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Research article

Analyzing fractional PDE system with the Caputo operator and Mohand transform techniques

Azzh Saad Alshehry[1], **Humaira Yasmin**[2,3,*] **and Ali M. Mahnashi**[4]

[1] Department of Mathematical Sciences, Faculty of Sciences, Princess Nourah Bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia

[2] Department of Basic Sciences, General Administration of Preparatory Year, King Faisal University, P.O. Box 400, Al Ahsa 31982, Saudi Arabia

[3] Department of Mathematics and Statistics, College of Science, King Faisal University, P.O. Box 400, Al Ahsa 31982, Saudi Arabia

[4] Department of Mathematics, Faculty of Science, Jazan University, P.O. Box 2097, Jazan 45142, Kingdom of Saudi Arabia

* **Correspondence:** Email: hhassain@kfu.edu.sa.

Abstract: In this paper, we explore advanced methods for solving partial differential equations (PDEs) and systems of PDEs, particularly those involving fractional-order derivatives. We apply the Mohand transform iterative method (MTIM) and the Mohand residual power series method (MRPSM) to address the complexities associated with fractional-order differential equations. Through several examples, we demonstrate the effectiveness and accuracy of MTIM and MRPSM in solving fractional PDEs. The results indicate that these methods simplify the solution process and enhance the solutions' precision. Our findings suggest that these approaches can be valuable tools for researchers dealing with complex PDE systems in various scientific and engineering fields.

Keywords: PDE and system of PDEs; MTIM; MRPSM; fractional order differential equation; Caputo operator
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1. Introduction

Mathematical methods have been used to represent a variety of real-world issues. For example, we can approximate a body's speed for a given distance and time by using the concept of rate of change. Specifically, we use the notion of differential calculus. Many complicated phenomena, such as chaos, solitons, asymptotic properties, singular formation, and others, are either poorly projected or have not yet been discovered [1–3]. Furthermore, the notion of calculus with differential operators and integrals is essential for describing physical phenomena and divining natural events connected to variation and changes. However, while researching issues with hereditary characteristics or memory, numerous researchers found numerous shortcomings and restrictions in integer-order calculus [4, 5]. Later, new operators defined with the aid of fractional-order were proposed by mathematicians and physicists. Many researchers are drawn to fractional calculus (FC) while looking at different models [6–8].

In many branches of physical science and engineering, fractional calculus, which deals with arbitrary order derivatives and integrals, is crucial [9]. Fractional calculus and its applications have developed rapidly in the last few years [10, 11]. Significant issues in acoustics, fluid mechanics, electromagnetic, analytical chemistry, signal processing, biology, and many other engineering and physical science branches are modeled by nonlinear and linear partial differential equations (PDEs) [12]. Both nonlinear and linear FDEs have been solved analytically and numerically in recent years using a variety of techniques, including the Yang-Laplace transform [13], the Adomian decomposition method [14], the homotopy analysis method [15], and the Laplace decomposition method [16]. Furthermore, nonlinear and linear FDEs are also subjected to the local fractional variational iteration approach [17, 18], the fractional complex transform method [19], the modified Laplace decomposition approach [16, 20], and the cylindrical-coordinate method [21].

Mathematical models known as fractional partial differential equations (FPDEs) depict physical processes that exhibit complicated dynamics and non-local effects. FPDEs represent an expansion of the traditional theory of partial differential equations. They enable non-integer orders of differentiation, which more accurately capture the non-local and nonlinear characteristics of an extensive range of physical systems [22, 23]. These formulas are being used more and more frequently in several fields, such as biology, engineering, economics, and physics. These formulas are valuable resources for constructing intricate systems and examining the behavior of those operations. This article explores the concept of fractional-order partial differential equations, their applications, and the challenges associated with their analysis and numerical solution [24–26].

In 2013, Al-Smadi proposed the Residual Power Series Method (RPSM) [27]. It is generated from the residual error function mixed with the Taylor series. The solution to the problem is an infinite convergence series [28–30]. Many DEs have inspired fresh RPSM algorithms [31–33]. Among these DEs are several Boussinesq DEs, fuzzy DEs, and KdV Burger's equation, among many others. These systems are built to generate exact and efficient approximations. We provide a technique to investigate the approximation of solutions to fractional PDEs and systems of PDEs using RPSM in the Mohand transform (MT) formulation. The computational series finds the exact solution after a few iterations [34–37].

The computational complexity and effort necessary to implement the methods that were previously discussed are among the most significant constraints. The Mohand distinguishes our work transform iterative methodology (MTIM), which we developed as an iterative approach to addressing fractional PDEs and systems of PDEs. This technique is highly effective in reducing the amount of computational work and complexity necessary due to integrating the MT with the new iterative process.

In this study, the Mohand residual power series method (MRPSM) and MTIM are used to solve fractional PDEs and systems of PDEs. The numerical values produced by these techniques are more precise when compared to those of other numerical procedures. This study includes a comparison study of the numerical data. A strong indicator of the efficacy and reliability of these methods is the fact that the results of the many approaches presented are compatible with one another. The attractiveness of fractional-order derivatives grows in direct correlation with their worth. Because of this, the algorithms can withstand spikes in computational error, are easy to use, and are quick and accurate. Discovering this will make solving many partial differential equations much easier for mathematicians.

2. Mohand transformation

The portions that follow will cover the basic elements and ideas of the MT, providing the foundation for this operation.

Definition 2.1. *The MT of the function* $\mathfrak{P}(\mathrm{t})$ *is defined as [38]*

$$M[\mathfrak{P}(\mathrm{t})] = R(s) = s^2 \int_0^{\mathrm{t}} \mathfrak{P}(\mathrm{t}) e^{-s\mathrm{t}} d\mathrm{t}, \;\; k_1 \leq s \leq k_2.$$

The inverse Mohand transform (IMT) is defined as

$$M^{-1}[R(s)] = \mathfrak{P}(\mathrm{t}).$$

Definition 2.2 ([39])**.** *The derivative of fractional-order in the framework of MT is defined as*

$$M[\mathfrak{P}^{\mathfrak{F}}(\mathrm{t})] = s^{\mathfrak{F}} R(s) - \sum_{k=0}^{n-1} \frac{\mathfrak{P}^{k}(0)}{s^{k-(\mathfrak{F}+1)}}, \; 0 < \mathfrak{F} \leq n.$$

Definition 2.3. *The properties of MT are given as follows:*

(1) $M[\mathfrak{P}'(\mathrm{t})] = sR(s) - s^2 R(0)$.

(2) $M[\mathfrak{P}''(\mathrm{t})] = s^2 R(s) - s^3 R(0) - s^2 R'(0)$.

(3) $M[\mathfrak{P}^{n}(\mathrm{t})] = s^n R(s) - s^{n+1} R(0) - s^n R'(0) - \cdots - s^n R^{n-1}(0)$.

Lemma 2.4. *Suppose there exists a function represented by* $\mathfrak{P}(\Re, \mathrm{t})$*, having exponential order.* $M[R(s)] = \mathfrak{P}(\Re, \mathrm{t})$ *denotes the MT in this case:*

$$M[D_{\mathrm{t}}^{r\mathfrak{F}} \mathfrak{P}(\Re, \mathrm{t})] = s^{r\mathfrak{F}} R(s) - \sum_{j=0}^{r-1} s^{\mathfrak{F}(r-j)-1} D_{\mathrm{t}}^{j\mathfrak{F}} \mathfrak{P}(\Re, 0), 0 < \mathfrak{F} \leq 1, \tag{2.1}$$

where $\Re = (\Re_1, \Re_2, \cdots, \Re_{\mathfrak{F}}) \in \mathbb{R}^{\mathfrak{F}}$*,* $\mathfrak{F} \in \mathbb{N}$*, and* $D_{\mathrm{t}}^{r\mathfrak{F}} = D_{\mathrm{t}}^{\mathfrak{F}}.D_{\mathrm{t}}^{\mathfrak{F}}.\cdots.D_{\mathrm{t}}^{\mathfrak{F}}(r - times)$.

Proof. To validate Eq (2.4), we use the induction method. Taking $r = 1$ in Eq (2.4):

$$M[D_{\mathrm{t}}^{2\mathfrak{F}} \mathfrak{P}(\Re, \mathrm{t})] = s^{2\mathfrak{F}} R(s) - s^{2\mathfrak{F}-1} \mathfrak{P}(\Re, 0) - s^{\mathfrak{F}-1} D_{\mathrm{t}}^{\mathfrak{F}} \mathfrak{P}(\Re, 0).$$

Equation (2.4) is true for $r = 1$ on the basis of Definition 2.2. Now, put $r = 2$ in Eq (2.4) to obtain the following outcome:

$$M[D_{r}^{2\mathfrak{F}} \mathfrak{P}(\Re, \mathrm{t})] = s^{2\mathfrak{F}} R(s) - s^{2\mathfrak{F}-1} \mathfrak{P}(\Re, 0) - s^{\mathfrak{F}-1} D_{\mathrm{t}}^{\mathfrak{F}} \mathfrak{P}(\Re, 0). \tag{2.2}$$

We obtain the next result from the LHS of Eq (2.2):

$$L.H.S = M[D_t^{2\mathfrak{F}}\mathfrak{P}(\mathfrak{R},t)]. \tag{2.3}$$

We may also write Eq (2.3) as

$$L.H.S = M[D_t^{\mathfrak{F}}D_t^{\mathfrak{F}}\mathfrak{P}(\mathfrak{R},t)]. \tag{2.4}$$

Assume

$$z(\mathfrak{R},t) = D_t^{\mathfrak{F}}\mathfrak{P}(\mathfrak{R},t). \tag{2.5}$$

Putting Eq (2.5) in Eq (2.4),

$$L.H.S = M[D_t^{\mathfrak{F}}z(\mathfrak{R},t)]. \tag{2.6}$$

Using the derivative of Caputo, Eq (2.6) becomes

$$L.H.S = M[J^{1-\mathfrak{F}}z'(\mathfrak{R},t)]. \tag{2.7}$$

Applying the RL integral on Eq (2.7),

$$L.H.S = \frac{M[z'(\mathfrak{R},t)]}{s^{1-\mathfrak{F}}}. \tag{2.8}$$

The derivative property of MT is applied on Eq (2.8) to obtain the following result:

$$L.H.S = s^{\mathfrak{F}}Z(\mathfrak{R},s) - \frac{z(\mathfrak{R},0)}{s^{1-\mathfrak{F}}}. \tag{2.9}$$

Using Eq (2.5), we obtain

$$Z(\mathfrak{R},s) = s^{\mathfrak{F}}R(s) - \frac{\mathfrak{P}(\mathfrak{R},0)}{s^{1-\mathfrak{F}}}.$$

As $M[z(t,\mathfrak{R})] = Z(\mathfrak{R},s)$, we can write Eq (2.9) as

$$L.H.S = s^{2\mathfrak{F}}R(s) - \frac{\mathfrak{P}(\mathfrak{R},0)}{s^{1-2\mathfrak{F}}} - \frac{D_t^{\mathfrak{F}}\mathfrak{P}(\mathfrak{R},0)}{s^{1-\mathfrak{F}}}. \tag{2.10}$$

Assume that Eq (2.4) is true for $r = K$. Taking $r = K$ in Eq (2.4),

$$M[D_t^{K\mathfrak{F}}\mathfrak{P}(\mathfrak{R},t)] = s^{K\mathfrak{F}}R(s) - \sum_{j=0}^{K-1} s^{\mathfrak{F}(K-j)-1}D_t^{j\mathfrak{F}}D_t^{j\mathfrak{F}}\mathfrak{P}(\mathfrak{R},0),\ 0 < \mathfrak{F} \leq 1. \tag{2.11}$$

Next, we will have to show that Eq (2.4) for $r = K+1$ holds. Taking $r = K+1$ in Eq (2.4),

$$M[D_t^{(K+1)\mathfrak{F}}\mathfrak{P}(\mathfrak{R},t)] = s^{(K+1)\mathfrak{F}}R(s) - \sum_{j=0}^{K} s^{\mathfrak{F}((K+1)-j)-1}D_t^{j\mathfrak{F}}\mathfrak{P}(\mathfrak{R},0). \tag{2.12}$$

From the left-hand side of Eq (2.12), we derive

$$L.H.S = M[D_t^{K\mathfrak{F}}(D_t^{K\mathfrak{F}})]. \tag{2.13}$$

Letting $D_t^{K\mathfrak{F}} = g(\Re, t)$, Eq (2.13) gives us

$$L.H.S = M[D_t^{\mathfrak{F}} g(\Re, t)]. \tag{2.14}$$

Applying Caputo's derivative and the RL integral on Eq (2.14),

$$L.H.S = s^{\mathfrak{F}} M[D_t^{K\mathfrak{F}} \mathfrak{P}(\Re, t)] - \frac{g(\Re, 0)}{s^{1-\mathfrak{F}}}. \tag{2.15}$$

On the basis of Eq (2.11), we can write Eq (2.15) as

$$L.H.S = s^{r\mathfrak{F}} R(s) - \sum_{j=0}^{r-1} s^{\mathfrak{F}(r-j)-1} D_t^{j\mathfrak{F}} \mathfrak{P}(\Re, 0). \tag{2.16}$$

Equation (2.16) can also be written as

$$L.H.S = M[D_t^{r\mathfrak{F}} \mathfrak{P}(\Re, 0)].$$

Using mathematical induction, Eq (2.4) is true for $r = K + 1$. Hence, it is proved that for all positive integers, Eq (2.4) holds. □

Lemma 2.5. *Let assume that there exists an exponential-order function $\mathfrak{P}(\Re, t)$. $M[\mathfrak{P}(\Re, t)] = R(s)$ denotes the MT of $\mathfrak{P}(\Re, t)$. The multiple fractional power series (MFPS) in MT is given as*

$$R(s) = \sum_{r=0}^{\infty} \frac{\hbar_r(\Re)}{s^{r\mathfrak{F}+1}}, s > 0, \tag{2.17}$$

where, $\Re = (s_1, \Re_2, \cdots, \Re_{\mathfrak{F}}) \in \mathbb{R}^{\mathfrak{F}}$, $\mathfrak{F} \in \mathbb{N}$.

Proof. Let us consider the Taylor series

$$\mathfrak{P}(\Re, t) = \hbar_0(\Re) + \hbar_1(\Re) \frac{t^{\mathfrak{F}}}{\Gamma[\mathfrak{F} + 1]} + +\hbar_2(\Re) \frac{t^{2\mathfrak{F}}}{\Gamma[2\mathfrak{F} + 1]} + \cdots. \tag{2.18}$$

MT is subjected to Eq (2.18) to obtain

$$M[\mathfrak{P}(\Re, t)] = M[\hbar_0(\Re)] + M\left[\hbar_1(\Re) \frac{t^{\mathfrak{F}}}{\Gamma[\mathfrak{F} + 1]}\right] + M\left[\hbar_1(\Re) \frac{t^{2\mathfrak{F}}}{\Gamma[2\mathfrak{F} + 1]}\right] + \cdots.$$

Utilizing the features of MT, we derive

$$M[\mathfrak{P}(\Re, t)] = \hbar_0(\Re) \frac{1}{s} + \hbar_1(\Re) \frac{\Gamma[\mathfrak{F} + 1]}{\Gamma[\mathfrak{F} + 1]} \frac{1}{s^{\mathfrak{F}+1}} + \hbar_2(\Re) \frac{\Gamma[2\mathfrak{F} + 1]}{\Gamma[2\mathfrak{F} + 1]} \frac{1}{s^{2\mathfrak{F}+1}} \cdots.$$

Thus, a new Taylor series form is obtained. □

Lemma 2.6. *If $M[\mathfrak{P}(\Re, t)] = R(s)$ denotes MT, then the new Taylor series form in MFPS is given as*

$$\hbar_0(\Re) = \lim_{s\to\infty} sR(s) = \mathfrak{P}(\Re, 0). \tag{2.19}$$

Proof. Assume the Taylor's series

$$\hbar_0(\Re) = sR(s) - \frac{\hbar_1(\Re)}{s^{\mathfrak{F}}} - \frac{\hbar_2(\Re)}{s^{2\mathfrak{F}}} - \cdots. \tag{2.20}$$

When the limit in Eq (2.19) is calculated and simplified, we get Eq (2.20). □

3. Mohand residual power series method

In this part, we construct the framework of the proposed method for the solution of fractional PDEs.

Step 1. Let us assume the fractional PDE

$$D_{\mathrm{t}}^{\mathfrak{F}}\mathfrak{P}(\mathfrak{R},\mathrm{t})+\mathfrak{K}(\mathfrak{R})N(\mathfrak{P})-\delta(\mathfrak{R},\mathfrak{P})=0. \tag{3.1}$$

Step 2. Applying the MT on both sides of Eq (3.1),

$$M[D_{\mathrm{t}}^{\mathfrak{F}}\mathfrak{P}(\mathfrak{R},\mathrm{t})+\mathfrak{K}(\mathfrak{R})N(\mathfrak{P})-\delta(\mathfrak{R},\mathfrak{P})]=0. \tag{3.2}$$

On the basis of Lemma 2.4, we derive

$$R(s)=\sum_{j=0}^{q-1}\frac{D_{\mathrm{t}}^{j}\mathfrak{P}(\mathfrak{R},0)}{s^{j\mathfrak{F}+1}}-\frac{\mathfrak{K}(\mathfrak{R})Y(s)}{s^{j\mathfrak{F}}}+\frac{F(\mathfrak{R},s)}{s^{j\mathfrak{F}}}, \tag{3.3}$$

where, $M[\delta(\mathfrak{R},\mathfrak{P})]=F(\mathfrak{R},s), M[N(\mathfrak{P})]=Y(s)$.

Step 3. The subsequent result is derived from Eq (3.3):

$$R(s)=\sum_{r=0}^{\infty}\frac{\hbar_r(\mathfrak{R})}{s^{r\mathfrak{F}+1}},\ s>0.$$

Step 4. To obtain series form solution use the following procedure step by step:

$$\hbar_0(\mathfrak{R})=\lim_{s\to\infty}sR(s)=\mathfrak{P}(\mathfrak{R},0).$$

Subsequently, we obtain

$$\begin{aligned}\hbar_1(\mathfrak{R})&=D_{\mathrm{t}}^{\mathfrak{F}}\mathfrak{P}(\mathfrak{R},0),\\ \hbar_2(\mathfrak{R})&=D_{\mathrm{t}}^{2\mathfrak{F}}\mathfrak{P}(\mathfrak{R},0),\\ &\vdots\\ \hbar_w(\mathfrak{R})&=D_{\mathrm{t}}^{w\mathfrak{F}}\mathfrak{P}(\mathfrak{R},0).\end{aligned}$$

Step 5. To obtain $R(s)$ as a K^{th} truncated series, we use

$$R_K(s)=\sum_{r=0}^{K}\frac{\hbar_r(\mathfrak{R})}{s^{r\mathfrak{F}+1}},\ s>0,$$

$$R_K(s)=\frac{\hbar_0(\mathfrak{R})}{s}+\frac{\hbar_1(\mathfrak{R})}{s^{\mathfrak{F}+1}}+\cdots+\frac{\hbar_w(\mathfrak{R})}{s^{w\mathfrak{F}+1}}+\sum_{r=w+1}^{K}\frac{\hbar_r(\mathfrak{R})}{s^{r\mathfrak{F}+1}}.$$

Step 6. The Mohand residual function (MRF) from (3.3) is solved separately from the K^{th}-truncated Mohand residual function

$$MRes(\mathfrak{R},s)=R(s)-\sum_{j=0}^{q-1}\frac{D_{\mathrm{t}}^{j}\mathfrak{P}(\mathfrak{R},0)}{s^{j\mathfrak{F}+1}}+\frac{\mathfrak{K}(\mathfrak{R})Y(s)}{s^{j\mathfrak{F}}}-\frac{F(\mathfrak{R},s)}{s^{j\mathfrak{F}}},$$

and

$$MRes_K(\Re, s) = R_K(s) - \sum_{j=0}^{q-1} \frac{D_t^j \mathfrak{P}(\Re, 0)}{s^{j\mathfrak{F}+1}} + \frac{\mathfrak{K}(\Re)Y(s)}{s^{j\mathfrak{F}}} - \frac{F(\Re, s)}{s^{j\mathfrak{F}}}. \tag{3.4}$$

Step 7. In Eq (3.4), use $R_K(s)$ in place of its expansion form:

$$\begin{aligned} MRes_K(\Re, s) = &\Big(\frac{\hbar_0(\Re)}{s} + \frac{\hbar_1(\Re)}{s^{\mathfrak{F}+1}} + \cdots + \frac{\hbar_w(\Re)}{s^{w\mathfrak{F}+1}} + \sum_{r=w+1}^{K} \frac{\hbar_r(\Re)}{s^{r\mathfrak{F}+1}}\Big) \\ &- \sum_{j=0}^{q-1} \frac{D_t^j \mathfrak{P}(\Re, 0)}{s^{j\mathfrak{F}+1}} + \frac{\mathfrak{K}(\Re)Y(s)}{s^{j\mathfrak{F}}} - \frac{F(\Re, s)}{s^{j\mathfrak{F}}}. \end{aligned} \tag{3.5}$$

Step 8. Multiplying $s^{K\mathfrak{F}+1}$ with Eq (3.5),

$$\begin{aligned} s^{K\mathfrak{F}+1} MRes_K(\Re, s) = s^{K\mathfrak{F}+1}&\Big(\frac{\hbar_0(\Re)}{s} + \frac{\hbar_1(\Re)}{s^{\mathfrak{F}+1}} + \cdots + \frac{\hbar_w(\Re)}{s^{w\mathfrak{F}+1}} + \sum_{r=w+1}^{K} \frac{\hbar_r(\Re)}{s^{r\mathfrak{F}+1}} \\ &- \sum_{j=0}^{q-1} \frac{D_t^j \mathfrak{P}(\Re, 0)}{s^{j\mathfrak{F}+1}} + \frac{\mathfrak{K}(\Re)Y(s)}{s^{j\mathfrak{F}}} - \frac{F(\Re, s)}{s^{j\mathfrak{F}}}\Big). \end{aligned} \tag{3.6}$$

Step 9. Taking the limit $s \to \infty$ of Eq (3.6),

$$\begin{aligned} \lim_{s\to\infty} s^{K\mathfrak{F}+1} MRes_K(\Re, s) = \lim_{s\to\infty} s^{K\mathfrak{F}+1}&\Big(\frac{\hbar_0(\Re)}{s} + \frac{\hbar_1(\Re)}{s^{\mathfrak{F}+1}} + \cdots + \frac{\hbar_w(\Re)}{s^{w\mathfrak{F}+1}} + \sum_{r=w+1}^{K} \frac{\hbar_r(\Re)}{s^{r\mathfrak{F}+1}} \\ &- \sum_{j=0}^{q-1} \frac{D_t^j \mathfrak{P}(\Re, 0)}{s^{j\mathfrak{F}+1}} + \frac{\mathfrak{K}(\Re)Y(s)}{s^{j\mathfrak{F}}} - \frac{F(\Re, s)}{s^{j\mathfrak{F}}}\Big). \end{aligned}$$

Step 10. The values of $\hbar_K(\Re)$ are obtained by solving the following expression:

$$\lim_{s\to\infty}(s^{K\mathfrak{F}+1} MRes_K(\Re, s)) = 0,$$

where $K = 1 + w, 2 + w, \cdots$.

Step 11. Put $\hbar_K(\Re)$ in Eq (3.3).

Step 12. To determine the required solution, take IMT to obtain $R_K(s)$ as $\mathfrak{P}_K(\Re, t)$.

Mohand transform iterative method

Suppose the PDE

$$D_t^{\mathfrak{F}} \mathfrak{P}(\Re, t) = \Upsilon\Big(\mathfrak{P}(\Re, t), D_{\Re}^{\eta} \mathfrak{P}(\Re, t), D_{\Re}^{2\eta} \mathfrak{P}(\Re, t), D_{\Re}^{3\eta} \mathfrak{P}(\Re, t)\Big),\ 0 < \mathfrak{F}, \eta \le 1, \tag{3.7}$$

with IC's

$$\mathfrak{P}^{(k)}(\Re, 0) = h_k,\ k = 0, 1, 2, \cdots, m - 1, \tag{3.8}$$

where $\mathfrak{P}(\Re,\mathrm{t})$ is a function to be determined and $\Upsilon\Big(\mathfrak{P}(\Re,\mathrm{t}), D_{\Re}^{\eta}\mathfrak{P}(\Re,\mathrm{t}), D_{\Re}^{2\eta}\mathfrak{P}(\Re,\mathrm{t}), D_{\Re}^{3\eta}\mathfrak{P}(\Re,\mathrm{t})\Big)$ is operator of $\mathfrak{P}(\Re,\mathrm{t}), D_{\Re}^{\eta}\mathfrak{P}(\Re,\mathrm{t}), D_{\Re}^{2\eta}\mathfrak{P}(\Re,\mathrm{t})$ and $D_{\Re}^{3\eta}\mathfrak{P}(\Re,\mathrm{t})$. Equation (3.7) is subjected to MT to obtain

$$M[\mathfrak{P}(\Re,\mathrm{t})] = \frac{1}{s^{\mathfrak{F}}}\Big(\sum_{k=0}^{m-1}\frac{\mathfrak{P}^{(k)}(\Re,0)}{s^{1-\mathfrak{F}+k}} + M\Big[\Upsilon\Big(\mathfrak{P}(\Re,\mathrm{t}), D_{\Re}^{\eta}\mathfrak{P}(\Re,\mathrm{t}), D_{\Re}^{2\eta}\mathfrak{P}(\Re,\mathrm{t}), D_{\Re}^{3\eta}\mathfrak{P}(\Re,\mathrm{t})\Big)\Big]\Big). \tag{3.9}$$

The IMT gives us

$$\mathfrak{P}(\Re,\mathrm{t}) = M^{-1}\Big[\frac{1}{s^{\mathfrak{F}}}\Big(\sum_{k=0}^{m-1}\frac{\mathfrak{P}^{(k)}(\Re,0)}{s^{1-\mathfrak{F}+k}} + M\Big[\Upsilon\Big(\mathfrak{P}(\Re,\mathrm{t}), D_{\Re}^{\eta}\mathfrak{P}(\Re,\mathrm{t}), D_{\Re}^{2\eta}\mathfrak{P}(\Re,\mathrm{t}), D_{\Re}^{3\eta}\mathfrak{P}(\Re,\mathrm{t})\Big)\Big]\Big)\Big]. \tag{3.10}$$

The solution via the MTIM technique is represented as

$$\mathfrak{P}(\Re,\mathrm{t}) = \sum_{i=0}^{\infty}\mathfrak{P}_i. \tag{3.11}$$

The decomposition of the operator $\Upsilon\Big(\mathfrak{P}, D_{\Re}^{\eta}\mathfrak{P}, D_{\Re}^{2\eta}\mathfrak{P}, D_{\Re}^{3\eta}\mathfrak{P}\Big)$ is

$$\begin{aligned}\Upsilon\Big(\mathfrak{P}, D_{\Re}^{\eta}\mathfrak{P}, D_{\Re}^{2\eta}\mathfrak{P}, D_{\Re}^{3\eta}\mathfrak{P}\Big) &= \Upsilon\Big(\mathfrak{P}_0, D_{\Re}^{\eta}\mathfrak{P}_0, D_{\Re}^{2\eta}\mathfrak{P}_0, D_{\Re}^{3\eta}\mathfrak{P}_0\Big)\\ &+ \sum_{i=0}^{\infty}\Bigg(\Upsilon\Big(\sum_{k=0}^{i}\Big(\mathfrak{P}_k, D_{\Re}^{\eta}\mathfrak{P}_k, D_{\Re}^{2\eta}\mathfrak{P}_k, D_{\Re}^{3\eta}\mathfrak{P}_k\Big)\Big) - \Upsilon\Big(\sum_{k=1}^{i-1}\Big(\mathfrak{P}_k, D_{\Re}^{\eta}\mathfrak{P}_k, D_{\Re}^{2\eta}\mathfrak{P}_k, D_{\Re}^{3\eta}\mathfrak{P}_k\Big)\Big)\Bigg).\end{aligned} \tag{3.12}$$

Putting Eqs (3.11) and (3.12) into Eq (3.10), we obtain

$$\begin{aligned}\sum_{i=0}^{\infty}\mathfrak{P}_i(\Re,\mathrm{t}) &= M^{-1}\Big[\frac{1}{s^{\mathfrak{F}}}\Big(\sum_{k=0}^{m-1}\frac{\mathfrak{P}^{(k)}(\Re,0)}{s^{2-\mathfrak{F}+k}} + M[\Upsilon(\mathfrak{P}_0, D_{\Re}^{\eta}\mathfrak{P}_0, D_{\Re}^{2\eta}\mathfrak{P}_0, D_{\Re}^{3\eta}\mathfrak{P}_0)]\Big)\Big]\\ &+ M^{-1}\Big[\frac{1}{s^{\mathfrak{F}}}\Big(M\Big[\sum_{i=0}^{\infty}\Big(\Upsilon\sum_{k=0}^{i}(\mathfrak{P}_k, D_{\Re}^{\eta}\mathfrak{P}_k, D_{\Re}^{2\eta}\mathfrak{P}_k, D_{\Re}^{3\eta}\mathfrak{P}_k)\Big)\Big]\Big)\Big]\\ &- M^{-1}\Big[\frac{1}{s^{\mathfrak{F}}}\Big(M\Big[\Big(\Upsilon\sum_{k=1}^{i-1}(\mathfrak{P}_k, D_{\Re}^{\eta}\mathfrak{P}_k, D_{\Re}^{2\eta}\mathfrak{P}_k, D_{\Re}^{3\eta}\mathfrak{P}_k)\Big)\Big]\Big)\Big]\end{aligned} \tag{3.13}$$

$$\begin{aligned}\mathfrak{P}_0(\Re,\mathrm{t}) &= M^{-1}\Big[\frac{1}{s^{\mathfrak{F}}}\Big(\sum_{k=0}^{m-1}\frac{\mathfrak{P}^{(k)}(\Re,0)}{s^{2-\mathfrak{F}+k}}\Big)\Big],\\ \mathfrak{P}_1(\Re,\mathrm{t}) &= M^{-1}\Big[\frac{1}{s^{\mathfrak{F}}}\Big(M[\Upsilon(\mathfrak{P}_0, D_{\Re}^{\eta}\mathfrak{P}_0, D_{\Re}^{2\eta}\mathfrak{P}_0, D_{\Re}^{3\eta}\mathfrak{P}_0)]\Big)\Big],\\ &\vdots\\ \mathfrak{P}_{m+1}(\Re,\mathrm{t}) &= M^{-1}\Big[\frac{1}{s^{\mathfrak{F}}}\Big(M\Big[\sum_{i=0}^{\infty}\Big(\Upsilon\sum_{k=0}^{i}(\mathfrak{P}_k, D_{\Re}^{\eta}\mathfrak{P}_k, D_{\Re}^{2\eta}\mathfrak{P}_k, D_{\Re}^{3\eta}\mathfrak{P}_k)\Big)\Big]\Big)\Big]\\ &- M^{-1}\Big[\frac{1}{s^{\mathfrak{F}}}\Big(M\Big[\Big(\Upsilon\sum_{k=1}^{i-1}(\mathfrak{P}_k, D_{\Re}^{\eta}\mathfrak{P}_k, D_{\Re}^{2\eta}\mathfrak{P}_k, D_{\Re}^{3\eta}\mathfrak{P}_k)\Big)\Big]\Big)\Big],\ m = 1, 2, \cdots.\end{aligned} \tag{3.14}$$

The general solution of Eq (3.7) is given as

$$\mathfrak{P}(\Re,\mathrm{t}) = \sum_{i=0}^{m-1}\mathfrak{P}_i. \tag{3.15}$$

4. Uses of proposed methods

Example 4.1. • Implementation of MRPSM

Let us consider the fractional PDE

$$D_{t}^{\mathfrak{F}}\mathfrak{P}(\mathfrak{R},\mathfrak{K},\mathfrak{Z},t)+\frac{\partial^2\mathfrak{P}(\mathfrak{R},\mathfrak{K},\mathfrak{Z},t)}{\partial\mathfrak{R}^2}+\frac{\partial^2\mathfrak{P}(\mathfrak{R},\mathfrak{K},\mathfrak{Z},t)}{\partial\mathfrak{K}^2}+\frac{\partial^2\mathfrak{P}(\mathfrak{R},\mathfrak{K},\mathfrak{Z},t)}{\partial\mathfrak{Z}^2}=0, \tag{4.1}$$

where $0<\mathfrak{F}\leq 1$.

The initial condition is

$$\mathfrak{P}(\mathfrak{R},\mathfrak{K},\mathfrak{Z},0)=e^{\mathfrak{R}+\mathfrak{K}+\mathfrak{Z}}, \tag{4.2}$$

with exact solution

$$\mathfrak{P}(\mathfrak{R},\mathfrak{K},t)=e^{\mathfrak{R}+\mathfrak{K}+\mathfrak{Z}-3t}. \tag{4.3}$$

Equation (4.1) is subjected to MT, and using Eq (4.2) we get the following result:

$$\mathfrak{P}(\mathfrak{R},\mathfrak{K},\mathfrak{Z},s)-\frac{e^{\mathfrak{R}+\mathfrak{K}+\mathfrak{Z}}}{s}+\frac{1}{s^{\mathfrak{F}}}\Big[\frac{\partial^2\mathfrak{P}(\mathfrak{R},\mathfrak{K},\mathfrak{Z},s)}{\partial\mathfrak{R}^2}\Big]+\frac{1}{s^{\mathfrak{F}}}\Big[\frac{\partial^2\mathfrak{P}(\mathfrak{R},\mathfrak{K},\mathfrak{Z},s)}{\partial\mathfrak{K}^2}\Big]+\frac{1}{s^{\mathfrak{F}}}\Big[\frac{\partial^2\mathfrak{P}(\mathfrak{R},\mathfrak{K},\mathfrak{Z},s)}{\partial\mathfrak{Z}^2}\Big]=0. \tag{4.4}$$

The k^{th} term's series is represented as

$$\mathfrak{P}(\mathfrak{R},\mathfrak{K},\mathfrak{Z},s)=\frac{e^{\mathfrak{R}+\mathfrak{K}+\mathfrak{Z}}}{s}+\sum_{r=1}^{k}\frac{f_r(\mathfrak{R},\mathfrak{K},\mathfrak{Z},s)}{s^{r\mathfrak{F}+1}},\quad r=1,2,3,4\cdots. \tag{4.5}$$

The residual function of Mohand is given by

$$\begin{aligned}\mathcal{M}_{t}Res(\mathfrak{R},\mathfrak{K},\mathfrak{Z},s)=\mathfrak{P}(\mathfrak{R},\mathfrak{K},\mathfrak{Z},s)-\frac{e^{\mathfrak{R}+\mathfrak{K}+\mathfrak{Z}}}{s}+\frac{1}{s^{\mathfrak{F}}}\Big[\frac{\partial^2\mathfrak{P}(\mathfrak{R},\mathfrak{K},\mathfrak{Z},s)}{\partial\mathfrak{R}^2}\Big]+\frac{1}{s^{\mathfrak{F}}}\Big[\frac{\partial^2\mathfrak{P}(\mathfrak{R},\mathfrak{K},\mathfrak{Z},s)}{\partial\mathfrak{K}^2}\Big]\\+\frac{1}{s^{\mathfrak{F}}}\Big[\frac{\partial^2\mathfrak{P}(\mathfrak{R},\mathfrak{K},\mathfrak{Z},s)}{\partial\mathfrak{Z}^2}\Big]=0,\end{aligned} \tag{4.6}$$

and the k^{th}-MRFs as

$$\begin{aligned}\mathcal{M}_{t}Res_k(\mathfrak{R},\mathfrak{K},\mathfrak{Z},s)=\mathfrak{P}_k(\mathfrak{R},\mathfrak{K},\mathfrak{Z},s)-\frac{e^{\mathfrak{R}+\mathfrak{K}+\mathfrak{Z}}}{s}+\frac{1}{s^{\mathfrak{F}}}\Big[\frac{\partial^2\mathfrak{P}_k(\mathfrak{R},\mathfrak{K},\mathfrak{Z},s)}{\partial\mathfrak{R}^2}\Big]+\frac{1}{s^{\mathfrak{F}}}\Big[\frac{\partial^2\mathfrak{P}_k(\mathfrak{R},\mathfrak{K},\mathfrak{Z},s)}{\partial\mathfrak{K}^2}\Big]\\+\frac{1}{s^{\mathfrak{F}}}\Big[\frac{\partial^2\mathfrak{P}_k(\mathfrak{R},\mathfrak{K},\mathfrak{Z},s)}{\partial\mathfrak{Z}^2}\Big]=0.\end{aligned} \tag{4.7}$$

Now, we use these steps to find the values of $f_r(\mathfrak{R},\mathfrak{K},\mathfrak{Z},s)$ for $r=1,2,3,...$: Take the r^{th}-Mohand residual function Eq (4.7) for the r^{th}-truncated series Eq (4.5), and then multiply the equation by $s^{r\mathfrak{F}+1}$ and solve $\lim_{s\to\infty}(s^{r\mathfrak{F}+1})M_{t}Res_{\mathfrak{P},r}(\mathfrak{R},\mathfrak{K},\mathfrak{Z},s))=0$ for $r=1,2,3,\cdots$. Using this procedure, we obtain the following terms:

$$f_1(\mathfrak{R},\mathfrak{K},\mathfrak{Z},s)=-3e^{\mathfrak{R}+\mathfrak{K}+\mathfrak{Z}}, \tag{4.8}$$

$$f_2(\mathfrak{R},\mathfrak{K},\mathfrak{Z},s) = 9e^{\mathfrak{R}+\mathfrak{K}+\mathfrak{Z}}, \tag{4.9}$$

$$f_3(\mathfrak{R},\mathfrak{K},\mathfrak{Z},s) = -27e^{\mathfrak{R}+\mathfrak{K}+\mathfrak{Z}}, \tag{4.10}$$

and so on.

The values of Eqs (4.9) and (4.10) are inserted in Eq (4.5) to obtain the following result:

$$\mathfrak{P}(\mathfrak{R},\mathfrak{K},\mathfrak{Z},s) = \frac{e^{\mathfrak{R}+\mathfrak{K}+\mathfrak{Z}}}{s} - \frac{3e^{\mathfrak{R}+\mathfrak{K}+\mathfrak{Z}}}{s^{\mathfrak{F}+1}} + \frac{9e^{\mathfrak{R}+\mathfrak{K}+\mathfrak{Z}}}{s^{2\mathfrak{F}+1}} - \frac{27e^{\mathfrak{R}+\mathfrak{K}+\mathfrak{Z}}}{s^{3\mathfrak{F}+1}} + \cdots. \tag{4.11}$$

Using IMT, we obtain the final solution

$$\mathfrak{P}_1(\mathfrak{R},\mathfrak{K},\mathfrak{Z},\mathrm{t}) = e^{\mathfrak{R}+\mathfrak{K}+\mathfrak{Z}} - \frac{3\mathrm{t}^{\mathfrak{F}}e^{\mathfrak{R}+\mathfrak{K}+\mathfrak{Z}}}{\Gamma(\mathfrak{F}+1)} + \frac{9\mathrm{t}^{2\mathfrak{F}}e^{\mathfrak{R}+\mathfrak{K}+\mathfrak{Z}}}{\Gamma(2\mathfrak{F}+1)} - \frac{27\mathrm{t}^{3\mathfrak{F}}e^{\mathfrak{R}+\mathfrak{K}+\mathfrak{Z}}}{\Gamma(3\mathfrak{F}+1)}. \tag{4.12}$$

• **Implementation of MTIM**

Consider the fractional PDE

$$D_{\mathrm{t}}^{\mathfrak{F}}\mathfrak{P}(\mathfrak{R},\mathfrak{K},\mathfrak{Z},\mathrm{t}) = -\frac{\partial^2\mathfrak{P}(\mathfrak{R},\mathfrak{K},\mathfrak{Z},\mathrm{t})}{\partial\mathfrak{R}^2} - \frac{\partial^2\mathfrak{P}(\mathfrak{R},\mathfrak{K},\mathfrak{Z},\mathrm{t})}{\partial\mathfrak{K}^2} - \frac{\partial^2\mathfrak{P}(\mathfrak{R},\mathfrak{K},\mathfrak{Z},\mathrm{t})}{\partial\mathfrak{Z}^2}, \quad \text{where } 0 < \mathfrak{F} \le 1. \tag{4.13}$$

The initial condition is

$$\mathfrak{P}(\mathfrak{R},\mathfrak{K},\mathfrak{Z},0) = e^{\mathfrak{R}+\mathfrak{K}+\mathfrak{Z}}. \tag{4.14}$$

MT is used on Eq (4.13), giving

$$M[D_{\mathrm{t}}^{\mathfrak{F}}\mathfrak{P}(\mathfrak{R},\mathfrak{K},\mathfrak{Z},\mathrm{t})] = \frac{1}{s^{\mathfrak{F}}}\Big(\sum_{k=0}^{m-1}\frac{\mathfrak{P}^{(k)}(\mathfrak{R},\mathfrak{K},\mathfrak{Z},0)}{s^{2-\mathfrak{F}+k}} + M\Big[-\frac{\partial^2\mathfrak{P}(\mathfrak{R},\mathfrak{K},\mathfrak{Z},\mathrm{t})}{\partial\mathfrak{R}^2} - \frac{\partial^2\mathfrak{P}(\mathfrak{R},\mathfrak{K},\mathfrak{Z},\mathrm{t})}{\partial\mathfrak{K}^2} - \frac{\partial^2\mathfrak{P}(\mathfrak{R},\mathfrak{K},\mathfrak{Z},\mathrm{t})}{\partial\mathfrak{Z}^2}\Big)\Big]\Big). \tag{4.15}$$

Applying IMT on Eq (4.15), we obtain

$$\mathfrak{P}(\mathfrak{R},\mathfrak{K},\mathfrak{Z},\mathrm{t}) = M^{-1}\Big[\frac{1}{s^{\mathfrak{F}}}\Big(\sum_{k=0}^{m-1}\frac{\mathfrak{P}^{(k)}(\mathfrak{R},\mathfrak{K},\mathfrak{Z},0)}{s^{2-\mathfrak{F}+k}} + M\Big[-\frac{\partial^2\mathfrak{P}(\mathfrak{R},\mathfrak{K},\mathfrak{Z},\mathrm{t})}{\partial\mathfrak{R}^2} - \frac{\partial^2\mathfrak{P}(\mathfrak{R},\mathfrak{K},\mathfrak{Z},\mathrm{t})}{\partial\mathfrak{K}^2} - \frac{\partial^2\mathfrak{P}(\mathfrak{R},\mathfrak{K},\mathfrak{Z},\mathrm{t})}{\partial\mathfrak{Z}^2}\Big]\Big)\Big]. \tag{4.16}$$

Recursively applying the MT, we obtain

$$\mathfrak{P}_0(\mathfrak{R},\mathfrak{K},\mathfrak{Z},\mathrm{t}) = M^{-1}\Big[\frac{1}{s^{\mathfrak{F}}}\Big(\sum_{k=0}^{m-1}\frac{\mathfrak{P}^{(k)}(\mathfrak{R},\mathfrak{K},\mathfrak{Z},0)}{s^{2-\mathfrak{F}+k}}\Big)\Big] = M^{-1}\Big[\frac{\mathfrak{P}(\mathfrak{R},\mathfrak{K},\mathfrak{Z},0)}{s^2}\Big] = e^{\mathfrak{R}+\mathfrak{K}+\mathfrak{Z}}.$$

The RL integral is implemented on Eq (4.13), giving

$$\mathfrak{P}(\mathfrak{R},\mathfrak{K},\mathfrak{Z},\mathrm{t}) = e^{\mathfrak{R}+\mathfrak{K}+\mathfrak{Z}} + M\Big[-\frac{\partial^2\mathfrak{P}(\mathfrak{R},\mathfrak{K},\mathfrak{Z},\mathrm{t})}{\partial\mathfrak{R}^2} - \frac{\partial^2\mathfrak{P}(\mathfrak{R},\mathfrak{K},\mathfrak{Z},\mathrm{t})}{\partial\mathfrak{K}^2} - \frac{\partial^2\mathfrak{P}(\mathfrak{R},\mathfrak{K},\mathfrak{Z},\mathrm{t})}{\partial\mathfrak{Z}^2}\Big]. \tag{4.17}$$

By using MTIM approach, we obtain the following terms:

$$\mathfrak{P}_0(\mathfrak{R},\mathfrak{K},\mathfrak{Z},\mathrm{t}) = e^{\mathfrak{R}+\mathfrak{K}+3}, \tag{4.18}$$

$$\mathfrak{P}_1(\mathfrak{R},\mathfrak{K},\mathfrak{Z},\mathrm{t}) = -\frac{3\mathrm{t}^{\mathfrak{F}}e^{\mathfrak{R}+\mathfrak{K}+3}}{\Gamma(\mathfrak{F}+1)}, \tag{4.19}$$

$$\mathfrak{P}_2(\mathfrak{R},\mathfrak{K},\mathfrak{Z},\mathrm{t}) = \frac{9\mathrm{t}^{2\mathfrak{F}}e^{\mathfrak{R}+\mathfrak{K}+3}}{\Gamma(2\mathfrak{F}+1)}, \tag{4.20}$$

$$\mathfrak{P}_3(\mathfrak{R},\mathfrak{K},\mathfrak{Z},\mathrm{t}) = -\frac{27\mathrm{t}^{3\mathfrak{F}}e^{\mathfrak{R}+\mathfrak{K}+3}}{\Gamma(3\mathfrak{F}+1)}. \tag{4.21}$$

The final solution is represented as follows:

$$\mathfrak{P}(\mathfrak{R},\mathfrak{K},\mathfrak{Z},\mathrm{t}) = \mathfrak{P}_0(\mathfrak{R},\mathfrak{K},\mathfrak{Z},\mathrm{t}) + \mathfrak{P}_1(\mathfrak{R},\mathfrak{Z},\mathfrak{K},\mathrm{t}) + \mathfrak{P}_2(\mathfrak{R},\mathfrak{K},\mathfrak{Z},\mathrm{t}) + \mathfrak{P}_3(\mathfrak{R},\mathfrak{K},\mathfrak{Z},\mathrm{t}) + \cdots, \tag{4.22}$$

$$\mathfrak{P}(\mathfrak{R},\mathfrak{K},\mathfrak{Z},\mathrm{t}) = e^{\mathfrak{R}+\mathfrak{K}+3} - \frac{3\mathrm{t}^{\mathfrak{F}}e^{\mathfrak{R}+\mathfrak{K}+3}}{\Gamma(\mathfrak{F}+1)} + \frac{9\mathrm{t}^{2\mathfrak{F}}e^{\mathfrak{R}+\mathfrak{K}+3}}{\Gamma(2\mathfrak{F}+1)} - \frac{27\mathrm{t}^{3\mathfrak{F}}e^{\mathfrak{R}+\mathfrak{K}+3}}{\Gamma(3\mathfrak{F}+1)} + \cdots. \tag{4.23}$$

Example 4.2. • Implementation of MRPSM

Consider the system of nonlinear PDEs

$$\begin{aligned} &D_{\mathrm{t}}^{\mathfrak{F}}\mathfrak{P}_1(\mathfrak{R},\mathrm{t}) + \frac{\partial^3\mathfrak{P}_1(\mathfrak{R},\mathrm{t})}{\partial\mathfrak{R}^3} - 3\mathfrak{P}_2(\mathfrak{R},\mathrm{t})\frac{\partial\mathfrak{P}_2(\mathfrak{R},\mathrm{t})}{\partial\mathfrak{R}} + 6\mathfrak{P}_1(\mathfrak{R},\mathrm{t})\frac{\partial\mathfrak{P}_1(\mathfrak{R},\mathrm{t})}{\partial\mathfrak{R}} = 0,\\ &D_{\mathrm{t}}^{\mathfrak{F}}\mathfrak{P}_2(\mathfrak{R},\mathrm{t}) + \frac{\partial^3\mathfrak{P}_2(\mathfrak{R},\mathrm{t})}{\partial\mathfrak{R}^3} + 3\mathfrak{P}_1(\mathfrak{R},\mathrm{t})\frac{\partial\mathfrak{P}_2(\mathfrak{R},\mathrm{t})}{\partial\mathfrak{R}} = 0 \quad \text{where} \quad 0 < \mathfrak{F} \le 1. \end{aligned} \tag{4.24}$$

The IC's are given as

$$\begin{aligned} \mathfrak{P}_1(\mathfrak{R},0) &= \frac{4c^2e^{c\mathfrak{R}}}{(e^{c\mathfrak{R}}+1)^2},\\ \mathfrak{P}_2(\mathfrak{R},0) &= \frac{4c^2e^{c\mathfrak{R}}}{(e^{c\mathfrak{R}}+1)^2}. \end{aligned} \tag{4.25}$$

Equation (4.24) is subjected to MT, and using Eq (4.25), we get the following result:

$$\begin{aligned} &\mathfrak{P}_1(\mathfrak{R},s) + \frac{\frac{4c^2e^{c\mathfrak{R}}}{(e^{c\mathfrak{R}}+1)^2}}{s} + \frac{1}{s^{\mathfrak{F}}}\Big[\frac{\partial^3\mathfrak{P}_1(\mathfrak{R},s)}{\partial\mathfrak{R}^3}\Big] - \frac{3}{s^{\mathfrak{F}}}\mathcal{M}_{\mathrm{t}}\Big[\mathcal{M}_{\mathrm{t}}^{-1}\mathfrak{P}_2(\mathfrak{R},s) \times \frac{\partial\mathcal{M}_{\mathrm{t}}^{-1}\mathfrak{P}_2(\mathfrak{R},s)}{\partial\mathfrak{R}}\Big]\\ &+ \frac{6}{s^{\mathfrak{F}}}\mathcal{M}_{\mathrm{t}}\Big[\mathcal{M}_{\mathrm{t}}^{-1}\mathfrak{P}_1(\mathfrak{R},s) \times \frac{\partial\mathcal{M}_{\mathrm{t}}^{-1}\mathfrak{P}_1(\mathfrak{R},s)}{\partial\mathfrak{R}}\Big] = 0,\\ &\mathfrak{P}_2(\mathfrak{R},s) - \frac{\frac{4c^2e^{c\mathfrak{R}}}{(e^{c\mathfrak{R}}+1)^2}}{s} + \frac{1}{s^{\mathfrak{F}}}\Big[\frac{\partial^3\mathfrak{P}_2(\mathfrak{R},s)}{\partial\mathfrak{R}^3}\Big] + \frac{3}{s^{\mathfrak{F}}}\mathcal{M}_{\mathrm{t}}\Big[\mathcal{M}_{\mathrm{t}}^{-1}\mathfrak{P}_1(\mathfrak{R},s) \times \frac{\partial\mathcal{M}_{\mathrm{t}}^{-1}\mathfrak{P}_2(\mathfrak{R},s)}{\partial\mathfrak{R}}\Big] = 0. \end{aligned} \tag{4.26}$$

The k^{th} term's series is represented as

$$\begin{aligned} \mathfrak{P}_1(\mathfrak{R},s) &= \frac{\frac{4c^2e^{c\mathfrak{R}}}{(e^{c\mathfrak{R}}+1)^2}}{s} + \sum_{r=1}^{k}\frac{f_r(\mathfrak{R},s)}{s^{r\mathfrak{F}+1}},\\ \mathfrak{P}_2(\mathfrak{R},s) &= \frac{\frac{4c^2e^{c\mathfrak{R}}}{(e^{c\mathfrak{R}}+1)^2}}{s} + \sum_{r=1}^{k}\frac{g_r(\mathfrak{R},s)}{s^{r\mathfrak{F}+1}}, \quad r = 1,2,3,4\cdots. \end{aligned} \tag{4.27}$$

The residual function of Mohand is given by

$$A_{t}Res(\Re,s)=\mathfrak{P}_1(\Re,s)+\frac{\frac{4c^2e^{c\Re}}{(e^{c\Re}+1)^2}}{s}+\frac{1}{s^{\mathfrak{F}}}\Big[\frac{\partial^3\mathfrak{P}_1(\Re,s)}{\partial\Re^3}\Big]-\frac{3}{s^{\mathfrak{F}}}\mathcal{M}_t\Big[\mathcal{M}_t^{-1}\mathfrak{P}_2(\Re,s)\times\frac{\partial\mathcal{M}_t^{-1}\mathfrak{P}_2(\Re,s)}{\partial\Re}\Big]$$
$$+\frac{6}{s^{\mathfrak{F}}}\mathcal{M}_t\Big[\mathcal{M}_t^{-1}\mathfrak{P}_1(\Re,s)\times\frac{\partial\mathcal{M}_t^{-1}\mathfrak{P}_1(\Re,s)}{\partial\Re}\Big]=0,$$
$$A_{t}Res(\Re,s)=\mathfrak{P}_2(\Re,s)-\frac{\frac{4c^2e^{c\Re}}{(e^{c\Re}+1)^2}}{s}+\frac{1}{s^{\mathfrak{F}}}\Big[\frac{\partial^3\mathfrak{P}_2(\Re,s)}{\partial\Re^3}\Big]+\frac{3}{s^{\mathfrak{F}}}\mathcal{M}_t\Big[\mathcal{M}_t^{-1}\mathfrak{P}_1(\Re,s)\times\frac{\partial\mathcal{M}_t^{-1}\mathfrak{P}_2(\Re,s)}{\partial\Re}\Big]=0, \tag{4.28}$$

and the k^{th}-MRFs as

$$A_{t}Res_k(\Re,s)=\mathfrak{P}_{1k}(\Re,s)+\frac{\frac{4c^2e^{c\Re}}{(e^{c\Re}+1)^2}}{s}+\frac{1}{s^{\mathfrak{F}}}\Big[\frac{\partial^3\mathfrak{P}_{1k}(\Re,s)}{\partial\Re^3}\Big]-\frac{3}{s^{\mathfrak{F}}}\mathcal{M}_t\Big[\mathcal{M}_t^{-1}\mathfrak{P}_{2k}(\Re,s)\times\frac{\partial\mathcal{M}_t^{-1}\mathfrak{P}_{2k}(\Re,s)}{\partial\Re}\Big]$$
$$+\frac{6}{s^{\mathfrak{F}}}\mathcal{M}_t\Big[\mathcal{M}_t^{-1}\mathfrak{P}_{1k}(\Re,s)\times\frac{\partial\mathcal{M}_t^{-1}\mathfrak{P}_{1k}(\Re,s)}{\partial\Re}\Big]=0,$$
$$A_{t}Res_k(\Re,s)=\mathfrak{P}_{2k}(\Re,s)-\frac{\frac{4c^2e^{c\Re}}{(e^{c\Re}+1)^2}}{s}+\frac{1}{s^{\mathfrak{F}}}\Big[\frac{\partial^3\mathfrak{P}_{2k}(\Re,s)}{\partial\Re^3}\Big]+\frac{3}{s^{\mathfrak{F}}}\mathcal{M}_t\Big[\mathcal{M}_t^{-1}\mathfrak{P}_{1k}(\Re,s)\times\frac{\partial\mathcal{M}_t^{-1}\mathfrak{P}_{2k}(\Re,s)}{\partial\Re}\Big]=0. \tag{4.29}$$

Now, we use these steps to find the values of $f_r(\Re,\mathfrak{K},\mathfrak{Z},s)$ and $g_r(\Re,\mathfrak{K},\mathfrak{Z},s)$ for $r=1,2,3,...$: Put the r^{th}-Mohand residual function Eq (4.29) for the r^{th}-truncated series Eq (4.27), and then multiply the equation by $s^{r\mathfrak{F}+1}$, and solve $\mathcal{M}_tRes_{\mathfrak{P}_{1,r}}(\Re,s)=0$ and $\mathcal{M}_tRes_{\mathfrak{P}_{2,r}}(\Re,s)=0$ for $r=1,2,3,\cdots$. Using this procedure, we obtain

$$f_1(\Re,s)=\frac{4c^5e^{c\Re}\left(e^{c\Re}-1\right)}{(e^{c\Re}+1)^3},\qquad g_1(\Re,s)=\frac{4c^5e^{c\Re}\left(e^{c\Re}-1\right)}{(e^{c\Re}+1)^3}, \tag{4.30}$$

$$f_2(\Re,s)=\frac{4c^8e^{c\Re}\left(-4e^{c\Re}+e^{2c\Re}+1\right)}{(e^{c\Re}+1)^4},\qquad g_2(\Re,s)=\frac{4c^8e^{c\Re}\left(-4e^{c\Re}+e^{2c\Re}+1\right)}{(e^{c\Re}+1)^4}, \tag{4.31}$$

and so on.

The values of Eqs (4.30) and (4.31) are inserted into Eq (4.27) to obtain the following result:

$$\begin{aligned}\mathfrak{P}_1(\Re,s)&=\frac{4c^2e^{c\Re}}{s\,(e^{c\Re}+1)^2}+\frac{4c^5e^{c\Re}\left(e^{c\Re}-1\right)}{s^{\mathfrak{F}+1}\,(e^{c\Re}+1)^3}+\frac{4c^8e^{c\Re}\left(-4e^{c\Re}+e^{2c\Re}+1\right)}{s^{2\mathfrak{F}+1}\,(e^{c\Re}+1)^4}+\cdots,\\ \mathfrak{P}_2(\Re,s)&=\frac{4c^2e^{c\Re}}{s\,(e^{c\Re}+1)^2}+\frac{4c^5e^{c\Re}\left(e^{c\Re}-1\right)}{s^{\mathfrak{F}+1}\,(e^{c\Re}+1)^3}+\frac{4c^8e^{c\Re}\left(-4e^{c\Re}+e^{2c\Re}+1\right)}{s^{2\mathfrak{F}+1}\,(e^{c\Re}+1)^4}+\cdots.\end{aligned} \tag{4.32}$$

Using IMT, we obtain the final solution:

$$\begin{aligned}\mathfrak{P}_1(\Re,t)&=\frac{4c^2e^{c\Re}}{(e^{c\Re}+1)^2}+\frac{4c^5t^{\mathfrak{F}}e^{c\Re}\left(e^{c\Re}-1\right)}{\Gamma(\mathfrak{F}+1)\,(e^{c\Re}+1)^3}+\frac{4c^8t^{2\mathfrak{F}}e^{c\Re}\left(-4e^{c\Re}+e^{2c\Re}+1\right)}{\Gamma(2\mathfrak{F}+1)\,(e^{c\Re}+1)^4}+\cdots,\\ \mathfrak{P}_2(\Re,t)&=\frac{4c^2e^{c\Re}}{(e^{c\Re}+1)^2}+\frac{4c^5t^{\mathfrak{F}}e^{c\Re}\left(e^{c\Re}-1\right)}{\Gamma(\mathfrak{F}+1)\,(e^{c\Re}+1)^3}+\frac{4c^8t^{2\mathfrak{F}}e^{c\Re}\left(-4e^{c\Re}+e^{2c\Re}+1\right)}{\Gamma(2\mathfrak{F}+1)\,(e^{c\Re}+1)^4}+\cdots.\end{aligned} \tag{4.33}$$

• Implementation of MTIM

Consider the system of nonlinear PDEs

$$\begin{aligned} D_{t}^{\mathfrak{F}}\mathfrak{P}_1(\mathfrak{R},t) &= -\frac{\partial^3\mathfrak{P}_1(\mathfrak{R},t)}{\partial\mathfrak{R}^3} + 3\mathfrak{P}_2(\mathfrak{R},t)\frac{\partial\mathfrak{P}_2(\mathfrak{R},t)}{\partial\mathfrak{R}} - 6\mathfrak{P}_1(\mathfrak{R},t)\frac{\partial\mathfrak{P}_1(\mathfrak{R},t)}{\partial\mathfrak{R}}, \\ D_{t}^{\mathfrak{F}}\mathfrak{P}_2(\mathfrak{R},t) &= -\frac{\partial^3\mathfrak{P}_2(\mathfrak{R},t)}{\partial\mathfrak{R}^3} - 3\mathfrak{P}_1(\mathfrak{R},t)\frac{\partial\mathfrak{P}_2(\mathfrak{R},t)}{\partial\mathfrak{R}}, \quad \text{where} \quad 0 < \mathfrak{F} \leq 1. \end{aligned} \tag{4.34}$$

The IC's are given as

$$\mathfrak{P}_1(\mathfrak{R},0) = \frac{4c^2e^{c\mathfrak{R}}}{(e^{c\mathfrak{R}}+1)^2}, \qquad \mathfrak{P}_2(\mathfrak{R},0) = \frac{4c^2e^{c\mathfrak{R}}}{(e^{c\mathfrak{R}}+1)^2}. \tag{4.35}$$

MT is used on Eq (4.34), giving

$$\begin{aligned} \mathcal{M}[D_{t}^{\mathfrak{F}}\mathfrak{P}_1(\mathfrak{R},t)] &= \frac{1}{s^{\mathfrak{F}}}\Big(\sum_{k=0}^{m-1}\frac{\mathfrak{P}_1^{(k)}(\mathfrak{R},0)}{s^{2-\mathfrak{F}+k}} + \mathcal{M}\Big[-\frac{\partial^3\mathfrak{P}_1(\mathfrak{R},t)}{\partial\mathfrak{R}^3} + 3\mathfrak{P}_2(\mathfrak{R},t)\frac{\partial\mathfrak{P}_2(\mathfrak{R},t)}{\partial\mathfrak{R}} - 6\mathfrak{P}_1(\mathfrak{R},t)\frac{\partial\mathfrak{P}_1(\mathfrak{R},t)}{\partial\mathfrak{R}}\Big]\Big), \\ \mathcal{M}[D_{t}^{\mathfrak{F}}\mathfrak{P}_2(\mathfrak{R},t)] &= \frac{1}{s^{\mathfrak{F}}}\Big(\sum_{k=0}^{m-1}\frac{\mathfrak{P}_2^{(k)}(\mathfrak{R},0)}{s^{2-\mathfrak{F}+k}} + \mathcal{M}\Big[-\frac{\partial^3\mathfrak{P}_2(\mathfrak{R},t)}{\partial\mathfrak{R}^3} - 3\mathfrak{P}_1(\mathfrak{R},t)\frac{\partial\mathfrak{P}_2(\mathfrak{R},t)}{\partial\mathfrak{R}}\Big]\Big). \end{aligned} \tag{4.36}$$

Applying IMT on Eq (4.36), we obtain

$$\begin{aligned} \mathfrak{P}_1(\mathfrak{R},t) &= \mathcal{M}^{-1}\Big[\frac{1}{s^{\mathfrak{F}}}\Big(\sum_{k=0}^{m-1}\frac{\mathfrak{P}_1^{(k)}(\mathfrak{R},0)}{s^{2-\mathfrak{F}+k}} + \mathcal{M}\Big[-\frac{\partial^3\mathfrak{P}_1(\mathfrak{R},t)}{\partial\mathfrak{R}^3} + 3\mathfrak{P}_2(\mathfrak{R},t)\frac{\partial\mathfrak{P}_2(\mathfrak{R},t)}{\partial\mathfrak{R}} - 6\mathfrak{P}_1(\mathfrak{R},t)\frac{\partial\mathfrak{P}_1(\mathfrak{R},t)}{\partial\mathfrak{R}}\Big]\Big)\Big], \\ \mathfrak{P}_2(\mathfrak{R},t) &= \mathcal{M}^{-1}\Big[\frac{1}{s^{\mathfrak{F}}}\Big(\sum_{k=0}^{m-1}\frac{\mathfrak{P}_2^{(k)}(\mathfrak{R},0)}{s^{2-\mathfrak{F}+k}} + \mathcal{M}\Big[-\frac{\partial^3\mathfrak{P}_2(\mathfrak{R},t)}{\partial\mathfrak{R}^3} - 3\mathfrak{P}_1(\mathfrak{R},t)\frac{\partial\mathfrak{P}_2(\mathfrak{R},t)}{\partial\mathfrak{R}}\Big]\Big)\Big]. \end{aligned} \tag{4.37}$$

Recursively applying the MT, we obtain

$$\mathfrak{P}_{10}(\mathfrak{R},t) = \mathcal{M}^{-1}\Big[\frac{1}{s^{\mathfrak{F}}}\Big(\sum_{k=0}^{m-1}\frac{\mathfrak{P}_1^{(k)}(\mathfrak{R},0)}{s^{2-\mathfrak{F}+k}}\Big)\Big] = \mathcal{M}^{-1}\Big[\frac{\mathfrak{P}_1(\mathfrak{R},0)}{s^2}\Big] = \frac{4c^2e^{c\mathfrak{R}}}{(e^{c\mathfrak{R}}+1)^2},$$

$$\mathfrak{P}_{20}(\mathfrak{R},t) = \mathcal{M}^{-1}\Big[\frac{1}{s^{\mathfrak{F}}}\Big(\sum_{k=0}^{m-1}\frac{\mathfrak{P}_2^{(k)}(\mathfrak{R},0)}{s^{2-\mathfrak{F}+k}}\Big)\Big] = \mathcal{M}^{-1}\Big[\frac{\mathfrak{P}_2(\mathfrak{R},0)}{s^2}\Big] = \frac{4c^2e^{c\mathfrak{R}}}{(e^{c\mathfrak{R}}+1)^2}.$$

The RL integral is implemented on Eq (4.34), giving

$$\begin{aligned} \mathfrak{P}_1(\mathfrak{R},t) &= \frac{4c^2e^{c\mathfrak{R}}}{(e^{c\mathfrak{R}}+1)^2} - \mathcal{M}\Big[-\frac{\partial^3\mathfrak{P}_1(\mathfrak{R},t)}{\partial\mathfrak{R}^3} + 3\mathfrak{P}_2(\mathfrak{R},t)\frac{\partial\mathfrak{P}_2(\mathfrak{R},t)}{\partial\mathfrak{R}} - 6\mathfrak{P}_1(\mathfrak{R},t)\frac{\partial\mathfrak{P}_1(\mathfrak{R},t)}{\partial\mathfrak{R}}\Big], \\ \mathfrak{P}_2(\mathfrak{R},t) &= \frac{4c^2e^{c\mathfrak{R}}}{(e^{c\mathfrak{R}}+1)^2} - \mathcal{M}\Big[-\frac{\partial^3\mathfrak{P}_2(\mathfrak{R},t)}{\partial\mathfrak{R}^3} - 3\mathfrak{P}_1(\mathfrak{R},t)\frac{\partial\mathfrak{P}_2(\mathfrak{R},t)}{\partial\mathfrak{R}}\Big]. \end{aligned} \tag{4.38}$$

By using the MTIM approach, we obtain the following terms:

$$\begin{aligned}
\mathfrak{P}_{10}(\mathfrak{R},\mathrm{t}) &= \frac{4c^2 e^{c\mathfrak{R}}}{\left(e^{c\mathfrak{R}}+1\right)^2},\\
\mathfrak{P}_{11}(\mathfrak{R},\mathrm{t}) &= \frac{c^5 \mathrm{t}^{\mathfrak{F}} \tanh\left(\frac{c\mathfrak{R}}{2}\right)\operatorname{sech}^2\left(\frac{c\mathfrak{R}}{2}\right)}{\Gamma(\mathfrak{F}+1)},\\
\mathfrak{P}_{12}(\mathfrak{R},\mathrm{t}) &= -\frac{c^8 \mathrm{t}^{2\mathfrak{F}}\left(3\operatorname{sech}^2\left(\frac{c\mathfrak{R}}{2}\right)-2\right)\left(3c^3\mathrm{t}^{\mathfrak{F}}\Gamma(2\mathfrak{F}+1)^2\tanh\left(\frac{c\mathfrak{R}}{2}\right)\operatorname{sech}^2\left(\frac{c\mathfrak{R}}{2}\right)+\Gamma(\mathfrak{F}+1)^2\Gamma(3\mathfrak{F}+1)\right)}{\Gamma(\mathfrak{F}+1)^2\Gamma(2\mathfrak{F}+1)\Gamma(3\mathfrak{F}+1)(\cosh(c\mathfrak{R})+1)}.
\end{aligned} \tag{4.39}$$

$$\begin{aligned}
\mathfrak{P}_{20}(\mathfrak{R},\mathrm{t}) &= \frac{4c^2 e^{c\mathfrak{R}}}{\left(e^{c\mathfrak{R}}+1\right)^2},\\
\mathfrak{P}_{21}(\mathfrak{R},\mathrm{t}) &= \frac{c^5 \mathrm{t}^{\mathfrak{F}} \tanh\left(\frac{c\mathfrak{R}}{2}\right)\operatorname{sech}^2\left(\frac{c\mathfrak{R}}{2}\right)}{\Gamma(\mathfrak{F}+1)},\\
\mathfrak{P}_{22}(\mathfrak{R},\mathrm{t}) &= -\frac{c^8 \mathrm{t}^{2\mathfrak{F}}\left(3\operatorname{sech}^2\left(\frac{c\mathfrak{R}}{2}\right)-2\right)\left(3c^3\mathrm{t}^{\mathfrak{F}}\Gamma(2\mathfrak{F}+1)^2\tanh\left(\frac{c\mathfrak{R}}{2}\right)\operatorname{sech}^2\left(\frac{c\mathfrak{R}}{2}\right)+\Gamma(\mathfrak{F}+1)^2\Gamma(3\mathfrak{F}+1)\right)}{\Gamma(\mathfrak{F}+1)^2\Gamma(2\mathfrak{F}+1)\Gamma(3\mathfrak{F}+1)(\cosh(c\mathfrak{R})+1)}.
\end{aligned} \tag{4.40}$$

The final solution is represented as follows:

$$\mathfrak{P}_1(\mathfrak{R},\mathrm{t}) = \mathfrak{P}_{10}(\mathfrak{R},\mathrm{t}) + \mathfrak{P}_{11}(\mathfrak{R},\mathrm{t}) + \mathfrak{P}_{12}(\mathfrak{R},\mathrm{t}) + \cdots, \tag{4.41}$$

$$\mathfrak{P}_2(\mathfrak{R},\mathrm{t}) = \mathfrak{P}_{20}(\mathfrak{R},\mathrm{t}) + \mathfrak{P}_{21}(\mathfrak{R},\mathrm{t}) + \mathfrak{P}_{22}(\mathfrak{R},\mathrm{t}) + \cdots. \tag{4.42}$$

$$\begin{aligned}
\mathfrak{P}_1(\mathfrak{R},\mathrm{t}) &= \frac{4c^2 e^{c\mathfrak{R}}}{\left(e^{c\mathfrak{R}}+1\right)^2} + \frac{c^5 \mathrm{t}^{\mathfrak{F}} \tanh\left(\frac{c\mathfrak{R}}{2}\right)\operatorname{sech}^2\left(\frac{c\mathfrak{R}}{2}\right)}{\Gamma(\mathfrak{F}+1)}\\
&- \frac{c^8 \mathrm{t}^{2\mathfrak{F}}\left(3\operatorname{sech}^2\left(\frac{c\mathfrak{R}}{2}\right)-2\right)\left(3c^3\mathrm{t}^{\mathfrak{F}}\Gamma(2\mathfrak{F}+1)^2\tanh\left(\frac{c\mathfrak{R}}{2}\right)\operatorname{sech}^2\left(\frac{c\mathfrak{R}}{2}\right)+\Gamma(\mathfrak{F}+1)^2\Gamma(3\mathfrak{F}+1)\right)}{\Gamma(\mathfrak{F}+1)^2\Gamma(2\mathfrak{F}+1)\Gamma(3\mathfrak{F}+1)(\cosh(c\mathfrak{R})+1)} + \cdots,
\end{aligned} \tag{4.43}$$

$$\begin{aligned}
\mathfrak{P}_2(\mathfrak{R},\mathrm{t}) &= \frac{4c^2 e^{c\mathfrak{R}}}{\left(e^{c\mathfrak{R}}+1\right)^2} + \frac{c^5 \mathrm{t}^{\mathfrak{F}} \tanh\left(\frac{c\mathfrak{R}}{2}\right)\operatorname{sech}^2\left(\frac{c\mathfrak{R}}{2}\right)}{\Gamma(\mathfrak{F}+1)}\\
&- \frac{c^8 \mathrm{t}^{2\mathfrak{F}}\left(3\operatorname{sech}^2\left(\frac{c\mathfrak{R}}{2}\right)-2\right)\left(3c^3\mathrm{t}^{\mathfrak{F}}\Gamma(2\mathfrak{F}+1)^2\tanh\left(\frac{c\mathfrak{R}}{2}\right)\operatorname{sech}^2\left(\frac{c\mathfrak{R}}{2}\right)+\Gamma(\mathfrak{F}+1)^2\Gamma(3\mathfrak{F}+1)\right)}{\Gamma(\mathfrak{F}+1)^2\Gamma(2\mathfrak{F}+1)\Gamma(3\mathfrak{F}+1)(\cosh(c\mathfrak{R})+1)} + \cdots.
\end{aligned} \tag{4.44}$$

5. Results and discussion

The graphical and tabular results offer a comprehensive evaluation of the effectiveness and accuracy of the MTIM and MRPSM for solving FPDEs.

5.1. Graphical analysis

Figures 1 and 2 demonstrate the influence of fractional order on the solutions of the equation $\mathfrak{P}(\mathfrak{R},\mathfrak{K},\mathfrak{Z},\mathrm{t})$ for specific values of the parameters $\mathfrak{K}$, $\mathfrak{Z}$, and t. Both 3D and 2D plots highlight the effectiveness of MTIM and MRPSM in accurately capturing the behavior of the solution for varying fractional orders. Figures 3 and 4 provide a comparison between the exact solution and the solutions obtained via MTIM and MRPSM. The close agreement between these solutions demonstrates the reliability of the proposed methods for fractional PDEs. Figures 5 and 6 offer further insights into the effect of fractional order on MRPSM solutions for two distinct cases, $\mathfrak{P}1$ and $\mathfrak{P}2$. These plots indicate that MRPSM is highly sensitive to fractional-order variations, which enhances its adaptability for different problems. Figures 7 and 8 depict the results for MTIM in comparison with MRPSM for $\mathfrak{P}1$ and $\mathfrak{P}2$ under different conditions. The strong correlation between the solutions obtained by the two methods further validates the applicability of both techniques. Figures 9 and 10 show a direct comparison of MRPSM and MTIM for the same fractional order, demonstrating their consistency across both 2D and 3D perspectives. The graphical results affirm that the methods provide nearly identical results, emphasizing their robustness.

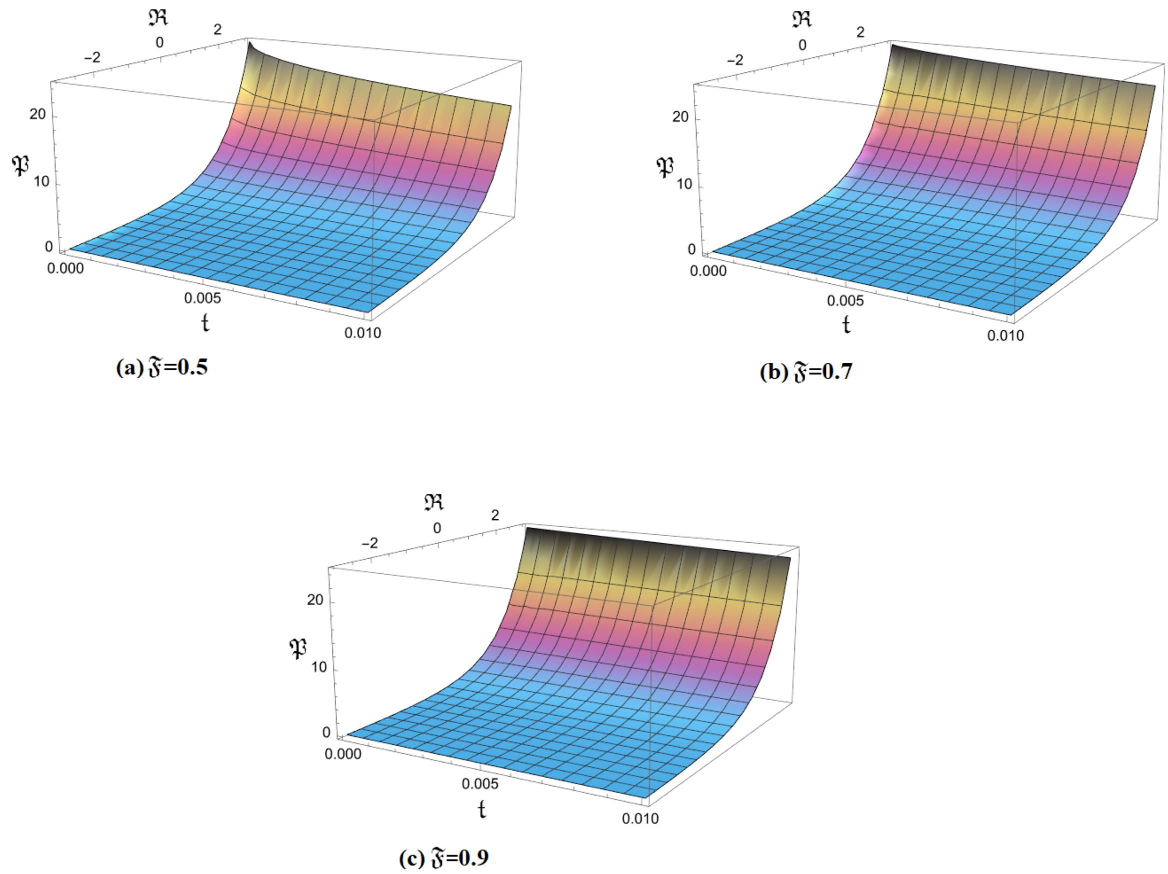


Figure 1. Graphical comparison of the fractional-order effect on our proposed method's solutions of $\mathfrak{P}(\mathfrak{R}, \mathfrak{K}, \mathfrak{Z}, \mathfrak{t})$ for $\mathfrak{K} = \mathfrak{Z} = 0.1$ and $\mathfrak{t} = 0.01$ in 3D.

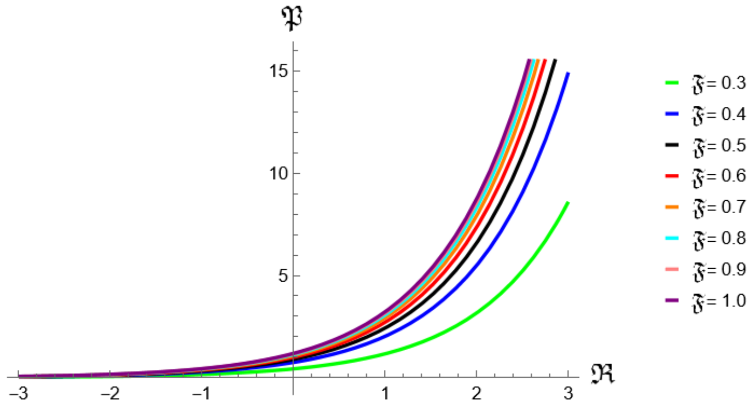


Figure 2. Graphical comparison of the fractional-order effect on our proposed method's solutions of $\mathfrak{P}(\mathfrak{N}, \mathfrak{K}, \mathfrak{Z}, t)$ for $\mathfrak{K} = \mathfrak{Z} = 0.1$ and $t = 0.01$ in 2D.

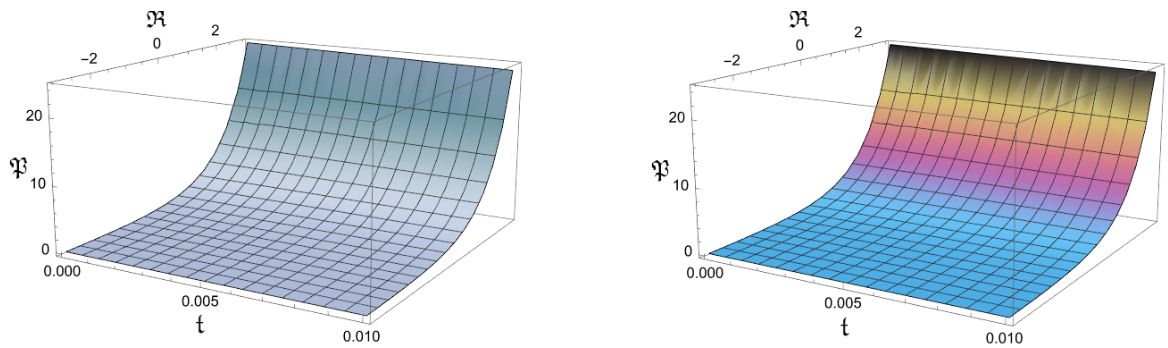


Figure 3. Graphical depiction of the exact solution and our proposed method's solutions of $\mathfrak{P}(\mathfrak{N}, \mathfrak{K}, \mathfrak{Z}, t)$ for $\mathfrak{K} = \mathfrak{Z} = 0.1$ and $t = 0.01$ in 3D.

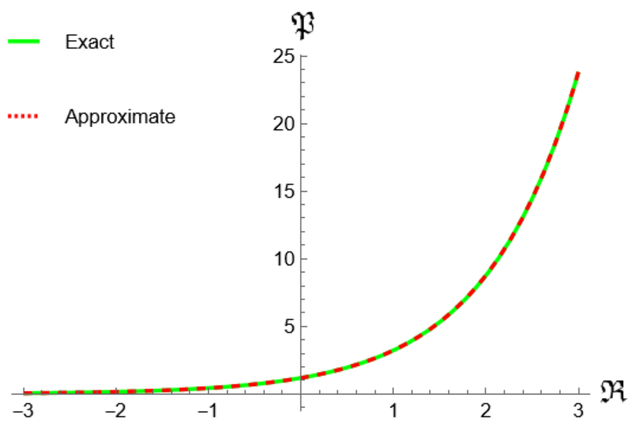


Figure 4. Graphical depiction of the exact solution and our proposed method's solutions of $\mathfrak{P}(\mathfrak{N}, \mathfrak{K}, \mathfrak{Z}, t)$ for $\mathfrak{K} = \mathfrak{Z} = 0.1$ and $t = 0.01$ in 2D.

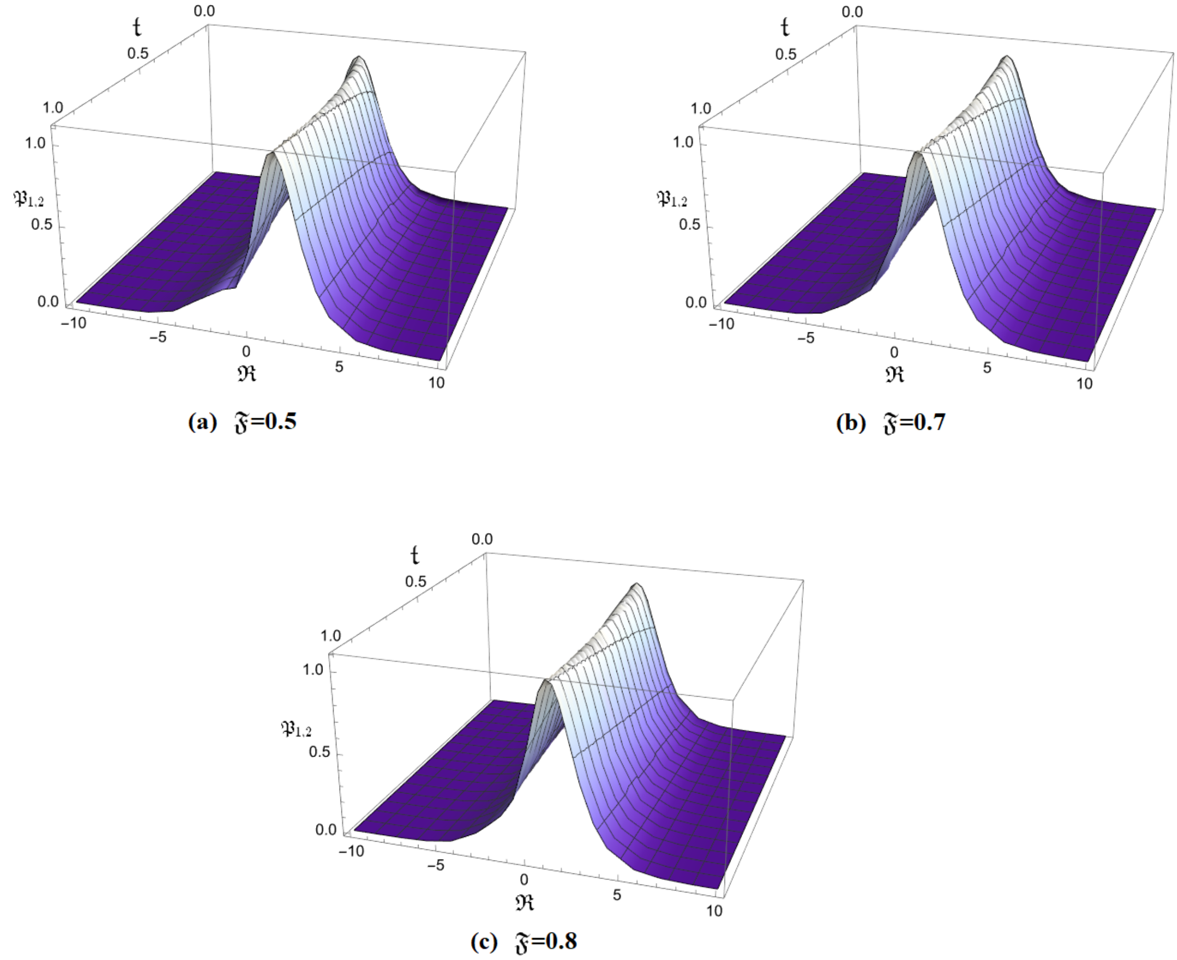


Figure 5. Graphical comparison of the fractional-order effect on the MRPSM solution of $\mathfrak{P}_1(\mathfrak{R}, \mathrm{t})$ and $\mathfrak{P}_2(\mathfrak{R}, \mathrm{t})$ for $\mathrm{t} = 0.1$ in 3D.

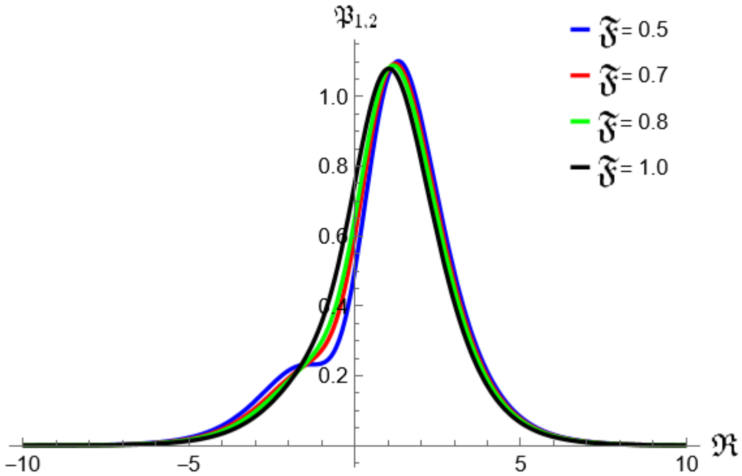
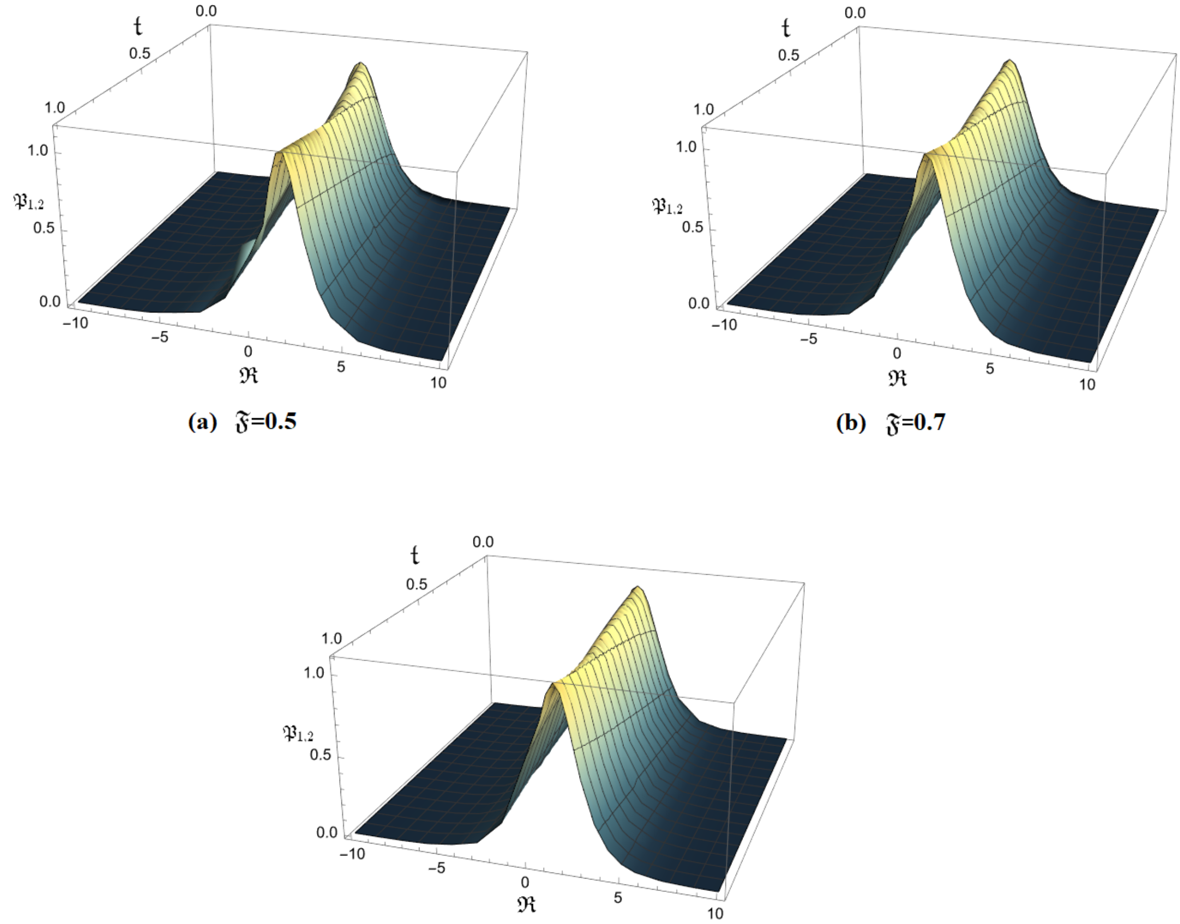


Figure 6. Graphical comparison of the fractional-order effect on the MRPSM solution of $\mathfrak{P}_1(\mathfrak{R}, \mathrm{t})$ and $\mathfrak{P}_2(\mathfrak{R}, \mathrm{t})$ for $\mathrm{t} = 0.1$ in 2D.



(a) $\mathfrak{F}$**=0.5** **(b)** $\mathfrak{F}$**=0.7** **(c)** $\mathfrak{F}$**=0.8**

Figure 7. Graphical comparison of the fractional-order effect on the MTIM solution of $\mathfrak{P}_1(\mathfrak{R},t)$ and $\mathfrak{P}_2(\mathfrak{R},t)$ for $t = 0.1$ in 3D.

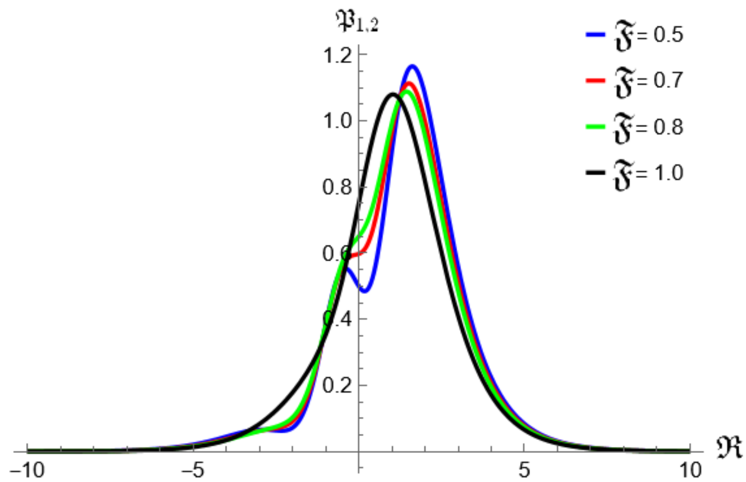


Figure 8. Graphical comparison of the fractional-order effect on the MTIM solution of $\mathfrak{P}_1(\mathfrak{R},t)$ and $\mathfrak{P}_2(\mathfrak{R},t)$ for $t = 0.1$ in 2D.

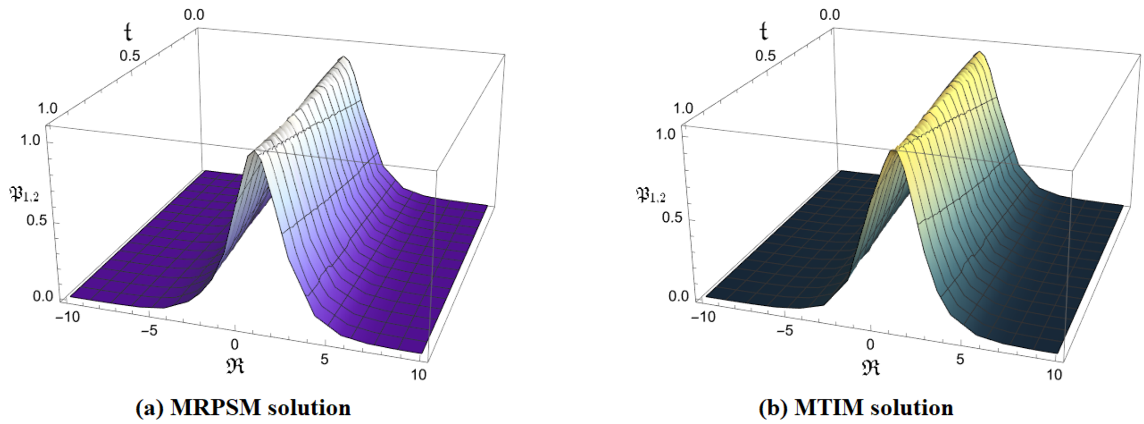


Figure 9. Graphical comparison of the MRPSM and MTIM solutions for $\mathfrak{t}$ = 0.1 in 3D.

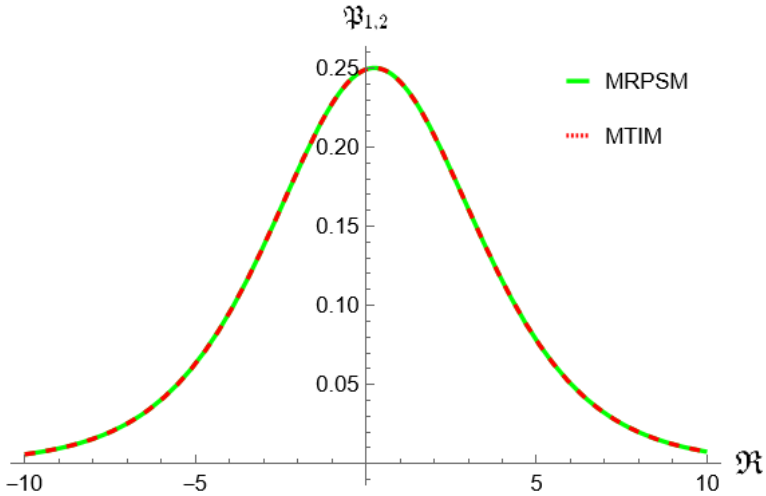


Figure 10. Graphical comparison of the MRPSM and MTIM solutions for $\mathfrak{t}$ = 0.1 in 2D.

5.2. *Tabular results*

Table 1 presents an absolute error comparison for the solutions of example 1 using MTIM and MRPSM. The minimal error values indicate that both methods provide highly accurate solutions, making them practical for solving complex fractional PDEs. Table 2 extends this error comparison to example 2, where both $\mathfrak{P}1$ and $\mathfrak{P}2$ are evaluated. Once again, the absolute errors are minimal, supporting the precision and reliability of MTIM and MRPSM for different cases. In conclusion, the graphical and tabular results strongly support the efficacy of the proposed methods. MTIM and MRPSM not only simplify the solution process for fractional PDEs, but also provide highly accurate and adaptable solutions. The methods demonstrate strong potential for application in various scientific and engineering fields requiring the analysis of fractional-order systems.

Table 1. Absolute error comparison for our proposed method's solutions for $\mathfrak{P}(\Re, \mathfrak{K}, \mathfrak{Z}, \mathfrak{t})$ for $\mathfrak{K} = \mathfrak{Z} = 0.1$.

t	$\Re$	$MRPSM_{\mathfrak{F}=1}$	$MTIM_{\mathfrak{F}=1}$	$Exact$	$MRPSM\ Error_{\mathfrak{F}=1}$	$MTIM\ Error_{\mathfrak{F}=1}$
0.01	0.10	1.30996440544749	1.30996440544749	1.30996445073324	$4.528574915063\times10^{-8}$	$4.528574915063\times10^{-8}$
	0.35	1.68202759155083	1.68202759155083	1.68202764969888	$5.814805303927\times10^{-8}$	$5.814805303927\times10^{-8}$
	0.60	2.15976617912133	2.15976617912133	2.15976625378491	$7.466357798691\times10^{-8}$	$7.466357798691\times10^{-8}$
	0.85	2.77319466809436	2.77319466809436	2.77319476396429	$9.586993199306\times10^{-8}$	$9.586993199306\times10^{-8}$
0.03	0.10	1.23367443521972	1.23367443521972	1.23367805995674	$3.624737017871\times10^{-6}$	$3.624737017871\times10^{-6}$
	0.35	1.58406933074002	1.58406933074002	1.58407398499448	$4.654254460056\times10^{-6}$	$4.654254460056\times10^{-6}$
	0.60	2.03398528246572	2.03398528246572	2.03399125864675	$5.976181022049\times10^{-6}$	$5.976181022049\times10^{-6}$
	0.85	2.61168879985479	2.61168879985479	2.61169647342311	$7.673568327426\times10^{-6}$	$7.673568327426\times10^{-6}$
0.05	0.10	1.16180660244557	1.16180660244557	1.16183424272828	$2.764028271173\times10^{-5}$	$2.764028271173\times10^{-5}$
	0.35	1.49178920681574	1.49178920681574	1.49182469764127	$3.549082552645\times10^{-5}$	$3.549082552645\times10^{-5}$
	0.60	1.91549525789186	1.91549525789186	1.91554082901389	$4.557112203484\times10^{-5}$	$4.557112203484\times10^{-5}$
	0.85	2.45954459667798	2.45954459667798	2.45960311115694	$5.851447895999\times10^{-5}$	$5.851447895999\times10^{-5}$
0.07	0.10	1.09406933762259	1.09406933762259	1.09417428370521	$1.049460826105\times10^{-4}$	$1.049460826105\times10^{-4}$
	0.35	1.40481283712613	1.40481283712613	1.40494759056359	$1.347534374536\times10^{-4}$	$1.347534374536\times10^{-4}$
	0.60	1.80381538855917	1.80381538855917	1.80398841539785	$1.730268386763\times10^{-4}$	$1.730268386763\times10^{-4}$
	0.85	2.31614480592246	2.31614480592246	2.31636697678109	$2.221708586298\times10^{-4}$	$2.221708586298\times10^{-4}$
0.10	0.10	0.99957044701003	0.99957044701003	1.00000000000000	$4.295529899697\times10^{-4}$	$4.295529899697\times10^{-4}$
	0.35	1.28347385973080	1.28347385973080	1.28402541668774	$5.515569569352\times10^{-4}$	$5.515569569352\times10^{-4}$
	0.60	1.64801305754867	1.64801305754867	1.64872127070012	$7.082131514557\times10^{-4}$	$7.082131514557\times10^{-4}$
	0.85	2.11609065292577	2.11609065292577	2.11700001661267	$9.093636869019\times10^{-4}$	$9.093636869019\times10^{-4}$

Table 2. Absolute error comparison for our proposed method's solutions for $\mathfrak{P}_1(\mathfrak{R}, \mathfrak{t})$ and $\mathfrak{P}_2(\mathfrak{R}, \mathfrak{t})$.

$\mathfrak{t}$	$\mathfrak{R}$	$MRPSM_{\mathfrak{F}=0.7}$	$MTIM_{\mathfrak{F}=0.7}$	$MRPSM_{\mathfrak{F}=1.0}$	$MTIM_{\mathfrak{F}=1.0}$	$\lvert MRPSM - MTIM\rvert_{\mathfrak{F}=0.7}$	$\lvert MRPSM - MTIM\rvert_{\mathfrak{F}=1.0}$
0.1	0.10	0.2499527853	0.2499526861	0.2499121337	0.2499121276	$9.9242792095\times10^{-8}$	$6.0831994996\times10^{-8}$
	0.35	0.2486293113	0.2486289768	0.2483568594	0.2483568389	$3.3452402567\times10^{-7}$	$2.0505029479\times10^{-8}$
	0.60	0.2454001942	0.2453996649	0.2449043498	0.2449043174	$5.2929990035\times10^{-7}$	$3.2444037623\times10^{-8}$
	0.85	0.2403639798	0.2403633177	0.2396597476	0.2396597070	$6.6212486968\times10^{-7}$	$4.0585694638\times10^{-8}$
0.3	0.10	0.2499229699	0.2499219730	0.2499903237	0.2499901595	$9.9690462096\times10^{-7}$	$1.6424638585\times10^{-7}$
	0.35	0.2490954767	0.2490921164	0.2488224297	0.2488218760	$3.3603301558\times10^{-6}$	$5.5363579581\times10^{-7}$
	0.60	0.2463478586	0.2463425417	0.2457429414	0.2457420654	$5.3168749624\times10^{-6}$	$8.7598901432\times10^{-7}$
	0.85	0.2417642150	0.2417575638	0.2408460059	0.2408449101	$6.6511162011\times10^{-6}$	$1.0958137564\times10^{-6}$
0.5	0.10	0.2497783238	0.2497754095	0.2499905839	0.2499898235	$2.9143115888\times10^{-6}$	$7.6039993426\times10^{-7}$
	0.35	0.2493515651	0.2493417416	0.2492122418	0.2492096786	$9.8234564366\times10^{-6}$	$2.5631286842\times10^{-6}$
	0.60	0.2469966017	0.2469810585	0.2465102198	0.2465061643	$1.5543142236\times10^{-5}$	$4.0555046959\times10^{-6}$
	0.85	0.2427855197	0.2427660760	0.2419673860	0.2419623128	$1.9443610367\times10^{-5}$	$5.0732118353\times10^{-6}$
0.7	0.10	0.2495611717	0.2495552642	0.2499129141	0.2499108276	$5.9075153423\times10^{-6}$	$2.0865374195\times10^{-6}$
	0.35	0.2494909149	0.2494710021	0.2495262957	0.2495192624	$1.9912839737\times10^{-5}$	$7.0332251095\times10^{-6}$
	0.60	0.2474879230	0.2474564159	0.2472061853	0.2471950570	$3.1507046666\times10^{-5}$	$1.1128304885\times10^{-5}$
	0.85	0.2436133438	0.2435739303	0.2430238880	0.2430099671	$3.9413570944\times10^{-5}$	$1.3920893276\times10^{-5}$
1.0	0.10	0.2491344789	0.2491219850	0.2496502911	0.2496442079	$1.2493927289\times10^{-5}$	$6.0831994737\times10^{-6}$
	0.35	0.2495497378	0.2495076238	0.2498553300	0.2498348249	$4.2114079676\times10^{-5}$	$2.0505029473\times10^{-5}$
	0.60	0.2480300251	0.2479633902	0.2481164215	0.2480839774	$6.6634909492\times10^{-5}$	$3.2444037567\times10^{-5}$
	0.85	0.2446212213	0.2445378648	0.2444869946	0.2444464089	$8.3356582432\times10^{-5}$	$4.0585694683\times10^{-5}$

6. Conclusions

In conclusion, this study has demonstrated the efficacy of MTIM and MRPSM in solving fractional-order PDEs and systems of PDEs. By employing the Caputo operator to define fractional derivatives, we have shown that these methods provide a powerful and flexible framework for tackling complex problems in applied mathematics. The results obtained through our examples confirm that MTIM and MRPSM not only simplify the solution process, but also improve the accuracy and reliability of the solutions. These methods hold significant potential for application in various scientific and engineering disciplines, particularly in situations where traditional approaches may fall short. Future work can extend these techniques to more complex and higher-dimensional problems, further establishing their utility in the field of fractional calculus and beyond.

Author contributions

A.S.A and H.Y Conceptualization, A.M.M; formal analysis, A.S.A. investigation, A.S.A. project administration, A.S.A. validation, visualization, H.Y. writing-review & editing; A.M.M; Data curation, H.Y. resources, A.M.M; validation, H.Y. software, H.Y. visualization, A.M.M; resources, H.Y. project administration, H.Y. writing-review & editing, H.Y. funding. All authors have read and agreed to the published version of the manuscript.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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