integer. Kumar and Chen's [24] scalar product and addition operations of the Lq-ROFNs $\sigma_1 = \langle a_{\xi_1}, a_{\eta_1} \rangle, \sigma_2 = \langle \mathfrak{a}_{\xi_2}, \mathfrak{a}_{\eta_2} \rangle$ and $\sigma = \langle \mathfrak{a}_{\xi}, \mathfrak{a}_{\eta} \rangle$ are defined as follows:

$$\sigma_1 \bigoplus \sigma_2 = \left\langle \mathfrak{a}_{\mathfrak{h}\left(\sqrt[q]{\frac{1}{\varepsilon}(1-\prod_{i=1}^{2}(1-\varepsilon(\frac{\xi_i}{\mathfrak{h}})^q))}\right)}, \mathfrak{a}_{\mathfrak{h}\left(\sqrt[q]{(1-\frac{1}{\varepsilon}(1-\prod_{i=1}^{2}(1-\varepsilon(1-(\frac{\eta_i}{\mathfrak{h}})^q))))}\right)} \right\rangle,$$

$$\psi\sigma = \left\langle \mathfrak{a}_{\mathfrak{h}\sqrt[q]{\frac{1}{\varepsilon}(1-(1-\varepsilon(\frac{\xi}{\mathfrak{h}})^q)^{\psi})}}, \ \mathfrak{a}_{\mathfrak{h}\sqrt[q]{1-\frac{1}{\varepsilon}(1-(1-\varepsilon(1-(\frac{\eta}{\mathfrak{h}})^q))^{\psi})}} \right\rangle,$$

where $\psi > 0$ and $0 < \varepsilon < 1$.

Some shortcomings of Kumar and Chen's [28] addition and scalar multiplication operations are shown in the following examples, respectively.

Example 2.1. Suppose $\sigma_1 = \langle \mathfrak{a}_{7.5}, \mathfrak{a}_{0.35} \rangle$, and $\sigma_2 = \langle \mathfrak{a}_{7.4}, \mathfrak{a}_{0.5} \rangle$ be Lq-ROFNs, where $\sigma_1, \sigma_2 \in \Gamma_{[0,8]}$ and $h = 8$. By Kumar and Chen's addition operation defined in Definition 2.6 of Lq-ROFNs $\sigma_1 = \langle a_{\xi_1}, a_{\eta_1} \rangle$, and $\sigma_2 = \langle \mathfrak{a}_{\xi_2}, \mathfrak{a}_{\eta_2} \rangle$ with q=1 and $\varepsilon = 0.99$, we get

$$\sigma_1 \bigoplus \sigma_2 = \left\langle \mathfrak{a}_{\mathfrak{h}\left(\sqrt[q]{\frac{1}{\varepsilon}(1-\prod_{i=1}^{2}(1-\varepsilon(\frac{\xi_i}{\mathfrak{h}})^q))}\right)}, \mathfrak{a}_{\mathfrak{h}\left(\sqrt[q]{(1-\frac{1}{\varepsilon}(1-\prod_{i=1}^{2}(1-\varepsilon(1-(\frac{\eta_i}{\mathfrak{h}})^q))))}\right)} \right\rangle.$$

$$\sigma_1 \bigoplus \sigma_2 = \left\langle \mathfrak{a}_{8\left(\frac{1}{0.99}(1-(1-0.99(\frac{7.5}{8}))(1-0.99(\frac{7.4}{8}))))\right)}, \right.$$
$$\left. \mathfrak{a}_{8\left((1-\frac{1}{0.99}(1-(1-0.99(1-(0.358)(1-0.99(1-\frac{0.5}{8})))\right)} \right\rangle$$
$$= \langle \mathfrak{a}_{8.03} \mathfrak{a}_{-0.04} \rangle.$$

The MD of accumulated Lq-ROFN $\langle \mathfrak{a}_{8.03}, \mathfrak{a}_{-0.04} \rangle$ with $q = 1$ is $\mathfrak{a}_{8.03} > \mathfrak{a}_8$, which is not suitable because the MD of this Lq-ROFN with $q = 1$ must be lies in the $\Gamma_{[0,8]}$ and the NMD of this Lq-ROFN with $q = 1$ must be lies in the $\Gamma_{[0,8]}$ but $\mathfrak{a}_{-0.04} < \mathfrak{a}_0$. Hence, the Kumar and Chen's addition operation [24] has the drawback that is observed in the above example.

Example 2.2. Suppose $\sigma = \langle \mathfrak{a}_{5.20}, \mathfrak{a}_{1.60} \rangle$ is a Lq-ROFN with $q = 1$, where $\sigma \in \Gamma_{[0,8]}$ and $h = 8$. Now, the Kumar and Chen's [24] scalar product operation defined in Definition 2.6 of Lq-ROFN with $q = 1, \psi = 5$ and $\varepsilon = 0.99$ is defined as follows:

$$\psi\sigma = \left\langle \mathfrak{a}_{\mathfrak{h}\sqrt[q]{\frac{1}{\varepsilon}(1-(1-\varepsilon(\frac{\xi}{\mathfrak{h}})^q)^{\psi})}}, \ \mathfrak{a}_{\mathfrak{h}\sqrt[q]{1-\frac{1}{\varepsilon}(1-(1-\varepsilon(1-(\frac{\eta}{\mathfrak{h}})^q))^{\psi})}} \right\rangle,$$
$$\psi\sigma = \left\langle \mathfrak{a}_{8(\frac{1}{0.99}(1-(1-0.99(\frac{5.20}{8}))^5))}, \ \mathfrak{a}_{8(1-\frac{1}{0.99}(1-(1-0.99(1-(\frac{1.60}{8})))^5))} \right\rangle$$
$$= \langle \mathfrak{a}_{8.03}, \mathfrak{a}_{-0.07} \rangle.$$

The MD of accumulated Lq-ROFN $\langle \mathfrak{a}_{8.03}, \mathfrak{a}_{-0.07} \rangle$ with $q = 1$ is $\mathfrak{a}_{8.03} > \mathfrak{a}_8$, which is not suitable because the MD of this Lq-ROFN with $q = 1$ should be lies in the $\Gamma_{[0,8]}$ and the NMD of this Lq-ROFN with $q = 1$ should be lies in the $\Gamma_{[0,8]}$ but $\mathfrak{a}_{-0.07} < \mathfrak{a}_0$. Hence, Kumar and Chen's [24] operation has some drawbacks which are mentioned in the above example.

3. Operations for Lq-ROFNs under Aczel Alsina's TN and TCN

In this section, we first develop basic operations based on Aczel Alsina's TN and TCN for Lq-ROFNs, then, by applying them on Examples 2.1 and 2.2 as provided in previous section (see following Examples 3.1 and 3.2), we verify that the proposed operations overcome the difficulties in Kumar and Chen's [24] scalar product and addition operations. Moreover, we present linguistic q-ROF Aczel-Alsina weighted arithmetic AgOps with their basic properties.

Definition 3.1. Suppose $\sigma_1 = \langle \mathfrak{a}_{\xi_1}, \mathfrak{a}_{\eta_1} \rangle, \sigma_2 = \langle \mathfrak{a}_{\xi_2}, \mathfrak{a}_{\eta_2} \rangle$ and $\sigma = \langle \mathfrak{a}_{\xi}, \mathfrak{a}_{\eta} \rangle$ are Lq-ROFNs, where $\sigma_1, \sigma_2, \sigma \in \Gamma_{[0,\mathfrak{h}]}$ and $\mathfrak{h}$ is an even positive integer. The suggested addition and scalar multiplication operation of the Lq-ROFNs $\sigma_1 = \langle a_{\xi_1}, a_{\eta_1} \rangle, \sigma_2 = \langle \mathfrak{a}_{\xi_2}, \mathfrak{a}_{\eta_2} \rangle$ and $\sigma = \langle \mathfrak{a}_{\xi}, \mathfrak{a}_{\eta} \rangle$ are defined as follows:

$$\sigma_1 \bigoplus \sigma_2 = \left\langle \mathfrak{a}_{\mathfrak{h}\left(\sqrt[q]{(1-\exp^{-((-\log(1-(\frac{\xi_{\sigma_1}}{\mathfrak{h}})^q))^\lambda+(-\log(1-(\frac{\xi_{\sigma_2}}{\mathfrak{h}})^q))^\lambda)^{1/\lambda})}}\right)}, \mathfrak{a}_{\mathfrak{h}\left(\exp^{-((-\log(\frac{\eta_{\sigma_1}}{\mathfrak{h}}))^\lambda+(-\log(\frac{\eta_{\sigma_2}}{\mathfrak{h}}))^\lambda)^{1/\lambda}}\right)} \right\rangle, \tag{3.1}$$

$$\sigma_1 \bigotimes \sigma_2 = \left\langle \mathfrak{a}_{\mathfrak{h}\left(\exp^{-((-\log(\frac{\xi_{\sigma_1}}{\mathfrak{h}}))^\lambda+(-\log(\frac{\xi_{\sigma_2}}{\mathfrak{h}}))^\lambda)^{1/\lambda}}\right)}, \mathfrak{a}_{\mathfrak{h}\left(\sqrt[q]{(1-\exp^{-((-\log(1-(\frac{\eta_{\sigma_1}}{\mathfrak{h}})^q))^\lambda+(-\log(1-(\frac{\eta_{\sigma_2}}{\mathfrak{h}})^q))^\lambda)^{1/\lambda})}}\right)} \right\rangle, \tag{3.2}$$

$$\psi\sigma = \left\langle \mathfrak{a}_{\mathfrak{h}\sqrt[q]{1-\exp^{-(\psi(-\log(1-(\frac{\xi_\sigma}{\mathfrak{h}})^q))^\lambda)^{1/\lambda}}}}, \mathfrak{a}_{\mathfrak{h}(\exp^{-(\psi(-\log(\frac{\eta_\sigma}{\mathfrak{h}}))^\lambda)^{1/\lambda}})} \right\rangle, \tag{3.3}$$

$$\sigma^\psi = \left\langle \mathfrak{a}_{\mathfrak{h}(\exp^{-(\psi(-\log(\frac{\xi_\sigma}{\mathfrak{h}}))^\lambda)^{1/\lambda}})}, \mathfrak{a}_{\mathfrak{h}\sqrt[q]{1-\exp^{-(\psi(-\log(1-(\frac{\eta_\sigma}{\mathfrak{h}})^q))^\lambda)^{1/\lambda}}}} \right\rangle, \tag{3.4}$$

where $\lambda > 0$ and $\psi > 0$.

Example 3.1. Let $\sigma_1 = \langle 7.50,\ 0.35 \rangle$ and $\sigma_2 = \langle 7.40,\ 0.50 \rangle$ be two Lq-ROFNs, where $\sigma_1, \sigma_2 \in \Gamma_{[0,8]}$. Let $\lambda = 2$, and $q = 1$. Based on the suggested operations provided in Definition 3.1, we get

$$\sigma_1 \bigoplus \sigma_2 = \left\langle \mathfrak{a}_{\mathfrak{h}\left(\sqrt[q]{(1-\exp^{-((-\log(1-(\frac{\xi_{\sigma_1}}{\mathfrak{h}})^q))^\lambda+(-\log(1-(\frac{\xi_{\sigma_2}}{\mathfrak{h}})^q))^\lambda)^{1/\lambda})}}\right)}, \mathfrak{a}_{\mathfrak{h}\left(\exp^{-((-\log(\frac{\eta_{\sigma_1}}{\mathfrak{h}}))^\lambda+(-\log(\frac{\eta_{\sigma_2}}{\mathfrak{h}}))^\lambda)^{1/\lambda}}\right)} \right\rangle,$$

$$\sigma_1 \bigoplus \sigma_2 = \left\langle \mathfrak{a}_{8\left((1-\exp^{-((-\log(1-\frac{7.5}{8}))^2+(-\log(1-\frac{7.4}{8})))^2)^{1/2})}\right)}, \right.$$

$$\mathfrak{a}_{8\left(\exp^{-\sqrt{(-\log(\frac{0.35}{8}))^2+(-\log(\frac{0.5}{8}))^2}}\right)}\Big\rangle,$$
$$= \langle \mathfrak{a}_{7.82}\mathfrak{a}_{0.12}\rangle.$$

The above Example 3.1 shows that the suggested addition operation of Lq-ROFNs based on the Aczel Alsina's TN and TCN overcomes the deficiencies of addition operation proposed by Kumar and Chen [24] for Lq-ROFNs in Example 2.1 with $q = 1$.

Example 3.2. Let $\sigma = \langle 5.20,\ 1.60\rangle$ be a Lq-ROFN where $\sigma \in \Gamma_{[0,8]}$. Let $\lambda = 2$, $\psi = 5$ and $q = 1$. Using the suggested scalar multiplication operation given in Definition 3.1, we get

$$\psi\sigma = \left\langle \mathfrak{a}_{\mathfrak{h}\sqrt[q]{1-\exp^{-(\psi(-\log(1-(\frac{\xi_\sigma}{\mathfrak{h}})^q))^\lambda)^{1/\lambda}}}},\ \mathfrak{a}_{\mathfrak{h}(\exp^{-(\psi(-\log(\frac{\eta_\sigma}{\mathfrak{h}}))^\lambda)^{1/\lambda}})}\right\rangle,$$
$$5\sigma = \left\langle \mathfrak{a}_{8(1-\exp^{-(5(-\log(1-\frac{5.20}{8}))^2)^{1/2}})},\ \mathfrak{a}_{\mathfrak{h}(\exp^{-(5(-\log(\frac{1.60}{8}))^2)^{1/2}})}\right\rangle,$$
$$= \langle \mathfrak{a}_{7.24}, \mathfrak{a}_{0.21}\rangle.$$

The above Example 3.2 shows that the suggested concept of scalar product with Lq-ROFNs based on the Aczel Alsina's TN and TCN overcomes the difficulties of Kumar and Chen's [24] scalar product of Lq-ROFNs as investigated in Example 2.2 with $q = 1$.

Theorem 3.1. Suppose $\sigma_1 = \langle \mathfrak{a}_{\xi_1}, \mathfrak{a}_{\eta_1}\rangle$, $\sigma_2 = \langle \mathfrak{a}_{\xi_2}, \mathfrak{a}_{\eta_2}\rangle$ and $\sigma = \langle \mathfrak{a}_{\xi}, \mathfrak{a}_{\eta}\rangle$ are Lq-ROFNs with $\sigma_1, \sigma_2, \sigma \in \Gamma_{[0,\mathfrak{h}]}$ and $\mathfrak{h}$ is an even positive integer. The suggested addition and scalar product of Lq-ROFkNs defined in Definition 3.1 verify the following conditions:

1) $\sigma_1 \bigoplus \sigma_2 = \sigma_2 \bigoplus \sigma_1$,
2) $\psi(\sigma_1 \bigoplus \sigma_2) = \psi\sigma_1 \bigoplus \psi\sigma_2$,
3) $(\psi_1 \bigoplus \psi_2)\sigma = \psi_1\sigma \bigoplus \psi_2\sigma$.

3.1. The suggested linguistic q-rung orthopair fuzzy Aczel-Alsina weighted arithmetic aggregation operator of linguistic q-rung orthopair fuzzy numbers

Now, we introduce the Lq-ROFAAWA operator based on suggested addition and scalar product of Lq-ROFNs.

Definition 3.2. Suppose $\sigma_1 = \langle \mathfrak{a}_{\xi_1}, \mathfrak{a}_{\eta_1}\rangle$, $\sigma_2 = \langle \mathfrak{a}_{\xi_2},\ \mathfrak{a}_{\eta_2}\rangle\ldots$, and $\sigma_t = \langle \mathfrak{a}_{\xi_t},\ \mathfrak{a}_{\eta_t}\rangle$ are Lq-ROFNs, where $\sigma_1, \sigma_2, \ldots, \sigma_t \in \Gamma_{[0,\mathfrak{h}]}$ and $\mathfrak{h}$ is an even positive integer. The suggested Lq-ROFAAWA AgOp is defined as:

$$Lq-ROFAAWA(\sigma_1, \sigma_2, \ldots, \sigma_t) = \mathfrak{w}_1\sigma_1 \bigoplus \mathfrak{w}_2\sigma_2 \bigoplus \ldots \bigoplus \mathfrak{w}_t\sigma_t$$
$$= \left\langle \mathfrak{a}_{\mathfrak{h}\sqrt[q]{1-\exp^{-(\sum_{r=1}^{t}\mathfrak{w}_r(-\log(1-(\frac{\xi_{\sigma_r}}{\mathfrak{h}})^q))^\lambda)^{1/\lambda}}}},\ \mathfrak{a}_{\mathfrak{h}\exp^{-(\sum_{r=1}^{t}\mathfrak{w}_r(-\log(\frac{\eta_{\sigma_r}}{\mathfrak{h}}))^\lambda)^{1/\lambda}}}\right\rangle, \tag{3.5}$$

where $\mathfrak{w}_r$ represents the weight for each Lq-ROFN σ_t, $\mathfrak{w}_r \in [0, \mathfrak{h}]$, $r = 1, 2, \ldots, t$, with $\sum_{r=1}^{t} \mathfrak{w}_r = 1$ and $\lambda > 0$.

Theorem 3.2. The aggregated value of the Lq-ROFNs $\sigma_1 = \langle \mathfrak{a}_{\xi_1}, \mathfrak{a}_{\eta_1} \rangle$, $\sigma_2 = \langle \mathfrak{a}_{\xi_2}, \ \mathfrak{a}_{\eta_2} \rangle \ldots$, $\sigma_t = \langle \mathfrak{a}_{\xi_t}, \ \mathfrak{a}_{\eta_t} \rangle$ using the suggested Lq-ROFAAWA AgOp defined in Eq (3.5) is a Lq-ROFN, where $\sigma_1, \sigma_2, \ldots, \sigma_t \in \Gamma_{[0,h]}$.

Now, we discuss some basic notions based on suggested Lq-ROFAAWA AgOp as follows:

Theorem 3.3. (Idempotency). Suppose $\sigma_1 = \langle \mathfrak{a}_{\xi_1}, \mathfrak{a}_{\eta_1} \rangle, \sigma_2 = \langle \mathfrak{a}_{\xi_2}, \ \mathfrak{a}_{\eta_2} \rangle \ldots, \ \sigma_n = \langle \mathfrak{a}_{\xi_n}, \ \mathfrak{a}_{\eta_n} \rangle$ are Lq-ROFNs and $\mathfrak{w}_1, \mathfrak{w}_2, \ldots, \mathfrak{w}_n$ are the weights of Lq-ROFNs, respectively, and $\mathfrak{w}_r \in [0, \mathfrak{h}]$ where $r = 1, 2, \ldots, n$ with $\sum_{r=1}^{n} \mathfrak{w}_r = 1$. If $\sigma_1 = \sigma_2 = \ldots = \sigma_n = \sigma$, then, L$q$-ROFAAWA$(\sigma_1, \sigma_2, \ldots, \sigma_n) = \sigma$.

Theorem 3.4. (Boundedness). Suppose $\sigma_1 = \langle \mathfrak{a}_{\xi_1}, \mathfrak{a}_{\eta_1} \rangle, \sigma_2 = \langle \mathfrak{a}_{\xi_2}, \ \mathfrak{a}_{\eta_2} \rangle \ldots, \ \sigma_n = \langle \mathfrak{a}_{\xi_n}, \ \mathfrak{a}_{\eta_n} \rangle$ are Lq-ROFNs, and let $\sigma_- = min\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ and let $\sigma_+ = max\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$. Then, $\sigma_- \leq L\text{q}{-}ROFAAWA(\sigma_1, \sigma_2, \ldots, \sigma_n) \leq \sigma_+$.

Theorem 3.5. (Monotonicity). Suppose $\sigma_1, \sigma_2, \ldots, \sigma_n$ and $\sigma_1^{'}, \sigma_2^{'}, \ldots, \sigma_n^{'}$ are Lq-ROFNs. If $\sigma_r \leq \sigma_r^{'}$ where $r = 1, 2, \ldots, n$. Then,

$$Lq - ROFAAWA(\sigma_1, \sigma_2, \ldots, \sigma_n) \leq Lq - ROFAAWA(\sigma_1^{'}, \sigma_2^{'}, \ldots, \sigma_n^{'}).$$

4. Two GDM methods based on the suggested Lq-ROFAAWA AgOp

This section provides two GDM approaches based on the suggested Lq-ROFAAWA AgOp. The first recommended GDM approach assumes the condition that weights of experts and weights of attributes are provided, and the other GDM model assumes the condition that weights of experts and weights of attributes are not provided.

4.1. First GDM approach for completely known weights

The suggested first GDM method is based on the proposed Lq-ROFAAWA AgOp in which weights of experts and weights of attributes are provided. Let $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_m$ be alternatives and let $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n$ be attributes where $\mathfrak{w}_1, \mathfrak{w}_2, \ldots, \mathfrak{w}_n$ are the weights of the attributes $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n$, respectively with $\sum_{l=1}^{n} \mathfrak{w}_l = 1$ where $l = 1, 2, \ldots, n$ and $\mathfrak{w}_l \geq 0$. Suppose $\mathcal{E}\mathfrak{x}_1, \mathcal{E}\mathfrak{x}_2, \ldots, \mathcal{E}\mathfrak{x}_t$ are experts and let $\beta_1, \beta_2, \ldots, \beta_t$ be the weights of the experts $\mathcal{E}\mathfrak{x}_1, \mathcal{E}\mathfrak{x}_2, \ldots, \mathcal{E}\mathfrak{x}_t$, respectively with $\sum_{i=1}^{t} \beta_i = 1$, where $\beta_i \geq 0, i = 1, 2, \ldots, t$. Every expert $\mathcal{E}\mathfrak{x}_i$ estimates alternative $\mathcal{R}_k$ related to attribute $\mathcal{S}_l$ to construct the decision matrix $\mathbb{D}^{'p} = \left(\varpi_{kl}^{'p}\right)_{m \times n}$ by the Lq-ROFN $\varpi_{kl}^{'p} = \left\langle \mathfrak{a}_{\xi_{kl}^{'p}}, \ \mathfrak{a}_{\eta_{kl}^{'p}} \right\rangle$ as below:

$$\mathbb{D}^{'p} = \begin{array}{c} \\ \mathcal{R}_1 \\ \mathcal{R}_2 \\ \vdots \\ \mathcal{R}_m \end{array} \begin{array}{c} \begin{array}{cccc} \mathcal{S}_1 & \mathcal{S}_2 & \ldots & \mathcal{S}_n \end{array} \\ \begin{pmatrix} \varpi_{11}^{'p} & \varpi_{12}^{p} & \cdots & \varpi_{1n}^{'p} \\ \varpi_{21}^{'p} & \varpi_{22}^{'p} & \cdots & \varpi_{2n}^{'p} \\ \vdots & \vdots & \ddots & \vdots \\ \varpi_{m1}^{'p} & \varpi_{m2}^{'p} & \cdots & \varpi_{mn}^{'p} \end{pmatrix} \end{array},$$

where $p = 1, 2, \ldots, t$.

Step 1: First of all convert decision matrix $\mathbb{D}'^{p} = \left(\varpi'^{p}_{kl}\right)_{m\times n} = \left(\left\langle \mathfrak{a}_{\xi'^{p}_{kl}},\ \mathfrak{a}_{\eta'^{p}_{kl}}\right\rangle\right)_{m\times n}$ into the new-decision matrix $\mathbb{D}^{p} = \left(\varpi^{p}_{kl}\right)_{m\times n} = \left(\left\langle \mathfrak{a}_{\xi^{p}_{kl}},\ \mathfrak{a}_{\eta^{p}_{kl}}\right\rangle\right)_{m\times n}$ as given below:

$$\varpi^{p}_{kl} = \begin{cases} \langle \mathfrak{a}_{\xi'^{p}_{kl}},\ \mathfrak{a}_{\eta'^{p}_{kl}}\rangle, & \text{if } \mathcal{S}_l \text{ is a asset type}, \\ \langle \mathfrak{a}_{\xi'^{p}_{kl}},\ \mathfrak{a}_{\eta'^{p}_{kl}}\rangle, & \text{if } \mathcal{S}_l \text{ is a expense type}, \end{cases} \tag{4.1}$$

where $k = 1, 2, \ldots, m, l = 1, 2, \ldots, n$ and $p = 1, 2, \ldots, t$.

Step 2: Now using Eq (3.5), we accumulate the Lq-ROFNs $\varpi^{1}_{kl}, \varpi^{2}_{kl}, \ldots, \varpi^{t}_{kl}$ defined in decision matrices $\mathbb{D}^1 = \left(\varpi^{1}_{kl}\right)_{m\times n}, \mathbb{D}^2 = \left(\varpi^{2}_{kl}\right)_{m\times n}, \ldots, \mathbb{D}^t = \left(\varpi^{t}_{kl}\right)_{m\times n}$, respectively, where the accumulated Lq-ROFN ϖ_{kl} for constructing the cumulative decision matrix $\mathbb{D} = \left(\varpi_{kl}\right)_{m\times n} = \left(\left\langle \mathfrak{a}_{\xi_{kl}},\ \mathfrak{a}_{\eta_{kl}}\right\rangle\right)_{m\times n}$ by applying the suggested Lq-ROFAAWA AgOp are given as:

$$\begin{aligned} \varpi_{kl} =&\ Lq - ROFAAWA(\varpi^{1}_{kl}, \varpi^{2}_{kl}, \ldots, \varpi^{t}_{kl}), \\ =&\ \left\langle \mathfrak{a}_{\mathfrak{h}\sqrt[q]{1-\exp^{-\left(\sum_{i=1}^{t}\beta_i\left(-\log(1-(\frac{\xi'^{p}_{kl}}{\mathfrak{h}})^q)\right)^{\lambda}\right)^{1/\lambda}}}},\ \mathfrak{a}_{\mathfrak{h}\exp^{-\left(\sum_{i=1}^{t}\beta_i\left(-\log(\frac{\eta'^{p}_{kl}}{\mathfrak{h}})\right)^{\lambda}\right)^{1/\lambda}}} \right\rangle, \end{aligned} \tag{4.2}$$

where $k = 1, 2, \ldots, m, l = 1, 2, \ldots, n$ and $p = 1, 2, \ldots, t$ with $\lambda > 0$.

Step 3: By the Lq-ROFAAWA AgOp, accumulate the Lq-ROFNs $\varpi_{k1}, \varpi_{k2}, \ldots, \varpi_{kn}$ defined in kth row of cumulative decision matrix $\mathbb{D} = \left(\varpi_{kl}\right)_{m\times n} = \left(\left\langle \mathfrak{a}_{\xi_{kl}},\ \mathfrak{a}_{\eta_{kl}}\right\rangle\right)_{m\times n}$ to get the overall accumulated Lq-ROFN $\varpi_k = \langle \mathfrak{a}_{\xi_k},\ \mathfrak{a}_{\eta_k}\rangle$ of alternative $\mathcal{R}_k$, which is calculated by the following formula:

$$\varpi_k = \left\langle \mathfrak{a}_{\mathfrak{h}\sqrt[q]{1-\exp^{-\left(\sum_{l=1}^{n}\mathfrak{w}_l\left(-\log(1-(\frac{\xi_{kl}}{\mathfrak{h}})^q)\right)^{\lambda}\right)^{1/\lambda}}}},\ \mathfrak{a}_{\mathfrak{h}\exp^{-\left(\sum_{l=1}^{n}\mathfrak{w}_l\left(-\log(\frac{\eta_{kl}}{\mathfrak{h}})\right)^{\lambda}\right)^{1/\lambda}}} \right\rangle, \tag{4.3}$$

where $k = 1, 2, \ldots, m$ and $\lambda > 0$.

Step 4: Next, we compute the ranking value $\mathcal{Y}(\varpi_k)$ of the overall accumulated Lq-ROFNs of corresponding alternative $\mathcal{R}_k$ using the formula given below:

$$\mathcal{Y}(\varpi_k) = \frac{1}{\mathfrak{h}}(\mathfrak{h} - \frac{\pi_k}{2})(\xi_k + \frac{\pi_k}{2}), \tag{4.4}$$

where $\pi_k = (\mathfrak{h}^q - \xi^q - \eta^q)^{1/q}$ and $k = 1, 2, \ldots, m$.

Step 5: By comparing the ranking values $\mathcal{Y}(\varpi_1), \mathcal{Y}(\varpi_2), \ldots, \mathcal{Y}(\varpi_k)$ of the alternatives $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_k$, the larger ranking value $\mathcal{Y}(\varpi_k)$ gives the better ranking order ($\mathcal{RO}$) of the corresponding alternative $\mathcal{R}_k$. If the ranking values of alternatives $\mathcal{R}_x$ and $\mathcal{R}_y$ are the same, i.e., $\mathcal{Y}(\varpi_x) = \mathcal{Y}(\varpi_y)$, then calculate the values of accuracy function $\mathcal{J}(\varpi_x)$ and $\mathcal{J}(\varpi_y)$ using Definition 2.4. The larger accuracy value represents the better $\mathcal{RO}$ of alternative $\mathcal{R}_k$. If $\mathcal{Y}(\varpi_x) = \mathcal{Y}(\varpi_y)$ and $\mathcal{J}(\varpi_x) = \mathcal{J}(\varpi_y)$, then the alternatives $\mathcal{R}_x$ and $\mathcal{R}_y$ have the same $\mathcal{RO}s$.

In the following three examples, we verify the applicability of our first suggested G^DM method based on Lq-ROFNs.

Example 4.1. Air pollution is a crucial environmental issue that poses serious risks to human health, climate, and the ecosystems. Air pollution contains harmful substances in the atmosphere from different sources, including agriculture, vehicle exhaust, and industrial emissions. It substantially impacts health, leading to cardiovascular issues, premature death, and respiratory diseases, specifically among vulnerable populations like the elderly and children. Combatting air pollution needs stricter emission regulations, increased public transportation use, and cleaner technologies. A coordinated effort among individuals, industries, and governments is compulsory to create a healthier environment. Suppose that the government of Pakistan wants to rank the most air-polluted areas/cities of the country to make some rules or to announce public holidays in the cities where the situation is very severe. For this crucial task, the Government of Pakistan hired a group of experts $\mathcal{E}\mathfrak{x}_1, \mathcal{E}\mathfrak{x}_2, \mathcal{E}\mathfrak{x}_3$, and $\mathcal{E}\mathfrak{x}_4$ to evaluate all the alternatives concerning significant parameters. Consider the alternatives representing different cities of Pakistan, and the attributes that serve as the factors that increase the concentration of air pollution. The alternatives $\mathcal{R}_1$ (Karachi), $\mathcal{R}_2$ (Lahore), $\mathcal{R}_3$ (Faisalabad), and $\mathcal{R}_4$ (Sarghoda) are estimated by experts $\mathcal{E}\mathfrak{x}_1, \mathcal{E}\mathfrak{x}_2, \mathcal{E}\mathfrak{x}_3$, and $\mathcal{E}\mathfrak{x}_4$ over the attributes $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4$, and $\mathcal{S}_5$. The attributes $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4$, and $\mathcal{S}_5$ represent the burning of fossil fuels, increase in factories, transportation, agricultural practices, and power plants, respectively, where $\mathfrak{w} = \{\mathfrak{w}_1(0.25), \mathfrak{w}_2(0.20), \mathfrak{w}_3(0.15), \mathfrak{w}_4(0.18), \mathfrak{w}_5(0.22)\}$ are weights allocated to attributes, and $\beta_1 = 0.25$, $\beta_2 = 0.30$, $\beta_3 = 0.20$, and $\beta_4 = 0.25$ are the weights of experts. The experts estimate every alternative $\mathcal{R}_k$ using the Lq-ROFNs $\varpi_{kl}^{'p}$ on the basis of $L^TS = \{\mathfrak{a}_0$=very hazardous, $\mathfrak{a}_1$= hazardous, $\mathfrak{a}_2$= very unhealthy, $\mathfrak{a}_3$=unhealthy, $\mathfrak{a}_4$= unhealthy for easily affected groups, $\mathfrak{a}_5$= moderate, $\mathfrak{a}_6$= good, $\mathfrak{a}_7$= very good, $\mathfrak{a}_8$= extremely good$\}$ in relation to attribute $\mathcal{S}_l$ to create decision-matrices $\mathbb{D}^{'1} = \left(\varpi_{kl}^{'1}\right)_{4\times5}, \mathbb{D}^{'2} = \left(\varpi_{kl}^{'2}\right)_{4\times5}, \mathbb{D}^{'3} = \left(\varpi_{kl}^{'3}\right)_{4\times5}$, and $\mathbb{D}^{'4} = \left(\varpi_{kl}^{'4}\right)_{4\times5}$, which are given as follows:

$$\mathbb{D}^{'1} = \begin{array}{c} \\ \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_3 \\ \mathcal{R}_4 \end{array} \begin{array}{c} \begin{array}{ccccc} \mathcal{S}_1 & \mathcal{S}_2 & \mathcal{S}_3 & \mathcal{S}_4 & \mathcal{S}_5 \end{array} \\ \left(\begin{array}{ccccc} \langle\mathfrak{a}_7, \mathfrak{a}_1\rangle & \langle\mathfrak{a}_6, \mathfrak{a}_2\rangle & \langle\mathfrak{a}_4, \mathfrak{a}_3\rangle & \langle\mathfrak{a}_7, \mathfrak{a}_1\rangle & \langle\mathfrak{a}_5, \mathfrak{a}_2\rangle \\ \langle\mathfrak{a}_6, \mathfrak{a}_2\rangle & \langle\mathfrak{a}_5, \mathfrak{a}_2\rangle & \langle\mathfrak{a}_6, \mathfrak{a}_1\rangle & \langle\mathfrak{a}_6, \mathfrak{a}_2\rangle & \langle\mathfrak{a}_7, \mathfrak{a}_1\rangle \\ \langle\mathfrak{a}_6, \mathfrak{a}_1\rangle & \langle\mathfrak{a}_5, \mathfrak{a}_3\rangle & \langle\mathfrak{a}_7, \mathfrak{a}_1\rangle & \langle\mathfrak{a}_5, \mathfrak{a}_1\rangle & \langle\mathfrak{a}_3, \mathfrak{a}_4\rangle \\ \langle\mathfrak{a}_5, \mathfrak{a}_2\rangle & \langle\mathfrak{a}_7, \mathfrak{a}_1\rangle & \langle\mathfrak{a}_4, \mathfrak{a}_3\rangle & \langle\mathfrak{a}_6, \mathfrak{a}_1\rangle & \langle\mathfrak{a}_4, \mathfrak{a}_4\rangle \end{array} \right) \end{array},$$

$$\mathbb{D}^{'2} = \begin{array}{c} \\ \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_3 \\ \mathcal{R}_4 \end{array} \begin{array}{c} \begin{array}{ccccc} \mathcal{S}_1 & \mathcal{S}_2 & \mathcal{S}_3 & \mathcal{S}_4 & \mathcal{S}_5 \end{array} \\ \left(\begin{array}{ccccc} \langle\mathfrak{a}_7, \mathfrak{a}_1\rangle & \langle\mathfrak{a}_4, \mathfrak{a}_4\rangle & \langle\mathfrak{a}_6, \mathfrak{a}_2\rangle & \langle\mathfrak{a}_5, \mathfrak{a}_2\rangle & \langle\mathfrak{a}_3, \mathfrak{a}_5\rangle \\ \langle\mathfrak{a}_7, \mathfrak{a}_1\rangle & \langle\mathfrak{a}_5, \mathfrak{a}_1\rangle & \langle\mathfrak{a}_6, \mathfrak{a}_1\rangle & \langle\mathfrak{a}_5, \mathfrak{a}_2\rangle & \langle\mathfrak{a}_4, \mathfrak{a}_3\rangle \\ \langle\mathfrak{a}_5, \mathfrak{a}_2\rangle & \langle\mathfrak{a}_6, \mathfrak{a}_1\rangle & \langle\mathfrak{a}_7, \mathfrak{a}_1\rangle & \langle\mathfrak{a}_5, \mathfrak{a}_3\rangle & \langle\mathfrak{a}_4, \mathfrak{a}_4\rangle \\ \langle\mathfrak{a}_6, \mathfrak{a}_2\rangle & \langle\mathfrak{a}_4, \mathfrak{a}_3\rangle & \langle\mathfrak{a}_5, \mathfrak{a}_2\rangle & \langle\mathfrak{a}_7, \mathfrak{a}_1\rangle & \langle\mathfrak{a}_5, \mathfrak{a}_3\rangle \end{array} \right) \end{array},$$

$$\mathbb{D}^{'3} = \begin{array}{c} \\ \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_3 \\ \mathcal{R}_4 \end{array} \begin{array}{c} \begin{array}{ccccc} \mathcal{S}_1 & \mathcal{S}_2 & \mathcal{S}_3 & \mathcal{S}_4 & \mathcal{S}_5 \end{array} \\ \left(\begin{array}{ccccc} \langle\mathfrak{a}_6, \mathfrak{a}_1\rangle & \langle\mathfrak{a}_5, \mathfrak{a}_2\rangle & \langle\mathfrak{a}_3, \mathfrak{a}_4\rangle & \langle\mathfrak{a}_7, \mathfrak{a}_1\rangle & \langle\mathfrak{a}_5, \mathfrak{a}_2\rangle \\ \langle\mathfrak{a}_7, \mathfrak{a}_2\rangle & \langle\mathfrak{a}_6, \mathfrak{a}_2\rangle & \langle\mathfrak{a}_7, \mathfrak{a}_1\rangle & \langle\mathfrak{a}_6, \mathfrak{a}_2\rangle & \langle\mathfrak{a}_5, \mathfrak{a}_1\rangle \\ \langle\mathfrak{a}_5, \mathfrak{a}_2\rangle & \langle\mathfrak{a}_5, \mathfrak{a}_2\rangle & \langle\mathfrak{a}_6, \mathfrak{a}_1\rangle & \langle\mathfrak{a}_4, \mathfrak{a}_3\rangle & \langle\mathfrak{a}_3, \mathfrak{a}_1\rangle \\ \langle\mathfrak{a}_6, \mathfrak{a}_2\rangle & \langle\mathfrak{a}_7, \mathfrak{a}_1\rangle & \langle\mathfrak{a}_5, \mathfrak{a}_1\rangle & \langle\mathfrak{a}_5, \mathfrak{a}_2\rangle & \langle\mathfrak{a}_4, \mathfrak{a}_4\rangle \end{array} \right) \end{array},$$

$$\mathbb{D}'^4 = \begin{array}{c} \\ \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_3 \\ \mathcal{R}_4 \end{array} \begin{array}{c} \begin{array}{ccccc} \mathcal{S}_1 & \mathcal{S}_2 & \mathcal{S}_3 & \mathcal{S}_4 & \mathcal{S}_5 \end{array} \\ \left(\begin{array}{ccccc} \langle \mathfrak{a}_5,\ \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_4,\ \mathfrak{a}_4 \rangle & \langle \mathfrak{a}_7,\ \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5,\ \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_4,\ \mathfrak{a}_2 \rangle \\ \langle \mathfrak{a}_6,\ \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_7,\ \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_6,\ \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5,\ \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_6,\ \mathfrak{a}_1 \rangle \\ \langle \mathfrak{a}_5,\ \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_3,\ \mathfrak{a}_4 \rangle & \langle \mathfrak{a}_6,\ \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_3,\ \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_5,\ \mathfrak{a}_2 \rangle \\ \langle \mathfrak{a}_4,\ \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_5,\ \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_4,\ \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_6,\ \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_5,\ \mathfrak{a}_2 \rangle \end{array} \right) \end{array}.$$

Now, we utilize the suggested first GDM method as below:

Step 1: Since all the attributes are asset type, we get the new-decision matrices $\mathbb{D}^1 = \left(\varpi^1_{kl}\right)_{4\times5} = \left(\left\langle \mathfrak{a}_{\xi^1_{kl}},\ \mathfrak{a}_{\eta'^1_{kl}} \right\rangle\right)_{4\times5}$, $\mathbb{D}^2 = \left(\varpi^2_{kl}\right)_{4\times5} = \left(\left\langle \mathfrak{a}_{\xi^2_{kl}},\ \mathfrak{a}_{\eta'^2_{kl}} \right\rangle\right)_{4\times5}$, $\mathbb{D}^3 = \left(\varpi^3_{kl}\right)_{4\times5} = \left(\left\langle \mathfrak{a}_{\xi^3_{kl}},\ \mathfrak{a}_{\eta'^3_{kl}} \right\rangle\right)_{4\times5}$, and $\mathbb{D}^4 = \left(\varpi^4_{kl}\right)_{4\times5} = \left(\left\langle \mathfrak{a}_{\xi^4_{kl}},\ \mathfrak{a}_{\eta'^4_{kl}} \right\rangle\right)_{4\times5}$.

Step 2: By the Eq (4.2), we accumulate the Lq-ROFNs ϖ_{kl} using the Lq-ROFNsl $\varpi^1_{kl}, \varpi^2_{kl}, \varpi^3_{kl}$, and ϖ^4_{kl} that belongs to the decision matrices $\mathbb{D}^1 = \left(\varpi^1_{kl}\right)_{4\times5} = \left(\left\langle \mathfrak{a}_{\xi^1_{kl}},\ \mathfrak{a}_{\eta'^1_{kl}} \right\rangle\right)_{4\times5}$, $\mathbb{D}^2 = \left(\varpi^2_{kl}\right)_{4\times5} = \left(\left\langle \mathfrak{a}_{\xi^2_{kl}},\ \mathfrak{a}_{\eta'^2_{kl}} \right\rangle\right)_{4\times5}$,$\mathbb{D}^3 = \left(\varpi^3_{kl}\right)_{4\times5} = \left(\left\langle \mathfrak{a}_{\xi^3_{kl}},\ \mathfrak{a}_{\eta'^3_{kl}} \right\rangle\right)_{4\times5}$, and $\mathbb{D}^4 = \left(\varpi^4_{kl}\right)_{4\times5} = \left(\left\langle \mathfrak{a}_{\xi^4_{kl}},\ \mathfrak{a}_{\eta'^4_{kl}} \right\rangle\right)_{4\times5}$ to construct the cumulative decision matrix $\mathbb{D} = \left(\varpi_{kl}\right)_{4\times5} = \left(\left\langle \mathfrak{a}_{\xi_{kl}},\ \mathfrak{a}_{\eta_{kl}} \right\rangle\right)_{4\times5}$ with $k = 1, 2, \ldots, 4$ and $l = 1, 2, \ldots, 5$ as given below for $q = 3$:

$$\mathbb{D} = \begin{array}{c} \\ \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_3 \\ \mathcal{R}_4 \end{array} \begin{array}{c} \begin{array}{ccccc} \mathcal{S}_1 & \mathcal{S}_2 & \mathcal{S}_3 & \mathcal{S}_4 & \mathcal{S}_5 \end{array} \\ \left(\begin{array}{ccccc} \langle \mathfrak{a}_{6.67},\ \mathfrak{a}_{1.23} \rangle & \langle \mathfrak{a}_{5.18},\ \mathfrak{a}_{2.76} \rangle & \langle \mathfrak{a}_{6.21},\ \mathfrak{a}_{1.94} \rangle & \langle \mathfrak{a}_{6.50},\ \mathfrak{a}_{1.19} \rangle & \langle \mathfrak{a}_{4.54},\ \mathfrak{a}_{2.44} \rangle \\ \langle \mathfrak{a}_{6.68},\ \mathfrak{a}_{1.16} \rangle & \langle \mathfrak{a}_{6.23},\ \mathfrak{a}_{1.32} \rangle & \langle \mathfrak{a}_{6.35},\ \mathfrak{a}_{1.00} \rangle & \langle \mathfrak{a}_{5.60},\ \mathfrak{a}_{2.00} \rangle & \langle \mathfrak{a}_{6.22},\ \mathfrak{a}_{1.29} \rangle \\ \langle \mathfrak{a}_{5.39},\ \mathfrak{a}_{1.63} \rangle & \langle \mathfrak{a}_{5.34},\ \mathfrak{a}_{1.91} \rangle & \langle \mathfrak{a}_{6.72},\ \mathfrak{a}_{1.16} \rangle & \langle \mathfrak{a}_{4.64},\ \mathfrak{a}_{2.08} \rangle & \langle \mathfrak{a}_{4.24},\ \mathfrak{a}_{2.24} \rangle \\ \langle \mathfrak{a}_{5.59},\ \mathfrak{a}_{2.18} \rangle & \langle \mathfrak{a}_{6.48},\ \mathfrak{a}_{1.29} \rangle & \langle \mathfrak{a}_{4.65},\ \mathfrak{a}_{1.83} \rangle & \langle \mathfrak{a}_{6.42},\ \mathfrak{a}_{1.32} \rangle & \langle \mathfrak{a}_{4.69},\ \mathfrak{a}_{1.96} \rangle \end{array} \right) \end{array}.$$

Step 3: Using Eq (4.3), we combine every accumulative Lq-ROFN $\varpi_k = \langle \mathfrak{a}_{\xi_k},\ \mathfrak{a}_{\eta_k} \rangle$ of corresponding alternative $\mathcal{R}_k$, with $k = 1, 2, 3, 4$ that are: $\varpi_1 = \langle \mathfrak{a}_{6.18},\ \mathfrak{a}_{1.74} \rangle$, $\varpi_2 = \langle \mathfrak{a}_{6.32},\ \mathfrak{a}_{1.42} \rangle$, $\varpi_3 = \langle \mathfrak{a}_{5.66},\ \mathfrak{a}_{1.80} \rangle$, and $\varpi_4 = \langle \mathfrak{a}_{5.92},\ \mathfrak{a}_{1.75} \rangle$.

Step 4: Next, by applying Eq (4.4), we calculate the ranking values $\mathcal{Y}(\varpi_1)$, $\mathcal{Y}(\varpi_2)$, $\mathcal{Y}(\varpi_3)$, and $\mathcal{Y}(\varpi_4)$, which are $\mathcal{Y}(\varpi_1) = 5.64$, $\mathcal{Y}(\varpi_2) = 5.76$, $\mathcal{Y}(\varpi_3) = 5.22$, and $\mathcal{Y}(\varpi_4) = 5.42$.

Step 5: From Step 4, we see that $\mathcal{Y}(\varpi_2) > \mathcal{Y}(\varpi_1) > \mathcal{Y}(\varpi_4) > \mathcal{Y}(\varpi_3)$, Hence, the more polluted city out of $\mathcal{R}_k$ cities with $k = 1, 2, 3, 4$ is $\mathcal{R}_2$.

Example 4.2. In today's interconnected world, marked by global unity and progress, students are increasingly prioritizing obtaining education in foreign countries. However, they often encounter various challenges in pursuing this path. Thus, finding a suitable foreign university for a student involves various challenges. Some common problems include: Financial constraints, academic requirements, language proficiency, and so on. To find a good foreign university, students seek assistance from various experts, including educational consultants, career counselors, admissions advisors, alumni networks, international student offices, standardized test prep centers, etc. Suppose a student wants to get admission for higher studies in a suitable foreign university and to find best option from the alternatives $\mathcal{R}_1$, $\mathcal{R}_2$, $\ldots$, $\mathcal{R}_6$, the student takes services of different experts such as $\mathcal{E}\mathfrak{x}_1, \mathcal{E}\mathfrak{x}_2, \ldots, \mathcal{E}\mathfrak{x}_6$. Consider the favorable attributes are $\mathcal{S}_1$ (Academic Reputation), $\mathcal{S}_2$ (Faculty Expertise), $\mathcal{S}_3$ (Research Opportunities), $\mathcal{S}_4$ (Financial Aid), $\mathcal{S}_5$ (Innovation), $\mathcal{S}_6$ (Student Services), and $\mathcal{S}_7$ (Cost), with the help of experts $\mathcal{E}\mathfrak{x}_1$, $\mathcal{E}\mathfrak{x}_2$,..., and $\mathcal{E}\mathfrak{x}_6$. Suppose the attributes $\mathcal{S}_1$, $\mathcal{S}_2$, ..., $\mathcal{S}_7$ having weights $\mathfrak{w} = \{\mathfrak{w}_1(0.20), \mathfrak{w}_2(0.15), \mathfrak{w}_3(0.05), \mathfrak{w}_4(0.03), \mathfrak{w}_5(0.12), \mathfrak{w}_6(0.32), \mathfrak{w}_7(0.13)\}$ and the experts' weights are $\beta_1 = 0.13$, $\beta_2 = 0.20$, $\beta_3 = 0.25$, $\beta_4 = 0.10$, $\beta_5 = 0.17$, $\beta_6 = 0.15$. The experts estimate each alternative $\mathcal{R}_k$ through the Lq-ROFNs $\varpi_{kl}^{'p}$ on the basis of $\mathrm{L^TS} = \{\alpha_0 = \text{extremely poor}, \alpha_1 = \text{very poor}, \alpha_2 = \text{poor}, \alpha_3 = \text{slightly poor}, \alpha_4 = \text{fair}, \alpha_5 = \text{slightly good}, \alpha_6 = \text{good}, \alpha_7 = \text{very good}, \alpha_8 = \text{extremely good}\}$ in relation to each attribute $\mathcal{S}_l$ to create the decision-matrices $\mathbb{D}'^1 = \left(\varpi_{kl}^{'1}\right)_{6\times7}$, $\mathbb{D}'^2 = \left(\varpi_{kl}^{'2}\right)_{6\times7}$, $\mathbb{D}'^3 = \left(\varpi_{kl}^{'3}\right)_{6\times7}$, $\mathbb{D}'^4 = \left(\varpi_{kl}^{'4}\right)_{6\times7}$, $\mathbb{D}'^5 = \left(\varpi_{kl}^{'5}\right)_{6\times7}$, and $\mathbb{D}'^6 = \left(\varpi_{kl}^{'6}\right)_{6\times7}$ that is given as follows:

$$\mathbb{D}'^1 = \begin{array}{c} \\ \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_3 \\ \mathcal{R}_4 \\ \mathcal{R}_5 \\ \mathcal{R}_6 \end{array} \begin{array}{c} \begin{array}{ccccccc} \mathcal{S}_1 & \mathcal{S}_2 & \mathcal{S}_3 & \mathcal{S}_4 & \mathcal{S}_5 & \mathcal{S}_6 & \mathcal{S}_7 \end{array} \\ \begin{pmatrix} \langle\alpha_5,\alpha_2\rangle & \langle\alpha_7,\alpha_1\rangle & \langle\alpha_4,\alpha_3\rangle & \langle\alpha_6,\alpha_1\rangle & \langle\alpha_4,\alpha_4\rangle & \langle\alpha_5,\alpha_2\rangle & \langle\alpha_4,\alpha_3\rangle \\ \langle\alpha_6,\alpha_1\rangle & \langle\alpha_5,\alpha_3\rangle & \langle\alpha_7,\alpha_1\rangle & \langle\alpha_5,\alpha_1\rangle & \langle\alpha_5,\alpha_1\rangle & \langle\alpha_4,\alpha_4\rangle & \langle\alpha_7,\alpha_1\rangle \\ \langle\alpha_6,\alpha_2\rangle & \langle\alpha_5,\alpha_2\rangle & \langle\alpha_6,\alpha_1\rangle & \langle\alpha_6,\alpha_2\rangle & \langle\alpha_7,\alpha_1\rangle & \langle\alpha_5,\alpha_1\rangle & \langle\alpha_3,\alpha_2\rangle \\ \langle\alpha_7,\alpha_1\rangle & \langle\alpha_6,\alpha_2\rangle & \langle\alpha_4,\alpha_3\rangle & \langle\alpha_7,\alpha_1\rangle & \langle\alpha_5,\alpha_2\rangle & \langle\alpha_6,\alpha_1\rangle & \langle\alpha_5,\alpha_3\rangle \\ \langle\alpha_6,\alpha_1\rangle & \langle\alpha_5,\alpha_2\rangle & \langle\alpha_4,\alpha_4\rangle & \langle\alpha_5,\alpha_1\rangle & \langle\alpha_5,\alpha_3\rangle & \langle\alpha_5,\alpha_1\rangle & \langle\alpha_5,\alpha_3\rangle \\ \langle\alpha_5,\alpha_1\rangle & \langle\alpha_6,\alpha_1\rangle & \langle\alpha_5,\alpha_3\rangle & \langle\alpha_6,\alpha_1\rangle & \langle\alpha_3,\alpha_2\rangle & \langle\alpha_5,\alpha_2\rangle & \langle\alpha_4,\alpha_3\rangle \end{pmatrix} \end{array},$$

$$\mathbb{D}'^2 = \begin{array}{c} \\ \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_3 \\ \mathcal{R}_4 \\ \mathcal{R}_5 \\ \mathcal{R}_6 \end{array} \begin{array}{c} \begin{array}{ccccccc} \mathcal{S}_1 & \mathcal{S}_2 & \mathcal{S}_3 & \mathcal{S}_4 & \mathcal{S}_5 & \mathcal{S}_6 & \mathcal{S}_7 \end{array} \\ \begin{pmatrix} \langle\alpha_4,\alpha_3\rangle & \langle\alpha_5,\alpha_2\rangle & \langle\alpha_4,\alpha_4\rangle & \langle\alpha_6,\alpha_1\rangle & \langle\alpha_4,\alpha_3\rangle & \langle\alpha_7,\alpha_1\rangle & \langle\alpha_5,\alpha_2\rangle \\ \langle\alpha_4,\alpha_1\rangle & \langle\alpha_6,\alpha_3\rangle & \langle\alpha_5,\alpha_1\rangle & \langle\alpha_4,\alpha_2\rangle & \langle\alpha_4,\alpha_4\rangle & \langle\alpha_6,\alpha_1\rangle & \langle\alpha_4,\alpha_1\rangle \\ \langle\alpha_6,\alpha_1\rangle & \langle\alpha_5,\alpha_3\rangle & \langle\alpha_6,\alpha_2\rangle & \langle\alpha_6,\alpha_1\rangle & \langle\alpha_4,\alpha_3\rangle & \langle\alpha_5,\alpha_2\rangle & \langle\alpha_5,\alpha_1\rangle \\ \langle\alpha_6,\alpha_1\rangle & \langle\alpha_5,\alpha_2\rangle & \langle\alpha_6,\alpha_2\rangle & \langle\alpha_5,\alpha_1\rangle & \langle\alpha_5,\alpha_2\rangle & \langle\alpha_4,\alpha_2\rangle & \langle\alpha_5,\alpha_3\rangle \\ \langle\alpha_5,\alpha_1\rangle & \langle\alpha_4,\alpha_2\rangle & \langle\alpha_6,\alpha_2\rangle & \langle\alpha_5,\alpha_3\rangle & \langle\alpha_4,\alpha_3\rangle & \langle\alpha_5,\alpha_1\rangle & \langle\alpha_4,\alpha_3\rangle \\ \langle\alpha_5,\alpha_2\rangle & \langle\alpha_4,\alpha_2\rangle & \langle\alpha_5,\alpha_3\rangle & \langle\alpha_6,\alpha_2\rangle & \langle\alpha_4,\alpha_1\rangle & \langle\alpha_5,\alpha_2\rangle & \langle\alpha_6,\alpha_1\rangle \end{pmatrix} \end{array},$$

$$\mathbb{D}'^3 = \begin{array}{c} \\ \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_3 \\ \mathcal{R}_4 \\ \mathcal{R}_5 \\ \mathcal{R}_6 \end{array} \begin{array}{c} \begin{array}{ccccccc} \mathcal{S}_1 & \mathcal{S}_2 & \mathcal{S}_3 & \mathcal{S}_4 & \mathcal{S}_5 & \mathcal{S}_6 & \mathcal{S}_7 \end{array} \\ \begin{pmatrix} \langle\alpha_7,\alpha_1\rangle & \langle\alpha_5,\alpha_1\rangle & \langle\alpha_4,\alpha_3\rangle & \langle\alpha_4,\alpha_1\rangle & \langle\alpha_6,\alpha_1\rangle & \langle\alpha_5,\alpha_2\rangle & \langle\alpha_4,\alpha_2\rangle \\ \langle\alpha_5,\alpha_2\rangle & \langle\alpha_4,\alpha_3\rangle & \langle\alpha_6,\alpha_1\rangle & \langle\alpha_4,\alpha_1\rangle & \langle\alpha_6,\alpha_2\rangle & \langle\alpha_4,\alpha_1\rangle & \langle\alpha_5,\alpha_2\rangle \\ \langle\alpha_6,\alpha_2\rangle & \langle\alpha_4,\alpha_2\rangle & \langle\alpha_5,\alpha_1\rangle & \langle\alpha_3,\alpha_2\rangle & \langle\alpha_6,\alpha_1\rangle & \langle\alpha_4,\alpha_1\rangle & \langle\alpha_6,\alpha_2\rangle \\ \langle\alpha_4,\alpha_2\rangle & \langle\alpha_5,\alpha_2\rangle & \langle\alpha_4,\alpha_1\rangle & \langle\alpha_5,\alpha_1\rangle & \langle\alpha_4,\alpha_2\rangle & \langle\alpha_5,\alpha_1\rangle & \langle\alpha_5,\alpha_3\rangle \\ \langle\alpha_4,\alpha_2\rangle & \langle\alpha_6,\alpha_2\rangle & \langle\alpha_4,\alpha_4\rangle & \langle\alpha_3,\alpha_1\rangle & \langle\alpha_5,\alpha_2\rangle & \langle\alpha_6,\alpha_1\rangle & \langle\alpha_4,\alpha_3\rangle \\ \langle\alpha_4,\alpha_2\rangle & \langle\alpha_5,\alpha_2\rangle & \langle\alpha_6,\alpha_1\rangle & \langle\alpha_4,\alpha_1\rangle & \langle\alpha_3,\alpha_2\rangle & \langle\alpha_6,\alpha_2\rangle & \langle\alpha_5,\alpha_3\rangle \end{pmatrix} \end{array},$$

$$\mathbb{D}'^4 = \begin{array}{c} \\ \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_3 \\ \mathcal{R}_4 \\ \mathcal{R}_5 \\ \mathcal{R}_6 \end{array} \begin{array}{c} \begin{array}{ccccccc} \mathcal{S}_1 & \mathcal{S}_2 & \mathcal{S}_3 & \mathcal{S}_4 & \mathcal{S}_5 & \mathcal{S}_6 & \mathcal{S}_7 \end{array} \\ \begin{pmatrix} \langle\alpha_4,\alpha_1\rangle & \langle\alpha_6,\alpha_1\rangle & \langle\alpha_5,\alpha_3\rangle & \langle\alpha_5,\alpha_5\rangle & \langle\alpha_4,\alpha_1\rangle & \langle\alpha_6,\alpha_2\rangle & \langle\alpha_4,\alpha_2\rangle \\ \langle\alpha_4,\alpha_1\rangle & \langle\alpha_5,\alpha_2\rangle & \langle\alpha_6,\alpha_1\rangle & \langle\alpha_5,\alpha_2\rangle & \langle\alpha_6,\alpha_1\rangle & \langle\alpha_4,\alpha_1\rangle & \langle\alpha_5,\alpha_1\rangle \\ \langle\alpha_6,\alpha_2\rangle & \langle\alpha_5,\alpha_2\rangle & \langle\alpha_6,\alpha_1\rangle & \langle\alpha_6,\alpha_2\rangle & \langle\alpha_7,\alpha_1\rangle & \langle\alpha_5,\alpha_1\rangle & \langle\alpha_3,\alpha_2\rangle \\ \langle\alpha_6,\alpha_1\rangle & \langle\alpha_6,\alpha_2\rangle & \langle\alpha_5,\alpha_3\rangle & \langle\alpha_4,\alpha_1\rangle & \langle\alpha_6,\alpha_2\rangle & \langle\alpha_5,\alpha_2\rangle & \langle\alpha_7,\alpha_1\rangle \\ \langle\alpha_5,\alpha_1\rangle & \langle\alpha_6,\alpha_2\rangle & \langle\alpha_4,\alpha_4\rangle & \langle\alpha_7,\alpha_1\rangle & \langle\alpha_6,\alpha_2\rangle & \langle\alpha_5,\alpha_2\rangle & \langle\alpha_5,\alpha_3\rangle \\ \langle\alpha_6,\alpha_2\rangle & \langle\alpha_5,\alpha_1\rangle & \langle\alpha_6,\alpha_2\rangle & \langle\alpha_7,\alpha_1\rangle & \langle\alpha_4,\alpha_1\rangle & \langle\alpha_5,\alpha_2\rangle & \langle\alpha_6,\alpha_1\rangle \end{pmatrix} \end{array},$$

$$\mathbb{D}'^{5} = \begin{array}{c} \\ \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_3 \\ \mathcal{R}_4 \\ \mathcal{R}_5 \\ \mathcal{R}_6 \end{array} \begin{array}{c} \begin{array}{ccccccc} \mathcal{S}_1 & \mathcal{S}_2 & \mathcal{S}_3 & \mathcal{S}_4 & \mathcal{S}_5 & \mathcal{S}_6 & \mathcal{S}_7 \end{array} \\ \left(\begin{array}{ccccccc} \langle \mathfrak{a}_4, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle \\ \langle \mathfrak{a}_4, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_5 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_2 \rangle \\ \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_3, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle \\ \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle \\ \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_7, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_4 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_3 \rangle \\ \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_4 \rangle & \langle \mathfrak{a}_7, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_3 \rangle \end{array} \right) \end{array},$$

$$\mathbb{D}'^{6} = \begin{array}{c} \\ \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_3 \\ \mathcal{R}_4 \\ \mathcal{R}_5 \\ \mathcal{R}_6 \end{array} \begin{array}{c} \begin{array}{ccccccc} \mathcal{S}_1 & \mathcal{S}_2 & \mathcal{S}_3 & \mathcal{S}_4 & \mathcal{S}_5 & \mathcal{S}_6 & \mathcal{S}_7 \end{array} \\ \left(\begin{array}{ccccccc} \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_7, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_3 \rangle \\ \langle \mathfrak{a}_5, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_1 \rangle \\ \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_1 \rangle \\ \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_4 \rangle & \langle \mathfrak{a}_7, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_3 \rangle \\ \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_7, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_4 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_3 \rangle \\ \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle \end{array} \right) \end{array}.$$

Now, we again utilize the suggested first GDM method to solve this example as below:

Step 1: Since all the attributes are asset type, so, we get the new-decision matrices as: $\mathbb{D}^1 = \left(\varpi^1_{kl}\right)_{6\times 7} = \left(\left\langle \mathfrak{a}_{\xi^1_{kl}}, \mathfrak{a}_{\eta'^1_{kl}} \right\rangle\right)_{6\times 7}$, $\mathbb{D}^2 = \left(\varpi^2_{kl}\right)_{6\times 7} = \left(\left\langle \mathfrak{a}_{\xi^2_{kl}}, \mathfrak{a}_{\eta'^2_{kl}} \right\rangle\right)_{6\times 7}$, $\mathbb{D}^3 = \left(\varpi^3_{kl}\right)_{6\times 7} = \left(\left\langle \mathfrak{a}_{\xi^3_{kl}}, \mathfrak{a}_{\eta'^3_{kl}} \right\rangle\right)_{6\times 7}$, $\mathbb{D}^4 = \left(\varpi^4_{kl}\right)_{6\times 7} = \left(\left\langle \mathfrak{a}_{\xi^4_{kl}}, \mathfrak{a}_{\eta'^4_{kl}} \right\rangle\right)_{6\times 7}$, $\mathbb{D}^5 = \left(\varpi^5_{kl}\right)_{6\times 7} = \left(\left\langle \mathfrak{a}_{\xi^5_{kl}}, \mathfrak{a}_{\eta'^5_{kl}} \right\rangle\right)_{6\times 7}$ and $\mathbb{D}^6 = \left(\varpi^6_{kl}\right)_{6\times 7} = \left(\left\langle \mathfrak{a}_{\xi^6_{kl}}, \mathfrak{a}_{\eta'^6_{kl}} \right\rangle\right)_{6\times 7}$.

Step 2: Using the Eq (4.2) we accumulate the Lq-ROFNs ϖ_{kl} by the Lq-ROFNs $\varpi^1_{kl}, \varpi^2_{kl}, \varpi^3_{kl}, \varpi^4_{kl}, \varpi^5_{kl}$ and ϖ^6_{kl} that belongs to the decision matrices $\mathbb{D}^1 = \left(\varpi^1_{kl}\right)_{6\times 7} = \left(\left\langle \mathfrak{a}_{\xi^1_{kl}}, \mathfrak{a}_{\eta'^1_{kl}} \right\rangle\right)_{6\times 7}$, $\mathbb{D}^2 = \left(\varpi^2_{kl}\right)_{6\times 7} = \left(\left\langle \mathfrak{a}_{\xi^2_{kl}}, \mathfrak{a}_{\eta'^2_{kl}} \right\rangle\right)_{6\times 7}$, $\mathbb{D}^3 = \left(\varpi^3_{kl}\right)_{6\times 7} = \left(\left\langle \mathfrak{a}_{\xi^3_{kl}}, \mathfrak{a}_{\eta'^3_{kl}} \right\rangle\right)_{6\times 7}$, $\mathbb{D}^4 = \left(\varpi^4_{kl}\right)_{6\times 7} = \left(\left\langle \mathfrak{a}_{\xi^4_{kl}}, \mathfrak{a}_{\eta'^4_{kl}} \right\rangle\right)_{6\times 7}$, $\mathbb{D}^5 = \left(\varpi^5_{kl}\right)_{6\times 7} = \left(\left\langle \mathfrak{a}_{\xi^5_{kl}}, \mathfrak{a}_{\eta'^5_{kl}} \right\rangle\right)_{6\times 7}$, and $\mathbb{D}^6 = \left(\varpi^6_{kl}\right)_{6\times 7} = \left(\left\langle \mathfrak{a}_{\xi^6_{kl}}, \mathfrak{a}_{\eta'^6_{kl}} \right\rangle\right)_{6\times 7}$ to construct the cumulative decision matrix $\mathbb{D} = \left(\varpi_{kl}\right)_{6\times 7} = \left(\left\langle \mathfrak{a}_{\xi_{kl}}, \mathfrak{a}_{\eta_{kl}} \right\rangle\right)_{6\times 7}$ with $k = 1, 2, \ldots 6$, and $l = 1, 2, \ldots 7$ as given below for $q = 3$:

$$\mathbb{D} = \begin{array}{c} \\ \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_3 \\ \mathcal{R}_4 \\ \mathcal{R}_5 \\ \mathcal{R}_6 \end{array} \begin{array}{c} \begin{array}{ccccccc} \mathcal{S}_1 & \mathcal{S}_2 & \mathcal{S}_3 & \mathcal{S}_4 & \mathcal{S}_5 & \mathcal{S}_6 & \mathcal{S}_7 \end{array} \\ \left(\begin{array}{ccccccc} \langle \mathfrak{a}_{6.08}, \mathfrak{a}_{1.42} \rangle & \langle \mathfrak{a}_{5.95}, \mathfrak{a}_{1.25} \rangle & \langle \mathfrak{a}_{5.02}, \mathfrak{a}_{2.27} \rangle & \langle \mathfrak{a}_{6.02}, \mathfrak{a}_{1.23} \rangle & \langle \mathfrak{a}_{5.45}, \mathfrak{a}_{1.50} \rangle & \langle \mathfrak{a}_{6.12}, \mathfrak{a}_{1.34} \rangle & \langle \mathfrak{a}_{4.67}, \mathfrak{a}_{1.90} \rangle \\ \langle \mathfrak{a}_{5.00}, \mathfrak{a}_{1.33} \rangle & \langle \mathfrak{a}_{5.60}, \mathfrak{a}_{2.11} \rangle & \langle \mathfrak{a}_{5.99}, \mathfrak{a}_{1.26} \rangle & \langle \mathfrak{a}_{4.69}, \mathfrak{a}_{1.45} \rangle & \langle \mathfrak{a}_{5.55}, \mathfrak{a}_{1.59} \rangle & \langle \mathfrak{a}_{5.37}, \mathfrak{a}_{1.45} \rangle & \langle \mathfrak{a}_{5.68}, \mathfrak{a}_{1.29} \rangle \\ \langle \mathfrak{a}_{5.91}, \mathfrak{a}_{1.51} \rangle & \langle \mathfrak{a}_{5.06}, \mathfrak{a}_{1.88} \rangle & \langle \mathfrak{a}_{5.54}, \mathfrak{a}_{1.23} \rangle & \langle \mathfrak{a}_{5.43}, \mathfrak{a}_{1.78} \rangle & \langle \mathfrak{a}_{6.31}, \mathfrak{a}_{1.30} \rangle & \langle \mathfrak{a}_{4.72}, \mathfrak{a}_{1.23} \rangle & \langle \mathfrak{a}_{5.45}, \mathfrak{a}_{1.51} \rangle \\ \langle \mathfrak{a}_{6.04}, \mathfrak{a}_{1.16} \rangle & \langle \mathfrak{a}_{5.53}, \mathfrak{a}_{2.12} \rangle & \langle \mathfrak{a}_{5.36}, \mathfrak{a}_{1.88} \rangle & \langle \mathfrak{a}_{6.27}, \mathfrak{a}_{1.00} \rangle & \langle \mathfrak{a}_{5.23}, \mathfrak{a}_{2.12} \rangle & \langle \mathfrak{a}_{5.14}, \mathfrak{a}_{1.48} \rangle & \langle \mathfrak{a}_{5.66}, \mathfrak{a}_{2.03} \rangle \\ \langle \mathfrak{a}_{5.11}, \mathfrak{a}_{1.43} \rangle & \langle \mathfrak{a}_{6.39}, \mathfrak{a}_{1.55} \rangle & \langle \mathfrak{a}_{4.96}, \mathfrak{a}_{3.06} \rangle & \langle \mathfrak{a}_{5.84}, \mathfrak{a}_{1.18} \rangle & \langle \mathfrak{a}_{4.90}, \mathfrak{a}_{2.74} \rangle & \langle \mathfrak{a}_{5.39}, \mathfrak{a}_{1.29} \rangle & \langle \mathfrak{a}_{4.37}, \mathfrak{a}_{3.00} \rangle \\ \langle \mathfrak{a}_{5.30}, \mathfrak{a}_{1.41} \rangle & \langle \mathfrak{a}_{5.38}, \mathfrak{a}_{1.70} \rangle & \langle \mathfrak{a}_{5.61}, \mathfrak{a}_{1.97} \rangle & \langle \mathfrak{a}_{6.34}, \mathfrak{a}_{1.12} \rangle & \langle \mathfrak{a}_{4.81}, \mathfrak{a}_{1.64} \rangle & \langle \mathfrak{a}_{5.39}, \mathfrak{a}_{2.00} \rangle & \langle \mathfrak{a}_{5.41}, \mathfrak{a}_{1.65} \rangle \end{array} \right) \end{array}.$$

Step 3: By applying Eq (4.3), we combine each accumulative Lq-ROFN $\varpi_k = \langle \mathfrak{a}_{\xi_k}, \mathfrak{a}_{\eta_k} \rangle$ of respective alternative $\mathcal{R}_k$, with $k = 1, 2, \ldots, 6$ that are: $\varpi_1 = \langle \mathfrak{a}_{5.87}, \mathfrak{a}_{1.45} \rangle$, $\varpi_2 = \langle \mathfrak{a}_{5.44}, \mathfrak{a}_{1.49} \rangle$, $\varpi_3 = \langle \mathfrak{a}_{5.56}, \mathfrak{a}_{1.43} \rangle$, $\varpi_4 = \langle \mathfrak{a}_{5.58}, \mathfrak{a}_{1.60} \rangle$, $\varpi_5 = \langle \mathfrak{a}_{5.52}, \mathfrak{a}_{1.67} \rangle$, and $\varpi_6 = \langle \mathfrak{a}_{5.38}, \mathfrak{a}_{1.70} \rangle$.

Step 4: Further, using Eq (4.4), we calculate the ranking values $\mathcal{Y}(\varpi_1)$, $\mathcal{Y}(\varpi_2)$, $\mathcal{Y}(\varpi_3)$, $\mathcal{Y}(\varpi_4)$, $\mathcal{Y}(\varpi_5)$ and $\mathcal{Y}(\varpi_6)$ that are provided as: $\mathcal{Y}(\varpi_1) = 5.38$, $\mathcal{Y}(\varpi_2) = 5.06$, $\mathcal{Y}(\varpi_3) = 5.14$, $\mathcal{Y}(\varpi_4) = 5.16$, $\mathcal{Y}(\varpi_5) = 5.12$, and $\mathcal{Y}(\varpi_6) = 5.02$.

Step 5: From previous Step, we observe that $\mathcal{Y}(\varpi_1) > \mathcal{Y}(\varpi_4) > \mathcal{Y}(\varpi_3) > \mathcal{Y}(\varpi_5) > \mathcal{Y}(\varpi_2) > \mathcal{Y}(\varpi_6)$. Thus, the best university out of all universities $\mathcal{R}_k$ with $k = 1, 2, \ldots, 6$ is $\mathcal{R}_1$.

Example 4.3. These days, banks strategically invest in diverse businesses as a means to mitigate losses. However, selecting the right businesses is no easy task. To address this challenge and safeguard their interests, banks rely on the opinions and evaluation reports of various experts. These experts specialize in the specific fields in which the bank intends to invest. Their insights provide crucial guidance, allowing the bank to make well-informed decisions, minimize risks, and enhance the overall effectiveness of their investment strategies. Consider a bank ABC wants to put money into a most profitable factory from different factories like a car factory ($\mathcal{R}_1$), a food factory ($\mathcal{R}_2$), a computer factory ($\mathcal{R}_3$) and an arms factory ($\mathcal{R}_4$). For this crucial task, the experts $\mathcal{E}\mathfrak{x}_1, \mathcal{E}\mathfrak{x}_3$, and $\mathcal{E}\mathfrak{x}_3$ evaluate the alternatives $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$, and $\mathcal{R}_4$ under the attributes of benefit related such as $\mathcal{S}_1$ (analysis of risk), $\mathcal{S}_2$(development analysis), $\mathcal{S}_3$(sociopolitical influence analysis), and $\mathcal{S}_4$(analysis of environmental effect). The weights of attributes are $\mathfrak{w} = \{\mathfrak{w}_1(0.30), \mathfrak{w}_2(0.10), \mathfrak{w}_3(0.20), \mathfrak{w}_4(0.40)\}$ and weights of experts are $\beta_1 = 0.243$, $\beta_2 = 0.514$, and $\beta_3 = 0.243$. The experts estimate every alternative $\mathcal{R}_k$ using Lq-ROFNs ϖ'^{p}_{kl} on the grounds of $L^T S = \{\mathfrak{a}_0 =$ extremely poor, $\mathfrak{a}_1 =$ very poor, $\mathfrak{a}_2 =$ poor, $\mathfrak{a}_3 =$ slightly poor, $\mathfrak{a}_4 =$ fair, $\mathfrak{a}_5 =$ slightly good, $\mathfrak{a}_6 =$ good, $\mathfrak{a}_7 =$ very good, $\mathfrak{a}_8 =$ extremely good$\}$ in relation to each attribute $\mathcal{S}_l$ to create decision-matrices $\mathbb{D}'^1 = \left(\varpi'^1_{kl}\right)_{4\times5}$, $\mathbb{D}'^2 = \left(\varpi'^2_{kl}\right)_{4\times5}$, and $\mathbb{D}'^3 = \left(\varpi'^3_{kl}\right)_{4\times5}$, which are given as follows:

$$\mathbb{D}'^1 = \begin{array}{c} \\ \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_3 \\ \mathcal{R}_4 \end{array} \begin{array}{c} \begin{array}{cccc} \quad\mathcal{S}_1 & \qquad\mathcal{S}_2 & \qquad\mathcal{S}_3 & \qquad\mathcal{S}_4 \end{array} \\ \left(\begin{array}{cccc} \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_3, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_3, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_1, \mathfrak{a}_6 \rangle \\ \langle \mathfrak{a}_3, \mathfrak{a}_4 \rangle & \langle \mathfrak{a}_3, \mathfrak{a}_4 \rangle & \langle \mathfrak{a}_2, \mathfrak{a}_5 \rangle & \langle \mathfrak{a}_2, \mathfrak{a}_4 \rangle \\ \langle \mathfrak{a}_1, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_2, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_3, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle \\ \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_7, \mathfrak{a}_1 \rangle \end{array}\right) \end{array},$$

$$\mathbb{D}'^2 = \begin{array}{c} \\ \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_3 \\ \mathcal{R}_4 \end{array} \begin{array}{c} \begin{array}{cccc} \quad\mathcal{S}_1 & \qquad\mathcal{S}_2 & \qquad\mathcal{S}_3 & \qquad\mathcal{S}_4 \end{array} \\ \left(\begin{array}{cccc} \langle \mathfrak{a}_3, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_3, \mathfrak{a}_4 \rangle & \langle \mathfrak{a}_2, \mathfrak{a}_3 \rangle \\ \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_2, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_3, \mathfrak{a}_4 \rangle & \langle \mathfrak{a}_2, \mathfrak{a}_5 \rangle \\ \langle \mathfrak{a}_2, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_3, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_1, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_3, \mathfrak{a}_3 \rangle \\ \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_3, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_1 \rangle \end{array}\right) \end{array},$$

$$\mathbb{D}'^3 = \begin{array}{c} \\ \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_3 \\ \mathcal{R}_4 \end{array} \begin{array}{c} \begin{array}{cccc} \quad\mathcal{S}_1 & \qquad\mathcal{S}_2 & \qquad\mathcal{S}_3 & \qquad\mathcal{S}_4 \end{array} \\ \left(\begin{array}{cccc} \langle \mathfrak{a}_3, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_3, \mathfrak{a}_5 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_2, \mathfrak{a}_6 \rangle \\ \langle \mathfrak{a}_3, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_2, \mathfrak{a}_4 \rangle & \langle \mathfrak{a}_2, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_3, \mathfrak{a}_4 \rangle \\ \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_2, \mathfrak{a}_5 \rangle & \langle \mathfrak{a}_3, \mathfrak{a}_4 \rangle & \langle \mathfrak{a}_1, \mathfrak{a}_3 \rangle \\ \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_4 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle \end{array}\right) \end{array}.$$

We now repeat the suggested first GDM method on this example as below:

Step 1: Since all the attributes are asset type, we get the new-decision matrices $\mathbb{D}^1 = \left(\varpi^1_{kl}\right)_{4\times4} = \left(\left\langle \mathfrak{a}_{\xi^1_{kl}}, \mathfrak{a}_{\eta'^1_{kl}} \right\rangle\right)_{4\times4}$, $\mathbb{D}^2 = \left(\varpi^2_{kl}\right)_{4\times4} = \left(\left\langle \mathfrak{a}_{\xi^2_{kl}}, \mathfrak{a}_{\eta'^2_{kl}} \right\rangle\right)_{4\times4}$, $\mathbb{D}^3 = \left(\varpi^3_{kl}\right)_{4\times4} = \left(\left\langle \mathfrak{a}_{\xi^3_{kl}}, \mathfrak{a}_{\eta'^3_{kl}} \right\rangle\right)_{4\times4}$, and $\mathbb{D}^4 = \left(\varpi^4_{kl}\right)_{4\times4} = \left(\left\langle \mathfrak{a}_{\xi^4_{kl}}, \mathfrak{a}_{\eta'^4_{kl}} \right\rangle\right)_{4\times4}$.

Step 2: From Eq (4.2), we accumulate Lq-ROFNs ϖ_{kl} using the Lq-ROFNs $\varpi^1_{kl}, \varpi^2_{kl}, \varpi^3_{kl}$, and ϖ^4_{kl} that belongs to the decision matrices $\mathbb{D}^1 = \left(\varpi^1_{kl}\right)_{4\times4} = \left(\left\langle \mathfrak{a}_{\xi^1_{kl}}, \mathfrak{a}_{\eta'^1_{kl}} \right\rangle\right)_{4\times4}$, $\mathbb{D}^2 = \left(\varpi^2_{kl}\right)_{4\times4} = \left(\left\langle \mathfrak{a}_{\xi^2_{kl}}, \mathfrak{a}_{\eta'^2_{kl}} \right\rangle\right)_{4\times4}$,

$\mathbb{D}^3 = \left(\varpi^3_{kl}\right)_{4\times4} = \left(\left\langle \mathfrak{a}_{\xi^3_{kl}},\ \mathfrak{a}_{\eta'^3_{kl}}\right\rangle\right)_{4\times4}$, and $\mathbb{D}^4 = \left(\varpi_k kl^4\right)_{4\times4} = \left(\left\langle \mathfrak{a}_{\xi^4_{kl}},\ \mathfrak{a}_{\eta'^4_{kl}}\right\rangle\right)_{4\times4}$. to construct the cumulative decision matrix $\mathbb{D} = \left(\varpi_{kl}\right)_{4\times4} = \left(\left\langle \mathfrak{a}_{\xi_{kl}},\ \mathfrak{a}_{\eta_{kl}} k\right\rangle\right)_{4\times4}$ with $k = 1, 2, \ldots 4$ and $l = 1, 2, \ldots 4$ that is given below for $q = 3$:

$$\mathbb{D} = \begin{array}{c} \\ \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_3 \\ \mathcal{R}_4 \end{array} \begin{array}{c} \begin{array}{cccc} \mathcal{S}_1 & \mathcal{S}_2 & \mathcal{S}_3 & \mathcal{S}_4 \end{array} \\ \left(\begin{array}{cccc} \langle \mathfrak{a}_{4.97},\ \mathfrak{a}_{1.78}\rangle & \langle \mathfrak{a}_{3.68},\ \mathfrak{a}_{1.37}\rangle & \langle \mathfrak{a}_{4.97},\ \mathfrak{a}_{2.32}\rangle & \langle \mathfrak{a}_{1.90},\ \mathfrak{a}_{3.86}\rangle \\ \langle \mathfrak{a}_{4.55},\ \mathfrak{a}_{2.29}\rangle & \langle \mathfrak{a}_{2.47},\ \mathfrak{a}_{1.67}\rangle & \langle \mathfrak{a}_{2.72},\ \mathfrak{a}_{2.50}\rangle & \langle \mathfrak{a}_{2.47},\ \mathfrak{a}_{4.44}\rangle \\ \langle \mathfrak{a}_{4.95},\ \mathfrak{a}_{2.11}\rangle & \langle \mathfrak{a}_{2.72},\ \mathfrak{a}_{3.31}\rangle & \langle \mathfrak{a}_{2.67},\ \mathfrak{a}_{2.29}\rangle & \langle \mathfrak{a}_{4.96},\ \mathfrak{a}_{2.11}\rangle \\ \langle \mathfrak{a}_{5.38},\ \mathfrak{a}_{1.64}\rangle & \langle \mathfrak{a}_{3.66},\ \mathfrak{a}_{3.19}\rangle & \langle \mathfrak{a}_{5.38},\ \mathfrak{a}_{1.64}\rangle & \langle \mathfrak{a}_{6.06},\ \mathfrak{a}_{1.23}\rangle \end{array}\right) \end{array}.$$

Step 3: Using Eq (4.3), we combine every accumulative Lq-ROFN $\varpi_k = \langle \mathfrak{a}_{\xi_k},\ \mathfrak{a}_{\eta_k}\rangle$ of corresponding alternative $\mathcal{R}_k$ with $k = 1, 2, \ldots, 4$ are $\varpi_1 = \langle \mathfrak{a}_{4.50},\ \mathfrak{a}_{2.34}\rangle$, $\varpi_2 = \langle \mathfrak{a}_{3.81},\ \mathfrak{a}_{2.77}\rangle$, $\varpi_3 = \langle \mathfrak{a}_{4.71},\ \mathfrak{a}_{2.22}\rangle$, and $\varpi_4 = \langle \mathfrak{a}_{5.68},\ \mathfrak{a}_{1.52}\rangle$.
Step 4: Next, by Eq (4.4), we calculate the ranking values $\mathcal{Y}(\varpi_1)$, $\mathcal{Y}(\varpi_2)$, $\mathcal{Y}(\varpi_3)$, and $\mathcal{Y}(\varpi_4)$ that are: $\mathcal{Y}(\varpi_1) = 4.44$, $\mathcal{Y}(\varpi_2) = 4.03$, $\mathcal{Y}(\varpi_3) = 4.57$, and $\mathcal{Y}(\varpi_4) = 5.23$.
Step 5: From preceding Step, we see that $\mathcal{Y}(\varpi_4) > \mathcal{Y}(\varpi_3) > \mathcal{Y}(\varpi_1) > \mathcal{Y}(\varpi_2)$. Hence, the best business out of all businesses $\mathcal{R}_k$ with $k = 1, 2, \ldots, 4$ is $\mathcal{R}_4$.

4.2. Second GDM approach for completely unknown weights

This subsection is devoted to designing the second G^DM approach based on the suggested Lq-ROFAAWA AgOp with the completely unknown weights of the experts and attributes.

Let $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_m$ be alternatives and let $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n$ be attributes. Suppose $\mathcal{E}\mathfrak{x}_1, \mathcal{E}\mathfrak{x}_2, \ldots, \mathcal{E}\mathfrak{x}_t$ are experts and every expert $\mathcal{E}\mathfrak{x}_i$ estimate each alternative $\mathcal{R}_k$ regarding each attribute $\mathcal{S}_l$ to construct the decision matrix $\mathbb{D}'^p = \left(\varpi'^p_{kl}\right)_{m\times n}$ using Lq-ROFN $\varpi'^p_{kl} = \left\langle \mathfrak{a}_{\xi'^p_{kl}},\ \mathfrak{a}_{\eta'^p_{kl}}\right\rangle$ as below:

$$\mathbb{D}'^p = \begin{array}{c} \\ \mathcal{R}_1 \\ \mathcal{R}_2 \\ \vdots \\ \mathcal{R}_m \end{array} \begin{array}{c} \begin{array}{cccc} \mathcal{S}_1 & \mathcal{S}_2 & \ldots & \mathcal{S}_n \end{array} \\ \left(\begin{array}{cccc} \varpi'^p_{11} & \varpi^p_{12} & \cdots & \varpi'^p_{1n} \\ \varpi'^p_{21} & \varpi'^p_{22} & \cdots & \varpi'^p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \varpi'^p_{m1} & \varpi'^p_{m2} & \cdots & \varpi'^p_{mn} \end{array}\right) \end{array},$$

where $p = 1, 2, \ldots, t$.
Step 1: First of all covert each decision matrix $\mathbb{D}'^p = \left(\varpi'^p_{kl}\right)_{m\times n} = \left(\left\langle \mathfrak{a}_{\xi'^p_{kl}},\ \mathfrak{a}_{\eta'^p_{kl}}\right\rangle\right)_{m\times n}$ into the new-decision matrix $\mathbb{D}^p = \left(\varpi^p_{kl}\right)_{m\times n} = \left(\left\langle \mathfrak{a}_{\xi^p_{kl}},\ \mathfrak{a}_{\eta'^p_{kl}}\right\rangle\right)_{m\times n}$ as given below:

$$\varpi^p_{kl} = \begin{cases} \langle \mathfrak{a}_{\xi'^p_{kl}},\ \mathfrak{a}_{\eta'^p_{kl}}\rangle, & \text{if } \mathcal{S}_l \text{ is an asset type,} \\ \langle \mathfrak{a}_{\xi'^p_{kl}},\ \mathfrak{a}_{\eta'^p_{kl}}\rangle, & \text{if } \mathcal{S}_l \text{ is an expense type,} \end{cases} \tag{4.5}$$

here $k = 1, 2, \ldots, m$, $l = 1, 2, \ldots, n$ and $p = 1, 2, \ldots, t$.
Step 2: The entropy ς^p of the decision-maker $\mathcal{E}\mathfrak{x}_i$ is calculated using the Lq-ROFNs provided in the

new-decision matrix $\mathbb{D}^p = \left(\varpi_{kl}^p\right)_{m\times n} = \left(\left\langle \mathfrak{a}_{\xi_{kl}^p},\ \mathfrak{a}_{\eta_{kl}^p} \right\rangle\right)_{m\times n}$ as below:

$$\varsigma^p = \frac{1}{mn}\sum_{k=1}^{m}\sum_{l=1}^{n}\left\{1-\frac{1}{\mathfrak{h}^2}(\xi_{kl}^p-\eta_{kl}^p)(\xi_{kl}^p+\eta_{kl}^p)\right\}, \tag{4.6}$$

where $k = 1,2,\ldots,m, l = 1,2,\ldots,n$ and $p = 1,2,\ldots,t$.
Next, the weights β_p of experts $\mathcal{E}x_i$ are calculated as:

$$\beta_p = \frac{\varsigma^p}{\sum_{p=1}^{t}\varsigma^p}, \tag{4.7}$$

where $\beta_p \in [0,1]$ and $\sum_{p=1}^{t}\beta_p = 1$.
Step 3: Further, we accumulate the Lq-ROFNs $\varpi_{kl}^1, \varpi_{kl}^2, \ldots, \varpi_{kl}^t$ given in decision matrices $\mathbb{D}^1 = \left(\varpi_{kl}^1\right)_{m\times n}, \mathbb{D}^2 = \left(\varpi_{kl}^2\right)_{m\times n}, \ldots, \mathbb{D}^t = \left(\varpi_{kl}^t\right)_{m\times n}$, respectively, by applying the suggested Lq-ROFAAWA AgOp in the form of an accumulated Lq-ROFN ϖ_{kl} that is helpful in the construction of cumulative decision matrix $\mathbb{D} = \left(\varpi_{kl}\right)_{m\times n} = \left(\left\langle \mathfrak{a}_{\xi_{kl}},\ \mathfrak{a}_{\eta_{kl}} \right\rangle\right)_{m\times n}$, where

$$\begin{aligned} \varpi_{kl} &= Lq-ROFAAWA(\varpi_{kl}^1, \varpi_{kl}^2, \ldots, \varpi_{kl}^t), \\ &= \left\langle \mathfrak{a}_{\mathfrak{h}\sqrt[q]{1-\exp^{-\left(\sum_{p=1}^{t}\beta_p\left(-\log(1-(\frac{\xi_{kl}'^p}{\mathfrak{h}})^q)\right)^\lambda\right)^{1/\lambda}}}},\ \mathfrak{a}_{\mathfrak{h}\exp^{-\left(\sum_{p=1}^{t}\beta_p\left(-\log(\frac{\eta_{kl}'^p}{\mathfrak{h}})\right)^\lambda\right)^{1/\lambda}}} \right\rangle, \end{aligned} \tag{4.8}$$

where $k = 1,2,\ldots,m, l = 1,2,\ldots,n$, and $p = 1,2,\ldots,t$ with $\lambda > 0$.
Step 4: The entropy ς_l of the attribute $\mathcal{S}_l$ is calculated using the Lq-ROFNs used in the non-decision matrix $\mathbb{D}^p = \left(\varpi_{kl}^p\right)_{m\times n} = \left(\left\langle \mathfrak{a}_{\xi_{kl}^p},\ \mathfrak{a}_{\eta_{kl}'^p} \right\rangle\right)_{m\times n}$ as below:

$$\varsigma_l = \frac{1}{m}\sum_{k=1}^{m}\left\{1-\frac{1}{\mathfrak{h}^2}(\xi_{kl}-\eta_{kl})(\xi_{kl}+\eta_{kl})\right\}, \tag{4.9}$$

where $k = 1,2,\ldots,m$ and $l = 1,2,\ldots,n$.

The weights $\mathfrak{w}_l$ of attribute $\mathcal{S}_l$ are calculated by

$$\mathfrak{w}_l = \frac{\varsigma_l}{\sum_{l=1}^{n}\varsigma_l}, \tag{4.10}$$

where $\mathfrak{w}_l \in [0,1]$ and $\sum_{l=1}^{n}\mathfrak{w}_l = 1$.
Step 5: Using the following formula of Lq-ROFAAWA AgOp, we accumulate the Lq-ROFNs

$\varpi_{k1}, \varpi_{k2}, \ldots, \varpi_{kn}$ given in kth row of cumulative decision matrix $\mathbb{D} = \left(\varpi_{kl}\right)_{m\times n} = \left(\left\langle \mathfrak{a}_{\xi_{kl}},\ \mathfrak{a}_{\eta_{kl}} \right\rangle\right)_{m\times n}$ to obtain the overall accumulated Lq-ROFN $\varpi_k = \langle \mathfrak{a}_{\xi_k},\ \mathfrak{a}_{\eta_k}\rangle$ of alternative $\mathcal{R}_k$.

$$\varpi_k = \left\langle \mathfrak{a}_{\mathfrak{h}\sqrt[q]{1-\exp^{-\left(\sum_{l=1}^{n} \mathfrak{w}_l\left(-\log(1-(\frac{\xi_{kl}}{\mathfrak{h}})^q)\right)^{\lambda}\right)^{1/\lambda}}}},\ \mathfrak{a}_{\mathfrak{h}\exp^{-\left(\sum_{l=1}^{n} \mathfrak{w}_l\left(-\log(\frac{\eta_{kl}}{\mathfrak{h}})\right)^{\lambda}\right)^{1/\lambda}}} \right\rangle, \tag{4.11}$$

where $k = 1, 2, \ldots, m$ and $\lambda > 0$.

Step 6: We now find the ranking value $\mathcal{Y}(\varpi_k)$ of corresponding alternative $\mathcal{R}_k$ using the overall accumulated Lq-ROFNs as follows:

$$\mathcal{Y}(\varpi_k) = \frac{1}{\mathfrak{h}}(\mathfrak{h} - \frac{\pi_k}{2})(\xi_k + \frac{\pi_k}{2}), \tag{4.12}$$

where $\pi_k = (\mathfrak{h}^q - \xi^q - \eta^q)^{1/q}$ and $k = 1, 2, \ldots, m$.

Step 7: At last, by comparing the ranking values $\mathcal{Y}(\varpi_1), \mathcal{Y}(\varpi_2), \ldots, \mathcal{Y}(\varpi_k)$ of the alternatives $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_k$, respectively, find the larger ranking value $\mathcal{Y}(\varpi_k)$ that gives the better $\mathcal{RO}$ of the corresponding alternative $\mathcal{R}_k$. If the ranking values of alternatives $\mathcal{R}_x$ and $\mathcal{R}_y$ are same, i.e., $\mathcal{Y}(\varpi_x) = \mathcal{Y}(\varpi_y)$, then calculate the values of accuracy function $\mathcal{J}(\varpi_x)$ and $\mathcal{J}(\varpi_y)$. The larger accuracy value represents the better $\mathcal{RO}$. For two or more alternatives, if $\mathcal{Y}(\varpi_x) = \mathcal{Y}(\varpi_y)$ and $\mathcal{J}(\varpi_x) = \mathcal{J}(\varpi_y)$, then the alternatives $\mathcal{R}_x$ and $\mathcal{R}_y$ have the same $\mathcal{RO}s$.

In the following, we apply the developed second GDM method on three examples.

Example 4.4. The selection of a mobile phone requires a complex decision-making exercise. The users usually consider different features, including processing power, battery life, and camera quality. Moreover, the operating system, and software updates are also crucial characteristics in the selection procedure. There is another critical factor, i.e., budget constraints. The user's reviews, recommendations of experts, and brand reputation also assist the decision-making process. Consider a user who wants to buy the best mobile phone from the alternatives $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_3$, and $\mathcal{R}_4$, and the attributes are $\mathcal{S}_1$ (camera quality), $\mathcal{S}_2$ (security alerts), $\mathcal{S}_3$ (innovation), $\mathcal{S}_4$ (reliability), and $\mathcal{S}_5$ (brand reputation), which are estimated by the experts $\mathcal{E}x_1$, $\mathcal{E}x_2$, and $\mathcal{E}x_3$. The experts estimate each alternative $\mathcal{R}_k$ using Lq-ROFNs $\varpi_{kl}^{'p}$ on the basis of $L^TS = \{\mathfrak{a}_0 =$ extremely poor, $\mathfrak{a}_1 =$ too poor $\mathfrak{a}_2 =$ very poor, $\mathfrak{a}_3 =$ poor, $\mathfrak{a}_4 =$ fair, $\mathfrak{a}_5 =$ little poor, $\mathfrak{a}_6 =$ slightly good, $\mathfrak{a}_7 =$ good, $\mathfrak{a}_8 =$ very good, $\mathfrak{a}_9 =$ too good, $\mathfrak{a}_{10} =$ extremely good$\}$ in relation to attribute $\mathcal{S}_l$ to create decision-matrices $\mathbb{D}^{'1} = \left(\varpi_{kl}^{'1}\right)_{6\times 7}$, $\mathbb{D}^{'2} = \left(\varpi_{kl}^{'2}\right)_{6\times 7}$, and $\mathbb{D}^{'3} = \left(\varpi_{kl}^{'3}\right)_{6\times 7}$ that are given as below:

$$\mathbb{D}^{'1} = \begin{array}{c} \\ \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_3 \\ \mathcal{R}_4 \end{array} \begin{array}{c} \begin{array}{ccccc} \mathcal{S}_1 & \mathcal{S}_2 & \mathcal{S}_3 & \mathcal{S}_4 & \mathcal{S}_5 \end{array} \\ \left(\begin{array}{ccccc} \langle \mathfrak{a}_5,\ \mathfrak{a}_4\rangle & \langle \mathfrak{a}_5,\ \mathfrak{a}_3\rangle & \langle \mathfrak{a}_6,\ \mathfrak{a}_2\rangle & \langle \mathfrak{a}_4,\ \mathfrak{a}_5\rangle & \langle \mathfrak{a}_6,\ \mathfrak{a}_3\rangle \\ \langle \mathfrak{a}_7,\ \mathfrak{a}_2\rangle & \langle \mathfrak{a}_3,\ \mathfrak{a}_6\rangle & \langle \mathfrak{a}_5,\ \mathfrak{a}_3\rangle & \langle \mathfrak{a}_3,\ \mathfrak{a}_6\rangle & \langle \mathfrak{a}_5,\ \mathfrak{a}_4\rangle \\ \langle \mathfrak{a}_6,\ \mathfrak{a}_3\rangle & \langle \mathfrak{a}_4,\ \mathfrak{a}_4\rangle & \langle \mathfrak{a}_4,\ \mathfrak{a}_5\rangle & \langle \mathfrak{a}_5,\ \mathfrak{a}_4\rangle & \langle \mathfrak{a}_7,\ \mathfrak{a}_2\rangle \\ \langle \mathfrak{a}_4,\ \mathfrak{a}_4\rangle & \langle \mathfrak{a}_6,\ \mathfrak{a}_3\rangle & \langle \mathfrak{a}_3,\ \mathfrak{a}_6\rangle & \langle \mathfrak{a}_7,\ \mathfrak{a}_2\rangle & \langle \mathfrak{a}_5,\ \mathfrak{a}_4\rangle \end{array}\right) \end{array},$$

$$\mathbb{D}'^2 = \begin{matrix} & \mathcal{S}_1 & \mathcal{S}_2 & \mathcal{S}_3 & \mathcal{S}_4 & \mathcal{S}_5 \\ \mathcal{R}_1 & \langle \mathfrak{a}_6,\ \mathfrak{a}_2\rangle & \langle \mathfrak{a}_4,\ \mathfrak{a}_5\rangle & \langle \mathfrak{a}_6,\ \mathfrak{a}_3\rangle & \langle \mathfrak{a}_5,\ \mathfrak{a}_3\rangle & \langle \mathfrak{a}_6,\ \mathfrak{a}_2\rangle \\ \mathcal{R}_2 & \langle \mathfrak{a}_5,\ \mathfrak{a}_3\rangle & \langle \mathfrak{a}_3,\ \mathfrak{a}_6\rangle & \langle \mathfrak{a}_5,\ \mathfrak{a}_4\rangle & \langle \mathfrak{a}_3,\ \mathfrak{a}_6\rangle & \langle \mathfrak{a}_5,\ \mathfrak{a}_3\rangle \\ \mathcal{R}_3 & \langle \mathfrak{a}_4,\ \mathfrak{a}_5\rangle & \langle \mathfrak{a}_5,\ \mathfrak{a}_4\rangle & \langle \mathfrak{a}_7,\ \mathfrak{a}_2\rangle & \langle \mathfrak{a}_4,\ \mathfrak{a}_4\rangle & \langle \mathfrak{a}_4,\ \mathfrak{a}_5\rangle \\ \mathcal{R}_4 & \langle \mathfrak{a}_3,\ \mathfrak{a}_6\rangle & \langle \mathfrak{a}_7,\ \mathfrak{a}_2\rangle & \langle \mathfrak{a}_5,\ \mathfrak{a}_4\rangle & \langle \mathfrak{a}_6,\ \mathfrak{a}_3\rangle & \langle \mathfrak{a}_3,\ \mathfrak{a}_6\rangle \end{matrix},$$

$$\mathbb{D}'^3 = \begin{matrix} & \mathcal{S}_1 & \mathcal{S}_2 & \mathcal{S}_3 & \mathcal{S}_4 & \mathcal{S}_5 \\ \mathcal{R}_1 & \langle \mathfrak{a}_3,\ \mathfrak{a}_6\rangle & \langle \mathfrak{a}_7,\ \mathfrak{a}_2\rangle & \langle \mathfrak{a}_5,\ \mathfrak{a}_4\rangle & \langle \mathfrak{a}_4,\ \mathfrak{a}_4\rangle & \langle \mathfrak{a}_4,\ \mathfrak{a}_5\rangle \\ \mathcal{R}_2 & \langle \mathfrak{a}_4,\ \mathfrak{a}_5\rangle & \langle \mathfrak{a}_5,\ \mathfrak{a}_4\rangle & \langle \mathfrak{a}_7,\ \mathfrak{a}_2\rangle & \langle \mathfrak{a}_5,\ \mathfrak{a}_4\rangle & \langle \mathfrak{a}_6,\ \mathfrak{a}_3\rangle \\ \mathcal{R}_3 & \langle \mathfrak{a}_7,\ \mathfrak{a}_2\rangle & \langle \mathfrak{a}_4,\ \mathfrak{a}_4\rangle & \langle \mathfrak{a}_4,\ \mathfrak{a}_5\rangle & \langle \mathfrak{a}_5,\ \mathfrak{a}_4\rangle & \langle \mathfrak{a}_7,\ \mathfrak{a}_2\rangle \\ \mathcal{R}_4 & \langle \mathfrak{a}_5,\ \mathfrak{a}_4\rangle & \langle \mathfrak{a}_6,\ \mathfrak{a}_3\rangle & \langle \mathfrak{a}_3,\ \mathfrak{a}_6\rangle & \langle \mathfrak{a}_3,\ \mathfrak{a}_6\rangle & \langle \mathfrak{a}_7,\ \mathfrak{a}_2\rangle \end{matrix}.$$

We now utilize the suggested second GDM method to solve this example as below:

Step 1: As we know all the attributes are asset type, we get the new-decision matrices $\mathbb{D}^1 = \left(\varpi^1_{kl}\right)_{4\times5} = \left(\left\langle \mathfrak{a}_{\xi^1_{kl}},\ \mathfrak{a}_{\eta'^1_{kl}}\right\rangle\right)_{4\times5}$, $\mathbb{D}^2 = \left(\varpi^2_{kl}\right)_{4\times5} = \left(\left\langle \mathfrak{a}_{\xi^2_{kl}},\ \mathfrak{a}_{\eta'^2_{kl}}\right\rangle\right)_{4\times5}$, and $\mathbb{D}^3 = \left(\varpi^3_{kl}\right)_{4\times5} = \left(\left\langle \mathfrak{a}_{\xi^3_{kl}},\ \mathfrak{a}_{\eta'^3_{kl}}\right\rangle\right)_{4\times5}$.

Step 2: Using Eq (4.6), we get the entropies ς^1, ς^2, and ς^3 of the experts $\mathcal{E}x_1, \mathcal{E}x_2$, and $\mathcal{E}x_3$, respectively, which are $\varsigma^1 = 0.7925$, $\varsigma^2 = 0.7910$, and $\varsigma^3 = 0.7840$. The weights of experts are calculated using Eq (4.7) such that $\beta_1 = 0.3347, \beta_2 = 0.3341$ and $\beta_3 = 0.3312$.

Step 3: By applying Eq (4.8), we find the accumulated Lq-ROFNs ϖ_{kl} from the Lq-ROFNs $\varpi^1_{kl}, \varpi^2_{kl}$, and ϖ^3_{kl} belongs to the new-decision matrices $\mathbb{D}^1 = \left(\varpi^1_{kl}\right)_{4\times5} = \left(\left\langle \mathfrak{a}_{\xi^1_{kl}},\ \mathfrak{a}_{\eta'^1_{kl}}\right\rangle\right)_{4\times5}$, $\mathbb{D}^2 = \left(\varpi^2_{kl}\right)_{4\times5} = \left(\left\langle \mathfrak{a}_{\xi^2_{kl}},\ \mathfrak{a}_{\eta'^2_{kl}}\right\rangle\right)_{4\times5}$, and $\mathbb{D}^3 = \left(\varpi^3_{kl}\right)_{4\times5} = \left(\left\langle \mathfrak{a}_{\xi^3_{kl}},\ \mathfrak{a}_{\eta'^3_{kl}}\right\rangle\right)_{4\times5}$ to construct the cumulative decision matrix $\mathbb{D} = \left(\varpi_{kl}\right)_{4\times5} = \left(\left\langle \mathfrak{a}_{\xi_{kl}},\ \mathfrak{a}_{\eta_{kl}}\right\rangle\right)_{4\times5}$ with $k = 1, 2, \ldots, 4$, and $l = 1, 2, \ldots, 5$ as below:

$$\mathbb{D} = \begin{matrix} & \mathcal{S}_1 & \mathcal{S}_2 & \mathcal{S}_3 & \mathcal{S}_4 & \mathcal{S}_5 \\ \mathcal{R}_1 & \langle \mathfrak{a}_{5.29},\ \mathfrak{a}_{3.29}\rangle & \langle \mathfrak{a}_{6.09},\ \mathfrak{a}_{2.93}\rangle & \langle \mathfrak{a}_{5.76},\ \mathfrak{a}_{2.79}\rangle & \langle \mathfrak{a}_{4.47},\ \mathfrak{a}_{3.82}\rangle & \langle \mathfrak{a}_{5.68},\ \mathfrak{a}_{3.37}\rangle \\ \mathcal{R}_2 & \langle \mathfrak{a}_{6.10},\ \mathfrak{a}_{2.93}\rangle & \langle \mathfrak{a}_{4.25},\ \mathfrak{a}_{5.10}\rangle & \langle \mathfrak{a}_{6.15},\ \mathfrak{a}_{2.79}\rangle & \langle \mathfrak{a}_{4.25},\ \mathfrak{a}_{5.10}\rangle & \langle \mathfrak{a}_{5.46},\ \mathfrak{a}_{3.56}\rangle \\ \mathcal{R}_3 & \langle \mathfrak{a}_{6.26},\ \mathfrak{a}_{2.93}\rangle & \langle \mathfrak{a}_{4.47},\ \mathfrak{a}_{4.00}\rangle & \langle \mathfrak{a}_{6.03},\ \mathfrak{a}_{3.36}\rangle & \langle \mathfrak{a}_{4.77},\ \mathfrak{a}_{4.00}\rangle & \langle \mathfrak{a}_{6.63},\ \mathfrak{a}_{2.65}\rangle \\ \mathcal{R}_4 & \langle \mathfrak{a}_{4.37},\ \mathfrak{a}_{4.47}\rangle & \langle \mathfrak{a}_{6.46},\ \mathfrak{a}_{2.58}\rangle & \langle \mathfrak{a}_{4.27},\ \mathfrak{a}_{5.16}\rangle & \langle \mathfrak{a}_{6.26},\ \mathfrak{a}_{3.01}\rangle & \langle \mathfrak{a}_{6.16},\ \mathfrak{a}_{3.30}\rangle \end{matrix}.$$

Step 4: Using Eq (4.9), the entropies $\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4$, and ς_5 of the attributes $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4$, and $\mathcal{S}_5$ are calculated as: $\varsigma_1 = 0.81, \varsigma_2 = 0.85, \varsigma_3 = 0.82, \varsigma_4 = 0.91$, and $\varsigma_5 = 0.75$. Moreover, the weights of attributes $\mathfrak{w} = \{\mathfrak{w}_1(0.20), \mathfrak{w}_2(0.21), \mathfrak{w}_3(0.19), \mathfrak{w}_4(0.22), \mathfrak{w}_5(0.18)\}$ are computed by Eq (4.10).

Step 5: Next, from the Eq (4.11), we combine accumulated Lq-ROFNs $\varpi_k = \langle \mathfrak{a}_{\xi_k},\ \mathfrak{a}_{\eta_k}\rangle$ of corresponding alternative $\mathcal{R}_k$, with $k = 1, 2, 3, 4$, which are given as $\varpi_1 = \langle \mathfrak{a}_{5.59},\ \mathfrak{a}_{3.24}\rangle$, $\varpi_2 = \langle \mathfrak{a}_{5.55},\ \mathfrak{a}_{3,69}\rangle$, $\varpi_3 = \langle \mathfrak{a}_{5.91},\ \mathfrak{a}_{3.46}\rangle$, and $\varpi_4 = \langle \mathfrak{a}_{5.88},\ \mathfrak{a}_{3.47}\rangle$.

Step 6: Using Eq (4.12), we calculate the ranking values $\mathcal{Y}(\varpi_1)$, $\mathcal{Y}(\varpi_2)$, $\mathcal{Y}(\varpi_3)$, and $\mathcal{Y}(\varpi_4)$ as follows: $\mathcal{Y}(\varpi_1) = 5.50$, $\mathcal{Y}(\varpi_2) = 5.49$, $\mathcal{Y}(\varpi_3) = 5.71$, and $\mathcal{Y}(\varpi_4) = 5.69$.

Step 7: From Step 6, we see that $\mathcal{Y}(\varpi_3) > \mathcal{Y}(\varpi_4) > \mathcal{Y}(\varpi_1) > \mathcal{Y}(\varpi_2)$, Hence, the best mobile phone from the collection $\mathcal{R}_k$ with $k = 1, 2, 3, 4$ is $\mathcal{R}_3$.

Example 4.5. Everyone can observe the rapid progress occurring globally in various fields. Similarly, the car manufacturing industry is undergoing a transformative shift from conventional fuel to electronic technology. The escalating concerns related to global warming, primarily fueled by high fuel consumption, have led experts to believe that oil reserves may deplete in the coming years. This realization has prompted car manufacturing companies to swiftly transition towards the production of electric vehicles (EVs). The performance of electric vehicles depends on various factors, including battery efficiency, charging infrastructure, and advancements in electric propulsion systems. This paradigm shift reflects a collective effort to address environmental challenges and align with a sustainable future. Assume that the Engineering Council of Pakistan is organizing a car exhibition featuring a competition for electronic cars, where prizes will be awarded to the top three position holders. To select the best designs, the services of Pakistan's leading automobile experts will be enlisted. Suppose $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_3$, and $\mathcal{R}_4$ are the alternatives (samples) and the attributes are $\mathcal{S}_1$ (battery performance), $\mathcal{S}_2$ (power and acceleration), $\mathcal{S}_3$ (design and aesthetics), $\mathcal{S}_4$ (charging infrastructure compatibility), and $\mathcal{S}_5$ (energy efficiency), which are estimated by the experts $\mathcal{E}\mathfrak{x}_1$, $\mathcal{E}\mathfrak{x}_2$, and $\mathcal{E}\mathfrak{x}_3$. The experts estimate each alternative $\mathcal{R}_k$ in the form of Lq-ROFNs ϖ'^{p}_{kl} on the basis of $L^T S = \{\mathfrak{a}_0 =$ extremely poor, $\mathfrak{a}_1 =$ very poor, $\mathfrak{a}_2 =$ poor, $\mathfrak{a}_3 =$ slightly poor, $\mathfrak{a}_4 =$ fair, $\mathfrak{a}_5 =$ good, $\mathfrak{a}_6 =$ very good, $\mathfrak{a}_7 =$ too good, $\mathfrak{a}_8 =$ extremely good$\}$ regarding every attribute $\mathcal{S}_l$ to create decision-matrices $\mathbb{D}'^1 = \left(\varpi'^1_{kl}\right)_{6\times 7}$, $\mathbb{D}'^2 = \left(\varpi'^2_{kl}\right)_{6\times 7}$, and $\mathbb{D}'^3 = \left(\varpi'^3_{kl}\right)_{6\times 7}$, which are given as:

$$\mathbb{D}'^1 = \begin{array}{c} \\ \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_3 \\ \mathcal{R}_4 \end{array} \begin{array}{c} \begin{array}{ccccc} \mathcal{S}_1 & \mathcal{S}_2 & \mathcal{S}_3 & \mathcal{S}_4 & \mathcal{S}_5 \end{array} \\ \begin{pmatrix} \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_3, \mathfrak{a}_4 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle \\ \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_1 \rangle \\ \langle \mathfrak{a}_8, \mathfrak{a}_0 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle \\ \langle \mathfrak{a}_7, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle \end{pmatrix} \end{array},$$

$$\mathbb{D}'^2 = \begin{array}{c} \\ \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_3 \\ \mathcal{R}_4 \end{array} \begin{array}{c} \begin{array}{ccccc} \mathcal{S}_1 & \mathcal{S}_2 & \mathcal{S}_3 & \mathcal{S}_4 & \mathcal{S}_5 \end{array} \\ \begin{pmatrix} \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_7, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle \\ \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_2, \mathfrak{a}_6 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle \\ \langle \mathfrak{a}_7, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_8, \mathfrak{a}_0 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_4 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle \\ \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_3 \rangle \end{pmatrix} \end{array},$$

$$\mathbb{D}'^3 = \begin{array}{c} \\ \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_3 \\ \mathcal{R}_4 \end{array} \begin{array}{c} \begin{array}{ccccc} \mathcal{S}_1 & \mathcal{S}_2 & \mathcal{S}_3 & \mathcal{S}_4 & \mathcal{S}_5 \end{array} \\ \begin{pmatrix} \langle \mathfrak{a}_4, \mathfrak{a}_4 \rangle & \langle \mathfrak{a}_7, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle \\ \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_7, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_3 \rangle \\ \langle \mathfrak{a}_7, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_7, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_2 \rangle \\ \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_7, \mathfrak{a}_1 \rangle \end{pmatrix} \end{array}.$$

We again utilize the proposed second GDM method on this example as below:

Step 1: Since all the attributes are asset type, we get the new-decision matrices $\mathbb{D}^1 = \left(\varpi^1_{kl}\right)_{4\times 5} = \left(\left\langle \mathfrak{a}_{\xi^1_{kl}}, \mathfrak{a}_{\eta'^1_{kl}} \right\rangle\right)_{4\times 5}$, $\mathbb{D}^2 = \left(\varpi^2_{kl}\right)_{4\times 5} = \left(\left\langle \mathfrak{a}_{\xi^2_{kl}}, \mathfrak{a}_{\eta'^2_{kl}} \right\rangle\right)_{4\times 5}$, and $\mathbb{D}^3 = \left(\varpi^3_{kl}\right)_{4\times 5} = \left(\left\langle \mathfrak{a}_{\xi^3_{kl}}, \mathfrak{a}_{\eta'^3_{kl}} \right\rangle\right)_{4\times 5}$.

Step 2: Using Eq (4.6), we get the entropies ς^1, ς^2, and ς^3 of the experts $\mathcal{E}\mathfrak{x}_1$, $\mathcal{E}\mathfrak{x}_2$, and $\mathcal{E}\mathfrak{x}_3$, respectively, which are $\varsigma^1 = 0.6161$, $\varsigma^2 = 0.6490$, and $\varsigma^3 = 0.5802$. The weights of experts are find out by applying

the Eq (4.7), which are provided by $\beta_1 = 0.34, \beta_2 = 0.35$, and $\beta_3 = 0.31$.
Step 3: Based on Eq (4.8), we calculate the accumulated Lq-ROFNs ϖ_{kl} using the Lq-ROFNs $\varpi^1_{kl}, \varpi^2_{kl}$, and ϖ^3_{kl}, which belongs to the decision matrices $\mathbb{D}^1 = \left(\varpi^1_{kl}\right)_{4\times5} = \left(\left\langle \mathfrak{a}_{\xi^1_{kl}}, \ \mathfrak{a}_{\eta'^1_{kl}} \right\rangle\right)_{4\times5}$, $\mathbb{D}^2 = \left(\varpi^2_{kl}\right)_{4\times5} = \left(\left\langle \mathfrak{a}_{\xi^2_{kl}}, \ \mathfrak{a}_{\eta'^2_{kl}} \right\rangle\right)_{4\times5}$, and $\mathbb{D}^3 = \left(\varpi^3_{kl}\right)_{4\times5} = \left(\left\langle \mathfrak{a}_{\xi^3_{kl}}, \ \mathfrak{a}_{\eta'^3_{kl}} \right\rangle\right)_{4\times5}$ to construct the cumulative decision matrix $\mathbb{D} = \left(\varpi_{kl}\right)_{4\times5} = \left(\left\langle \mathfrak{a}_{\xi_{kl}}, \ \mathfrak{a}_{\eta_{kl}} \right\rangle\right)_{4\times5}$ with $k = 1, 2, \ldots, 4$, and $l = 1, 2, \ldots, 5$ as given below:

$$\mathbb{D} = \begin{array}{c} \\ \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_3 \\ \mathcal{R}_4 \end{array} \begin{array}{c} \begin{array}{ccccc} \mathcal{S}_1 & \mathcal{S}_2 & \mathcal{S}_3 & \mathcal{S}_4 & \mathcal{S}_5 \end{array} \\ \begin{pmatrix} \langle \mathfrak{a}_{5.74}, \ \mathfrak{a}_{2.37} \rangle & \langle \mathfrak{a}_{6.24}, \ \mathfrak{a}_{1.73} \rangle & \langle \mathfrak{a}_{5.71}, \ \mathfrak{a}_{1.00} \rangle & \langle \mathfrak{a}_{6.31}, \ \mathfrak{a}_{1.96} \rangle & \langle \mathfrak{a}_{5.00}, \ \mathfrak{a}_{2.00} \rangle \\ \langle \mathfrak{a}_{5.00}, \ \mathfrak{a}_{1.00} \rangle & \langle \mathfrak{a}_{3.74}, \ \mathfrak{a}_{3.02} \rangle & \langle \mathfrak{a}_{6.39}, \ \mathfrak{a}_{1.23} \rangle & \langle \mathfrak{a}_{4.77}, \ \mathfrak{a}_{2.27} \rangle & \langle \mathfrak{a}_{4.51}, \ \mathfrak{a}_{1.31} \rangle \\ \langle \mathfrak{a}_{8.00}, \ \mathfrak{a}_{0.00} \rangle & \langle \mathfrak{a}_{5.00}, \ \mathfrak{a}_{1.31} \rangle & \langle \mathfrak{a}_{8.00}, \ \mathfrak{a}_{0.00} \rangle & \langle \mathfrak{a}_{6.36}, \ \mathfrak{a}_{1.83} \rangle & \langle \mathfrak{a}_{5.39}, \ \mathfrak{a}_{1.52} \rangle \\ \langle \mathfrak{a}_{6.43}, \ \mathfrak{a}_{1.23} \rangle & \langle \mathfrak{a}_{5.78}, \ \mathfrak{a}_{1.56} \rangle & \langle \mathfrak{a}_{4.78}, \ \mathfrak{a}_{1.30} \rangle & \langle \mathfrak{a}_{5.27}, \ \mathfrak{a}_{2.27} \rangle & \langle \mathfrak{a}_{6.39}, \ \mathfrak{a}_{1.35} \rangle \end{pmatrix} \end{array}.$$

Step 4: The entropies $\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4$, and ς_5 of the attributes $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4$, and $\mathcal{S}_5$ are respectively calculated as $\varsigma_1 = 0.39, \varsigma_2 = 0.63, \varsigma_3 = 0.39, \varsigma_4 = 0.56$, and $\varsigma_5 = 0.58$ using the Eq (4.9). The weights of attributes $\mathfrak{w} = \{\mathfrak{w}_1(0.15), \mathfrak{w}_2(0.25), \mathfrak{w}_3(0.15), \mathfrak{w}_4(0.22), \mathfrak{w}_5(0.23)\}$ are find out by applying the Eq (4.10).
Step 5: Based on Eq (4.11), we combine the accumulated Lq-ROFNs $\varpi_k = \langle \mathfrak{a}_{\xi_k}, \ \mathfrak{a}_{\eta_k} \rangle$ of each alternative $\mathcal{R}_k$ with $k = 1, 2, 3, 4$, which are provided as $\varpi_1 = \langle \mathfrak{a}_{5.96}, \ \mathfrak{a}_{1.81} \rangle$, $\varpi_2 = \langle \mathfrak{a}_{5.26}, \ \mathfrak{a}_{1.75} \rangle$, $\varpi_3 = \langle \mathfrak{a}_{8.00}, \ \mathfrak{a}_{0.00} \rangle$, and $\varpi_4 = \langle \mathfrak{a}_{5.97}, \ \mathfrak{a}_{1.65} \rangle$
Step 6: Next, using Eq (4.12), we calculate the ranking values $\mathcal{Y}(\varpi_1)$, $\mathcal{Y}(\varpi_2)$, $\mathcal{Y}(\varpi_3)$, and $\mathcal{Y}(\varpi_4)$ of each object as follows: $\mathcal{Y}(\varpi_1) = 5.43$, $\mathcal{Y}(\varpi_2) = 4.89$, $\mathcal{Y}(\varpi_3) = 8.00$, and $\mathcal{Y}(\varpi_4) = 5.42$.
Step 7: From Step 6, we see that $\mathcal{Y}(\varpi_3) > \mathcal{Y}(\varpi_1) > \mathcal{Y}(\varpi_4) > \mathcal{Y}(\varpi_2)$, Hence, the best car out of collection $\mathcal{R}_k$ with $k = 1, 2, 3, 4$ is $\mathcal{R}_3$.

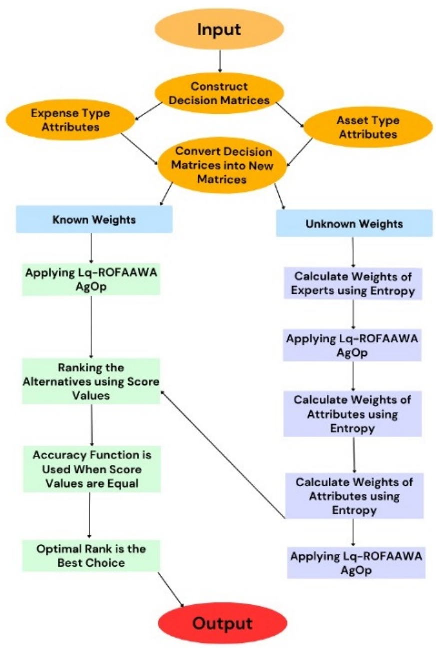

Figure 1. Flowchart diagram.

The flowchart diagram shown in Figure 1 is suggested to the reader for a better understanding of the process discussed in the above GDM methodologies.

5. Tests to check validity of the GDM methods

To estimate the authenticity of decision-making approaches, Wang and Triantaphyllou [38] discussed three test criteria as below:

Test 1: In the case of an efficient GDM process, the rank of the optimal object is not affected by changing the sub-optimal object by any bad object without altering the significance of corresponding decision attribute.

Test 2: The transitive property must be held to get an effective decision making approach.

Test 3: If a GDM situation is divided into a sub-GDM parts, and applied the suggested GDM approach to both scenarios, then the ranking of alternatives of sub-GDM problems should be same as the ranking of alternatives as original GDM problem.

First, we verify the reliability and authenticity of our suggested first GDM method by implementing these test criteria to Example 4.2 as given in the previous section.

Test 1: For estimating the authenticity of our suggested first GDM technique using Test 1, we alternate the sub-optimal object $\mathcal{R}_3$ in the provided decision matrices $\mathbb{D}'^1, \mathbb{D}'^2, \mathbb{D}'^3, \mathbb{D}'^4, \mathbb{D}'^5$, and $\mathbb{D}'^6$ of Example 4.2, which are estimated from the experts $\mathcal{E}\mathfrak{x}_s$, where $s = 1, 2, 3, \ldots, 6$ as given below:

$$\mathbb{D}'^1 = \begin{array}{c} \\ \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_3 \\ \mathcal{R}_4 \\ \mathcal{R}_5 \\ \mathcal{R}_6 \end{array} \begin{array}{c} \begin{array}{ccccccc} \mathcal{S}_1 & \mathcal{S}_2 & \mathcal{S}_3 & \mathcal{S}_4 & \mathcal{S}_5 & \mathcal{S}_6 & \mathcal{S}_7 \end{array} \\ \left(\begin{array}{ccccccc}
\langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_7, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_4 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_3 \rangle \\
\langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_7, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_4 \rangle & \langle \mathfrak{a}_7, \mathfrak{a}_1 \rangle \\
\langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_7, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_3, \mathfrak{a}_2 \rangle \\
\langle \mathfrak{a}_7, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_7, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_3 \rangle \\
\langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_4 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_3 \rangle \\
\langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_3, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_3 \rangle
\end{array} \right) \end{array},$$

$$\mathbb{D}'^2 = \begin{array}{c} \\ \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_3 \\ \mathcal{R}_4 \\ \mathcal{R}_5 \\ \mathcal{R}_6 \end{array} \begin{array}{c} \begin{array}{ccccccc} \mathcal{S}_1 & \mathcal{S}_2 & \mathcal{S}_3 & \mathcal{S}_4 & \mathcal{S}_5 & \mathcal{S}_6 & \mathcal{S}_7 \end{array} \\ \left(\begin{array}{ccccccc}
\langle \mathfrak{a}_4, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_4 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_7, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle \\
\langle \mathfrak{a}_4, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_4 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_1 \rangle \\
\langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle \\
\langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_3 \rangle \\
\langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_3 \rangle \\
\langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle
\end{array} \right) \end{array},$$

$$\mathbb{D}'^3 = \begin{array}{c} \\ \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_3 \\ \mathcal{R}_4 \\ \mathcal{R}_5 \\ \mathcal{R}_6 \end{array} \begin{array}{c} \begin{array}{ccccccc} \mathcal{S}_1 & \mathcal{S}_2 & \mathcal{S}_3 & \mathcal{S}_4 & \mathcal{S}_5 & \mathcal{S}_6 & \mathcal{S}_7 \end{array} \\ \left(\begin{array}{ccccccc}
\langle \mathfrak{a}_7, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_2 \rangle \\
\langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle \\
\langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_3, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle \\
\langle \mathfrak{a}_4, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_3 \rangle \\
\langle \mathfrak{a}_4, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_4 \rangle & \langle \mathfrak{a}_3, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_3 \rangle \\
\langle \mathfrak{a}_4, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_3, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_3 \rangle
\end{array} \right) \end{array},$$

$$\mathbb{D}'^{4} = \begin{array}{c} \\ \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_3 \\ \mathcal{R}_4 \\ \mathcal{R}_5 \\ \mathcal{R}_6 \end{array} \begin{array}{c} \begin{array}{ccccccc} \mathcal{S}_1 & \mathcal{S}_2 & \mathcal{S}_3 & \mathcal{S}_4 & \mathcal{S}_5 & \mathcal{S}_6 & \mathcal{S}_7 \end{array} \\ \left(\begin{array}{ccccccc} \langle \mathfrak{a}_4, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_5 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_2 \rangle \\ \langle \mathfrak{a}_4, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle \\ \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_7, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_3, \mathfrak{a}_2 \rangle \\ \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_7, \mathfrak{a}_1 \rangle \\ \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_4 \rangle & \langle \mathfrak{a}_7, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_3 \rangle \\ \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_7, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle \end{array} \right) \end{array},$$

$$\mathbb{D}'^{5} = \begin{array}{c} \\ \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_3 \\ \mathcal{R}_4 \\ \mathcal{R}_5 \\ \mathcal{R}_6 \end{array} \begin{array}{c} \begin{array}{ccccccc} \mathcal{S}_1 & \mathcal{S}_2 & \mathcal{S}_3 & \mathcal{S}_4 & \mathcal{S}_5 & \mathcal{S}_6 & \mathcal{S}_7 \end{array} \\ \left(\begin{array}{ccccccc} \langle \mathfrak{a}_4, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle \\ \langle \mathfrak{a}_4, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_5 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_2 \rangle \\ \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_3, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle \\ \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle \\ \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_7, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_4 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_3 \rangle \\ \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_4 \rangle & \langle \mathfrak{a}_7, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_3 \rangle \end{array} \right) \end{array},$$

$$\mathbb{D}'^{6} = \begin{array}{c} \\ \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_3 \\ \mathcal{R}_4 \\ \mathcal{R}_5 \\ \mathcal{R}_6 \end{array} \begin{array}{c} \begin{array}{ccccccc} \mathcal{S}_1 & \mathcal{S}_2 & \mathcal{S}_3 & \mathcal{S}_4 & \mathcal{S}_5 & \mathcal{S}_6 & \mathcal{S}_7 \end{array} \\ \left(\begin{array}{ccccccc} \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_7, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_3 \rangle \\ \langle \mathfrak{a}_5, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_1 \rangle \\ \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_1 \rangle \\ \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_4 \rangle & \langle \mathfrak{a}_7, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_3 \rangle \\ \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_7, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_4 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_3 \rangle \\ \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle \end{array} \right) \end{array}.$$

The accumulated Lq-ROFNs of the bad alternative $\mathcal{R}_3'$ in the given decision making matrices $\mathbb{D}'^1, \mathbb{D}'^2, \mathbb{D}'^3, \mathbb{D}'^4, \mathbb{D}'^5$, and $\mathbb{D}'^6$ are chosen arbitrary. Thus, the developed decision matrices are $\mathbb{D}'^{*1}, \mathbb{D}'^{*2}, \mathbb{D}'^{*3}, \mathbb{D}'^{*4}, \mathbb{D}'^{*5}$, and $\mathbb{D}'^{*6}$ as below:

$$\mathbb{D}'^{*1} = \begin{array}{c} \\ \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_3' \\ \mathcal{R}_4 \\ \mathcal{R}_5 \\ \mathcal{R}_6 \end{array} \begin{array}{c} \begin{array}{ccccccc} \mathcal{S}_1 & \mathcal{S}_2 & \mathcal{S}_3 & \mathcal{S}_4 & \mathcal{S}_5 & \mathcal{S}_6 & \mathcal{S}_7 \end{array} \\ \left(\begin{array}{ccccccc} \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_7, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_4 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_3 \rangle \\ \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_7, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_4 \rangle & \langle \mathfrak{a}_7, \mathfrak{a}_1 \rangle \\ \langle \mathfrak{a}_4, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_7, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_1 \rangle \\ \langle \mathfrak{a}_7, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_7, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_3 \rangle \\ \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_4 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_3 \rangle \\ \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_3, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_3 \rangle \end{array} \right) \end{array},$$

$$\mathbb{D}'^{*2} = \begin{array}{c} \\ \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_3' \\ \mathcal{R}_4 \\ \mathcal{R}_5 \\ \mathcal{R}_6 \end{array} \begin{array}{c} \begin{array}{ccccccc} \mathcal{S}_1 & \mathcal{S}_2 & \mathcal{S}_3 & \mathcal{S}_4 & \mathcal{S}_5 & \mathcal{S}_6 & \mathcal{S}_7 \end{array} \\ \left(\begin{array}{ccccccc} \langle \mathfrak{a}_4, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_4 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_7, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle \\ \langle \mathfrak{a}_4, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_4 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_1 \rangle \\ \langle \mathfrak{a}_5, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_7, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle \\ \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_3 \rangle \\ \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_3 \rangle \\ \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle \end{array} \right) \end{array},$$

$$\mathbb{D}'^{*3} = \begin{array}{c} \\ \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_3' \\ \mathcal{R}_4 \\ \mathcal{R}_5 \\ \mathcal{R}_6 \end{array} \begin{array}{c} \begin{array}{ccccccc} \mathcal{S}_1 & \mathcal{S}_2 & \mathcal{S}_3 & \mathcal{S}_4 & \mathcal{S}_5 & \mathcal{S}_6 & \mathcal{S}_7 \end{array} \\ \begin{pmatrix} \langle a_7, a_1\rangle & \langle a_5, a_1\rangle & \langle a_4, a_3\rangle & \langle a_4, a_1\rangle & \langle a_6, a_1\rangle & \langle a_5, a_2\rangle & \langle a_4, a_2\rangle \\ \langle a_5, a_2\rangle & \langle a_4, a_3\rangle & \langle a_6, a_1\rangle & \langle a_4, a_1\rangle & \langle a_6, a_2\rangle & \langle a_4, a_1\rangle & \langle a_5, a_2\rangle \\ \langle a_3, a_1\rangle & \langle a_5, a_2\rangle & \langle a_6, a_2\rangle & \langle a_7, a_1\rangle & \langle a_4, a_2\rangle & \langle a_5, a_2\rangle & \langle a_6, a_1\rangle \\ \langle a_4, a_2\rangle & \langle a_5, a_2\rangle & \langle a_4, a_1\rangle & \langle a_5, a_1\rangle & \langle a_4, a_2\rangle & \langle a_5, a_1\rangle & \langle a_5, a_3\rangle \\ \langle a_4, a_2\rangle & \langle a_6, a_2\rangle & \langle a_4, a_4\rangle & \langle a_3, a_1\rangle & \langle a_5, a_2\rangle & \langle a_6, a_1\rangle & \langle a_4, a_3\rangle \\ \langle a_4, a_2\rangle & \langle a_5, a_2\rangle & \langle a_6, a_1\rangle & \langle a_4, a_1\rangle & \langle a_3, a_2\rangle & \langle a_6, a_2\rangle & \langle a_5, a_3\rangle \end{pmatrix} \end{array},$$

$$\mathbb{D}'^{*4} = \begin{array}{c} \\ \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_3' \\ \mathcal{R}_4 \\ \mathcal{R}_5 \\ \mathcal{R}_6 \end{array} \begin{array}{c} \begin{array}{ccccccc} \mathcal{S}_1 & \mathcal{S}_2 & \mathcal{S}_3 & \mathcal{S}_4 & \mathcal{S}_5 & \mathcal{S}_6 & \mathcal{S}_7 \end{array} \\ \begin{pmatrix} \langle a_4, a_1\rangle & \langle a_6, a_1\rangle & \langle a_5, a_3\rangle & \langle a_5, a_5\rangle & \langle a_4, a_1\rangle & \langle a_6, a_2\rangle & \langle a_4, a_2\rangle \\ \langle a_4, a_1\rangle & \langle a_5, a_2\rangle & \langle a_6, a_1\rangle & \langle a_5, a_2\rangle & \langle a_6, a_1\rangle & \langle a_4, a_1\rangle & \langle a_5, a_1\rangle \\ \langle a_4, a_2\rangle & \langle a_7, a_1\rangle & \langle a_5, a_1\rangle & \langle a_4, a_3\rangle & \langle a_6, a_1\rangle & \langle a_6, a_2\rangle & \langle a_5, a_2\rangle \\ \langle a_6, a_1\rangle & \langle a_6, a_2\rangle & \langle a_5, a_3\rangle & \langle a_4, a_1\rangle & \langle a_6, a_2\rangle & \langle a_5, a_2\rangle & \langle a_7, a_1\rangle \\ \langle a_5, a_1\rangle & \langle a_6, a_2\rangle & \langle a_4, a_4\rangle & \langle a_7, a_1\rangle & \langle a_6, a_2\rangle & \langle a_5, a_2\rangle & \langle a_5, a_3\rangle \\ \langle a_6, a_2\rangle & \langle a_5, a_1\rangle & \langle a_6, a_2\rangle & \langle a_7, a_1\rangle & \langle a_4, a_1\rangle & \langle a_5, a_2\rangle & \langle a_6, a_1\rangle \end{pmatrix} \end{array},$$

$$\mathbb{D}'^{*5} = \begin{array}{c} \\ \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_3' \\ \mathcal{R}_4 \\ \mathcal{R}_5 \\ \mathcal{R}_6 \end{array} \begin{array}{c} \begin{array}{ccccccc} \mathcal{S}_1 & \mathcal{S}_2 & \mathcal{S}_3 & \mathcal{S}_4 & \mathcal{S}_5 & \mathcal{S}_6 & \mathcal{S}_7 \end{array} \\ \begin{pmatrix} \langle a_4, a_1\rangle & \langle a_5, a_2\rangle & \langle a_6, a_1\rangle & \langle a_5, a_2\rangle & \langle a_6, a_1\rangle & \langle a_4, a_1\rangle & \langle a_5, a_1\rangle \\ \langle a_4, a_1\rangle & \langle a_6, a_1\rangle & \langle a_5, a_3\rangle & \langle a_5, a_5\rangle & \langle a_4, a_1\rangle & \langle a_6, a_2\rangle & \langle a_4, a_2\rangle \\ \langle a_5, a_3\rangle & \langle a_4, a_3\rangle & \langle a_6, a_2\rangle & \langle a_3, a_2\rangle & \langle a_5, a_2\rangle & \langle a_5, a_1\rangle & \langle a_7, a_1\rangle \\ \langle a_6, a_1\rangle & \langle a_5, a_3\rangle & \langle a_6, a_2\rangle & \langle a_6, a_1\rangle & \langle a_4, a_3\rangle & \langle a_5, a_2\rangle & \langle a_5, a_1\rangle \\ \langle a_5, a_2\rangle & \langle a_7, a_1\rangle & \langle a_4, a_3\rangle & \langle a_6, a_1\rangle & \langle a_4, a_4\rangle & \langle a_5, a_2\rangle & \langle a_4, a_3\rangle \\ \langle a_5, a_1\rangle & \langle a_6, a_2\rangle & \langle a_4, a_4\rangle & \langle a_7, a_1\rangle & \langle a_6, a_2\rangle & \langle a_5, a_2\rangle & \langle a_5, a_3\rangle \end{pmatrix} \end{array},$$

$$\mathbb{D}'^{*6} = \begin{array}{c} \\ \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_3' \\ \mathcal{R}_4 \\ \mathcal{R}_5 \\ \mathcal{R}_6 \end{array} \begin{array}{c} \begin{array}{ccccccc} \mathcal{S}_1 & \mathcal{S}_2 & \mathcal{S}_3 & \mathcal{S}_4 & \mathcal{S}_5 & \mathcal{S}_6 & \mathcal{S}_7 \end{array} \\ \begin{pmatrix} \langle a_5, a_2\rangle & \langle a_6, a_1\rangle & \langle a_5, a_2\rangle & \langle a_7, a_1\rangle & \langle a_5, a_2\rangle & \langle a_6, a_1\rangle & \langle a_5, a_3\rangle \\ \langle a_5, a_3\rangle & \langle a_6, a_2\rangle & \langle a_4, a_2\rangle & \langle a_5, a_1\rangle & \langle a_6, a_2\rangle & \langle a_5, a_3\rangle & \langle a_4, a_1\rangle \\ \langle a_6, a_2\rangle & \langle a_5, a_2\rangle & \langle a_4, a_3\rangle & \langle a_5, a_3\rangle & \langle a_7, a_1\rangle & \langle a_6, a_2\rangle & \langle a_3, a_2\rangle \\ \langle a_5, a_1\rangle & \langle a_6, a_2\rangle & \langle a_4, a_4\rangle & \langle a_7, a_1\rangle & \langle a_6, a_2\rangle & \langle a_5, a_2\rangle & \langle a_5, a_3\rangle \\ \langle a_5, a_2\rangle & \langle a_7, a_1\rangle & \langle a_4, a_3\rangle & \langle a_6, a_1\rangle & \langle a_4, a_4\rangle & \langle a_5, a_2\rangle & \langle a_4, a_3\rangle \\ \langle a_6, a_1\rangle & \langle a_5, a_3\rangle & \langle a_6, a_2\rangle & \langle a_6, a_1\rangle & \langle a_4, a_3\rangle & \langle a_5, a_2\rangle & \langle a_5, a_1\rangle \end{pmatrix} \end{array}.$$

The ranking values of $\mathcal{Y}(\varpi_1)$, $\mathcal{Y}(\varpi_2)$, $\mathcal{Y}(\varpi_3)$, $\mathcal{Y}(\varpi_4)$, $\mathcal{Y}(\varpi_5)$, and $\mathcal{Y}(\varpi_6)$ are calculated as $\mathcal{Y}(\varpi_1) = 5.38$, $\mathcal{Y}(\varpi_2) = 5.06$, $\mathcal{Y}(\varpi_3) = 5.23$, $\mathcal{Y}(\varpi_4) = 5.16$, $\mathcal{Y}(\varpi_5) = 5.12$, and $\mathcal{Y}(\varpi_6) = 5.02$. Next, by applying the suggested first GDM approach on the above determined decision matrices $\mathbb{D}'^{*1}, \mathbb{D}'^{*2}, \mathbb{D}'^{*3}, \mathbb{D}'^{*4}, \mathbb{D}'^{*5}$, and $\mathbb{D}'^{*6}$. Clearly, $\mathcal{Y}(\varpi_1) > \mathcal{Y}(\varpi_3) > \mathcal{Y}(\varpi_4) > \mathcal{Y}(\varpi_5) > \mathcal{Y}(\varpi_2) > \mathcal{Y}(\varpi_6)$. So, the most suitable choice from all the other choices $\mathcal{R}_k$ with $k = 1, 2, \ldots, 6$ is $\mathcal{R}_1$. Note that the object $\mathcal{R}_1$ is the best alternative by applying the suggested first GDM approach on both the original decision matrices and the substituted decision matrices. Hence, the suggested first GDM method verifies Test Criterion 1.

Tests Criteria 2$^{\text{nd}}$ and 3$^{\text{rd}}$: For investigating the authenticity of the first GDM method using 2nd and 3rd test criteria, we divide the original GDM situation of Example 4.2 into three sub-GDM parts where the objects are $\{\mathcal{R}_1, \mathcal{R}_3, \mathcal{R}_4, \mathcal{R}_5\}$, $\{\mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_4, \mathcal{R}_5\}$, and $\{\mathcal{R}_3, \mathcal{R}_5, \mathcal{R}_2, \mathcal{R}_6\}$. Now, by employing the suggested first GDM method on these sub-problems, we obtain the rankings as "$\mathcal{R}_1 > \mathcal{R}_4 > \mathcal{R}_3 > \mathcal{R}_5$",

"$\mathcal{R}_4 > \mathcal{R}_3 > \mathcal{R}_5 > \mathcal{R}_2$", and "$\mathcal{R}_3 > \mathcal{R}_5 > \mathcal{R}_2 > \mathcal{R}_6$". To verify the transitivity, these three rankings of sub-GDM problems are merged, and the final ranking is provided by "$\mathcal{R}_1 > \mathcal{R}_4 > \mathcal{R}_3 > \mathcal{R}_5, \mathcal{R}_2 > \mathcal{R}_6$", which is similar to the ranking orders of alternatives $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_4, \mathcal{R}_5$ and $\mathcal{R}_6$ as determined in the Example 4.2. Thus, the suggested 1st GDM method verifies the Tests Criteria 2nd and 3rd.

We are now ready to check the reliability of the suggested second GDM method.

Test 1: For estimating the authenticity of our suggested 1st GDM method with Test 1, we put a substitute of the sub-optimal object $\mathcal{R}_3$ in the decision matrices $\mathbb{D}'^1, \mathbb{D}'^2$, and $\mathbb{D}'^3$ of Example 4.5, so, we get

$$\mathbb{D}'^1 = \begin{array}{c} \\ \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_3 \\ \mathcal{R}_4 \end{array} \begin{array}{c} \begin{array}{ccccc} \mathcal{S}_1 & \mathcal{S}_2 & \mathcal{S}_3 & \mathcal{S}_4 & \mathcal{S}_5 \end{array} \\ \begin{pmatrix} \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_3, \mathfrak{a}_4 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle \\ \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_1 \rangle \\ \langle \mathfrak{a}_8, \mathfrak{a}_0 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle \\ \langle \mathfrak{a}_7, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle \end{pmatrix} \end{array},$$

$$\mathbb{D}'^2 = \begin{array}{c} \\ \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_3 \\ \mathcal{R}_4 \end{array} \begin{array}{c} \begin{array}{ccccc} \mathcal{S}_1 & \mathcal{S}_2 & \mathcal{S}_3 & \mathcal{S}_4 & \mathcal{S}_5 \end{array} \\ \begin{pmatrix} \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_7, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle \\ \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_2, \mathfrak{a}_6 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle \\ \langle \mathfrak{a}_7, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_8, \mathfrak{a}_0 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_4 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle \\ \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_3 \rangle \end{pmatrix} \end{array},$$

$$\mathbb{D}'^3 = \begin{array}{c} \\ \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_3 \\ \mathcal{R}_4 \end{array} \begin{array}{c} \begin{array}{ccccc} \mathcal{S}_1 & \mathcal{S}_2 & \mathcal{S}_3 & \mathcal{S}_4 & \mathcal{S}_5 \end{array} \\ \begin{pmatrix} \langle \mathfrak{a}_4, \mathfrak{a}_4 \rangle & \langle \mathfrak{a}_7, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle \\ \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_7, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_3 \rangle \\ \langle \mathfrak{a}_7, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_7, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_2 \rangle \\ \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_7, \mathfrak{a}_1 \rangle \end{pmatrix} \end{array}.$$

The accumulated Lq-ROFNs of the bad alternative $\mathcal{R}_3'$ are chosen arbitrarily, and are respectively inserted in the given matrices $\mathbb{D}'^1, \mathbb{D}'^2$, and $\mathbb{D}'^3$. Thus, the computed decision matrices $\mathbb{D}'^{*1}, \mathbb{D}'^{*2}$, and $\mathbb{D}'^{*3}$ are given as:

$$\mathbb{D}'^{*1} = \begin{array}{c} \\ \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_{3'} \\ \mathcal{R}_4 \end{array} \begin{array}{c} \begin{array}{ccccc} \mathcal{S}_1 & \mathcal{S}_2 & \mathcal{S}_3 & \mathcal{S}_4 & \mathcal{S}_5 \end{array} \\ \begin{pmatrix} \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_3, \mathfrak{a}_4 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle \\ \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_1 \rangle \\ \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_8, \mathfrak{a}_0 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_4 \rangle & \langle \mathfrak{a}_7, \mathfrak{a}_1 \rangle \\ \langle \mathfrak{a}_7, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle \end{pmatrix} \end{array},$$

$$\mathbb{D}'^{*2} = \begin{array}{c} \\ \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_{3'} \\ \mathcal{R}_4 \end{array} \begin{array}{c} \begin{array}{ccccc} \mathcal{S}_1 & \mathcal{S}_2 & \mathcal{S}_3 & \mathcal{S}_4 & \mathcal{S}_5 \end{array} \\ \begin{pmatrix} \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_7, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle \\ \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_2, \mathfrak{a}_6 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle \\ \langle \mathfrak{a}_4, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_8, \mathfrak{a}_0 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle \\ \langle \mathfrak{a}_5, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_6, \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_4, \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_5, \mathfrak{a}_3 \rangle \end{pmatrix} \end{array},$$

$$\mathbb{D}'^{*3} = \begin{matrix} & \mathcal{S}_1 & \mathcal{S}_2 & \mathcal{S}_3 & \mathcal{S}_4 & \mathcal{S}_5 \\ \mathcal{R}_1 & \langle \mathfrak{a}_4,\ \mathfrak{a}_4 \rangle & \langle \mathfrak{a}_7,\ \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_6,\ \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5,\ \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_5,\ \mathfrak{a}_2 \rangle \\ \mathcal{R}_2 & \langle \mathfrak{a}_5,\ \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_4,\ \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_7,\ \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5,\ \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4,\ \mathfrak{a}_3 \rangle \\ \mathcal{R}_{3'} & \langle \mathfrak{a}_5,\ \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_5,\ \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_4,\ \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_6,\ \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_6,\ \mathfrak{a}_2 \rangle \\ \mathcal{R}_4 & \langle \mathfrak{a}_6,\ \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_6,\ \mathfrak{a}_1 \rangle & \langle \mathfrak{a}_5,\ \mathfrak{a}_3 \rangle & \langle \mathfrak{a}_4,\ \mathfrak{a}_2 \rangle & \langle \mathfrak{a}_7,\ \mathfrak{a}_1 \rangle \end{matrix}.$$

Consequently, by applying the suggested second GDM method on the above developed decision matrices $\mathbb{D}'^{*1}, \mathbb{D}'^{*2}$, and $\mathbb{D}'^{*3}$, the ranking values $\mathcal{Y}(\varpi_1)$, $\mathcal{Y}(\varpi_2)$, $\mathcal{Y}(\varpi_3)$, and $\mathcal{Y}(\varpi_4)$ are calculated as: $\mathcal{Y}(\varpi_1) = 5.41$, $\mathcal{Y}(\varpi_2) = 4.94$, $\mathcal{Y}(\varpi_3) = 8.00$, and $\mathcal{Y}(\varpi_4) = 5.48$. As $\mathcal{Y}(\varpi_3) > \mathcal{Y}(\varpi_4) > \mathcal{Y}(\varpi_1) > \mathcal{Y}(\varpi_2)$, so, the most suitable alternative from the alternatives $\mathcal{R}_k$ with $k = 1, 2, \ldots, 4$ is $\mathcal{R}_3$. It can be observed that the optimal object is $\mathcal{R}_3$ in both the original and transformed decision problems. Hence, it is verified that the suggested second GDM method holds the requirements of Test Pattern 1.

Test Criteria 2nd and 3rd: To investigate the reliability of the second GDM method, we divide the genuine GDM dilemma of Example 4.5 into three dependent GDM questions that include the objects as: $\{\mathcal{R}_3, \mathcal{R}_2, \mathcal{R}_4\}$, $\{\mathcal{R}_3, \mathcal{R}_1, \mathcal{R}_4\}$, and $\{\mathcal{R}_2, \mathcal{R}_1, \mathcal{R}_4\}$. Next, by employing the suggested 2nd GDM method on these sub-problems, we obtain the ranks as follows: '$\mathcal{R}_3 > \mathcal{R}_4 > \mathcal{R}_2$', '$\mathcal{R}_3 > \mathcal{R}_1 > \mathcal{R}_4$' and '$\mathcal{R}_1 > \mathcal{R}_4 > \mathcal{R}_2$'. Further, we combine these determined ranks and find the final rankings as '$\mathcal{R}_3 > \mathcal{R}_1 > \mathcal{R}_4 > \mathcal{R}_2$', which is the same as computed in the Example 4.5, and thus, it showed the transitive property. Thus, the suggested second GDM method verifies the Test Criteria 2nd and 3rd.

5.1. Discussion

In this section, we compare our suggested GDM methods with some approaches, including Kumar and Chen's GDM technique [24]. In addition, we investigate the effect of q in the Lq-ROFNs for Example 4.3 as presented in the previous section.

One may verify from Table 1 that the optimal decision object is invariant in the case of the first GDM method and preexisting approaches [10, 20, 24, 28, 40, 57], that is, $\mathcal{R}_4$. Moreover, there is a minor change in the rankings of sub-optimal objects. In [60], a GDM method is presented based on q-ROF numbers using Aczel-Alsina TN and TCN operations. The current study extended the method introduced in [60] using linguistic terms, and proposed Lq-ROFAAWA operators. In certain situations, Lq-ROFNs perform better than q-ROFNs for the following reason: Lq-ROFNs simplify the process for decision-makers by representing uncertainty with linguistic terms (such as high, medium, and low), making it easier to express preferences. In contrast, q-ROFNs rely on numerical information, which can be more challenging for non-experts to interpret.

Table 1. The comparison of suggested first GDM approach with some existing methods for Example 4.3.

1st-GDM method	Ranking orders
Liu and Wang's GDM technique [57]	$\mathcal{R}_4 > \mathcal{R}_3 > \mathcal{R}_2 > \mathcal{R}_1$
Chen et al.'s GDM method [10]	$\mathcal{R}_4 > \mathcal{R}_3 > \mathcal{R}_1 > \mathcal{R}_2$
Kumar and Chen's GDM technique [24]	$\mathcal{R}_4 > \mathcal{R}_3 > \mathcal{R}_1 > \mathcal{R}_2$
Garg and Kumar's GDM method [20]	$\mathcal{R}_4 > \mathcal{R}_1 > \mathcal{R}_3 > \mathcal{R}_2$
Nayana et al. GDM technique [40]	$\mathcal{R}_4 > \mathcal{R}_1 > \mathcal{R}_3 > \mathcal{R}_2$
Kumar and Chen's GDM method [28]	$\mathcal{R}_4 > \mathcal{R}_3 > \mathcal{R}_1 > \mathcal{R}_2$
The suggested first GDM approach	$\mathcal{R}_4 > \mathcal{R}_3 > \mathcal{R}_1 > \mathcal{R}_2$

Additionally, note that the parameter q in the Lq-ROFNs plays an important role in group decision making problems. In the preceding section, a GDM problem is solved in Example 4.3 by considering $q = 3$. Now, to examine the effect of q-rung parameter on this problem again, we solve for every $q \in \{1, 2, 4, 5, \ldots, 10\}$, where Aczel-Alsina parameter is $\lambda = 2$ in the algorithmic process. The calculated ranking value for each alternative is given in Table 2 for every $q \in \{1, 2, 3, \ldots, 10\}$. From Table 2, it is clear that the ranking values are changed for all $q \in \{1, 2, 3, \ldots, 10\}$, but the overall ranking orders are invariant, which yields a very interesting fact that the applicability scope of suggested approach for GDM using Lq-ROFAAWA aggregation operator is higher than existing models, e.g., [24].

Table 2. Effect of parameter q on the ranking orders of alternatives with suggested Lq-ROFAAWA operator on Example 4.3.

Parameter (q)	$\mathcal{Y}(\varpi_1)$	$\mathcal{Y}(\varpi_2)$	$\mathcal{Y}(\varpi_3)$	$\mathcal{Y}(\varpi_4)$	Ranking orders (or $\mathcal{RO}s$)
$q = 1$	4.08	3.60	4.21	5.50	$\mathcal{R}_4 > \mathcal{R}_3 > \mathcal{R}_1 > \mathcal{R}_2$
$q = 2$	4.39	4.01	4.54	4.76	$\mathcal{R}_4 > \mathcal{R}_3 > \mathcal{R}_1 > \mathcal{R}_2$
$q = 3$	4.44	4.03	4.57	5.23	$\mathcal{R}_4 > \mathcal{R}_3 > \mathcal{R}_1 > \mathcal{R}_2$
$q = 4$	4.45	4.04	4.56	5.12	$\mathcal{R}_4 > \mathcal{R}_3 > \mathcal{R}_1 > \mathcal{R}_2$
$q = 5$	4.51	4.11	4.60	5.10	$\mathcal{R}_4 > \mathcal{R}_3 > \mathcal{R}_1 > \mathcal{R}_2$
$q = 6$	4.75	4.30	4.82	5.31	$\mathcal{R}_4 > \mathcal{R}_3 > \mathcal{R}_1 > \mathcal{R}_2$
$q = 7$	4.71	4.27	4.77	5.23	$\mathcal{R}_4 > \mathcal{R}_3 > \mathcal{R}_1 > \mathcal{R}_2$
$q = 8$	4.65	4.22	4.71	5.14	$\mathcal{R}_4 > \mathcal{R}_3 > \mathcal{R}_1 > \mathcal{R}_2$
$q = 9$	4.73	4.29	4.78	5.22	$\mathcal{R}_4 > \mathcal{R}_3 > \mathcal{R}_1 > \mathcal{R}_2$
$q = 10$	4.69	4.27	4.75	5.18	$\mathcal{R}_4 > \mathcal{R}_3 > \mathcal{R}_1 > \mathcal{R}_2$

From Table 2, it can be observed that by varying the range of parameter q from 1 to 10, the ranking order $\mathcal{R}_4 > \mathcal{R}_3 > \mathcal{R}_1 > \mathcal{R}_2$ remains the same (see also Figure 2). This does not imply that the approach is generally insensitive. Sensitivity refers to the extent to which the output changes in response to variations in the input. The ranking order is determined by the input information, while q serves as a parameter. Additionally, it may indicate certain aspects of the proposed technique's sensitivity, as outlined below:

Robustness: If the approach is robust, it can yield consistent results even when the parameters are changed. This may be a good thing as it suggests reliability.

System stability: Stability may be inherent in the system or data-set being evaluated. If the characteristics or criteria that influence the ranking remain dominant despite changes in q, the key components of the ranking may remain robust and consistent across a range of q values. This would suggest a stable and potentially well-functioning system.

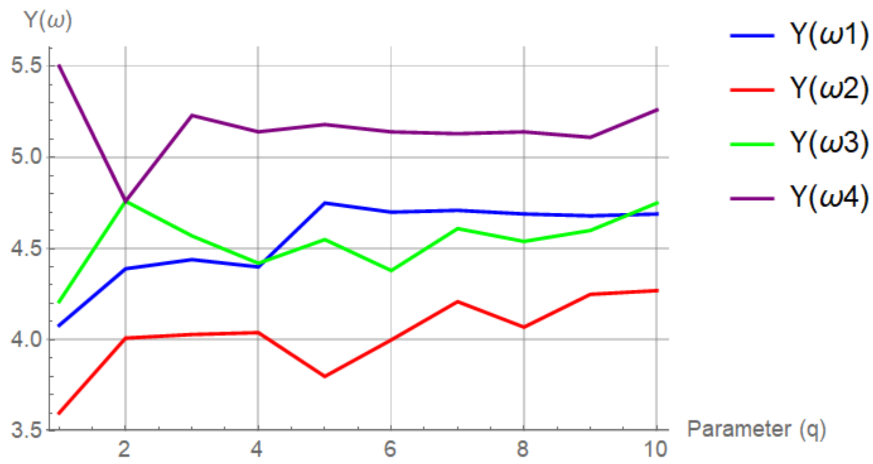

Figure 2. Effect of parameter q on the object's final score values with suggested Lq-ROFAAWA operator on Example 4.3.

5.2. Limitations of the proposed methodologies

Like every preexisting GDM model, our proposed methodologies have their pros and cons. The advantages are discussed in the previous sections, however, in this section, we will discuss certain limitations of the developed methods that we identified during this study. The first limitation of the presented work is the complexity in computations when dealing with large datasets. This limitation, however, may be resolved with the utilization of mathematical software like MATLAB, MAPLE, etc. The second limitation of the proposed study is its inefficiency in handling the neutral part of the data.

6. Conclusions and future directions

In this article, we developed two novel operations using Lq-ROFNs based on the Aczel-Alsina's t-conorm (TCN) and t-norm (TN), i.e., addition and scalar multiplication. These operations overcome the flaws of Kumar and Chen's addition and scalar product operations [24] of LIFNs. Moreover, we have introduced the linguistic q-rung orthopair fuzzy Aczel-Alsina weighted arithmetic (Lq-ROFAAWA) aggregation operator of Lq-ROFNs using presented addition and scalar multiplication operations. Further, we have suggested two types of new group decision-making (GDM) techniques based on the suggested Lq-ROFAAWA aggregation operator. The proposed first GDM method assumes the condition that the weights of experts and weights of attributes are known completely. However, the suggested second GDM method considers the condition that the weights of experts and the weights of attributes are unknown. Moreover, we applied the suggested methodologies to different environmental and economic real-world issues and successfully solved them, i.e., ranking of countries regarding air pollution, selection of best company for bank to invest, and selection of best electric vehicle design. In the end, we have validated our proposed GDM methods with three tests and compared them with preexisting GDM methods.

Consequently, the suggested GDM techniques overcome the deficiencies in different recent works, including Chen et al.'s GDM method [10], Kumar and Chen's GDM technique [24], Garg and Kumar's GDM technique [20], Liu and Wang's GDM technique [57], Tang and Meng's GDM technique [27], and Kumar and Chen's GDM method [28] because they cannot differentiate the ranking positions of objects. In future research, this study can be extended to different domains, such as Aczel-Alsina operators based on linguistic fuzzy measurements and rough attributes as studied in [4]; approximation of linguistic q-rung orthopair fuzzy information with covering-based rough sets using the concepts presented in [5]; and Aczel-Alsina operators based on linguistic quasi-rung fuzzy sets as proposed in [61, 62].

Author contributions

Ghous Ali: conceptualization, data curation, formal analysis, supervision, validation, methodology, visualization, writing-review and editing; Kholood Alsager: funding acquisition, investigation, writing-review and editing; Asad Ali: Methodology, writing-original draft. All authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

The authors declare no conflict of interest.

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A. Theorem 3.1

Proof. 1) Based on Definition 3.1, we get

$$\begin{aligned}
\sigma_1 \bigoplus \sigma_2 &= \Big\langle \mathfrak{a}_{\mathfrak{h}\left(\sqrt[q]{(1-\exp^{-((-\log(1-(\frac{\xi\sigma_1}{\mathfrak{h}})^q))^\lambda+(-\log(1-(\frac{\xi\sigma_2}{\mathfrak{h}})^q))^\lambda)^{1/\lambda}})}\right)}, \\
&\quad \mathfrak{a}_{\mathfrak{h}\left(\exp^{-((-\log(\frac{\eta\sigma_1}{\mathfrak{h}}))^\lambda+(-\log(\frac{\eta\sigma_2}{\mathfrak{h}}))^\lambda)^{1/\lambda}}\right)} \Big\rangle, \\
&= \Big\langle \mathfrak{a}_{\mathfrak{h}\left(\sqrt[q]{(1-\exp^{-((-\log(1-(\frac{\xi\sigma_2}{\mathfrak{h}})^q))^\lambda+(-\log(1-(\frac{\xi\sigma_1}{\mathfrak{h}})^q))^\lambda)^{1/\lambda}})}\right)}, \\
&\quad \mathfrak{a}_{\mathfrak{h}\left(\exp^{-((-\log(\frac{\eta\sigma_2}{\mathfrak{h}}))^\lambda+(-\log(\frac{\eta\sigma_1}{\mathfrak{h}}))^\lambda)^{1/\lambda}}\right)} \Big\rangle, \\
&= \sigma_2 \bigoplus \sigma_1,
\end{aligned}$$

where $\lambda > 0$.

2) Again using Definition 3.1, we have

$$\begin{aligned}
\psi(\sigma_1 \bigoplus \sigma_2) = \Big\langle & \mathfrak{a}_{\mathfrak{h}\left(\sqrt[q]{(1-\exp^{-\left(\psi\left((-\log(1-(\frac{\xi\sigma_1}{\mathfrak{h}})^q))^\lambda+(-\log(1-(\frac{\xi\sigma_2}{\mathfrak{h}})^q))^\lambda\right)\right)^{1/\lambda}})}\right)}, \\
& \mathfrak{a}_{\mathfrak{h}\left(\exp^{-\left(\psi\left((-\log(\frac{\eta\sigma_1}{\mathfrak{h}}))^\lambda+(-\log(\frac{\eta\sigma_2}{\mathfrak{h}}))^\lambda\right)\right)^{1/\lambda}}\right)} \Big\rangle,
\end{aligned}$$

$$= \left\langle \mathfrak{a}_{\mathfrak{h}\left(\sqrt[q]{(1-\exp^{-\left(\psi\left(-\log(1-(\frac{\xi_{\sigma_1}}{\mathfrak{h}})^q)\right)^\lambda+\psi\left(-\log(1-(\frac{\xi_{\sigma_2}}{\mathfrak{h}})^q)\right)^\lambda\right)^{1/\lambda}})}\right)},\right.$$

$$\left. \mathfrak{a}_{\mathfrak{h}\left(\exp^{-\left(\psi\left(-\log(\frac{\eta_{\sigma_1}}{\mathfrak{h}})\right)^\lambda+\psi\left(-\log(\frac{\eta_{\sigma_2}}{\mathfrak{h}})\right)^\lambda\right)^{1/\lambda}}\right)} \right\rangle,$$

$$= \psi\sigma_1 \bigoplus \psi\sigma_2,$$

where $\psi > 0$ and $\lambda > 0$.

3) Similarly, by Definition 3.1. We obtain:

$$(\psi_1 \bigoplus \psi_2)\sigma = \left\langle \mathfrak{a}_{\mathfrak{h}\sqrt[q]{1-\exp^{-\left((\psi_1+\psi_2)\left(-\log(1-(\frac{\xi_\sigma}{\mathfrak{h}})^q)\right)^\lambda\right)^{1/\lambda}}}}, \mathfrak{a}_{\exp^{-\left((\psi_1+\psi_2)\left(-\log(\frac{\eta_\sigma}{\mathfrak{h}})\right)^\lambda\right)^{1/\lambda}}} \right\rangle$$

$$= \left\langle \mathfrak{a}_{\mathfrak{h}\sqrt[q]{1-\exp^{-\left(\psi_1\left(-\log(1-(\frac{\xi_\sigma}{\mathfrak{h}})^q)\right)^\lambda+\psi_2\left(-\log(1-(\frac{\xi_\sigma}{\mathfrak{h}})^q)\right)^\lambda\right)^{1/\lambda}}}}, \mathfrak{a}_{\exp^{-\left(\psi_1\left(-\log(\frac{\eta_\sigma}{\mathfrak{h}})\right)^\lambda+\psi_2\left(-\log(\frac{\eta_\sigma}{\mathfrak{h}})\right)^\lambda\right)^{1/\lambda}}} \right\rangle$$

$$= \psi_1\sigma \bigoplus \psi_2\sigma,$$

where $\psi_1 > 0, \psi_2 > 0$ and $\lambda > 0$.

□

B. Theorem 3.2

Proof. Suppose $\sigma_1 = \langle \mathfrak{a}_{\xi_1}, \mathfrak{a}_{\eta_1} \rangle$, $\sigma_2 = \langle \mathfrak{a}_{\xi_2}, \mathfrak{a}_{\eta_2} \rangle \ldots$, and $\sigma_t = \langle \mathfrak{a}_{\xi_t}, \mathfrak{a}_{\eta_t} \rangle$ are Lq-ROFNs where $\sigma_1, \sigma_2, \ldots, \sigma_t \in \Gamma_{[0,\mathfrak{h}]}$ and $\mathfrak{h}$ is an even positive integer. We show this result using the mathematical induction principle as below:

1) When $t = 2$, we get,

$$\mathfrak{w}_1\sigma_1 = \left\langle \mathfrak{a}_{\mathfrak{h}\sqrt[q]{1-\exp^{-(\mathfrak{w}_1(-\log(1-(\frac{\xi_{\sigma_1}}{\mathfrak{h}})^q))^\lambda)^{1/\lambda}}}}, \mathfrak{a}_{\exp^{\mathfrak{w}_1(-\log(\frac{\eta_{\sigma_1}}{\mathfrak{h}}))^\lambda)^{1/\lambda}}} \right\rangle.$$

$$\mathfrak{w}_2\sigma_2 = \left\langle \mathfrak{a}_{\mathfrak{h}\sqrt[q]{1-\exp^{-(\mathfrak{w}_2(-\log(1-(\frac{\xi_{\sigma_2}}{\mathfrak{h}})^q))^\lambda)^{1/\lambda}}}}, \mathfrak{a}_{\exp^{\mathfrak{w}_2(-\log(\frac{\eta_{\sigma_2}}{\mathfrak{h}}))^\lambda)^{1/\lambda}}} \right\rangle.$$

$$Lq-ROFAAWA(\sigma_1, \sigma_2) = \left\langle \mathfrak{a}_{\mathfrak{h}\sqrt[q]{1-\exp^{-(\mathfrak{w}_1(-\log(1-(\frac{\xi_{\sigma_1}}{\mathfrak{h}})^q))^\lambda)^{1/\lambda}}}}, \mathfrak{a}_{\exp^{\mathfrak{w}_1(-\log(\frac{\eta_{\sigma_1}}{\mathfrak{h}}))^\lambda)^{1/\lambda}}} \right\rangle$$

$$\bigoplus \left\langle \mathfrak{a}_{\mathfrak{h}\sqrt[q]{1-\exp^{-(\mathfrak{w}_2(-\log(1-(\frac{\xi_{\sigma_2}}{\mathfrak{h}})^q))^\lambda)^{1/\lambda}}}}, \mathfrak{a}_{\exp^{\mathfrak{w}_2(-\log(\frac{\eta_{\sigma_2}}{\mathfrak{h}}))^\lambda)^{1/\lambda}}} \right\rangle,$$

$$= \left\langle \mathfrak{a}_{h\sqrt[q]{1-\exp^{-(\sum_{r=1}^{2}\mathfrak{w}_r(-\log(1-(\frac{\xi_{\sigma_r}}{\mathfrak{h}})^q))^\lambda)^{1/\lambda}}}},\ \mathfrak{a}_{\exp^{-(\sum_{r=1}^{2}\mathfrak{w}_r(-\log(\frac{\eta_{\sigma_r}}{\mathfrak{h}}))^\lambda)^{1/\lambda}}} \right\rangle.$$

Thus, the aggregated value Lq-ROFAAWA($\sigma_1,\ \sigma_2$) of Lq-ROFNs $\sigma_1 = \langle \mathfrak{a}_{\xi_1}, \mathfrak{a}_{\eta_1} \rangle$ and $\sigma_2 = \langle \mathfrak{a}_{\xi_2},\ \mathfrak{a}_{\eta_2} \rangle$ using the suggested Lq-ROFAAWA AgOp defined in Eq (3.5) is a Lq-ROFN.

2) Suppose the aggregated value Lq-ROFAAWA($\sigma_1, \sigma_2, \ldots, \sigma_k$) of the L$q$-ROFNs $\sigma_1 = \langle \mathfrak{a}_{\xi_1}, \mathfrak{a}_{\eta_1} \rangle$, $\sigma_2 = \langle \mathfrak{a}_{\xi_2},\ \mathfrak{a}_{\eta_2} \rangle \ldots$, and $\sigma_k = \langle \mathfrak{a}_{\xi_k},\ \mathfrak{a}_{\eta_k} \rangle$ by applying the suggested Lq-ROFAAWA AgOp given in Equation (3.5) is a Lq-ROFN for $t = k$, then

$$\begin{aligned} Lq-ROFAAWA(\sigma_1, \sigma_2, \ldots, \sigma_k) &= \mathfrak{w}_1\sigma_1 \bigoplus \mathfrak{w}_2\sigma_2 \bigoplus \ldots \bigoplus \mathfrak{w}_k\sigma_k \\ &= \left\langle \mathfrak{a}_{\mathfrak{h}\sqrt[q]{1-\exp^{-(\sum_{r=1}^{k}\mathfrak{w}_r(-\log(1-(\frac{\xi_{\sigma_r}}{\mathfrak{h}})^q))^\lambda)^{1/\lambda}}}}, \right. \\ &\quad \left. \mathfrak{a}_{\exp^{-(\sum_{r=1}^{k}\mathfrak{w}_r(-\log(\frac{\eta_{\sigma_r}}{\mathfrak{h}}))^\lambda)^{1/\lambda}}} \right\rangle. \end{aligned}$$

Thus, the aggregated value of Lq-ROFAAWA($\sigma_1, \sigma_2, \ldots, \sigma_k$) of the $\sigma_1 = \langle \mathfrak{a}_{\xi_1}, \mathfrak{a}_{\eta_1} \rangle$, $\sigma_2 = \langle \mathfrak{a}_{\xi_2},\ \mathfrak{a}_{\eta_2} \rangle \ldots$, and $\sigma_k = \langle \mathfrak{a}_{\xi_k},\ \mathfrak{a}_{\eta_k} \rangle$ is a Lq-ROFN.

3) For $t = k + 1$, we get

$$\begin{aligned} Lq-ROFAAWA(\sigma_1, \sigma_2, \ldots, \sigma_{k+1}) &= \mathfrak{w}_1\sigma_1 \bigoplus \mathfrak{w}_2\sigma_2 \bigoplus \ldots \bigoplus \mathfrak{w}_{k+1}\sigma_{k+1} \\ &= \left\langle \mathfrak{a}_{\mathfrak{h}\sqrt[q]{1-\exp^{-(\sum_{r=1}^{k}\mathfrak{w}_r(-\log(1-(\frac{\xi_{\sigma_r}}{\mathfrak{h}})^q))^\lambda)^{1/\lambda}}}}, \right. \\ &\quad \left. \mathfrak{a}_{\exp^{-(\sum_{r=1}^{k}\mathfrak{w}_r(-\log(\frac{\eta_{\sigma_r}}{\mathfrak{h}}))^\lambda)^{1/\lambda}}} \right\rangle \\ &\quad \bigoplus \left\langle \mathfrak{a}_{\mathfrak{h}\sqrt[q]{1-\exp^{-(\mathfrak{w}_1(-\log(1-(\frac{\xi_{\sigma_1}}{\mathfrak{h}})^q))^\lambda)^{1/\lambda}}}}, \right. \\ &\quad \left. \mathfrak{a}_{\exp^{-(\mathfrak{w}_1(-\log(\frac{\eta_{\sigma_1}}{\mathfrak{h}}))^\lambda)^{1/\lambda}}} \right\rangle \\ &= \left\langle \mathfrak{a}_{\mathfrak{h}\sqrt[q]{1-\exp^{-(\sum_{r=1}^{k+1}\mathfrak{w}_r(-\log(1-(\frac{\xi_{\sigma_r}}{\mathfrak{h}})^q))^\lambda)^{1/\lambda}}}}, \right. \\ &\quad \left. \mathfrak{a}_{\exp^{-(\sum_{r=1}^{k+1}\mathfrak{w}_r(-\log(\frac{\eta_{\sigma_r}}{\mathfrak{h}}))^\lambda)^{1/\lambda}}} \right\rangle. \end{aligned}$$

Hence, the computed value of the Lq-ROFNs $\sigma_1 = \langle \mathfrak{a}_{\xi_1}, \mathfrak{a}_{\eta_1} \rangle, \sigma_2 = \langle \mathfrak{a}_{\xi_2},\ \mathfrak{a}_{\eta_2} \rangle \ldots,\ \sigma_{k+1} = \langle \mathfrak{a}_{\xi_{k+1}},\ \mathfrak{a}_{\eta_{k+1}} \rangle$ using the suggested Lq-ROFAAWA AgOp defined in Eq (3.5) is a Lq-ROFN.

□

C. Theorem 3.3

Proof. As the weights of Lq-ROFNs $\sigma_1, \sigma_2, \ldots, \sigma_n$ are $\mathfrak{w}_1, \mathfrak{w}_2, \ldots, \mathfrak{w}_n$, respectively, where $\mathfrak{w}_r \in [0, \mathfrak{h}], r = 1, 2, \ldots, n$ with $\sum_{r=1}^{n} \mathfrak{w}_r = 1$. If $\sigma_1 = \sigma_2 = \ldots = \sigma_n = \sigma$, then, we have

$$Lq-ROFAAWA(\sigma_1, \sigma_2, \ldots, \sigma_n) = \mathfrak{w}_1\sigma_1 \bigoplus \mathfrak{w}_2\sigma_2 \bigoplus \ldots \bigoplus \mathfrak{w}_n\sigma_n,$$

$$= \mathfrak{w}_1\sigma \bigoplus \mathfrak{w}_2\sigma \bigoplus \ldots \bigoplus \mathfrak{w}_n\sigma,$$
$$= \sigma(\mathfrak{w}_1 + \mathfrak{w}_2 + \ldots + \mathfrak{w}_n),$$
$$= \sigma.$$

□

D. Theorem 3.4

Proof. Since $\sigma_- = \min\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ and $\sigma_+ = \max\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$, by implementing the suggested Lq-ROFAAWA AgOp, we obtain:

$$Lq - ROFAAWA(\sigma_1, \sigma_2, \ldots, \sigma_n) = \bigoplus_{r=1}^{n} \mathfrak{w}_r\sigma_r \leq \bigoplus_{r=1}^{n} \mathfrak{w}_r\sigma_+ = \sigma_+ \sum_{r=1}^{n} \mathfrak{w}_r, \tag{D.1}$$

$$Lq - ROFAAWA(\sigma_1, \sigma_2, \ldots, \sigma_n) = \bigoplus_{r=1}^{n} \mathfrak{w}_r\sigma_r \geq \bigoplus_{r=1}^{n} \mathfrak{w}_r\sigma_- = \sigma_- \sum_{r=1}^{n} \mathfrak{w}_r. \tag{D.2}$$

By putting $\sum_{r=1}^{n} \mathfrak{w}_r = 1$ in both Eqs (D.1) and (D.2), we get $\sigma_- \leq Lq{-}ROFAAWA(\sigma_1, \sigma_2, \ldots, \sigma_n) \leq \sigma_+$. □

E. Theorem 3.5

Proof. Using the suggested Lq-ROFAAWA AgOp, we get

$$Lq - ROFAAWA(\sigma_1, \sigma_2, \ldots, \sigma_n) = \bigoplus_{r=1}^{n} \mathfrak{w}_r\sigma_r,$$

$$Lq - ROFAAWA(\sigma_1', \sigma_2', \ldots, \sigma_n') = \bigoplus_{r=1}^{n} \mathfrak{w}_r\sigma_r',$$

since $\sigma_r \leq \sigma_r'$ with $r = 1, 2, \ldots, n$, and $\sum_{r=1}^{n} \mathfrak{w}_r = 1$, so, $\bigoplus_{r=1}^{n} \mathfrak{w}_r\sigma_r \leq \bigoplus_{r=1}^{n} \mathfrak{w}_r\sigma_r'$. Therefore, we get $Lq{-}ROFAAWA(\sigma_1, \sigma_2, \ldots, \sigma_n) \leq Lq{-}ROFAAWA(\sigma_1', \sigma_2', \ldots, \sigma_n')$. □