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Research article

Analytical and numerical techniques for solving a fractional integro-differential equation in complex space

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Abstract: In this article, we describe the existence and uniqueness of a solution to the nonlinear fractional Volterra integro differential equation in complex space using the fixed-point theory. We also examine the remarkably effective Euler wavelet method, which converts the model to a matrix structure that lines up with a system of algebraic linear equations; this method then provides approximate solutions for the given problem. The proposed technique demonstrates superior accuracy in numerical solutions when compared to the Euler wavelet method. Although we provide two cases of computational methods using MATLAB R2022b, which could be the final step in confirming the theoretical investigation.

Keywords: fractional calculus; complex plane; fixed point theorem; rationalized Haar wavelet; Euler wavelet method
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1. Introduction

The effects of complex functions are widely used in various fields of science, including physics, mathematical mechanics, electrical engineering, biological mechanisms, chemistry, AC voltage analysis, signal analysis, fluid dynamics, radio frequency transmission, and cell technology [1,2]. Additionally, fractional mathematics is a branch of mathematics that has been steadily refined over the past three centuries [3,4]. In the 19th century, Riemann and Liouville defined differentiation as a fractional order. Harmonic oscillators, hydrodynamics, optimal control, quantum physics, phase field systems, electromagnetism, and dispersion media are just a few of the fascinating and complex phenomena that have been extensively modeled with fractional mathematics in recent decades.

Finding unique solutions to different kinds of differential and integral equations is the primary focus of most academic journals. In 1922, the famous Polish mathematician Stefan Banach established the Banach fixed-point principle, a powerful and important technique. Recently, many researchers have investigated nonlinear fractional differential and integro-differential equations (NF/IDEqs) to determine their existence and unique solutions. For example, Schaefer's fixed-point theory was used in [5] to demonstrate the existence of a unique solution to fractional differential equations (FDEs). In [6], the existence of a unique fractional fuzzy system was examined through the lens of metric fixed-point theory. [7] introduces a nonlinear implicit random FIDE in the sense of the mean square and discusses the uniqueness and existence of its solutions. In [8], sufficient conditions for solving NFIDEq in complex space were provided. For additional analytical investigations, refer to [9–14].

Nevertheless, most fractional-order equations lack analytical solutions. Consequently, there has been significant interest in developing numerical techniques for solving FIDE. Several scientists have addressed the computational outcomes of these equations by employing strategies that manage them in a more practical setting. In [15], a differential transform scheme was devised to address a set of composite fractional oscillation problems. The Ritz approximation method was introduced in [16] to obtain the solutions for fractional control equations. Moreover, numerous academics have proposed robust computational strategies based on wavelet techniques for solving FDEs. For instance, Chebyshev cardinal wavelets were applied in [17], the Haar wavelet method was used in [18], the Euler wavelet method was introduced in [19], and the Legendre wavelet method was applied in [20]. Chebyshev wavelet operational matrices were also introduced in [21], while numerical techniques based on wavelet functions and collocation approaches were suggested in [22–25].

Based on previous research mentioned in the existing literature, we want to present a qualitative analysis to solve the following NFVIDEqs in complex space. Consider the problem:

$$ {}^{c}D^{\nu}\,\Omega\,(t) = \mathrm{W}\,(\mathrm{t}, \Omega(t)) + \lambda(t)\,\int_0^t \Xi(t,r,\Omega(r))\,dr\,,\ \mathrm{t} \in \mathrm{T} = [0,\tau]\,, \tag{1} $$

with initial condition

$$ \Omega\,(0) = \Omega_0 + \beta(t)\,\int_0^\tau \Omega(r)\,dr, \tag{2} $$

where $\Omega\,(t)$ is an unknown C^1 complex function, $\Omega : E \longrightarrow C$, $E \subset C$, ${}^{C}D^{\nu}$ is a Caputo derivative with $\upsilon \in (0,1)$, $r \in T$ such that $r \prec t$, W and Ξ are known and continuous functions such that $W: T \times E \longrightarrow C$, $\Xi: T \times T \times E \longrightarrow C$, the values of $\lambda(t)$ and $\beta(t)$ are real, and Ω_0 is prescribed constant. To resolve problems (1) and (2), both theoretical and numerical methods are used.

The article investigates an unexplored area of fractional calculus and analyzes the existence and originality of the solution. In addition, it is proposed to integrate the rationalization of the Haar wavelet method (RHM) and the Euler polynomial (EP) approach into a new numerical technique. This method is used to solve a first fractional model using variables specified on complex planes. The proposed method uses the power series to develop a numerical solution that shows rapid convergence and includes multiple terms that can be easily calculated. This methodology proves computational efficiency and makes it easy to implement in computer systems.

This article organizes its structure as follows: Section 2 covers crucial subjects. Section 3 outlined the necessary conditions for the existence and uniqueness of a solution to problems (1) and (2). Section 4 presents the numerical approach for problems (1) and (2) using the Euler wavelet method (EWM). Additionally, we provide a newly developed approach that merges the (RHM) with the (EP) approximation. In Section 5, we discuss numerical problems linked to what we established in Section 4 to present the precision of the proposed technique and calculate the case's abs. error. Section 6 contains the conclusion.

2. Basic concepts

We will elaborate on the definitions and preliminary information provided in this paper.

Definition 1. *[26]* The Riemann-Liouville operator of order $\upsilon > o$is represented as:

$$I^{\upsilon}\, \mathrm{V}\,(z) = \frac{1}{\Gamma(\upsilon)} \int_0^z (\mathrm{z} - \mathrm{u})\, V(u)\ \mathrm{du}\ \ , \mathrm{V} > 0,$$

where $\Gamma(\cdot)$ represents the gamma function.

Proposition 1. *[27]* Caputo's operator is related to the (R-L) operator in the following manner:

1) $D^{\upsilon}\, \mathrm{I}^{\upsilon}\, \mathrm{V}(z) = V(z)$, $\mathrm{z} > 0$;

2) $I^{\upsilon}\, \mathrm{D}^{\upsilon}\, \mathrm{V}(z) = V(z) - \sum_{k=0}^{n-1} \frac{V^{(k)}\,(0)\, \mathrm{z}^k}{k!}$, $\mathrm{z} > 0$.

Theorem 1. *[28]* Consider X be a Banach space. The set $G \subset X$ of functions is relatively compact if and only if it is bounded and equicontinuous.

Theorem 2. *[28]* In a Banach space, each contraction mapping admits a unique fixed point.

Definition 2. *[29]* The Euler polynomials of degree *m* are defined as:

$$\sum_{\ell=0}^{m} \binom{n}{\ell} E_1(z) + \mathrm{E}_m(z) = 2\, \mathrm{z}^m\ ,\ \mathrm{z}\ \in\ [0, 1],$$

which can be constructed by the following generating functions

$$\frac{2e^{zt}}{e^t + 1} = \sum_{m=0}^{\infty} E_m(z)\, \frac{t^m}{m!}\ .$$

Proposition 2. *[29]* The Euler polynomials constitute a comprehensive foundation throughout the interval $[0,1]$.

Definition 3. *[30]* Degenerate Euler polynomials for$n \in N$are defined as:

$$E\ (\mathrm{u}\ , v) = \sum_{k=0}^{n}\ \ \sum_{\ell=1}^{k} \binom{n}{k}\ \ (u)_\ell\, S^{(2)}\ (\mathrm{k}, \ell)\ .\ \mathrm{E}_{n-k}(v),$$

such that $z(t) = u\,(t) + i\, v(t), (u)_l$ is a falling factorial sequence defined as $(u)_l = u\,(u - 1) \dots (u - l + 1)$, $l \geq 1$, and $S^{(2)}$ is a Stirling numbers of the second kind.

Definition 4. *[31]* Rationalized Haar function$h_m(t)$for$m = 2^{\beta}$ + k , β = 1, 2, ...,and$k =$ 0, 1, 2, ..., 2^{β} - 1, are defined by

$$h_m(t) = H(2^{\beta} t - \mathrm{k})\ ,\ \ \mathrm{t}\ \in\ [0, 1),$$

where

$$H(t) = \begin{cases} 1\ ,\ \ 0 < \mathrm{t}\ \leq \frac{1}{2}; \\ -1\ ,\ \ \frac{1}{2} < \mathrm{t} < 1; \\ 0,\ \ other \text{ wise.} \end{cases}$$

3. Existence and uniqueness solution of NFIDEq

Before we begin our examination of the theoretical solution to the problems (1) and (2), we will apply the (R-L) fractional integral operator to the problems (1) and (2) to transform it into the following integral equation:

$$\Omega\ (t) = \Omega_0 + \beta(t)\ \int_0^{\tau} \Omega(r)\ dr + \frac{1}{\Gamma(\upsilon)} \int_0^{\tau} (t-r)^{\upsilon-1} \left[W\ (\mathrm{r}\ , \Omega(r)) + \lambda(t)\ \int_0^{r} \Xi\ (\mathrm{r, s}, \Omega(s))\ ds)\right] dr. \tag{3}$$

Consider these assumptions:

(C1) The function $\Omega : \mathrm{E} \longrightarrow \mathbb{C}$ is an analytical function;

(C2) For r, s $\in$ T, and $\Omega_1, \Omega_2 \in \mathbb{C}(T)$, there exists a non-negative constantp, q such that:

$$|W(\mathrm{r}, \Omega_1) - W(r, \Omega_2)|\ \leq\ \mathrm{p}\ |\Omega_1 - \Omega_2|,$$

$$|\Xi(\mathrm{r, s}, \Omega_1) - \Xi(r, \mathrm{s}, \Omega_2)|\ \leq\ \mathrm{q}\ |\Omega_1 - \Omega_2|;$$

(C3) For M, N $\in \mathrm{R}^{+}$, we have $\underset{0\leq t\leq \tau}{Sup}\ |\lambda(t) + \beta(t)| \leq$ M, and $\underset{0\leq t\leq \tau}{Sup}\ |\ \Omega(t)\ | \leq$ N.

This segment's primary goal is to establish the existence and uniqueness of solutions to NFVIDEq. Theorem 1 helps to analyze the compactness of function sets, which is important for ensuring the problem's well-posedness. Furthermore, Theorem 2 is critical for proving the solution's uniqueness with the aid of certain fractional calculus properties.

In order to examine whether a solution exists for problems (1) and (2), consider the integral operator Ψ: $(\mathbb{C}(T), \|.\|_{\infty}) \to (\mathbb{C}(T), \|\cdot\|_{\infty})$.

Where

$$\|x_i\|_{\infty} = \underset{t\in T}{Sup}|x_i(t)|, \forall x_i \in \mathbb{C}(T),$$

such that

$$(\Psi\Omega)(t) =\ \Omega_0 + \beta(t)\ \int_0^{\tau} \Omega(r)\ \ \mathrm{dr} + \frac{1}{\Gamma(\upsilon)} \int_0^{t} (t\ \text{- r})^{\upsilon-1} \left[W(r, \Omega(r)) + \lambda(t)\ \int_0^{r} \Xi\ (\mathrm{r, s}, \Omega(s))\ \mathrm{ds}\right] \mathrm{dr}.$$

Now, we will proceed to present the following theorems:

Theorem 3. Under conditions (C1)–(C3), the problems (1) and (2) possess at least one solution.

Proof. Here's how we are going to approach the proof:

Step 1. Suppose that we have a sequence of solutions $\{\Omega_K\}_{K\in N}$ that converges to Ω in $\mathbb{C}(T)$ for some $t \in T$, and by the aid of Theorem 1, we have:

$$|\Psi\Omega_K(t) - \Psi\Omega\,(t)| \le |\beta(t)| \int_0^\tau |\Omega_K(r) - \Omega(r)|\, dr$$

$$+\frac{1}{\Gamma(\upsilon)} \int_0^t (t - r)^{\upsilon-1} \, |W(r,\Omega_K(r)) - W(r,\Omega(r))|\, dr$$

$$+ \frac{1}{\Gamma(\upsilon)} \int_0^t (t - r)^{\upsilon-1} \, |\lambda(t)| \left[\int_0^r |\Xi\,(r, s, \Omega_K(s)) - \Xi\,(r, s, \Omega(s))|\, ds\right] dr\,.$$

So,

$$Sup\, \left|(\Psi\Omega_k)(t) - (\Psi\Omega)(t)\right| \le \left[MT + \frac{PT^{\upsilon}}{\Gamma(\upsilon+1)} + \frac{MqT^{\upsilon+1}}{\Gamma(\upsilon+2)}\right].\ Sup\, |\Omega_K(t) - \Omega(t)|.$$

Thus,

$$\|(\Psi\Omega_k)(t) - (\Psi\Omega)(t)\|_\infty \le Q\,(M, \upsilon)\, \|\Omega_K(t) - \Omega(t)\|_\infty,$$

where

$$Q\,(M,\upsilon) = MT + \frac{PT^{\upsilon}}{\Gamma(\upsilon+1)} + \frac{MqT^{\upsilon+1}}{\Gamma(\upsilon+2)}.$$

From (C1), we have $\|\Psi\Omega_K(t) - \Psi\Omega(t)\|_\infty \longrightarrow 0$, as $K \longrightarrow \infty$. This implies that Ψ is a continuous operator on $\mathbb{C}(T)$.

Step 2. Our goal here is to show that Ψ transforms bounded sets into bounded sets in $\mathbb{C}(T)$ such that $\Psi: \widetilde{B_a} \longrightarrow \mathbb{C}(T)$, where $\widetilde{B_a}$ is a closed bounded convex subset of $\mathbb{C}(T)$ such that $\widetilde{B_a} = \{\Omega(t) \in \mathbb{C}(T) : \|\Omega\|_\infty < a, a > 0\}$.

For all $\Omega(t) \in \widetilde{B_a}$, we have

$$|\Psi\,\Omega\,(t)| \le |\Omega_0| + |\beta(t)| \int_0^\tau |\Omega(r)|\, dr$$

$$+ \frac{1}{\Gamma(\upsilon)} \int_0^t (t - r)^{\upsilon-1} \, |W(r,\Omega(r)) - W(r, 0)|\, dr$$

$$+\frac{1}{\Gamma(\upsilon)} \int_0^t (t - r)^{\upsilon-1} \, |\lambda(t)| \left[\int_0^r |\Xi\,(r, s, \Omega(s) - \Xi\,(r, s, 0)|\, ds\right] dr$$

$$+ \frac{1}{\Gamma(\upsilon)} \int_0^t (t - r)^{\upsilon-1} \, |W(r, 0)|\, dr$$

$$+\frac{1}{\Gamma(\upsilon)}\int_0^t (t - \mathrm{r})^{\upsilon-1} \; |\lambda(t)| \int_0^r |\Xi\ (\mathrm{r, s, 0})| \ \mathrm{ds\ dr}.$$

For a_0 , $\mathrm{b}_0 > 0$, set $Sup\ |W(r,0)| \le \mathrm{a}_0$, and $Sup\ |\Xi(r,\mathrm{s},0)| \le \mathrm{b}_0$.

Thus, we have

$$\|(\Psi\Omega)\ (t)\|_\infty \le |\Omega_0| + \mathrm{MNT} + \frac{(PN+a_0)\ T^{\upsilon}}{\Gamma(\upsilon+1)} + \frac{(qN+b_0)\ T^{\upsilon+1}}{\Gamma(\upsilon+2)}.$$

So,

$$\|(\Psi\Omega)\ (t)\|_\infty < \mu, \text{ for all } t \in T,$$

where

$$\mu = |\Omega_0| + \mathrm{MNT} + \frac{(PN+a_0)T^{\upsilon}}{\Gamma(\upsilon+1)} + \frac{(qN+b_0)\ \mathrm{T}^{\upsilon+1}}{\Gamma(\upsilon+2)}.$$

Thus, for all $\Omega \in \widetilde{B_a}$, we have $\Psi\ \widetilde{B_a} \subset \widetilde{B_a}$.

Step 3. We will show that $\Psi$is completely continuous on $\mathbb{C}(T)$.

For $\Omega \in \widetilde{B_a}$, and θ_1 , $\theta_2 \in$ T such that $\theta_1 < t < \theta_2$,

$$|(\Psi\ \Omega)(\theta_1) - (\Psi\ \Omega)(\theta_2)| \le |\theta_1 - \theta_2|\ |\lambda\ (\theta)|\ N$$

$$+\frac{1}{\Gamma(\upsilon)}\int_0^t ((\theta_1 - \mathrm{r})^{\upsilon-1} - (\theta_2 - \mathrm{r})^{\upsilon-1})\ \left|W\big(r,\Omega(r)\big)\right| \mathrm{dr}$$

$$+\frac{1}{\Gamma(\upsilon)}\int_{\theta_1}^{\theta_2} (\theta_2 - \mathrm{r})^{\upsilon-1}\ \left|W\big(\mathrm{r},\Omega(r)\big)\right| \mathrm{dr}$$

$$+\frac{1}{\Gamma(\upsilon)}\int_{\theta_1}^{\theta_2} ((\theta_1 - \mathrm{r})^{\upsilon-1} - (\theta_2 - \mathrm{r})^{\upsilon-1})\ \int_0^t \left|\Xi\ \big(\mathrm{r, s},\Omega(s)\big)\right| \mathrm{ds\ dr}$$

$$+\frac{1}{\Gamma(\upsilon)}\int_{\theta_1}^{\theta_2} ((\theta_1 - \mathrm{r})^{\upsilon-1} - (\theta_2 - \mathrm{r})^{\upsilon-1})\ \int_0^t \left|\Xi\ \big(\mathrm{r, s},\Omega(s)\big)\right| \mathrm{ds\ dr}$$

$$+\frac{1}{\Gamma(\upsilon)}\int_{\theta_1}^{\theta_2} (\theta_1 - \mathrm{r})^{\upsilon-1}\ \int_0^t |\Xi\ (\mathrm{r, s},\Omega(s))|\ \mathrm{ds\ dr},$$

$$|(\Psi\ \Omega)(\theta_1) - (\Psi\ \Omega)(\theta_2)| \le |\theta_1 - \theta_2|\ |\lambda\ (\theta)|\ N$$

$$+\frac{1}{\Gamma(\upsilon+1)}\ [2\ (\theta_2 - \theta_1)^{\upsilon} + \theta_1^{\upsilon} - \theta_2^{\upsilon}]$$

$$+\frac{1}{\Gamma(\upsilon+2)}\ [2\ (\theta_2 - \theta_1)^{\upsilon+1} + \theta_1^{\upsilon} - \theta_2^{\upsilon}].$$

So, we obtain$\|\Psi\, \Omega\, (\theta_1)$ - $\Psi\, \Omega\, (\theta_2)\|_\infty \longrightarrow 0$, as$Q_1 \longrightarrow Q_2$, and hence $[\Psi\, \widetilde{B_a}]$ be equicontinuous for all $\Omega \in \widetilde{B_a}$.

By Theorem 1,Ψis relatively compact, and hence it is completely continuous. From Schafer's theory, see [32], we have validated that at least one solution exists for problems (1) and (2) on $\mathbb{C}(T)$.

Theorem 4. Under the conditions (C2) and (C3), the problems (1) and (2) provide a unique solution if

$$Q(M, \upsilon) < 1.$$

Proof. According to fixed point theory, it is evident that $\Omega\,(t)$ is a solution to the problems (1) and (2) only when $\Omega \in \mathbb{C}(T)$ becomes a fixed point of the operator Ψ.

For Ω_1 , $\Omega_2 \in \mathbb{C}(T)$, we have

$$|(\Psi\Omega_1)\,(t) - (\Psi\Omega_2)\,(t)| \le |\beta(t)| \int_0^\tau |\Omega_1(r) - \Omega_2(r)|\, \mathrm{dr}$$

$$+\frac{1}{\Gamma(\upsilon)} \int_0^t (t - \mathrm{r})^{\upsilon-1}\ |W(r, \Omega_1(r)) - W(r, \Omega_2(r))|\, \mathrm{dr}$$

$$+\frac{1}{\Gamma(\upsilon)} \int_0^t (t - \mathrm{r})^{\upsilon-1}\ |\lambda(t)| \left[\int_0^r |\Xi\ (\mathrm{r, s}, \Omega_1(s) - \Xi\ (\mathrm{r, s}, \Omega_2(s)|\right] \mathrm{dr}\,.$$

So, according to Theorem 3, we have

$$Sup\, \left|(\Psi\ \Omega_1)(t) - (\Psi\ \Omega_2)(t)\right| \le \left[\mathrm{MT} + \frac{PT^\upsilon}{\Gamma(\upsilon+1)} + \frac{MqT^{\upsilon+1}}{\Gamma(\upsilon+2)}\right] .Sup\, |\Omega_1(t) - \Omega_2(t)|.$$

Thus,

$$\|(\Psi\, \Omega_1)(t) - (\Psi\, \Omega_2)(t)\|_\infty \le Q\ (\mathrm{M}, \upsilon)\ \|\Omega_1(t) - \Omega_2(t)\|_\infty.$$

For $Q(M, \upsilon) < 1$, Ψ becomes a contraction mapping. So, according to Theorem 2, Ψhas a fixed point, which guarantees the uniqueness of the solution for problems (1) and (2).

4. Numerical approximation for solving problems (1) and (2)

This section introduces a modified method for solving the NFVIDEq after presenting a computational approach based on the Euler Wavelet Method (EWM).

4.2. Euler wavelets method

The Euler wavelet of degree m denoted by $\hat{\Psi}_{nm}(z)$, and defined on the interval $[0,1]$ as:

$$\hat{\Psi}_{nm}(z) \simeq 2^{\frac{\alpha-1}{2}}\, \hat{\mathrm{E}}_m\, (2^{\alpha-1}\, \mathrm{Z} - \mathrm{n} + 1),\ \frac{n-1}{2^{\alpha-1}} \le \mathrm{Z} \le \frac{n}{2^{\alpha-1}}, \tag{4}$$

where n = 1, 2, ... $2^{\alpha-1}$, $\alpha \in \mathbb{Z}^+$, and m = 0, 1, ... N -1.

$$\hat{E}_m(z) = \begin{cases} 1, & m = 0; \\ \left(\frac{2(-1)^{m-1}(m!)^2}{(2m)!} E_m(z)\right)^{\frac{-1}{2}}, & m > 0, \end{cases} \tag{5}$$

where $E_m(z)$ is the Euler polynomial defined in Definition 2.

A function $z(t)$ can be expressed in terms of Euler wavelets as a truncated series given by

$$Z(t) = \sum_{n=1}^{2^{\alpha-1}} \sum_{m=0}^{N-1} h_{nm} \hat{\Psi}_{nm}(t), \tag{6}$$

$$= H^T \Psi(t), \tag{7}$$

where H^T is the coefficient vector defined as

$$H^T = \left[h_{10}, h_{11}, ..., h_{1(N-1)}, h_{20}, ... h_{2(N-1)}, ... h_{2^{\alpha-1}0}, ... h_{2^{\alpha-1}(N-1)}\right]. \tag{8}$$

Using Eq (6), we obtain

$$\beta_{ij} = \int_0^1 \hat{\Psi}_{ij}(1) z(t) \, dt = \sum_{n=1}^{2^{\alpha-1}} \sum_{m=0}^{N-1} h_{nm} \int_0^1 \hat{\Psi}_{nm}(t) \, \hat{\Psi}_{ij}(t) \, z(t) \, dt,$$

$$= \sum_{n=1}^{2^{\alpha-1}} \sum_{m=0}^{N-1} h_{nm} \gamma_{nm}^{ij}, \tag{9}$$

where

$$\gamma_{nm}^{ij} = \int_0^1 \hat{\Psi}_{nm}(1) \hat{\Psi}_{ij}(t) z(t) \, dt.$$

$\Psi(t)$ in Eq (7) is Euler function vector, which is defined as

$$\Psi(t) = \left[\hat{\Psi}_{10}, \hat{\Psi}_{11}, ..., \hat{\Psi}_{1(N-1)}, \hat{\Psi}_{20}, ..., \hat{\Psi}_{2(N-1)}, \hat{\Psi}_{2^{\alpha-1}0}, ..., \hat{\Psi}_{2^{\alpha-1}(N-1)}\right]. \tag{10}$$

So, we can formulate a system of matrices as

$$B^T = H^T \Gamma, \tag{11}$$

with $\beta = \left[\beta_{10}, \beta_{11}, ..., \beta_{1(N-1)}, \beta_{20}, ..., \beta_{2(N-1)}, \beta_{2^{\alpha-1}0}, ..., \beta_{2^{\alpha-1}(N-1)}\right]^T$, and $\Gamma = \left[\gamma_{nm}^{ij}\right]_{M\times M}$, is a matrix of order $M = 2^{\alpha-1}$ N, and is given by:

$$\Gamma = \int_0^1 \Psi(t) \, . \, \Psi^T(t) \quad dt.$$

Similarly, we can approximate the function of two variables $F(t, s)$ in terms of Euler wavelets as

$$F(t,s) = \Psi(t) \hat{F} \Psi(s), \tag{12}$$

where $\hat{F}$ is a matrix of order m × m given by:

$$\hat{F} = \Gamma^{-1}\left[\int_0^1 \int_0^1 F(t,s) \, \Psi(t) \, \Psi(s) \, ds\right] \Gamma^{-1}.$$

The EW vector $\Psi(t)$, defined in Eq (10), can be determined as

$$\int_0^t \Psi(V)\ dv = \mathrm{F}\,\Psi(t), \tag{13}$$

where F is $M \times M$ dimensional matrix.

Now, we can define the fractional integration of $\Psi(t)$ as

$$I^{\upsilon}\,\Psi(t) = F^{\upsilon}\,\Psi(t), \tag{14}$$

where F^{υ} is $M \times M$ dimensional matrix.

So, from Definition 1, and Eq (12), F^{υ} obtained as follows:

$$F^{\upsilon} = \left[\int_0^1 \left(\frac{1}{\Gamma(\upsilon)_0} \int_0^t (t - \mathrm{v})^{\upsilon-1}\ \Psi(v)\,\mathrm{d\,v}\right) \Psi^T(t)\ \mathrm{dt}\right] . \Gamma^{-1}. \tag{15}$$

Numerical solution for solving problems (1) and (2) using EWM:

We will convert problems (1) and (2) to a set of algebraic equations by implementing the EWM.

We will approximate the following functions with the aid of Eqs (10)–(12) as follows:

Let

$$D^{\upsilon}\,\Omega(t) = \mathrm{H}_1^T\,\Psi(t)\,,\ 0 < \upsilon < 1, \tag{16}$$

such that

$$\Omega(0) = U^T\Psi(t), \tag{17}$$

$$W(\mathrm{t}\,,\Omega(t)) = \Psi^T(t)\ \hat{\mathrm{F}}. \tag{18}$$

Integrating Eq (16) and using Eq (14), we obtain

$$\Omega(t) = U^T\Psi(t) + H_1^T\ \mathrm{F}^{\upsilon}\,\Psi(t) = (U^T + H_1^T\ \mathrm{F}^{\upsilon})\,\Psi(t) = \Psi^T(t)\ \cdot H_2, \tag{19}$$

where

$$H_2 = \mathrm{U} + (F^{\upsilon})^T H_1.$$

The integral part of problems (1) and (2) can be defined as

$$\int_0^t \Xi\,(\mathrm{t, r}, \Omega(r))\ \mathrm{dr} = \int_0^1 \Psi^T(t)\ \hat{\mathrm{F}}\Psi(r)\ \Psi^T(r)\ \mathrm{H}_2\ \mathrm{dr}$$

$$= \Psi^T(t)\ \hat{\mathrm{F}}H_2 \cdot \int_0^1 \Psi(r)\ \Psi^T(r)\ \mathrm{dr}$$

$$= \Psi^T(t)\ \hat{\mathrm{F}}\,\Gamma\,\mathrm{H}_2$$

$$= \Psi^T(t)\ \mathrm{H}_3, \tag{20}$$

which implies that

$$H_2 = U + \hat{F} + \lambda(t)\,H_3. \tag{21}$$

Equation (21) is a linear system of $m = 2^{\alpha-1}N$ algebraic equations. By plugging the value of H_2 into Eq (19), the approximate solution of problems (1) and (2) can be computed numerically.

4.3. Proposed technique for solving problems (1) and (2)

4.2.1 Rationalized Haar Wavelet Method (RHW)

RHW is considered to be one of the essential categories among the various kinds of wavelets [22,23]. We can enlarge any function $u(x)$ as defined by Definition 4 as:

$$u(x) \simeq \sum_{k=0}^{\infty} e_k \, \mathrm{h}_k(x), \tag{22}$$

where

$$e_k = 2^i \int_0^1 u(z)\, h_k(z)\, dz = 2^i < u, h_k > h_r.$$

For $i = 1, 2, ..., \mathrm{r}$, the level of wavelet is2^i,$r$is a translation parameter.

Equation (22) could be expressed as

$$u(x) = \sum_{k=0}^{n-1} e_k \, \mathrm{h}_k(x) = \mathrm{e}^T \, \mathrm{h}(x), \tag{23}$$

where

$$e^T = [e_0, \mathrm{e}_1, ..., \mathrm{e}_{\text{n-1}}], \text{ and} h(x) = [h_0(x), h_1(x), ..., h_{n+1}(x)].$$

Also, any function $v(\mathrm{x}, \mathrm{y})$ of two variables in a complex space can be similarly approximated by RH functions as

$$v(\mathrm{x,y}) = \sum_{k=0}^{n-1} \sum_{\ell=0}^{n-1} e_{k\ell} \, \mathrm{h}_{k\ell}(\mathrm{x,y}) = \tilde{\mathrm{e}}^T \, \mathrm{h(x,y)}, \tag{24}$$

where

$$\tilde{e}^T = \left[e_{00}, \mathrm{e}_{01}, ..., \mathrm{e}_{\text{n-1,n-1}}\right]^T,$$

$$h(x,y) = \left[h_{00}, h_{01}, ..., h_{n+1,n-1}\right]^T_{(n-1)\times(n-1)} (x,y),$$

where

$$h_{k\ell}(x,y) = h_k(x)\, h_\ell(y), \tag{25}$$

the coefficients $e_{k\ell}$ are given by:

$$e_{k\ell} = \frac{\langle v(x,y), \mathrm{h}_{k\ell}(x,y)\rangle}{\|h_{k\ell}(x,y)\|_\infty^2}. \tag{26}$$

Consider $\Omega_n(t)$ be a sequence of functions derived iteratively from problems (1) and (2) as

$$CD^v\, \Omega_n(t) = \Omega_0 + \beta(t) \int_0^T \Omega_{n-1}(t)\ \mathrm{dr} + \mathrm{W}(\mathrm{t}, \Omega_{n-1}(t)) + \lambda(t) \int_0^t \Xi(\mathrm{t}, \Omega_{n-1}(r))\ dr.$$

Assume

$$F_{n-1}(t) = \mathrm{W}(\mathrm{t}, \Omega_{n-1}(t)),$$

and

$$G_{n-1}(\mathrm{t}, \mathrm{r}) = \Xi(\mathrm{t}, , \mathrm{r}, \Omega(r)).$$

Then, we have

$$CD^{v}\,\Omega_{n-1}(t) = \Omega_0 + \beta(t)\int_0^T \Omega_{n-1}(r)\ \mathrm{dr} + \mathrm{F}_{n\text{-}1}(t) + \lambda(t)\int_0^t G_{n-1}(\mathrm{t}, \mathrm{t})\ \mathrm{dr}. \tag{27}$$

Consider Q_n be the orthogonal projection with the following property (see [18]),

$$\int_0^t Q_n(\Omega_{\text{n-1}}(t))\ \mathrm{dr} = \sum_{i=1}^{n-1}\ \sum_{k=0}^{n-1} e_{ik}\,\mathrm{h}_k(t), \tag{28}$$

where

$$e_{ik} = \frac{\langle \Omega_{i-1}(t), h_k(t)\rangle}{\|h_k(t)\|_\infty^2},$$

$$\int_0^t Q_n(\Omega_{\text{n-1}}(\mathrm{t,r}))\ \mathrm{dr} = \sum_{i=1}^{n-1}\ \sum_{k=0}^{n-1}\ \sum_{\ell=0}^{n-1} e_{ik\ell}\,\mathrm{h}_{k\ell}(\mathrm{t}, \mathrm{r}), \tag{29}$$

where

$$e_{ik\ell} = \frac{\langle \Xi(t,r,\Omega_{n-1}(r)), h_{k\ell}(t,r)\rangle}{\|h_{k\ell}(t,r)\|_\infty^2}.$$

Equation (27), with the assistance of Eqs (28) and (29), will be:

$$CD^{v}\,\Omega_n(t) = \Omega_0 + \beta(t)\sum_{i=1}^{n-1}\ \sum_{k=0}^{n-1} e_{ik}\,\mathrm{h}_k(t) + \sum_{k=0}^{n-1} e_k\,\mathrm{h}_k(t)$$

$$+\lambda(t)\sum_{i=1}^{n-1}\ \sum_{k=0}^{n-1}\ \sum_{k\ell=0}^{n-1} e_{ik\ell}\,\mathrm{h}_{k\ell}(\mathrm{t}, \mathrm{r}). \tag{30}$$

4.2.2 Euler polynomial approximation

The fractional derivative part of problems (1) and (2) can be approximated by using the Euler polynomial approximation as follows:

Lemma 1. Consider the fractional derivative of a complex function $\Omega(t) \in \mathbb{C}[0,1]$ with respect to Euler polynomials as:

$$D_n^{v}\,\Omega(t) = \frac{1}{2}\left[\sum_{\ell=0}^{n}\binom{n}{\ell}(1)_{n-1}\,\mathrm{E}_\ell(a(t), \mathrm{b}(t)) + \mathrm{E}_n(b(t))\right], \tag{31}$$

where

$$a(t) = \mathrm{Re}(\Omega(t)), \text{and} b(t) = \mathrm{Im}(\Omega(t)).$$

then, we have

$$\Omega\ (t) = \sum_{\ell=0}^{n} \ \sum_{k=0}^{n} \ \sum_{q=0}^{k} \binom{n}{\ell} \binom{\ell}{k} (1)_{n-1} (a\ (t))_q \ \mathrm{S}^2\ (\mathrm{k, q})\ .\ \omega_{\ell-k,v} + \omega_{n,v}(t),$$

where

$$\omega_{n,v}\ (t) \ = \frac{1}{2\ \Gamma(v)} \int_0^t (t - \mu)^{v\text{-}1} \ \mathrm{E}_n\ (b\ (\mu))\ \mathrm{d}\mu\ .$$

Proof. As per Definition 1 and Proposition 1, applying I_t^v to both sides of formula (31) results in:

$$\Omega_n\ (t) - \sum_{k=0}^{n-1} \frac{\Omega(0)}{k!}\ \mathrm{t}^k = \frac{1}{2}\ \mathrm{I}_t^v \left[\sum_{\ell=0}^{n} \binom{n}{\ell} (1)_{n-1}\ \mathrm{E}_\ell\ (a\ (t), \mathrm{b}(t)) + E_n\ (b\ (t))\right].$$

According to Definition 3, we obtain

$$\Omega_n(t) - \sum_{k=0}^{n-1} \frac{\Omega\ (0)}{k!} t^k = \frac{1}{2} I_t^v \left[\sum_{\ell=0}^{n} \sum_{k=0}^{\ell} \ \sum_{q=0}^{k} \binom{n}{\ell}\binom{\ell}{k} (1)_{n-1}(a)_q S^2(\mathrm{k, q}) E_{\ell-k}(b(t)) + \mathrm{E}_n(b(t)) \right],$$

$$= \frac{1}{2}\left[\frac{1}{\Gamma(v)} \sum_{\ell=0}^{n} \sum_{k=0}^{\ell} \ \sum_{q=0}^{k} \binom{n}{\ell}\binom{\ell}{k} (1)_{n-1}(a)_q S^2(\mathrm{k, q}) \int_0^t (t-\mu)^{v-1}\ \mathrm{E}_{\ell-k}(b(\mu))\ \mathrm{d}\mu \right.$$

$$\left. + \frac{1}{\Gamma(v)} \int_0^t (t - \mu)^{v-1}\ \mathrm{E}_n\ (b\ (\mu))\ \mathrm{d}\mu\right].$$

Set

$$\omega_{n,v}\ (t) = \frac{1}{2\ \Gamma(v)} \int_0^t (t - \mu)^{v-1}\ \mathrm{E}_n\ (b\ (\mu))\ \mathrm{d}\mu,$$

and

$$\omega_{\ell-k,v}\ (t) = \frac{1}{2\ \Gamma(v)} \int_0^t (t - \mu)^{v-1}\ \mathrm{E}_{\ell\text{-k}}\ (b\ (\mu))\ \mathrm{d}\mu.$$

Therefore, as per Lemma 1, Eq (30) can be expressed as

$$\Omega_n(t) = \Omega_0 + \sum_{\ell=0}^{n} \ \sum_{k=0}^{n} \sum_{q=0}^{k} \binom{n}{\ell}\binom{\ell}{k} (1)_{n-1}\big(a\ (t)\big)_q S^2(\mathrm{k, q})\ .\ \omega_{\ell-k,v}\ (t) + \omega_{n,v}\ (t)$$

$$+ \beta\ (t) \sum_{i=1}^{n-1} \ \sum_{k=0}^{n-1} \mathrm{e}_{ik}\ \mathrm{h}_k\ (t) + \sum_{k=0}^{n-1} \mathrm{e}_k\ \mathrm{h}_k\ (t)$$

$$+ \lambda\ (t) \sum_{i=1}^{n-1} \ \sum_{k=0}^{n-1} \sum_{\ell=0}^{n-1} \ e_{ik\ell}\ \mathrm{h}_{k\ell}\ (\mathrm{t, r}). \tag{32}$$

Lemma 2. The suggested technique has a convergence rate of order $O(M^2(2d)^M)$.

Proof. Using Lemma 1, assumption (C3), and putting $d \leq \frac{1}{2}$, we get the proof. (See [8]).

5. Numerical experiments

This section will numerically conduct the proposed method for solving problems (1) and (2) to validate the theoretical work. To validate the approach and assess its effectiveness, we pose two problems that satisfy the assumptions (C1)–(C3).

Problem 1. Consider the following: NFVIDeq

$$D^{0.8}\,\Omega\,(t) = \frac{1 + e^{\Omega(t)}}{3} + t^2 \int_0^t e^{-\pi(r^2+t)}\;\Omega\,(r)\,dr\,,$$

with the initial condition $\Omega\,(0) = 2$.

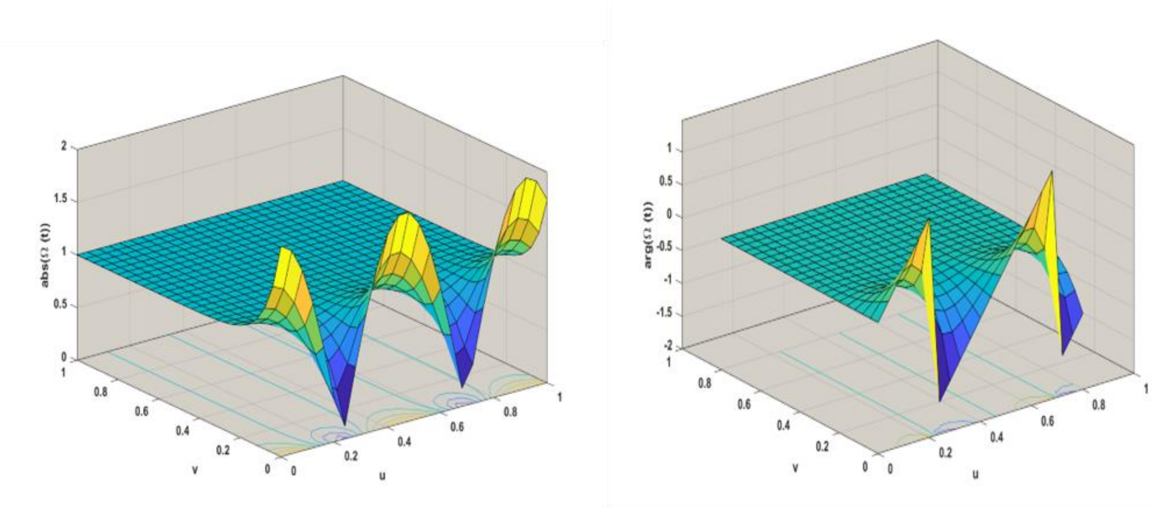
Here we have $\upsilon = 0.8$, $\lambda\,(t) = t^2$, $\Xi\,(t, r, \Omega\,(r)) = e^{-\pi(r^2+t)}\,\Omega\,(r)$, $\beta(t) = 0$, and the known function $W\,(t, \Omega\,(t)) = \frac{1 + e^{\Omega(t)}}{3}$ is employed to guarantee that the exact solution will be $\Omega\,(t) = e^{\sqrt{3}\Pi i t} + 1$. Also, we have that $P = 0.6$, $q = 0.87$.

So, we have $MT + \frac{P\,T^{\nu}}{\Gamma\,(\nu+1)} + \frac{M\,q\,T^{\nu+1}}{\Gamma\,(\nu+2)} = 0.864 < 1$. According to Theorem 2, we conclude that Problem 1, has a unique solution.

Using EWM and our proposed method, we have assessed and presented the estimated solutions in Table 1 with $t \in [0\,,1]$, as $t = 0.1 : 0.1 : 1$, by selecting two distinct $\tilde{n}$ values as $\tilde{n} = 10$, and $\tilde{n} = 20$. Figure 1 shows the numerical solutions using the suggested technique for $\Omega_n(t)$ together with its magnitude $abs\,(\Omega\,(t))$ and the argument $arg(\Omega\,(t))$ at $\tilde{n} = 20$.

Table 1. Shows the absolute errors in problem 1's exact and approximate values by using EWM and the proposed method at $\tilde{n} = 10$ and $\tilde{n} = 20$.

t_i	$\tilde{n} = 10$		$\tilde{n} = 20$	
	EWM	The proposed method	EWM	The proposed method
0.1	2.45×10^{-10}	2.22×10^{-10}	1.04×10^{-11}	4.04×10^{-11}
0.2	8.45×10^{-12}	4.32×10^{-12}	1.28×10^{-10}	2.32×10^{-12}
0.3	1.36×10^{-11}	1.54×10^{-11}	5.38×10^{-12}	1.36×10^{-13}
0.4	8.53×10^{-10}	6.04×10^{-10}	3.19×10^{-11}	5.46×10^{-12}
0.5	1.17×10^{-9}	7.25×10^{-10}	8.35×10^{-12}	8.93×10^{-12}
0.6	2.86×10^{-12}	1.39×10^{-12}	5.18×10^{-11}	3.47×10^{-11}
0.7	9.28×10^{-13}	8.23×10^{-13}	4.37×10^{-10}	6.45×10^{-13}
0.8	1.84×10^{-13}	3.39×10^{-13}	4.03×10^{-12}	7.02×10^{-13}
0.9	3.53×10^{-12}	2.28×10^{-13}	2.38×10^{-13}	1.39×10^{-15}
1	1.09×10^{-13}	5.04×10^{-13}	1.22×10^{-13}	7.43×10^{-15}



(I) **(II)**

Figure 1. Approximate solutions for Problem 1 by utilizing the magnitude of the solution in (I) and the argument of the solution in (II) through the proposed method atñ = 20.

Problem 2. Consider the following NFVIDeq:

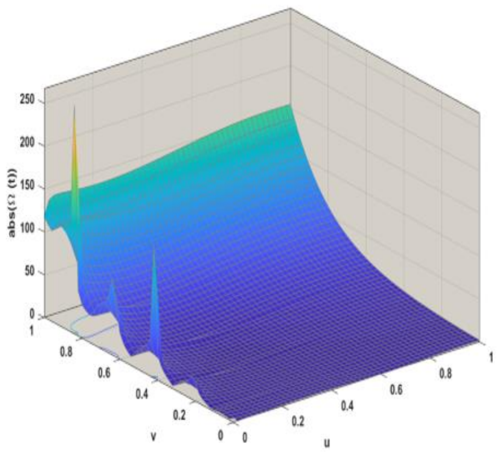
$$D^{0.2}\,\Omega\,(t) = \frac{t}{\sqrt{1+\Omega^2\,(t)}} + \int_0^t \mathrm{Cos}\ t^2\ e^{\Omega\,(r)}\ \ dr\,,$$

with the initial condition $\Omega\,(0) = 0$.

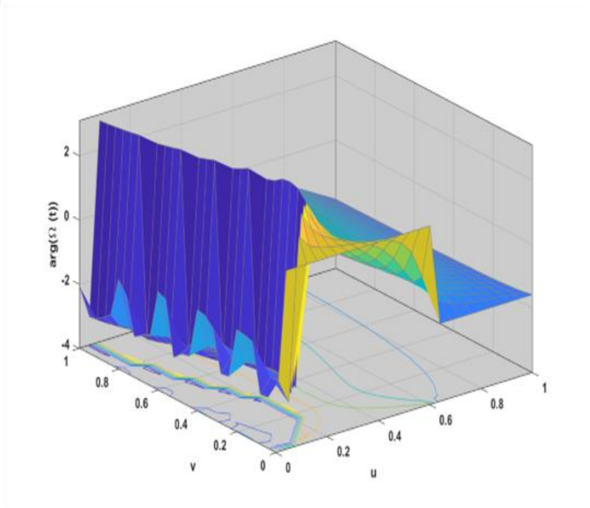
The known function $W\,(t, \Omega\,(t))$ used to ensure the exact solution given by $\Omega(t) = \frac{2\,i\,sin\,t - 3\,t^2}{\sqrt{1+t^3+tan\,t}}$, as we shown in Problem 1, the condition $MT + \frac{P\,T^{\nu}}{\Gamma\,(\nu+1)} + \frac{M\,q\,T^{\nu+1}}{\Gamma\,(\nu+2)} = 0.893<1$. So, this problem has a unique solution.

Table 2. Shows the absolute errors in problem 2's exact and approximate values by using EWM and the proposed method at $\tilde{n} = 10$ and $\tilde{n} = 20$.

t_i	$\tilde{n} = 20$		$\tilde{n} = 30$	
	EWM	The proposed method	EWM	The proposed method
0.1	8.37×10^{-12}	9.17×10^{-12}	3.11×10^{-11}	5.14×10^{-12}
0.2	1.28×10^{-12}	2.06×10^{-12}	3.53×10^{-12}	3.42×10^{-13}
0.3	4.02×10^{-11}	1.45×10^{-11}	6.73×10^{-13}	3.84×10^{-15}
0.4	1.46×10^{-11}	6.18×10^{-11}	3.02×10^{-11}	4.01×10^{-12}
0.5	7.83×10^{-13}	7.03×10^{-12}	7.94×10^{-13}	7.12×10^{-12}
0.6	6.25×10^{-15}	3.94×10^{-15}	2.89×10^{-12}	9.91×10^{-12}
0.7	2.05×10^{-13}	9.47×10^{-13}	7.37×10^{-13}	1.74×10^{-17}
0.8	8.47×10^{-13}	2.41×10^{-15}	8.28×10^{-13}	6.47×10^{-17}
0.9	1.19×10^{-14}	6.18×10^{-17}	1.45×10^{-19}	3.11×10^{-19}
1	3.38×10^{-17}	8.03×10^{-17}	8.36×10^{-18}	8.32×10^{-19}



(I)



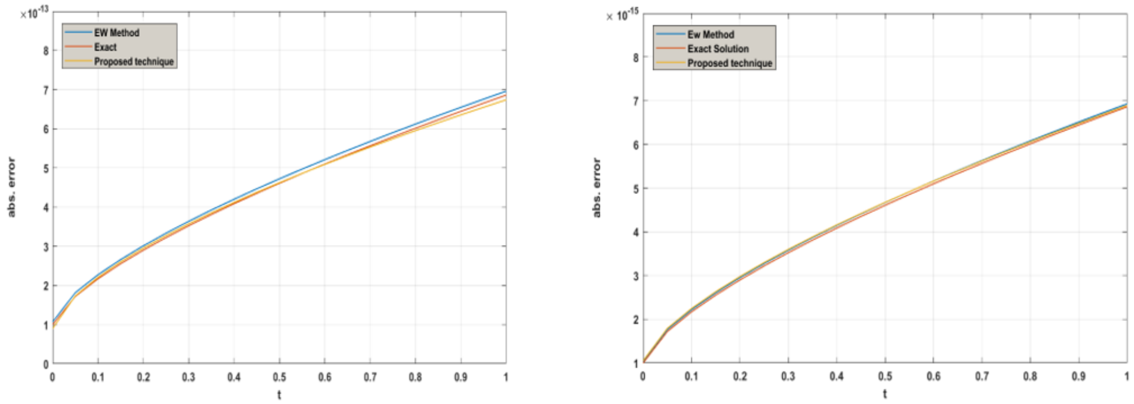
(II)

Figure 2. Approximate solutions for Problem 2 by utilizing the magnitude of the solution in (I) and the argument of the solution in (II) through the proposed method at$\tilde{n}$ = 30.

Table 2 shows the approximate solutions for problem 2 at t = 0.1 : 0.1 : 1 , and by taking $\tilde{n} = 20$, $\tilde{n} = 30$. Figure 2 shows the numerical solution by the proposed technique at $\tilde{n} = 30$ by the aid of $u = \text{Re}\ (\Omega\ (t)), v = \text{Im}\ (\Omega\ (t))$, the magnitude, and the argument of the solution.

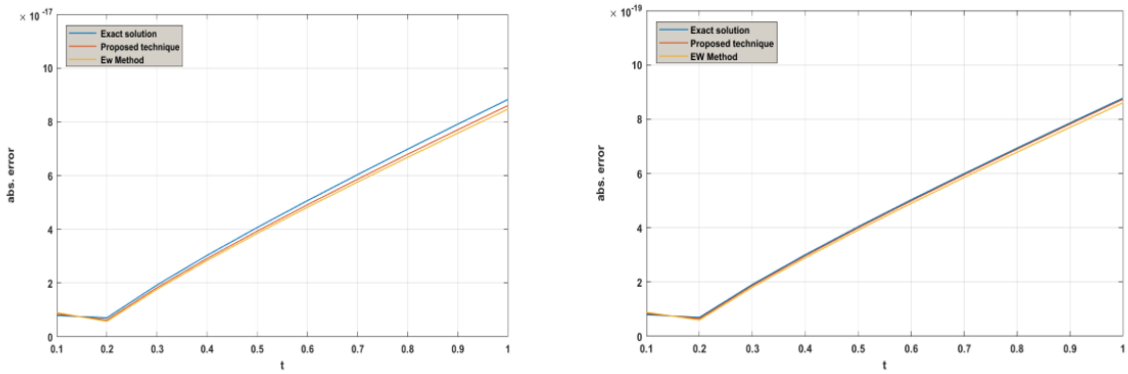
6. Conclusions

In general, the FIDEq solution is hard to study, especially if the unknown function is complex. We present the existence and uniqueness results of the solution for NFVIDEq in the given problem by applying the fixed-point theorem of Banach space with the contraction mapping principle and some properties of fractional calculus. In addition, we analyze the approximate solutions for solving NFVIDEq in problems (1) and (2), utilizing the EWM to a matrix representation that aligns with a system of algebraic linear equations. We demonstrate that this method is both highly efficient and effective. On the other hand, this research employs a novel approach by using Euler's polynomial method to construct the rationalized Haar wavelet method (RHM), which takes the form of convergent series with easily computed terms in the bases of Euler polynomials and Haar wavelet functions. In Section 5, we present two examples of numerical calculations using MATLAB R2022b. These mathematical calculations are the last stage in supporting the theoretical study. The problems supplied show the differences between exact and numerical solutions for various values of n. Furthermore, the absolute errors in every problem are shown in Tables 1 and 2. Figures 3 and 4 substantially converge the precise and numerical solutions. Based on what exists, we can deduce that increasing the value of n results in a longer time to attain $t \to 1$. When compared to the EWM, the proposed method gives more accurate numerical answers. Therefore, we can conclude that the suggested method is very good at finding exact numerical solutions and cuts down on processing time while keeping accuracy high. Also, the suggested approach is particularly effective and significantly reduces the time required for calculations while maintaining precision. This study's findings add to the existing literature on the topic, particularly for applied researchers in the sciences and engineering.



(I) (II)

Figure 3. The disparity that exists in the precise, approximate solutions of EWM and the proposed method of Problem 1 at $\tilde{n} = 10$ and $\tilde{n} = 20$in (I) and (II), respectively.



(I) (II)

Figure 4. The disparity that exists in the precise, approximate solutions of EWM and the proposed method of Problem 2 at $\tilde{n} = 20$ and $\tilde{n} = 30$in (I) and (II), respectively.

As a future work, we can explore the proposed numerical method to fractional integro-differential equations in higher-dimensional complex spaces, which are crucial for modeling multi-variable systems in physics and engineering. Also, we can make a comparative study of the proposed method with other advanced numerical techniques such as finite element methods, boundary element methods, or more recent machine learning-based approaches to determine which techniques offer better accuracy and computational efficiency.

Author Contributions

Amnah E. Alshammaky: Conceptualization, investigation, resources, and supervision; Eslam M. Youssef: methodology, software, formal analysis, writing-original draft preparation, and project administration. All authors have read and agreed to the published version of the manuscript.

Conflicts of interest

The authors declare no conflict of interest.

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