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Research article

KC-bitopological spaces

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Abstract: A topological space (X, τ) is called a KC-space when every compact subset of X is closed. The aim of this paper is to introduce new, namely KC-bitopological spaces and pairwise KC-topological spaces "P-KC-topological spaces". We examined the properties of these concepts and showed the relationships between these concepts and other bitopological spaces. We also discussed the effect of some types of functions on KC-bitopological spaces and pairwise KC-topological spaces. Several examples are discussed, and many well-known theories are generalized.

Keywords: bitopological space; KC-spaces; P-KC-spaces; K-function; compact function; P-Hausdorff; P-compact
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Abbreviations:

P: pairwise spaces; KC: closed compact spaces; P-KC: pairwise closed compact spaces; P-Hausdorff: pairwise Hausdorff spaces; KC-spaces: closed compact spaces; P-KC-spaces: pairwise closed compact spaces; LC-spaces: closed lindlöf spaces; P-LC-spaces: pairwise closed lindlöf spaces; $\mathbb{R}$: the set of all real numbers; $\mathbb{Q}$: the set of all rational numbers; $\mathbb{Z}$: the set of all integers; $\mathbb{N}$: the set of all natural numbers; τ_u: the usual topology; τ_s: the Sorgenfrey line topology; τ_{cof}: the coffinite topology; τ_{coc}: the cocountable topology; $\tau_{l.r}$: the left-ray topology; $\tau_{r.r}$: the right-ray topology; τ_{dis}: the discrete topology; τ_{ind}: the indiscrete topology; Cl_iA: the τ_i-closure of A

1. Introduction

The basic concepts of bitopological spaces began to be studied in 1963 by mathematician Kelly [1]. Several authors have since addressed the problem of defining compactness in bitopological spaces, as seen in Kim [2].

In 1969, Fletcher et al. [3] introduced the main definitions of $\tau_i\tau_j$-open covers and P-open covers in bitopological spaces. A cover $\tilde{U}$ of a bitopological space $\left(X, \tau_i, \tau_j\right)$ is called $\tau_i\tau_j$-open if

$$\tilde{U} \subset \tau_i \cup \tau_j.$$

If $\tilde{U}$ contains at least one non-empty member of τ_i and at least one non-empty member of τ_j, it is called P-open. They also defined the concepts of pairwise compact (P-compact) spaces. In 1972, Datta [4] studied the concept of semi-compact (S-compact) spaces in the bitopological space $\left(X, \tau_i, \tau_j\right)$. Cooke and Reilly [5] discussed the relationships between these previous definitions in 1975. In 1983, Fora and Hdieb [6] introduced the concepts of pairwise Lindelöf (P-Lindelöf) and semi-Lindelöf (S-Lindelöf) spaces. They also provided the definitions of certain types of functions as follows: function

$$g : \left(X, \tau_i, \tau_j\right) \longrightarrow \left(Y, \sigma_i, \sigma_j\right)$$

is called P-continuous (or P-closed, respectively) if both functions

$$g : (X, \tau_i) \longrightarrow (Y, \sigma_i)$$

and

$$g : \left(X, \tau_j\right) \longrightarrow \left(Y, \sigma_j\right)$$

are continuous (or closed, respectively).

This overview introduces KC-bitopological spaces, emphasizing their importance in relation to compactness and closedness. The concept of KC-topological spaces was first developed by Hewitt in the early 1940s, with foundational definitions and illustrative examples provided, see [7]. Hewitt introduced minimal topological spaces, establishing that every Hausdorff compact space is a minimal KC-space "mKC-space". By 1947, it was proven that every compact KC-space is an mKC-space. In 1965, the auther Aull, shows his contribution in [7] developed a space thats found between T_1 and T_2, namely, KC-space. The relationship between them is represented by the following diagram: $T_2 \rightarrow KC \rightarrow T_1$, researchers examined space in [8–11]. Wilansky [12] explored the relationships between separation axioms between T_1 and T_2 spaces, highlighting KC-topological spaces. It was established that every T_2-space is a KC-space, and every KC-space is a T_1-space, see also [13,14]. In 2004, Valdis showed that a minimal space where every compact subset is closed is countably compact [15], leading to the result that every minimal KC-space is countably compact [16–18]. Ali [8] in 2006 expanded on this by introducing KC-space, minimal KC-space, and minimal Hausdorff spaces, as well as minimal

LC-topological spaces. Ali found that the direct image of a KC-space under a continuous function is not necessarily a KC-space unless certain conditions are met. Further, the relationships between minimal KC-topological spaces and minimal Hausdorff or LC-topological spaces were studied.

In 2006, Alas and Wilson [19] examined the minimal properties of spaces between T_1 and T_2 spaces, noting that KC-spaces extend Hausdorff spaces. They introduced KC-closed spaces and discussed their relationship with KC-spaces, see also [9, 16, 19]. In 2008, Bella and Costantini [9] showed that any minimal KC-space must be compact, though not necessarily Hausdorff. In 2009, Oprsal [20] addressed the problem of whether every KC-space with a weaker KC-topology is compact, a question resolved by Bella and Costantini [9]. In 2010, Zarif and Razzak [11] linked KC-topological spaces with concepts such as connected functions, closed functions, and K-functions, yielding significant results. In 2011, Adnan [7] introduced Cooke-topological spaces and analyzed their properties and relationships with KC-spaces. Bella and Costantini [21] introduced SC-spaces, defined by closed convergent sequences and their limits, and explored the relationships between T_2, KC, SC, and T_1 spaces as $T_2 \longrightarrow KC \longrightarrow SC \longrightarrow T_1$. Jebour and Saleh [22] introduced K-(SC)-spaces as weaker versions of KC-topological spaces, developing new results, see also [23, 24]. In 2015, Ali and Abker [25] introduced minimal compact closed spaces "mKC-spaces" and examined their relations with other spaces, including KC-topological spaces. A new definition for α-KC-spaces was proposed, with results indicating that every KC-space is an α-KC-space, see the study [26]. For more studies about this topic, see [13, 27]. In 2018, Almohor and Hdieb [28] explored properties of pairwise L-closed spaces (LC-topological spaces), contributing to the field by generalizing relationships between LC-topological spaces and KC-topological spaces. Nadhim et al. introduced concepts of strong and weak forms of KC-topological spaces [29].

In this paper, we first introduce the concept of closed compactness, referred to as "KC-spaces" in bitopological spaces. We provide basic definitions for KC-bitopological spaces and pairwise KC-topological spaces, deriving many related results. Illustrative examples and theories of these two main concepts are discussed. We also explore how these concepts relate to other topological concepts within bitopological spaces.

Next, we study the effect of various types of functions on KC-bitopological spaces and pairwise KC-topological spaces. We examine the necessary conditions for the direct image of a KC-bitopological space and a pairwise KC-topological space to remain a KC-bitopological space and a pairwise KC-topological space, respectively. Additional conditions are established for the inverse image of these spaces to retain their properties.

Finally, we analyze the relationship between KC-bitopological spaces, pairwise KC-topological spaces, and other bitopological spaces, such as P-compact spaces. This involves studying the impact of different functions on KC-bitopological spaces and pairwise KC-topological spaces. For instance, a bitopological space $\left(X, \tau_i, \tau_j\right)$ is considered compact if both (X, τ_i) and $\left(X, \tau_j\right)$ are compact spaces.

2. KC-bitopological spaces

In this section, we introduce the concepts of KC-bitopological spaces and pairwise KC-topological spaces.

We will examine their properties and explore their relationships with other bitopological spaces.

Definition 1. *(1) A bitopological space (X, τ_1, τ_2) is said to be a P-KC-space if each τ_i-compact* *subset of X is τ_j-closed for all $(i \neq j,\ i, j = 1, 2)$.*

(2) *A bitopological space (X, τ_1, τ_2) is said to be a KC-space if each τ_i-compact subset of X is τ_i-closed for all ($i = 1$, 2).*

Remark 1. *Part (2) in the above definition is equivalent to that both (X, τ_1) and (X, τ_2) are KC-spaces.*

In general: If both (X, τ_i) and $\left(X, \tau_j\right)$ have the same topological property P, then the bitopological space $\left(X, \tau_i, \tau_j\right)$ has property P.

So, we say that a bitopological space $\left(X, \tau_i, \tau_j\right)$ is a KC-space, if both (X, τ_i) and $\left(X, \tau_j\right)$ are KC-spaces.

Example 1. (1) *The bitopological space $(\mathbb{R}, \tau_{ind}, \tau_{dis})$ is not a KC-space, since any compact subset of $(\mathbb{R}\ , \tau_{dis})$ is closed but (1,2) is a compact subset in $(\mathbb{R}\ , \tau_{ind})$, which is not closed.*

(2) *The bitopological space $(\mathbb{R}, \tau_{dis}, \tau_u)$ is KC-space since any compact subset of both $(\mathbb{R}\ , \tau_{dis})$ and $(\mathbb{R}\ , \tau_u)$ is closed.*

(3) *The bitopological space $\left(\mathbb{R}, \tau_{cof}, \tau_{coc}\right)$ is a KC-space.*

Example 2. (1) *The bitopological space $(\mathbb{R}, \tau_{ind}, \tau_{dis})$ is not a P-KC-space.*

(2) *The bitopological space $\left(\mathbb{R}, \tau_{cof}, \tau_{dis}\right)$ is a P-KC-space.*

(3) *The bitopological space $(\mathbb{R}, \tau_{coc}, \tau_{dis})$ is a P-KC-space.*

Example 3. (1) *The bitopological space $(\mathbb{R}, \tau_u, \tau_s)$ is a KC-space, but not a P-KC-space.*

(2) *The bitopological space $(\mathbb{R}, \tau_{l.r}, \tau_{r.r})$ is never a KC-space nor a P-KC-space.*

(3) *The bitopological space $(\mathbb{R}, \tau_{ind}, \tau_{l.r})$ is never a KC-space nor a P-KC-space.*

Definition 2. (1) *A bitopological space (X, τ_1, τ_2) is said to be a P-LC-space. Each τ_i-lindlöf subset of X is τ_i-closed for all $(i \neq j, i, j = 1, 2)$.*

(2) *A bitopological space (X, τ_1, τ_2) is said to be a LC-space. Each τ_i-lindlöf subset of X is τ_i-closed for all $(i = 1, 2)$.*

Since every compact (τ_i-compact, respectively) is lindelöf (τ_i-lindelöf, respectively) the prove of the following two theorems is clear.

Theorem 1. *Every P-LC-bitopological space is a P-KC-space.*

Theorem 2. *Every LC-bitopological space is a KC-space.*

Example 4. *$(\mathcal{R}, \tau_{dis}, \tau_{coc})$ is a P-LC-space.*

Example 5. *$(\mathcal{R}, \tau_{dis}, \tau_{ind})$ is not a LC-space.*

Example 6. *$(\mathcal{R}, \tau_{r.r}, \tau_{l.r})$ is never a LC-space nor a P-LC-space.*

It is easy to prove the following theorems:

Theorem 3. *Every discrete space is a KC-space.*

Theorem 4. *Every discrete bitopological space $(X, \tau_{dis}, \tau_{dis})$ is a KC-space.*

Definition 3. *Let* $\left(X, \overset{1}{\tau}_X\right)$ *and* $\left(X, \overset{2}{\tau}_X\right)$ *be two topological spaces defined on the same set X and* $A \subseteq X$. *If* $\left(A, \overset{1}{\tau}_A\right)$ *and* $\left(A, \overset{2}{\tau}_A\right)$ *are two subspaces of* $\left(X, \overset{1}{\tau}_X\right)$ *and* $\left(X, \overset{2}{\tau}_X\right)$, *respectively, then* $\left(A, \overset{1}{\tau}_A, \overset{2}{\tau}_A\right)$ *is a subspace of* $\left(X, \overset{1}{\tau}_X, \overset{2}{\tau}_X\right)$.

Theorem 5. *Every subspace of a P-KC-space is a P-KC-space.*

Proof. Assume that $\left(X, \overset{1}{\tau}_X, \overset{2}{\tau}_X\right)$ a *P-KC*-space and $\left(A, \overset{1}{\tau}_A, \overset{2}{\tau}_A\right)$ a subspace of $\left(X, \overset{1}{\tau}_X, \overset{2}{\tau}_X\right)$. Let K be a $\overset{1}{\tau}_A$ compact subset of $\left(A, \overset{1}{\tau}_A\right)$, K is a $\overset{1}{\tau}_X$ compact subset of $\left(X, \overset{1}{\tau}_X\right)$ but $\left(X, \overset{1}{\tau}_X, \overset{2}{\tau}_X\right)$ is a *P-KC*space, so K is $\overset{2}{\tau}_X$ closed. Hence,

$$K = K \cap A$$

is $\overset{2}{\tau}_A$ closed in $\left(A, \overset{2}{\tau}_A\right)$.

Therefore, any $\overset{1}{\tau}_A$-compact subset of A is $\overset{2}{\tau}_A$-closed. In the same way, we prove that any $\overset{2}{\tau}_A$-compact subset of A is $\overset{1}{\tau}_A$-closed. Thus, $\left(A, \overset{1}{\tau}_A, \overset{2}{\tau}_A\right)$ is a *P-KC*-space. □

Theorem 6. *Every subspace of a KC-bitopological space* $\left(X, \overset{1}{\tau}_X, \overset{2}{\tau}_X\right)$ *is a KC-space.*

Proof. Assume that $\left(X, \overset{1}{\tau}_X, \overset{2}{\tau}_X\right)$ is a *KC*-space and $\left(A, \overset{1}{\tau}_A, \overset{2}{\tau}_A\right)$ is a subspace of $\left(X, \overset{1}{\tau}_X, \overset{2}{\tau}_X\right)$. Let K be a $\overset{1}{\tau}_A$ compact subset of $\left(A, \overset{1}{\tau}_A\right)$, K is a $\overset{1}{\tau}_A$compact subset of $\left(X, \overset{1}{\tau}_X\right)$, but $\left(X, \overset{1}{\tau}_X, \overset{2}{\tau}_X\right)$ is a *KC*-space, so K is $\overset{1}{\tau}_X$ closed. Hence,

$$K = K \cap A$$

is $\overset{1}{\tau}_A$ closed in $\left(A, \overset{1}{\tau}_A\right)$.

Therefore, any $\overset{1}{\tau}_A$-compact subset of A is $\overset{1}{\tau}_A$-closed. In the same way we prove that any $\overset{2}{\tau}_A$-compact subset of A is $\overset{2}{\tau}_A$-closed. Thus $\left(A, \overset{1}{\tau}_A, \overset{2}{\tau}_A\right)$ is *KC*-space. □

Theorem 7. *The intersection of any two KC-spaces is a KC-space.*

Proof. Let $\left(X, \overset{1}{\tau}_X\right)$ and $\left(X, \overset{2}{\tau}_X\right)$ be two *KC*-spaces. Let

$$\overset{3}{\tau}_X = \overset{1}{\tau}_X \cap \overset{2}{\tau}_X$$

and K be a compact subset of $\left(X, \overset{3}{\tau}_X\right)$, K is a compact subset of both $\left(X, \overset{1}{\tau}_X\right)$ and $\left(X, \overset{2}{\tau}_X\right)$. Thus, K is closed in both $\overset{1}{\tau}_X$ and $\overset{2}{\tau}_X$, so K is closed in $\overset{3}{\tau}_X$. Therefore, $\left(X, \overset{3}{\tau}_X\right)$ is a *KC*-space. □

Definition 4. *The intersection of the bitopological spaces* $\left(X, \tau_i, \tau_j\right)$ *and* $\left(X, \overset{*}{\tau}_i, \overset{*}{\tau}_j\right)$ *is the bitopological space* $\left(X, \tau_i \cap \overset{*}{\tau}_i, \tau_j \cap \overset{*}{\tau}_j\right)$.

Theorem 8. *The intersection of any two KC-bitopological spaces defined on the same set is a KC-bitopological space.*

Proof. Let $\left(X, \tau_i, \tau_j\right)$ and $\left(X, \overset{*}{\tau}_i, \overset{*}{\tau}_j\right)$ be two KC-bitopological spaces, and

$$\left(X, \sigma_i, \sigma_j\right) = \left(X, \tau_i, \tau_j\right) \cap \left(X, \overset{*}{\tau}_i, \overset{*}{\tau}_j\right) = \left(X, \tau_i \cap \overset{*}{\tau}_i, \tau_j \cap \overset{*}{\tau}_j\right)$$

be the intersection of $\left(X, \tau_i, \tau_j\right)$ and $\left(X, \overset{*}{\tau}_i, \overset{*}{\tau}_j\right)$, where

$$\sigma_i = \tau_i \cap \overset{*}{\tau}_i$$

and

$$\sigma_j = \tau_j \cap \overset{*}{\tau}_j.$$

Let K be a σ_i-compact subset of (X, σ_i). By above theorem, K is σ_i-closed subset of (X, σ_i). Hence, (X, σ_i) is a KC-space. Similarly, $\left(X, \sigma_j\right)$ is KC-space. Therefore, $\left(X, \sigma_i, \sigma_j\right)$ is a KC-space. □

Theorem 9. *The intersection of any two P-KC-topological spaces defined on the same set is a P-KC-space.*

Proof. Let $\left(X, \tau_i, \tau_j\right)$ and $\left(X, \overset{*}{\tau}_i, \overset{*}{\tau}_j\right)$ be two P-KC-topological spaces, and

$$\left(X, \sigma_i, \sigma_j\right) = \left(X, \tau_i \cap \overset{*}{\tau}_i, \tau_j \cap \overset{*}{\tau}_j\right)$$

be the intersection of $\left(X, \tau_i, \tau_j\right)$ and $\left(X, \overset{*}{\tau}_i, \overset{*}{\tau}_j\right)$, where

$$\sigma_i = \tau_i \cap \overset{*}{\tau}_i$$

and

$$\sigma_j = \tau_j \cap \overset{*}{\tau}_j.$$

Let K be a σ_i-compact subset of (X, σ_i). K is τ_i-compact and τ_i^*-compact, but both $\left(X, \tau_i, \tau_j\right)$ and $\left(X, \overset{*}{\tau}_i, \overset{*}{\tau}_j\right)$ are P-KC-spaces, so K is closed in both τ_j and $\overset{*}{\tau}_j$. Hence, K is σ_j-closed. Therefore, $\left(X, \sigma_i, \sigma_j\right)$ is a P-KC-space. □

Definition 5. *A bitopological space $\left(X, \tau_i, \tau_j\right)$ is called P-Hausdorff if for distinct points a and b, there is a τ_i-open set U and a τ_j-open set V such that $a \in U, b \in V$, and*

$$U \cap V = \phi.$$

Definition 6. *A bitopological space $\left(X, \tau_i, \tau_j\right)$ is called Hausdorff (resp. compact) if both (X, τ_i) and $\left(X, \tau_j\right)$ are Hausdorff (resp. compact) spaces.*

Theorem 10. *Every Hausdorff space is a KC-space.*

Proof. Let A be a τ_i-compact subset of X. Every compact subspace of a Hausdorff space is closed, then A is τ_i-closed. Similarly, we can show that if A is a τ_j-compact subset of X, then A is τ_j-closed. □

Example 7. *The bitopological space $(\mathbb{R}, \tau_{dis}, \tau_u)$ is a Hausdorff space; so it is a KC-space.*

Remark 2. *The converse of above theorem is not true; see the following example.*

Example 8. *The bitopological space* $\left(\mathbb{R}, \tau_{cof}, \tau_{coc}\right)$ *is a KC-space but not a Hausdorff space.*

Recall that: In a topological space (X, τ_X), if every countable intersection of any collection of open sets is open, then X is called a p-space.

Theorem 11. *Every Hausdorff p-space is a LC-space.*

Proof. Assume that $\left(X, \tau_i, \tau_j\right)$ is a Hausdroff P-space. Let D be a τ_i-Lindelöf subset of X. D is a τ_i-Lindelöf subset of Hausdroff P-space (X, τ_i). Therefore, D is τ_i-closed.

Similarly, we can get if D is a τ_j-Lindelöf, then D is τ_j-closed. □

Example 9. *A space* $(\mathbb{N}, \tau_{dis}, \tau_{coc})$ *is a Hausdorff P-space, so it is a LC-space.*

Definition 7. *If* $\left(X, \overset{1}{\tau}_X\right)$ *and* $\left(X, \overset{2}{\tau}_X\right)$ *are two metric spaces, then* $\left(X, \overset{1}{\tau}_X, \overset{2}{\tau}_X\right)$ *is called the bitopological metric space.*

Corollary 1. *Every metric space is a KC-space.*

Proof. Since every metric space is a Hausdorff then it is a KC-space. □

Corollary 2. *Every bitopological metric space is a KC-space.*

The following theorem can be found in [6].

Theorem 12. *If a bitopological space* $\left(X, \tau_i, \tau_j\right)$ *is a P-Hausdorff space, then for each x in X we have:*

a)
$$\{x\} = \bigcap_{\alpha\in\Delta} \left\{Cl_i\ V_\alpha :\ V_\alpha \text{ is a } \tau_j\text{-open set contains } x\right\},\quad (i \neq j,\ i, j = 1, 2)\,,$$

where $Cl_i\ V_\alpha$ is a τ_i-closure of V_α.

b)
$$\{x\} = \bigcap_{\beta\in\Gamma} \left\{Cl_j\ U_\beta :\ U_\beta \text{ is a } \tau_i\text{-open set contains } x\right\},\quad (i \neq j,\ i, j = 1, 2)\,,$$

where $Cl_j\ U_\beta$ is a τ_j-closure of U_β.

Proof. Let $y \in X$ such that $x \neq y$. There exists a τ_i-open set U_i and a τ_j-open set U_j such that $y \in U_i$, $x \in U_j$, and

$$U_i \cap U_j = \phi.$$

Since

$$U_j \subseteq X - U_i,$$

then

$$x \in U_j \subseteq Cl_i\ U_j \subseteq X - U_i.$$

Hence, $x \in Cl_i U_j$, $\forall \tau_j$-open sets U_j and

$$\{x\} = \bigcap_{\alpha\in\Delta} \left\{Cl_i V_\alpha : V_\alpha \text{is a } \tau_j\text{-open set contains } x\right\},\ (i \neq j,\ i, j = 1, 2)\ \forall x \in X.$$

This proves part (a). The proof of part (b) is similar to (a).

□

Theorem 13. *Let $\left(X, \tau_i, \tau_j\right)$ be a P-Hausdorff space. Then, every τ_i-compact subset of X is τ_j-closed* $(i \neq j, i,\ j = 1, 2)$.

Proof. Let B be a τ_i-compact subset of X and $x \in X - B$. By above theorem,

$$\{x\} = \bigcap_{\alpha\in\Delta} \left\{Cl_i\ U_\alpha :\ U_\alpha \text{ is a } \tau_j\text{-open set contains } x\right\},\ (i \neq j, i,\ j = 1, 2).$$

Since

$$B \subseteq X - \{x\} = X - \bigcap_{\alpha\in\Delta} \left\{Cl_i\ U_\alpha :\ U_\alpha \text{ is a } \tau_j\text{-open set contains } x\right\} = \bigcup_{\alpha\in\Delta} (X - Cl_i\ U_\alpha),$$

$\{X - Cl_i\ U_\alpha : \alpha \in \Delta\}$ is a τ_i-open cover of a τ_i-compact set B. So, there exists a finite subset $\overset{*}{\Delta} \subseteq \Delta$ such that $\left\{X - Cl_i\ U_\alpha : \alpha \in \overset{*}{\Delta}\right\}$ is a cover of B.

Hence,

$$B \subseteq \bigcup_{\alpha\in\overset{*}{\Delta}} (X - Cl_i\ U_\alpha) = X - \bigcap_{\alpha\in\overset{*}{\Delta}} \{Cl_i\ U_\alpha\} \subseteq X - \bigcap_{\alpha\in\overset{*}{\Delta}} U_\alpha.$$

Letting

$$U = \bigcap_{\alpha\in\overset{*}{\Delta}} U_\alpha,$$

then U is a τ_j-open set such that

$$x \in U \subseteq X - B.$$

Hence, B is τ_j-closed. □

Theorem 14. *Every P-Hausdorff space $\left(X, \tau_i, \tau_j\right)$ is a p-KC-space.*

Proof. Let A be a τ_i-compact subset of a P-Hausdorff space X. By above theorem, A is τ_j-closed where $(i \neq j,\ i\ ,\ j = 1, 2)$. Hence, $\left(X, \tau_i, \tau_j\right)$ is a P-KC-space. □

Example 10. *The bitopological space $\left(\mathbb{R}, \tau_{cof}, \tau_{dis}\right)$ is a P-Hausdorff, so it is a P-KC-space.*

Remark 3. *The converse of above theorem is not true, see the following example:*

Example 11. *The bitopological space $\left(\mathbb{R}, \tau_{cof}, \tau_{coc}\right)$ is a P-KC-space but not a P-Hausdorff.*

Theorem 15. *Every P-Hausdorff P-space is a P-LC-space.*

Proof. Let B be a τ_i-lindelöf subset of X and $x \in X - B$.

$$\{x\} = \bigcap_{\alpha\in\Delta} \left\{Cl_i\ U_\alpha :\ U_\alpha \text{ is a } \tau_j\text{-open set contains } x\right\},\ (i \neq j, i,\ j = 1, 2).$$

Since

$$B \subseteq X - \{x\} = X - \bigcap_{\alpha \in \Delta} \left\{ Cl_i\ U_\alpha :\ U_\alpha \text{ is a } \tau_j\text{-open set contains } x \right\} = \bigcup_{\alpha \in \Delta} (X - Cl_i\ U_\alpha),$$

$\{X - Cl_i\ U_\alpha : \alpha \in \Delta\}$ is a τ_i-open cover of a τ_i-lindelöf set B. So, there exists a countable subset $\overset{*}{\Delta} \subseteq \Delta$ such that $\left\{X - Cl_i\ U_\alpha : \alpha \in \overset{*}{\Delta}\right\}$ is a cover of B. Hence,

$$B \subseteq \bigcup_{\alpha \in \overset{*}{\Delta}} (X - Cl_i\ U_\alpha) = X - \bigcap_{\alpha \in \overset{*}{\Delta}} \{Cl_i\ U_\alpha\} \subseteq X - \bigcap_{\alpha \in \overset{*}{\Delta}} U_\alpha.$$

Letting

$$U = \bigcap_{\alpha \in \overset{*}{\Delta}} U_\alpha,$$

then U is a τ_j-open set such that $x \in U \subseteq X - B$. Hence, B is τ_j-closed. □

Example 12. *$(\mathbb{Z}, \tau_{coc}, \tau_{dis})$ is a P-Hausdorff P-space, so it is a P-LC-space.*

Example 13. *$\left(\mathbb{N}, \tau_{coc}, \tau_{cof}\right)$ is a P-LC-space but not a P-Hausdorff and P-space.*

Theorem 16. *Let $\left(X, \tau_i, \tau_j\right)$ be a compact P-KC-space. Then, $\tau_i = \tau_j$.*

Proof. Let $\phi \neq W$ and $w \in \tau_i$. Then, $X - W$ is a τ_i-closed subset of a compact space (X, τ_i). So, $X - W$ is τ_i-compact subset of *P-KC*-space X. Therefore, $X - W$ is a τ_j-closed. Hence, W is a τ_j-open and $\tau_i \subseteq \tau_j$. Similarly, we can get $\tau_j \subseteq \tau_i$. Consequently, $\tau_i = \tau_j$. □

Corollary 3. *If a space $\left(X, \tau_i, \tau_j\right)$ is a compact P-Hausdorff, then $\tau_i = \tau_j$.*

Proof. Let $H \in \tau_i$. $X - H$ is a τ_i-closed subset of a compact space (X, τ_i). Therefore, $X - H$ is a τ_i-compact subset of *P*-Hausdorff space X. Hence, $X - H$ is τ_j-closed, so H is τ_j-open. So, $H \in \tau_j$, and then $\tau_i \subseteq \tau_j$. In the same way, we get $\tau_j \subseteq \tau_i$. Consequently, $\tau_i = \tau_j$. □

Theorem 17. *Let $\left(X, \tau_i, \tau_j\right)$ be a Lindelöf P-LC-space P-space. Then, $\tau_i = \tau_j$.*

Proof. Let $\phi \neq O \in \tau_i$. Then, $X - O$ is τ_i-closed subset of a Lindelöf *P*-space (X, τ_i). So, $X - O$ is τ_i-Lindelöf. But, X is a *P-LC*-space, so $X - O$ is τ_j-closed and then $O \in \tau_j$. Therefore, $\tau_i \subseteq \tau_j$. By the same technique, we obtained $\tau_j \subseteq \tau_i$. Consequently, $\tau_i = \tau_j$. □

Corollary 4. *If the space $\left(X, \tau_i, \tau_j\right)$ is a Lindelöf P-Hausdorff P-space, then $\tau_i = \tau_j$.*

The proof comes directly from the fact that "every *P*-Hausdorff *P*-space is a *P-LC*-space".

3. Mappings on *KC*-bitopological spaces

In this section, we discuss the effects of various types of functions on *KC*-bitopological spaces and pairwise *KC*-topological spaces.

Definition 8. *A function*

$$g : \left(X, \tau_i, \tau_j\right) \longrightarrow \left(Y, \sigma_i, \sigma_j\right)$$

is said to be a compact function, if $g^{-1}\{y\}$ *is* τ_i*-compact and* τ_j*-compact for each* $y \in Y$.

We can find the following definition in [11].

Definition 9. *A function*

$$g : (X, \tau_i) \longrightarrow (Y, \sigma_i)$$

is said to be a K-function, if $g^{-1}\{B\}$ *is a compact subset of X for all compact subsets B of Y and* $g(A)$ *is a compact subset of Y for all compact subsets A of X.*

Definition 10. *A function*

$$g : \left(X, \tau_i, \tau_j\right) \longrightarrow \left(Y, \sigma_i, \sigma_j\right)$$

is said to be a K-function, if both functions

$$g : (X, \tau_i) \longrightarrow (Y, \sigma_i)$$

and

$$g : \left(X, \tau_j\right) \longrightarrow \left(Y, \sigma_j\right)$$

are K-functions.

Theorem 18. *Let*

$$g : \left(X, \tau_i, \tau_j\right) \longrightarrow \left(Y, \sigma_i, \sigma_j\right)$$

be an onto closed K-function. If X is a P-KC-space, then so is Y.

Proof. Let B be a σ_i-compact subset of Y. To show that B is a σ_j-closed subset of Y. Since g is a K-function, then $g^{-1}\{B\}$ is a τ_i-compact subset of X.

But, X is a P-KC-space, so $g^{-1}\{B\}$ is a τ_j-closed. Since g is an onto closed, function then

$$g\left(g^{-1}\{B\}\right) = B$$

is σ_j-closed in Y. So, Y is a P-KC-space. □

We presented some definitions that will be used later.

Definition 11. *[6] A function*

$$g : \left(X, \tau_i, \tau_j\right) \longrightarrow \left(Y, \sigma_i\ , \sigma_j\right)$$

is called P-continuous (P-closed, respectively) if the functions

$$g : (X, \tau_i) \longrightarrow (Y, \sigma_i)$$

and

$$g : \left(X, \tau_j\right) \longrightarrow \left(Y, \sigma_j\right)$$

are continuous (closed, respectively).

Theorem 19. *Let*

$$g:\left(X,\tau_i,\tau_j\right)\longrightarrow\left(Y,\sigma_i,\sigma_j\right)$$

be an onto P-closed K-function. If X is P-Hausdorff, then Y is a P-KC-space.

Proof. Assume that

$$g:\left(X,\tau_i,\tau_j\right)\longrightarrow\left(Y,\sigma_i,\sigma_j\right)$$

be an onto P-closed K-function. Let A be a σ_i-compact subset of Y. Since g is a K-function, then $g^{-1}(A)$ is a τ_i-compact subset of X. Since X is P-Hausdorff, then $g^{-1}(A)$ is τ_j-closed. Hence,

$$g\left(g^{-1}(A)\right)=A$$

is σ_j-closed because

$$g:\left(X,\tau_j\right)\longrightarrow\left(Y,\sigma_j\right)$$

is closed. Hence, Y is a P-KC-space. □

Theorem 20. *Let*

$$g:\left(X,\tau_i,\tau_j\right)\longrightarrow\left(Y,\sigma_i,\sigma_j\right)$$

be an onto P-closed K-function. If X is Hausdorff, then Y is a KC-space.

Proof. Assume that

$$g:\left(X,\tau_i,\tau_j\right)\longrightarrow\left(Y,\sigma_i,\sigma_j\right)$$

be an onto P-closed K-function. Let A be a σ_i-compact subset of Y. Since g is a K-function, then $g^{-1}(A)$ is a τ_i-compact subset of X. Since X is Hausdorff, then $g^{-1}(A)$ is τ_i- closed. Hence,

$$g\left(g^{-1}(A)\right)=A$$

is σ_i-closed because

$$g:(X,\tau_i)\longrightarrow(Y,\sigma_i)$$

is closed. Hence, Y is a KC-space. □

Definition 12. *[3] A cover $\widetilde{U}$ of a bitopological space $\left(X,\tau_i,\tau_j\right)$ is called a P-open cover if $\widetilde{U}$ contains at least one non-empty element of τ_i and at least one non-empty element of τ_j.*

Definition 13. *[3] A bitopological space $\left(X,\tau_i,\tau_j\right)$ is called P-compact (resp. Lindelöf) if each P-open cover of X has a finite (resp. countable) subcover.*

Example 14. *The bitopological space $(\mathbb{R},\tau_{dis},\tau_{coc})$ is not P-compact. Since*

$$\widetilde{U}=\{\{x_i\}:x_i\in\mathbb{Q}\}\cup\{Irr\}$$

is a P-open cover of X which has no finite subcover if $\widetilde{U}$ has a finite subcover $\acute{U}$, then

$$\acute{U}=\{x_1,\ x_2,x_3,...,\ x_n\}\cup\{Irr\},$$

where $x_i\in\mathbb{Q}\ \forall\ i=1,2,3,...,n$.

This means

$$\mathbb{R}\subseteq\{x_1,\ x_2,x_3,...,\ x_n\}\cup\{Irr\},$$

which is a contradiction.

Example 15. *Consider $X = \mathbb{R}$. Let*

$$ß_1 = \{X, \phi, \{x\} : x > 5\}$$

and

$$ß_2 = \{X, \phi, \{x\} : x < 0\}.$$

Let τ_1 and τ_2 be the topologies on X induced by the bases $ß_1$ and $ß_2$, respectively. $(\mathbb{R}, \tau_1, \tau_2)$ is a P-compact space since any P-open cover of X must contain $\{X\}$. Hence, $\{X\}$ is a finite subcover of any P-open cover.

Theorem 21. *Let $\left(X, \tau_i, \tau_j\right)$ be a P-compact space, every τ_i-closed proper subset of X is τ_j-compact, where $(i \neq j,\ i, j = 1, 2)$.*

Proof. Let F be a τ_i-closed proper subset of a P-compact X, and

$$\widetilde{U} = \{U_\alpha : \alpha \in \Delta\}$$

be a τ_j-open cover of F.

For each $x \in X - F$, there exists a τ_i-open set $U(x)$ such that

$$x \in U(x) \subseteq X - F.$$

Now,

$$\{U_\alpha : \alpha \in \Delta\} \cup \{U(x) : x \in X - F\}$$

is a P-open cover of the P-compact space X, so there exists a finite set $\Delta_1 \subseteq \Delta$ and a finite set

$$\{x_1,\ x_2,\ x_3, ...,\ x_n\} \subseteq X - F$$

such that

$$\{U_\alpha : \alpha \in \Delta_1\} \cup \{U(x_1),\ U(x_2), U(x_3),\ ...,\ U(x_n)\}$$

is a finite cover of X. Since

$$U(x_i) \cap F = \phi,\ \forall i = 1, 2, ..., n,$$

then

$$\bigcup_{i=1}^{n} U(x_i) \cap F = \phi.$$

So, $\{U_\alpha : \alpha \in \Delta_1\}$ is a finite subcover of $\widetilde{U}$ for F. Therefore, F is τ_j-compact. □

Remark 4. *The expression (proper subset) in the previous theory can not be dispensed with or removed.*

Example 16. *Consider $(\mathbb{Z}, \tau_{coc}, \tau_{dis})$. Then, $\mathbb{Z}$ is τ_{coc}-closed but it is not τ_{dis}-compact.*

Theorem 22. *Let*

$$g : \left(X, \tau_i, \tau_j\right) \longrightarrow \left(Y, \sigma_i, \sigma_j\right)$$

be P-continuous and X be P-compact. If Y is a P-KC-space then, g is P-closed.

Proof. Let A be a τ_i-closed subset of a P-compact space $\left(X,\tau_i,\tau_j\right)$. A is a τ_j-compact, where $(i \neq j, i,\ j = 1,2)$. Since g is P-continuous, then $g\left(A\right)$ is a σ_j-compact subset of Y. But, Y is a P-KC-space, so $g\left(A\right)$ is a σ_i-closed subset of Y, where $(i \neq j,\ i, j = 1,2)$. Hence,

$$g : (X,\tau_i) \longrightarrow (Y,\sigma_i)$$

is closed. Similarly, we can show that

$$g : \left(X,\tau_j\right) \longrightarrow \left(Y,\sigma_j\right)$$

is closed. Therefore, g is P-closed. □

Theorem 23. *Let*

$$g : \left(X,\tau_i,\tau_j\right) \longrightarrow \left(Y,\sigma_i,\sigma_j\right)$$

be P-continuous and X be a compact space. If Y is a KC-space, then g is P-closed.

Proof. Let A be a τ_i-closed subset of a compact space $\left(X,\tau_i,\tau_j\right)$. Then, A is τ_i-compact. Since g is P-continuous, then $g\left(A\right)$ is a σ_i-compact subset of Y. But, Y is a KC-space, so $g\left(A\right)$ is a σ_i-closed subset of Y. Hence,

$$g : (X,\tau_i) \longrightarrow (Y,\sigma_i)$$

is closed. Similarly, we can show that

$$g : \left(X,\tau_j\right) \longrightarrow \left(Y,\sigma_j\right)$$

is closed. Therefore, g is P-closed. □

We now remind an important theory in single topological spaces.

Theorem 24. *Every locally compact KC-space is* T_2.

This previous theory can be generelized in bitopological spaces and proven in the same way as follows:

Theorem 25. *Every locally compact KC-bitopological space is* T_2.

Theorem 26. *Let*

$$g : \left(X,\tau_i,\tau_j\right) \longrightarrow \left(Y,\sigma_i,\sigma_j\right)$$

be an onto P-continuous function. If X is T_2 *compact, and Y is a KC-space, then Y is* T_2.

Proof. Since g is an onto, P-continuous function, then Y is compact, so Y is locally compact. But, Y is a KC-space, hence Y is T_2. □

Theorem 27. *Let*

$$g : \left(X,\tau_i,\tau_j\right) \longrightarrow \left(Y,\sigma_i,\sigma_j\right)$$

be a P-continuous bijective K-function. If Y is a P-KC-space, then

$$X = g^{-1}\left(Y\right)$$

is a P-KC-space.

Proof. Let A be a τ_i-compact subset of X. We prove that A is τ_j-closed subset of X. Since A is a τ_i-compact subset of X, then $g(A)$ is σ_i-compact subset of Y because g is a K-function. But, Y is a P-KC-space, so $g(A)$ is a σ_j-closed subset of Y. Since g is P-continuous bijective function, then

$$g^{-1}(g(A)) = A$$

is τ_j-closed in X. Similarly, if A is a τ_j-compact subset of X, then A is τ_i-closed. Hence,

$$X = g^{-1}(Y)$$

is a P-KC-space. □

Theorem 28. *Let*

$$g : \left(X, \tau_i, \tau_j\right) \longrightarrow \left(Y, \sigma_i, \sigma_j\right)$$

be a P-continuous bijective K-function. If Y is a KC-space, then

$$X = g^{-1}(Y)$$

is a KC-space.

Proof. Let A be a τ_i-compact subset of X. We prove that A is a τ_i-closed subset of X. Since A be a τ_i-compact subset of X, then $g(A)$ is a σ_i-compact subset of Y because g is a K-function. But, Y is a KC-space, so $g(A)$ is a σ_i-closed subset of Y. Since g is a P-continuous bijective function, then

$$g^{-1}(g(A)) = A$$

is τ_i-closed in X. Similarly, if A is a τ_j-compact subset of X, then A is τ_j-closed. Hence,

$$X = g^{-1}(Y)$$

is a KC-space. □

Theorem 29. *Let*

$$g : \left(X, \tau_i, \tau_j\right) \longrightarrow \left(Y, \sigma_i, \sigma_j\right)$$

be a K-function. If X and Y are P-compact P-KC-spaces, then g is P-continuous and P-closed.

Proof. First, we show that g is P-continuous. Let A be a σ_i-closed subset of Y. Since Y is P-compact, then A is σ_j-compact ($i \neq j,\ i, j = 1, 2$). Since g is a K-function, then $g^{-1}(A)$ is a τ_j-compact subset of X. Hence, $g^{-1}(A)$ is a τ_i-closed subset of X because X is a P-KC-space. So,

$$g : (X, \tau_i) \longrightarrow (Y, \sigma_i)$$

is continuous. Similarly, we can show that

$$g : \left(X, \tau_j\right) \longrightarrow \left(Y, \sigma_j\right)$$

is continuous. Hence, g is P-continuous.

Second, we show that g is P-closed.

Let C be a τ_i-closed subset of X, then, C is a τ_j-compact subset of X. Since g is a K-function, then $g(C)$ is a σ_j-compact subset of Y. So, $g(C)$ is σ_i-closed subset of Y because Y is P-KC-space. Hence,

$$g : (X, \tau_i) \longrightarrow (Y, \sigma_i)$$

is closed. Similarly, we can show that

$$g : \left(X, \tau_j\right) \longrightarrow \left(Y, \sigma_j\right)$$

is closed. Hence, g is P-closed. □

Theorem 30. *Let*

$$g : \left(X, \tau_i, \tau_j\right) \longrightarrow \left(Y, \sigma_i, \sigma_j\right)$$

be a K-function. If X and Y are compact KC-spaces, then g is P-continuous and P-closed.

Proof. First, we show that g is P-continuous.

Let A be a σ_i-closed subset of Y. Since Y is compact, then A is σ_i-compact. Since g is a K-function, then $g^{-1}(A)$ is a τ_i-compact subset of X. Hence, $g^{-1}(A)$ is a τ_i-closed subset of X because X is a KC-space. So,

$$g : (X, \tau_i) \longrightarrow (Y, \sigma_i)$$

is continuous. Similarly, we can show that

$$g : \left(X, \tau_j\right) \longrightarrow \left(Y, \sigma_j\right)$$

is continuous. Hence, g is P-continuous.

Second, we show that g is P-closed. Let C be a τ_i-closed subset of X. C is a τ_i-compact subset of X. Since, g is a K-function, then $g(C)$ is a σ_i-compact subset of Y. So, $g(C)$ is a σ_i-closed subset of Y because Y is a KC-space. Hence,

$$g : (X, \tau_i) \longrightarrow (Y, \sigma_i)$$

is closed. Similarly, we can show that

$$g : \left(X, \tau_j\right) \longrightarrow \left(Y, \sigma_j\right)$$

is closed. Hence, g is P-closed. □

4. Conclusions

In the introduction to this research we noted, this research carefully reviewed previous studies on the topic, highlighting the main results and contributions of those studies. At the end of the introduction to the study, we provided an overview and summary of the results and conclusions we reached. The key findings are as follows:

(1) The study introduces and develops the fundamental definitions of KC-bitopological spaces and pairwise KC-topological spaces.
(2) A variety of illustrative examples are provided to support and reinforce the study's subject matter.

(3) The concepts of KC-bitopological spaces and pairwise KC-topological spaces are linked to other important topological concepts within bitopological spaces, clarifying their interrelationships.
(4) The study examines the effects of different types of functions on the direct and indirect images of KC-bitopological spaces and pairwise KC-topological spaces.
(5) Necessary and sufficient conditions are established for certain functions to ensure that the direct and indirect images of KC-bitopological spaces and pairwise KC-topological spaces remain within these respective categories.

Author contributions

Hamza Qoqazeh: proposed and set the main title, wrote the basic definitions of the subject of the study, established and provd the basic theories contained, general supervision of the research implementation process; Ali Atoom: wrote the introduction, added a number of theories and proved them; Maryam Alholi: wrote a summary; enriched the subject with illustrative examples; Eman ALmuhur: developed the main results and conclusions, checked the examples; Eman Hussein: carried out a scientific audit on the correctness of the formulation of the theories contained in the research and their proof; Anas Owledat: checked the grammar and linguistics; Abeer Al-Nana: reviewed the previous studies, documented the main references, examined the percentage of scientific inference. All authors have read and agreed to the published version of the manuscript.

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Conflict of interest

The researchers declare no personal interests in the publication of this paper. This research is original, and its primary aim is to contribute to the advancement of scientific knowledge in the field of general topology.

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