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Research article

On the oscillation of solutions of third-order differential equations with non-positive neutral coefficients

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Abstract: The oscillation property of third-order differential equations with non-positive neutral coefficients is discussed. New sufficient conditions are provided to guarantee that every solution of the considered equation is almost oscillatory. Both the canonical and non-canonical cases are considered. Illustrative examples are introduced to support the obtained results.

Keywords: oscillation; third-order; nonpositive neutral coefficients
Mathematics Subject Classification: 34C10, 34K11

1. Introduction

In this article, we are concerned with the oscillatory behavior of solutions of a general class of third-order differential equations with non-positive neutral coefficients of the type

$$\left(d\left(l\right)\left(w''\left(l\right)\right)^{\alpha}\right)' + \sum_{i=1}^{n} q_i\left(l\right) y^{\beta_i}\left(\tau_i\left(l\right)\right) = 0, \qquad l \geq l_0 > 0, \tag{1.1}$$

where $w(l) = y(l) - \sum_{j=1}^{m} a_j(l)\, y\left(\delta_j(l)\right)$, α and β_i are quotients of odd positive integers, and m, n are positive integers. We shall consider the following two cases: the canonical case

$$D(l_0) = \int_{l_0}^{\infty} \frac{1}{d^{\frac{1}{\alpha}}(l)} dl = \infty, \tag{1.2}$$

and the non-canonical case

$$D(l_0) = \int_{l_0}^{\infty} \frac{1}{d^{\frac{1}{\alpha}}(l)} dl < \infty. \tag{1.3}$$

Throughout the paper, we assume that

(H_1) $d(l) \in C([l_0, \infty), (0, \infty))$, $a_j(l) \in C([l_0, \infty))$, $0 \le a_j(l) \le a_{0j}$, $\sum_{j=1}^{m} a_{0j} < 1$, $j = 1, 2, ..., m$;
(H_2) $\delta_j(l), \tau_i(l), q_i(l) \in C([l_0, \infty))$, $\delta_j(l) \le l$, $\lim_{l\to\infty} \delta_j(l) = \lim_{l\to\infty} \tau_i(l) = \infty$, $q_i(l) > 0$, $i = 1, 2, ..., n$, $j = 1, 2, ..., m$.

Any nontrivial function $y(l) \in C\left([l_y, \infty)\right)$, $l_y \ge l_0$, such that $w \in C^2\left([l_y, \infty)\right)$, $d(l)\left(w''(l)\right)^{\alpha} \in C^1\left([l_y, \infty)\right)$ and $y(l)$ satisfies (1.1) on $[l_y, \infty)$ is called a solution of (1.1). Our attention is restricted to those solutions $y(l)$ of (1.1) that satisfy $\sup\{|y(l)| : l \ge T\} > 0$ for $T \ge l_y$. We tacitly suppose that (1.1) possesses such a solution. A solution $y(l)$ of (1.1) is termed oscillatory if it has arbitrarily large zeros on $[l_y, \infty)$; otherwise, it is said to be non-oscillatory. The equation itself is termed oscillatory if all its solutions oscillate.

In dynamical models, delay and oscillation effects are often formulated by means of external sources and/or nonlinear diffusion, perturbing the natural evolution of related systems (see, e.g., [1–4]). Recently, there has been considerable interest in studying the qualitative properties of solutions of various types of differential equations, see, e.g., [5–7], for the oscillation of second-order differential equations, while [8, 9], for fourth-order differential equations, and the references [10–12], for the oscillation of nth-order differential equations. In fact, it is notable that the analysis of differential equations with non-positive neutral coefficients is more difficult in comparison with that of non-negative neutral coefficients. The same thing can be said also for the non-canonical case compared to the canonical case. Moreover, although there has been a lot of interest in the oscillatory behavior of solutions of third-order equations with non-negative neutral coefficients (see, e.g., [13–16]), for equations with non-positive neutral coefficients there are relatively fewer published results and so they are not very prevalent in the literature (see, e.g., [17–20]). For instance, we mention here some of the related works that motivate our work. In [20], Qiu established new oscillation criteria for the third-order nonlinear dynamic equation on time scales of the type,

$$\left(d_1(l)\left(\left[d_2(l)\left((z(l)) \pm b(l)\, z(\gamma(l))^{\Delta}\right)^{\gamma_2}\right]^{\Delta}\right)^{\gamma_1}\right)^{\Delta} + g(l, z(\tau(l))) = 0, \tag{1.4}$$

with $\int_{l_0}^{\infty} \frac{1}{d_1^{\frac{1}{\gamma_1}}(l)} \Delta l = \infty$, $\int_{l_0}^{\infty} \frac{1}{d_2^{\frac{1}{\gamma_2}}(l)} \Delta l = \infty$. The authors in [17] were concerned with the D.E.,

$$(d(l)\left[(v(l)) \pm b(l)\, v(\xi(l))''\right]^{\alpha})' + g(l)\, v^{\alpha}(\eta(l)) = 0, \quad l \ge l_0, \tag{1.5}$$

and established numerous criteria for the so-called Hile and Nehari type under the assumptions $0 \le b(l) \le 1$ and $d'(l) \ge 0$ in the canonical case (1.2). Meanwhile, Jiang et al. [19] were motivated by the work of Baculucova and Duzirina [17], and Li et al. [21], to establish an affirmative answer to the question about the possibility of obtaining asymptotic criteria for the D.E.,

$$\left(d(l)\left[\left(v(l)-\int_a^b p(l,\xi)\,v(l,\zeta)\,d\xi\right)''\right]^{\alpha}\right)'+\int_c^d g(l,\nu)\,v(l,\theta)\,d\theta=0, \tag{1.6}$$

where $\alpha > 0$ is a quotient of odd positive integers, $0 \leq \int_a^b p(l,\zeta)\,d\zeta \leq p_0 < 1$, without the need for the restrictive condition $d'(l) \geq 0$. More recently Garce [22] studied the oscillatory behavior of solutions of the nonlinear differential equation

$$(v(l)-p(l)\,v^{\alpha}(\tau(l)))'''+g(l)\,v^{\beta}(\sigma(l))=0, l \geq l_0 > 0, \tag{1.7}$$

with $\tau(l) \leq l, \sigma(l) \leq l, \tau'(l) > 0$, and $\sigma'(l) > 0$. Meanwhile, Garce et al. [18] were concerned with the oscillatory behavior of solutions of nonlinear differential equations of the type

$$\left(d(l)\left[(v(l)-p(l)\,v^{\alpha}(\tau(l)))''\right]^{\gamma}\right)'+g(l)\,v^{\beta}(\sigma(l)=0, l \geq l_0 > 0, \tag{1.8}$$

using comparison methods and integral conditions, with $\gamma \geq \beta, \tau(l) \leq l, \sigma(l) \leq l, \tau'(l) > 0$, and $\sigma'(l) > 0$ in the canonical case (1.2).

The principal goal of this paper is to study the oscillatory behavior of solutions of the nonlinear third-order differential equation with non-positive neutral coefficients (1.1) in the two cases canonical (1.2) and non-canonical (1.3) by using Riccati transformation without the need for the restrictive condition $d'(l) \geq 0$. Moreover, we do not need specific restrictions on the functions $\tau_i(l)$, that is, $\tau_i(l)$ may be delayed or advanced; furthermore, we considered the two cases $\beta_i \geq \alpha$ and $\beta_i \leq \alpha$.

2. Preliminaries

This section is devoted to present some notations and lemmas needed for our results.

Define

$$D_1(l,T)=\int_T^l \frac{1}{d^{\frac{1}{\alpha}}(u)}du, \quad \text{and} \quad D_2(l,T)=\int_T^l D_1(u,l_1)\,du.$$

We first start with the following two lemmas, which are very similar to Lemmas 2.1 and 2.2 of [19].

Lemma 2.1. *Assume that $y(l)$ is an eventually positive solution of* (1.1)*, such that* (1.2) *be satisfied. Then there exists $l_1 \geq l_0$, such that for all $l \geq l_1$ the corresponding function w satisfies one of the following four cases:*

$$i)\ w > 0, w' > 0, w'' > 0, (d(w'')^{\alpha})' \leq 0,$$

$$ii)\ w > 0, w' < 0, w'' > 0, (d(w'')^{\alpha})' \leq 0,$$

$$iii)\ w < 0, w' < 0, w'' > 0, (d(w'')^{\alpha})' \leq 0,$$

$$iv)\ w < 0, w' < 0, w'' < 0, (d(w'')^{\alpha})' \leq 0.$$

Lemma 2.2. *If for any eventually positive solution* $y(l)$ *of* (1.1)*, the corresponding* $w(l)$ *satisfies case (i) of Lemma 2.1, then for any* $l_2 > l_1 \geq l_0$*,*

$$w(l) \geq \frac{\int_{l_2}^{l} \int_{l_1}^{v} d^{\frac{-1}{\alpha}}(h)\, dh dv}{\int_{l_1}^{l} d^{\frac{-1}{\alpha}}(h)\, dh} w'(l),$$

and $\frac{w'(l)}{\int_{l_1}^{l} d^{\frac{-1}{\alpha}}(h) dh}$ *is nonincreasing eventually.*

Now, we introduce the following preliminary result:

Lemma 2.3. *If for any eventually positive solution* $y(l)$ *of* (1.1)*, the corresponding* $w(l)$ *satisfies case (ii) of Lemma 2.1, and*

$$\int_{l_0}^{\infty} \int_{v}^{\infty} \left(\frac{1}{d(u)} \int_{u}^{\infty} \sum_{i=1}^{n} q_i(s)\, ds \right)^{\frac{1}{\alpha}} du dv = \infty, \tag{2.1}$$

then $\lim_{l \to \infty} y(l) = 0$.

Proof. Since $w(l)$ satisfies property (ii), then there exists a finite constant $M \geq 0$ such that $\lim_{l \to \infty} w(l) = M$. We claim that $M = 0$. Otherwise, assume that $M > 0$. By the definition of w, $y(l) \geq w(l) > M$. Consequently, by (1.1), we have

$$\begin{aligned} \left(d(l)\left(w''(l)\right)^{\alpha}\right)' &= -\sum_{i=1}^{n} q_i(l)\, y^{\beta_i}(\tau_i(l)) \leq -\sum_{i=1}^{n} q_i(l)\, w^{\beta_i}(\tau_i(l)) \\ &\leq -\sum_{i=1}^{n} M^{\beta_i} q_i(l) \leq -M^{\kappa} \sum_{i=1}^{n} q_i(l), \end{aligned} \tag{2.2}$$

where $\kappa = \{^{\min \beta_i \;\; M \geq 1}_{\max \beta_i \;\; M < 1}$. Integrating (2.2) from l to ∞, we obtain

$$w''(l) \geq M^{\frac{\kappa}{\alpha}} \left(\frac{1}{d(l)} \int_{l}^{\infty} \sum_{i=1}^{n} q_i(s)\, ds \right)^{\frac{1}{\alpha}}.$$

Therefore, by integrating from l to ∞ and then integrating the result from l_1 to ∞, it follows that

$$w(l_1) \geq M^{\frac{\kappa}{\alpha}} \int_{l_1}^{\infty} \int_{v}^{\infty} \left(\frac{1}{d(u)} \int_{u}^{\infty} \sum_{i=1}^{n} q_i(s)\, ds \right)^{\frac{1}{\alpha}} du dv.$$

This contradicts (2.1). Hence, $M = 0$ and $\lim_{l \to \infty} w(l) = 0$. Next, we claim that $y(l)$ is bounded. If this is false, then there exists a sequence $\{l_m\}$ such that $\lim_{l \to \infty} l_m = \infty$ and $\lim_{m \to \infty} y(l_m) = \infty$, where $y(l_m) = \max\{y(s) : l_0 \leq s \leq l_m\}$. Since $\lim_{l \to \infty} \delta_j(l) = \infty, \delta_j(l_m) > l_0$ for sufficiently large m. By $\delta_j(l) \leq l$, we conclude that

$$y\left(\delta_j(l_m)\right) = \max\left\{y(s) : l_0 \leq s \leq \delta_j(l_m)\right\} \leq \max\{y(s) : l_0 \leq s \leq l_m\} = y(l_m),$$

and so

$$w(l_m) = y(l_m) - \sum_{j=1}^{m} a_j(l_m)\, y\left(\delta_j(l_m)\right)$$
$$\geq y(l_m) - \sum_{j=1}^{m} a_j(l_m)\, y(l_m)$$
$$\geq \left(1 - \sum_{j=1}^{m} a_{0j}\right) y(l_m),$$

which yields $\lim_{l\to\infty} w(l_m) = \infty$. This contradicts $\lim_{l\to\infty} w(l) = 0$, therefore $y(l)$ is bounded, and hence we may suppose that $\limsup_{l\to\infty} y(l) = b_0$, where $0 \leq b_0 < \infty$. Then there exists a sequence $\{l_k\}$ such that $\lim_{l\to\infty} l_k = \infty$ and $\lim_{l\to\infty} y(l_k) = b_0$. Now assuming that $b_0 > 0$, and letting

$$\epsilon = \frac{b_0\left(1 - \sum_{j=1}^{m} a_{0j}\right)}{2\sum_{j=1}^{m} a_{0j}},$$

we have $y\left(\delta_j(l_k)\right) < b_0 + \epsilon$ eventually, and thus

$$0 = \lim_{k\to\infty} w(l_k) \geq \lim_{k\to\infty}\left(y(l_k) - \sum_{j=1}^{m} a_{0j}(b_0 + \epsilon)\right)$$
$$= b_0 - \sum_{j=1}^{m} a_{0j}\left(b_0 + \frac{b_0\left(1 - \sum_{j=1}^{m} a_{0j}\right)}{2\sum_{j=1}^{m} a_{0j}}\right) = \frac{b_0}{2}\left(1 - \sum_{j=1}^{m} a_{0j}\right) > 0,$$

which is a contradiction. Thus $b_0 = 0$ and $\lim_{l\to\infty} y(l) = 0$. The proof is complete. □

3. Main results

Theorem 3.1. *Assume that $\beta_i \geq \alpha$, $i = 1, ..., n$, (1.2) and (2.1) hold. Suppose that there exists $I(l) \in C([l_0, \infty))$ such that*

$$I(l) \leq \inf\{l, \tau_i(l)\}, \quad \lim_{l\to\infty} I(l) = \infty. \tag{3.1}$$

If there exist a function $v(l) \in C^1([l_0, \infty), (0, \infty))$, and a constant $C_1 > 0$ such that, for all sufficiently large $l_1 \geq l_0$ and for some $l_3 > l_2 > l_1$,

$$\limsup_{l\to\infty} \int_{l_3}^{l} \left(v(u) \sum_{i=1}^{n} q_i(u)\, C_1^{\beta_i - \alpha} N(u) - \frac{d(u)\,[v'(u)]^{\alpha+1}}{(\alpha+1)^{\alpha+1} v^{\alpha}(u)} \right) du = \infty, \tag{3.2}$$

where

$$N(l) = \left[\frac{\int_{l_2}^{I(l)} \int_{l_1}^{s} d^{\frac{-1}{\alpha}}(\varkappa)\, d\varkappa ds}{\int_{l_1}^{l} d^{\frac{-1}{\alpha}}(\varkappa)\, d\varkappa} \right]^{\alpha}, \tag{3.3}$$

then Eq (1.1) is almost oscillatory.

Proof. Assume that $y(l)$ is an eventually positive solution of (1.1). Then there exists a $l_1 \geq l_0$ such that $y(l) > 0, y\left(\delta_j(l)\right) > 0$ and $y(\tau_i(l)) > 0$ for $l \geq l_1$. It is clear by Lemma 2.1 that the function $w(l)$ obeys one of four possible cases (i), (ii), (iii), or (iv). Assume first that case (i) is satisfied for $l \geq l_1$. Define the Riccati transformation $\phi(l)$ by

$$\phi(l) = v(l)\frac{d(l)(w''(l))^{\alpha}}{(w'(l))^{\alpha}},$$

then $\phi(l) > 0$ for $l \geq l_1$, and

$$\phi'(l) = \frac{v'(l)}{v(l)}\phi(l) + v(l)\frac{\left[d(l)(w''(l))^{\alpha}\right]'}{(w'(l))^{\alpha}} - \alpha v(l)\frac{d(l)(w''(l))^{\alpha+1}}{(w'(l))^{\alpha+1}}. \tag{3.4}$$

But since from (1.1) and the definition of w, we have

$$\begin{aligned}(d(l)(w''(l))^{\alpha})' &= -\sum_{i=1}^{n} q_i(l)\, y^{\beta_i}(\tau_i(l)) \\ &\leq -\sum_{i=1}^{n} q_i(l)\, w^{\beta_i}(\tau_i(l)).\end{aligned}$$

Hence, since $w'(l) > 0$ and $\tau_i(l) \geq I(l)$, then

$$(d(l)(w''(l))^{\alpha})' \leq -\sum_{i=1}^{n} q_i(l)\, w^{\beta_i}(I(l)). \tag{3.5}$$

Then from (3.4), we have

$$\phi'(l) \leq \frac{v'(l)}{v(l)}\phi(l) - v(l)\frac{\sum_{i=1}^{n} q_i(l)\, w^{\beta_i}(I(l))}{(w'(l))^{\alpha}} - \alpha\frac{(\phi(l))^{\frac{\alpha+1}{\alpha}}}{(d(l)v(l))^{\frac{1}{\alpha}}}. \tag{3.6}$$

Now since $w(l)$ is positive and increasing, then there exist a $l_2 \geq l_1$ and $C_1 > 0$ such that

$$w(l) \geq C_1, \quad l \geq l_2. \tag{3.7}$$

This, with (3.6), leads to

$$\phi'(l) \leq \frac{v'(l)}{v(l)}\phi(l) - v(l)\sum_{i=1}^{n} q_i(l)\, C_1^{\beta_i-\alpha}\left[\frac{w(I(l))}{w'(l)}\right]^{\alpha} - \alpha\frac{(\phi(l))^{\frac{\alpha+1}{\alpha}}}{(d(l)v(l))^{\frac{1}{\alpha}}}. \tag{3.8}$$

Since $I(l) \leq l$, then by using the nonincreasing property of $\frac{w'(l)}{\int_{l_1}^{l} d^{\frac{-1}{\alpha}}(h)dh}$ (see Lemma 2.2), we obtain

$$\frac{w'(I(l))}{w'(l)} \geq \frac{\int_{l_1}^{I(l)} d^{\frac{-1}{\alpha}}(h)\,dh}{\int_{l_1}^{l} d^{\frac{-1}{\alpha}}(h)\,dh}. \tag{3.9}$$

Now by using Lemma 2.2, we have

$$\left[\frac{w(I(l))}{w'(l)}\right]^{\alpha} = \left(\frac{w(I(l))\,w'(I(l))}{w'(I(l))\,w'(l)}\right)^{\alpha} \geq N(l), \tag{3.10}$$

and so, by substituting from (3.10) into (3.8), we obtain

$$\phi'(l) \leq \frac{v'(l)}{v(l)}\phi(l) - v(l)\sum_{i=1}^{n} q_i(l)\, C_1^{\beta_i-\alpha} N(l) - \alpha\frac{(\phi(l))^{\frac{\alpha+1}{\alpha}}}{(d(l)\, v(l))^{\frac{1}{\alpha}}}.$$

Applying the inequality

$$TV - RV^{\frac{\alpha+1}{\alpha}} \leq \frac{\alpha^{\alpha} T^{\alpha+1}}{(\alpha+1)^{\alpha+1} R^{\alpha}}, \quad R > 0, \tag{3.11}$$

with $V = \phi(l)$, $R = \frac{\alpha}{(d(l)v(l))^{\frac{1}{\alpha}}}$ and $T = \frac{v'(l)}{v(l)}$, we obtain

$$\phi'(l) \leq -v(l)\sum_{i=1}^{n} q_i(l)\, C_1^{\beta_i-\alpha} N(l) + \frac{d(l)\,[v'(l)]^{\alpha+1}}{(\alpha+1)^{\alpha+1}\, v^{\alpha}(l)}.$$

By integrating from l_3 ($l_3 > l_2$) to l, we arrive at

$$\int_{l_3}^{l} \left(v(u)\sum_{i=1}^{n} q_i(u)\, C_1^{\beta_i-\alpha} N(u) - \frac{d(u)\,[v'(u)]^{\alpha+1}}{(\alpha+1)^{\alpha+1}\, v^{\alpha}(u)} \right) du \leq \phi(l_3),$$

this contradicts (3.2). Now consider the case (*ii*), then by Lemma 2.3, $\lim_{l\to\infty} y(l) = 0$. In both cases (*iii*) and (*iv*), similar analysis to that in [19, Theorem 3.1], case (*iii*), and case (*iv*) can be used to arrive at the conclusion $\lim_{l\to\infty} y(l) = 0$. This completes the proof. □

Theorem 3.2. *Assume that* $\beta_i \leq \alpha$, $i = 1, ..., n$, (1.2) *and* (2.1) *hold. Suppose that there exists* $I(l) \in C([l_0, \infty))$ *satisfies* (3.1). *Suppose further that there exist a function* $k(l) \in C^1([l_0, \infty), (0, \infty))$, *and a constant* $C_2 > 0$, *for sufficiently large* $l_1 \geq l_0$ *and some* $l_3 > l_2 > l_1$. *If*

$$\limsup_{l\to\infty} \int_{l_3}^{l} \left[k(s)\sum_{i=1}^{n} q_i(s)\, C_2^{\beta_i-\alpha} \frac{D_2^{\beta_i}(I(s), l_2)}{D_2^{\alpha}(s, l_2)} - \frac{1}{(\alpha+1)^{\alpha+1}} \frac{(k'(s))^{\alpha+1}}{D_1^{\alpha}(s, l_1)\, k^{\alpha}(s)} \right] ds = \infty, \tag{3.12}$$

then Eq (1.1) *is almost oscillatory.*

Proof. For the sake of contradiction, suppose that (1.1) has an eventually positive solution $y(l)$. Then for any, $l_1 \geq l_0$, we have $y(l) > 0$, $y(\delta_j(l)) > 0$ and $y(\tau_i(l)) > 0$, $i = 1, 2, ..., n$, $j = 1, 2, ..., m$. It is clear by Lemma 2.1, that the function $w(l)$ obeys one of the four possible cases (*i*), (*ii*), (*iii*), or (*iv*). Assume first that case (*i*) is satisfied for $l \geq l_1$. Define

$$\Omega(l) = k(l)\frac{d(l)\,(w''(l))^{\alpha}}{(w(l))^{\alpha}}, \quad l \geq l_1,$$

then $\Omega(l) > 0$, and by using (3.5), we obtain

$$\Omega'(l) \leq \frac{k'(l)}{k(l)}\Omega(l) - k(l)\frac{\sum_{i=1}^{n} q_i(l)\, w^{\beta_i}(I(l))}{(w(l))^{\alpha}} - \alpha k(l)\frac{d(l)\,(w''(l))^{\alpha}\, w'(l)}{(w(l))^{\alpha+1}}. \tag{3.13}$$

But since $w'(l) > 0$ and $d(l)\,(w''(l))^{\alpha}$ is nonincreasing, we obtain

$$w'(l) \geq \left(d(l)\,(w''(l))^{\alpha}\right)^{\frac{1}{\alpha}} D_1(l, l_1). \tag{3.14}$$

Substituting into (3.13), we obtain

$$\Omega'(l) \leq \frac{k'(l)}{k(l)}\Omega(l) - k(l)\frac{\sum_{i=1}^{n} q_i(l)\, w^{\beta_i}(I(l))}{(w(l))^{\alpha}} - \alpha k(l)\, D_1(l,l_1)\frac{d^{1+\frac{1}{\alpha}}(l)\,(w''(l))^{\alpha+1}}{(w(l))^{\alpha+1}}.$$

By the definition of Ω, we have

$$\Omega'(l) \leq \frac{k'(l)}{k(l)}\Omega(l) - k(l)\frac{\sum_{i=1}^{n} q_i(l)\, w^{\beta_i}(I(l))}{(w(l))^{\alpha}} - \alpha D_1(l,l_1)\frac{(\Omega(l))^{\frac{\alpha+1}{\alpha}}}{k^{\frac{1}{\alpha}}(l)}.$$

By applying the inequality (3.11), with

$$R = \frac{\alpha D_1(l,l_1)}{k^{\frac{1}{\alpha}}(l)}, T = \frac{k'(l)}{k(l)} \quad \text{and } V = \Omega(l),$$

we obtain

$$\Omega'(l) \leq -k(l)\frac{\sum_{i=1}^{n} q_i(l)\, w^{\beta_i}(I(l))}{(w(l))^{\alpha}} + \frac{1}{(\alpha+1)^{\alpha+1}}\frac{(k'(l))^{\alpha+1}}{D_1^{\alpha}(l,l_1)\, k^{\alpha}(l)}. \tag{3.15}$$

But since from (3.14), we have

$$\left(\frac{w'(l)}{D_1(l,l_1)}\right)' \leq 0, \text{ for } l \geq l_2 > l_1,$$

which leads to

$$w(l) \geq \int_{l_2}^{l} \frac{D_1(s,l_1)\, w'(s)}{D_1(s,l_1)} ds \geq \frac{w'(l)}{D_1(l,l_1)} D_2(l,l_2).$$

Hence

$$\left(\frac{w(l)}{D_2(l,l_2)}\right)' \leq 0. \tag{3.16}$$

This with $I(l) \leq l$, yields

$$\frac{w(I(l))}{w(l)} \geq \frac{D_2(I(l),l_2)}{D_2(l,l_2)}. \tag{3.17}$$

Substituting from (3.17) into (3.15), we obtain

$$\Omega'(l) \leq -k(l)\sum_{i=1}^{n} q_i(l)\, w^{\beta_i-\alpha}(I(l))\left[\frac{D_2(I(l),l_2)}{D_2(l,l_2)}\right]^{\alpha} + \frac{1}{(\alpha+1)^{\alpha+1}}\frac{(k'(l))^{\alpha+1}}{D_1^{\alpha}(l,l_1)\, k^{\alpha}(l)}. \tag{3.18}$$

Now since by (3.16), $\frac{w(l)}{D_2(l,l_2)}$ is decreasing, there exists a constant $C_2 > 0$ such that for $l_3 > l_2$, we have

$$\frac{w(l)}{D_2(l,l_2)} \leq C_2, \quad \text{for } l \geq l_3.$$

Substituting into (3.18), we obtain

$$\Omega'(l) \leq -k(l)\sum_{i=1}^{n} q_i(l)\, C_2^{\beta_i-\alpha}\frac{D_2^{\beta_i}(I(l),l_2)}{D_2^{\alpha}(l,l_2)} + \frac{1}{(\alpha+1)^{\alpha+1}}\frac{(k'(l))^{\alpha+1}}{D_1^{\alpha}(l,l_1)\, k^{\alpha}(l)}. \tag{3.19}$$

Integrating (3.19) from l_3 to l, we obtain

$$\int_{l_3}^{l}\left[k(s)\sum_{i=1}^{n}q_i(s)\,C_2^{\beta_i-\alpha}\frac{D_2^{\beta_i}(I(s),l_2)}{D_2^{\alpha}(s,l_2)}-\frac{1}{(\alpha+1)^{\alpha+1}}\frac{(k'(s))^{\alpha+1}}{D_1^{\alpha}(s,l_1)\,k^{\alpha}(s)}\right]ds\le\Omega(l_3).$$

This contradicts (3.12). The proofs of the cases (*ii*)–(*iv*) are as in the proof of Theorem 3.1. □

Now, we discuss the oscillatory behavior of Eq (1.1) in the non-canonical case (1.3).

Theorem 3.3. *Assume that* $\beta_i \ge \alpha$, $i = 1,...,n$, (1.3) *and* (2.1) *hold. Assume that* $I(l)$ *be as in Theorem 3.1, for sufficiently large* $l_1 \ge l_0$ *and for some* $l_3 > l_2 > l_1$, (3.2) *is satisfied. Suppose further that there exist constants* $C_3 > 0$ *and* $0 < L < 1$*, such that*

$$\limsup_{l\to\infty}\int_{l_3}^{l}\left[D^{\alpha}(s)\sum_{i=1}^{n}q_i(s)\,C_3^{\beta_i-\alpha}L^{\alpha}s^{\alpha-\frac{\alpha}{L}}(I(s))^{\frac{\alpha}{L}}-\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{1}{D(s)\,d^{\frac{1}{\alpha}}(s)}\right]ds=\infty, \tag{3.20}$$

then Eq (1.1) *is almost oscillatory.*

Proof. For the sake of contradiction suppose that (1.1) has an eventually positive solution $y(l)$. In view of (1.3), there exist six possible cases including (*i*)–(*iv*) (as in Lemma 2.1), and the two extra cases:

$$v)\;\; w>0,\,w'>0,\,w''<0,\,(d\,(w'')^{\alpha})'\le 0,$$

$$vi)\;\; w<0,\,w'>0,\,w''<0,\,(d\,(w'')^{\alpha})'\le 0.$$

The proofs of the four cases (*i*)–(*iv*) follow the same arguments of Theorem 3.1. Now consider the case(v). Since $d(w'')^{\alpha}$ is decreasing, then

$$d(s)(w''(s))^{\alpha}\le d(l)(w''(l))^{\alpha},\quad s\ge l\ge l_1. \tag{3.21}$$

Integrating from l to g and letting $g\to\infty$, we have

$$w'(l)\ge -d^{\frac{1}{\alpha}}(l)\,w''(l)\,D(l). \tag{3.22}$$

In view of case (v), since $w(l)>0, w'(l)>0$ and $w''(l)<0$ on $[l_1,\infty]$, for any constant $L\in(0,1)$, we have

$$w(l)\ge Llw'(l),\;\text{ for } l\ge l_2\ge l_1. \tag{3.23}$$

Now define

$$\Phi(l)=\frac{d(l)\,[w''(l)]^{\alpha}}{[w'(l)]^{\alpha}}, \tag{3.24}$$

then $\Phi(l)<0$ for $l\ge l_2$, and

$$\Phi'(l)=\frac{[d(l)\,[w''(l)]^{\alpha}]'}{[w'(l)]^{\alpha}}-\alpha d(l)\left[\frac{\Phi(l)}{d(l)}\right]^{\frac{\alpha+1}{\alpha}}.$$

Thus, from (3.5), we have

$$\Phi'(l)\le\frac{-\sum_{i=1}^{n}q_i(l)\,w^{\beta_i}(I(l))}{[w'(l)]^{\alpha}}-\alpha d(l)\left[\frac{\Phi(l)}{d(l)}\right]^{\frac{\alpha+1}{\alpha}}. \tag{3.25}$$

Therefore, in view of (3.23), we obtain

$$\left(\frac{w(l)}{l^{\frac{1}{L}}}\right)' \leq 0. \tag{3.26}$$

But since $I(l) \leq l$, then

$$\frac{w(I(l))}{w(l)} \geq \left[\frac{I(l)}{l}\right]^{\frac{1}{L}}. \tag{3.27}$$

This with (3.23), leads to

$$\frac{w(I(l))}{w'(l)} \geq \frac{LlI^{\frac{1}{L}}(l)}{l^{\frac{1}{L}}}.$$

Consequently, by substituting in (3.25), we have

$$\Phi'(l) \leq -\sum_{i=1}^{n} q_i(l)\, w^{\beta_i-\alpha}(I(l))\, L^{\alpha} l^{\alpha-\frac{\alpha}{L}} (I(l))^{\frac{\alpha}{L}} - \alpha\frac{[\Phi(l)]^{\frac{\alpha+1}{\alpha}}}{d^{\frac{1}{\alpha}}(l)}. \tag{3.28}$$

Now since, from the positivity and increasing properties of $w(l)$, there exists a constant $C_3 > 0$ such that $w(l) \geq C_3$, then we have

$$\Phi'(l) \leq -\sum_{i=1}^{n} q_i(l)\, C_3^{\beta_i-\alpha} L^{\alpha} l^{\alpha-\frac{\alpha}{L}} (I(l))^{\frac{\alpha}{L}} - \alpha\frac{[\Phi(l)]^{\frac{\alpha+1}{\alpha}}}{d^{\frac{1}{\alpha}}(l)}. \tag{3.29}$$

It is clear by (3.22) and (3.24) that

$$-D^{\alpha}(l)\,\Phi(l) \leq 1. \tag{3.30}$$

Multiplying (3.29) by $D^{\alpha}(l)$ and integrating from l_3 to l, we have

$$\begin{aligned}&\int_{l_3}^{l}\left[D^{\alpha}(s)\sum_{i=1}^{n} q_i(s)\, C_3^{\beta_i-\alpha} L^{\alpha} s^{\alpha-\frac{\alpha}{L}} (I(s))^{\frac{\alpha}{L}} + \alpha D^{\alpha}(s)\frac{[\Phi(s)]^{\frac{\alpha+1}{\alpha}}}{d^{\frac{1}{\alpha}}(s)} + \alpha d^{-\frac{1}{\alpha}}(s)\, D^{\alpha-1}(s)\,\Phi(s)\right] ds \\ \leq\ & -D^{\alpha}(l)\,\Phi(l) + D^{\alpha}(l_3)\,\Phi(l_3).\end{aligned}$$

Set $R = \frac{D^{\alpha}(s)}{d^{\frac{1}{\alpha}}(s)}, T = d^{-\frac{1}{\alpha}}(s)\, D^{\alpha-1}(s)$ and $v = -\Phi(s)$, then using the inequality (3.11), we have

$$\begin{aligned}&\int_{l_3}^{l}\left[D^{\alpha}(s)\sum_{i=1}^{n} q_i(s)\, C_3^{\beta_i-\alpha} L^{\alpha} s^{\alpha-\frac{\alpha}{L}} (I(s))^{\frac{\alpha}{L}} - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{1}{D(s)\, d^{\frac{1}{\alpha}}(s)}\right] ds \\ \leq\ & 1 + D^{\alpha}(l_3)\,\Phi(l_3).\end{aligned}$$

This is a contradiction with (3.20). Assume case (vi) holds. Now, by using an argument similar to that used in Lemma 2.3, we arrive at the conclusion that $\lim_{l\to\infty} y(l) = 0$. The proof is complete. □

Theorem 3.4. *Assume that $\beta_i \leq \alpha$, $i = 1, ..., n$,* (1.3) *and* (2.1) *hold. Assume further that $I(l)$ be as in Theorem 3.1, for sufficiently large $l_1 \geq l_0$ and for some $l_3 > l_2 > l_1$,* (3.12) *is satisfied. If there exists a constant $C_4 > 0$, such that*

$$\limsup_{l\to\infty}\int_{l_3}^{l}\left[D^{\alpha}(s)\sum_{i=1}^{n} q_i(s)\, C_4^{\beta_i-\alpha} L^{\alpha} s^{\alpha-\frac{\alpha}{L}} (I(s))^{\frac{\beta_i}{L}} - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{1}{D(s)\, d^{\frac{1}{\alpha}}(s)}\right] ds = \infty, \tag{3.31}$$

then Eq (1.1) *is almost oscillatory.*

Proof. Let $y(l)$ be a non-oscillatory solution of (1.1) such that $y(l) > 0$, in view of (1.3), there exist six possible cases $(i)-(vi)$ (as in Theorem 3.3). The proofs in the four cases $(i)-(iv)$ are as in Theorem 3.2. Now assume that case (v) holds. Then following the same lines of the proof of Theorem 3.3, we arrive at (3.28), but since by (3.26), $\frac{w(l)}{l^{\frac{1}{L}}}$ is nonincreasing, then there exists a positive constant C_4 such that

$$\frac{w(l)}{l^{\frac{1}{L}}} \leq C_4,$$

$$l \geq l_3 \geq l_2.$$

This with (3.28), leads to

$$\Phi'(l) \leq -\sum_{i=1}^{n} q_i(l) C_4^{\beta_i-\alpha} L^\alpha l^{\alpha-\frac{\alpha}{L}} (I(l))^{\frac{\beta_i}{L}} - \alpha \frac{[\Phi(l)]^{\frac{\alpha+1}{\alpha}}}{d^{\frac{1}{\alpha}}(l)}. \tag{3.32}$$

Multiplying both sides of (3.32) by $D^\alpha(l)$ and integrating from $l_4 (> l_3)$ to l, and then applying the inequality (3.11), we obtain

$$\int_{l_4}^{l} \left[D^\alpha(s) \sum_{i=1}^{n} q_i(s) C_4^{\beta_i-\alpha} L^\alpha s^{\alpha-\frac{\alpha}{L}} (I(s))^{\frac{\beta_i}{L}} - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} \frac{1}{D(s) d^{\frac{1}{\alpha}}(s)} \right] ds \leq 1 + D^\alpha(l_4)\Phi(l_4).$$

This contradicts (3.31). Assume that case (vi) holds. By a similar argument to that used in Lemma 2.3, we arrive at the conclusion that $\lim_{l\to\infty} y(l) = 0$. The proof is complete. □

4. Examples

Example 4.1. *Consider the differential equation*

$$\left(y(l) - \frac{1}{l^2} y\left(\frac{l}{5}\right) - \frac{1}{l^4} y\left(\frac{l}{2}\right)\right)''' + \frac{1}{l^3} y(l) + \frac{1}{l^4} y(2l) = 0, \quad l \geq 2. \tag{4.1}$$

Here $a_1 = \frac{1}{l^2}, a_2 = \frac{1}{l^4}, d(l) = 1, q_1(l) = \frac{1}{l^3}, q_2(l) = \frac{1}{l^4}, \delta_1(l) = \frac{l}{5}, \delta_2(l) = \frac{l}{2}, \tau_1(l) = l, \tau_2(l) = 2l, \alpha = \beta_1 = \beta_2 = 1$. *Note that* $\int_{l_0}^{\infty} d^{\frac{1}{\alpha}}(s)\,ds = \int_2^\infty ds = \infty$,

$$\int_{l_0}^{\infty} \int_{v}^{\infty} \left(\frac{1}{d(u)} \int_{u}^{\infty} \sum_{i=1}^{n} q_i(s)\,ds \right)^{\frac{1}{\alpha}} du dv = \int_{2}^{\infty} \int_{v}^{\infty} \int_{u}^{\infty} \left(\frac{1}{s^3} + \frac{1}{s^4} \right) ds du dv$$
$$= \infty.$$

Choosing $v(l) = l$ *and* $I(l) = l$, *we have*

$$N(l) = \left[\frac{\int_{l_2}^{I(l)} \int_{l_1}^{s} d^{\frac{-1}{\alpha}}(h)\,dhds}{\int_{l_1}^{l} d^{\frac{-1}{\alpha}}(h)\,dh} \right]^\alpha = \frac{\int_{l_2}^{l}\int_{l_1}^{s} dhds}{\int_{l_1}^{l} dh} = \frac{l^2 - 2ll_1 + \lambda}{2(l - l_1)},$$
$$\lambda = 2l_1 l_2 - l_2^2,$$

and

$$\begin{aligned}
&\limsup_{l\to\infty}\int_{l_3}^{l}\left(v(\iota)\sum_{i=1}^{n}q_i(\iota)C_1^{\beta_i-\alpha}N(\iota)-\frac{d(\iota)[v'(\iota)]^{\alpha+1}}{(\alpha+1)^{\alpha+1}v^{\alpha}(\iota)}\right)d\iota\\
=&\limsup_{l\to\infty}\int_{l_3}^{l}\left(\iota\left(\frac{1}{\iota^3}+\frac{1}{\iota^4}\right)\left(\frac{\iota^2-2\iota l_1+\lambda}{2(\iota-l_1)}\right)-\frac{1}{4\iota}\right)d\iota\\
=&\limsup_{l\to\infty}\int_{l_3}^{l}\left[\frac{1}{2(\iota-l_1)}-\frac{l_1}{\iota(\iota-l_1)}+\frac{\lambda}{2\iota^2(\iota-l_1)}+\frac{1}{2\iota(\iota-l_1)}-\frac{l_1}{\iota^2(\iota-l_1)}+\frac{\lambda}{2\iota^3(\iota-l_1)}-\frac{1}{4\iota}\right]d\iota\\
=&\infty.
\end{aligned}$$

Thus, by Theorem 3.1, Eq (4.1) *is almost oscillatory.*

Example 4.2. *Consider the differential equation*

$$\left[l^{\frac{4}{3}}\left(y(l)-\frac{1}{2l}y\left(\frac{l}{2}\right)-\frac{1}{7l}y\left(\frac{l}{3}\right)\right)''\right]'+\frac{1}{l^{\frac{5}{3}}}y^3\left(\frac{l}{2}\right)+\frac{1}{l^4}y^5\left(\frac{l}{3}\right)=0,\quad l\geq 1. \tag{4.2}$$

Here $a_1=\frac{1}{2l}, a_2=\frac{1}{7l}, d(l)=l^{\frac{4}{3}}$, $q_1(l)=\frac{1}{l^{\frac{5}{3}}}, q_2(l)=\frac{1}{l^4}, \delta_1(l)=\frac{l}{2}, \delta_2(l)=\frac{l}{3}, \tau_1(l)=\frac{l}{2}, \tau_2(l)=\frac{l}{3}$, $\alpha=1, \beta_1=3, \beta_2=5$. *Note that* $\int_{l_0}^{\infty}d^{\frac{1}{\alpha}}(s)\,ds=\int_1^{\infty}l^{-\frac{4}{3}}ds<\infty$,

$$\begin{aligned}
&\int_{l_0}^{\infty}\int_{v}^{\infty}\left(\frac{1}{d(u)}\int_{u}^{\infty}\sum_{i=1}^{n}q_i(s)\,ds\right)^{\frac{1}{\alpha}}dudv\\
=&\int_{1}^{\infty}\int_{v}^{\infty}\frac{1}{u^{\frac{4}{3}}}\int_{u}^{\infty}\left(\frac{1}{s^{\frac{5}{3}}}+\frac{1}{s^4}\right)dsdudv=\infty.
\end{aligned}$$

Taking $I(l)=\frac{l}{3}$, *we have*

$$\begin{aligned}
N(l)&=\left[\frac{\int_{l_1}^{I(l)}\int_{l_2}^{s}d^{\frac{-1}{\alpha}}(h)\,dhds}{\int_1^{l}d^{\frac{-1}{\alpha}}(h)\,dh}\right]^{\alpha}=\frac{\int_{l_2}^{\frac{l}{3}}\int_{l_1}^{s}h^{-\frac{4}{3}}dhds}{\int_{l_1}^{l}h^{-\frac{4}{3}}dh}=\frac{-\frac{3\sqrt[3]{3}}{2}l+l^{\frac{4}{3}}l_1^{\frac{-1}{3}}+\gamma l^{\frac{1}{3}}}{3l^{\frac{1}{3}}l_1^{\frac{-1}{3}}-3},\\
\gamma&=\frac{9}{2}l_2^{\frac{2}{3}}-3l_2l_1^{\frac{-1}{3}},
\end{aligned}$$

choosing $l_1=1, l_2=(1,2), l_3>6$ *and* $v(l)=\frac{1}{l^{\frac{1}{3}}}$, *then*

$$\begin{aligned}
&\limsup_{l\to\infty}\int_{l_3}^{l}\left(v(\iota)\sum_{i=1}^{n}q_i(\iota)C_1^{\beta_i-\alpha}N(\iota)-\frac{d(\iota)[v'(\iota)]^{\alpha+1}}{(\alpha+1)^{\alpha+1}v^{\alpha}(\iota)}\right)d\iota\\
=&\limsup_{l\to\infty}\int_{l_3}^{l}\left(\frac{1}{\iota^{\frac{1}{3}}}\left(\frac{C_1^2}{\iota^{\frac{5}{3}}}+\frac{C_1^4}{\iota^4}\right)\left(\frac{-\frac{3\sqrt[3]{3}}{2}\iota+\iota^{\frac{4}{3}}l_1^{\frac{-1}{3}}+\gamma\iota^{\frac{1}{3}}}{3\iota^{\frac{1}{3}}l_1^{\frac{-1}{3}}-3}\right)-\frac{1}{36\iota}\right)d\iota\\
\geq&\limsup_{l\to\infty}\int_{l_3}^{l}\left(\frac{1}{\iota^{\frac{1}{3}}}\left(\frac{C_1^2}{\iota^{\frac{5}{3}}}+\frac{C_1^4}{\iota^4}\right)\left(\frac{-\frac{3\sqrt[3]{3}}{2}\iota+\iota^{\frac{4}{3}}l_1^{\frac{-1}{3}}+\gamma\iota^{\frac{1}{3}}}{3\iota^{\frac{1}{3}}l_1^{\frac{-1}{3}}}\right)-\frac{1}{36\iota}\right)d\iota
\end{aligned}$$

$$
\begin{aligned}
&= \limsup_{l\to\infty}\int_{l_3}^{l}\left(\frac{-\sqrt[3]{3}C_1^2}{2\iota^{\frac{4}{3}}}+\frac{\gamma C_1^2}{3\iota^2}+\frac{C_1^2}{3\iota}+\frac{C_1^4}{\iota^{\frac{13}{4}}}\left(\frac{-\frac{3\sqrt[3]{3}}{2}\iota+\iota^{\frac{4}{3}}+\gamma\iota^{\frac{1}{3}}}{3\iota^{\frac{1}{3}}}\right)-\frac{1}{36\iota}\right)d\iota\\
&= \infty,
\end{aligned}
$$

for $C_1^2 > \frac{1}{12}$, *and*

$$
\begin{aligned}
&\limsup_{l\to\infty}\int_{l_3}^{l}\left[D^{\alpha}(s)\sum_{i=1}^{n}q_i(s)C_3^{\beta_i-\alpha}L^{\alpha}s^{\alpha-\frac{\alpha}{L}}(I(s))^{\frac{\alpha}{L}}-\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{1}{D(s)\,d^{\frac{1}{\alpha}}(s)}\right]ds\\
= &\limsup_{l\to\infty}\int_{l_3}^{l}\left(\frac{3}{s}C_3^2L\left(\frac{1}{3}\right)^{\frac{1}{L}}+\frac{3}{s^{\frac{10}{3}}}C_3^4L\left(\frac{1}{3}\right)^{\frac{1}{L}}-\frac{1}{12s}\right)ds=\infty,
\end{aligned}
$$

for $C_3^2L\left(\frac{1}{3}\right)^{\frac{1}{L}} > \frac{1}{36}$. *Thus, by Theorem 3.3, Eq* (4.2) *is almost oscillatory.*

5. Conclusions

In this article, we discussed a general class of third-order differential equations with non-positive neutral coefficients (1.1) in the two cases of canonical and non-canonical conditions. Our criteria do not need to determine whether the functions $\tau_i(l)$ are delayed or advanced. Moreover, our new criteria do not need the restrictive condition $d'(l) \geq 0$.

Author contributions

A. A. El-Gaber: Investigation, Software, Supervision, Writing-original draft; M. M. A. El-Sheikh: Investigation, Software, Supervision, Writing-original draft; M. Zakarya: Writing-review editing and Funding; A.A. I Al-Thaqfan: Writing-review editing and Funding; H. M. Rezk: Investigation, Softwar, Writing-original draft. All authors have read and agreed to the published version of the manuscript.

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Conflict of interest

The authors declare that there are no conflicts of interest in this paper.

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