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Research article

On the extraction of complex behavior of generalized higher-order nonlinear Boussinesq dynamical wave equation and (1+1)-dimensional Van der Waals gas system

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Abstract: In this paper, we apply the powerful sine-Gordon expansion method (SGEM), along with a computational program, to construct some new traveling wave soliton solutions for two models, including the higher-order nonlinear Boussinesq dynamical wave equation, which is a well-known nonlinear evolution model in mathematical physics, and the (1+1)-dimensional framework of the Van der Waals gas system. This study presents some new complex traveling wave solutions, as well as logarithmic and complex function properties. The 3D and 2D graphical representations of all obtained solutions, unveiling new properties of the considered model are simulated. Additionally, several simulations, including contour surfaces of the results, are performed, and we discuss their physical implications. A comprehensive conclusion is provided at the end of this paper.

Keywords: the higher-order Boussinesq dynamical equation; the (1+1)-dimensional Van der Waals gas system; SGEM; soliton solution
Mathematics Subject Classification: 35A24, 35A99

1. Introduction

In modern times nonlinear evaluation equations (NLEEs) play an important role for describing natural phenomena arising in real world problems. Chen et al. studied the biochemical safety of drinking water by using NLEEs [1]. An air cushion vehicle in steady motion was numerically investigated by using the boundary integral equation method in [2]. Dental age evaluation in China was studied by Wang et al. in [3]. By using the traditional Demirjian method for dental age estimation, they predicted some important processes. A mathematical analysis of an HIV/AIDS infection model was studied by employing fractional order derivative methods in [4]. A new mathematical model regarding the transmission and control of rat-bite fever was introduced to the literature by using nonlinear partial differential equations (NLPDEs) in [5]. NLPDEs are often used to describe nonlinear complex models that arise in the various domains of applied science, such as chemical physics, quantum mechanics, optical fibers, and fluid mechanics. Accurate solutions are required in nonlinear sciences to validate experimental results. Recent advances have resulted in a variety of approaches for calculating exact solutions to nonlinear evolution equations which describe the propagation of nonlinear dispersive waves in inhomogeneous media. It is critical to find analytical solutions to these equations. Many experts from all over the world have used a variety of analytical and numerical methods to observe these models which have been newly presented to the literature. In this sense, developing and revising these methods is very important. Thus, different efficient methods have been designed by various scholars, such as analytical and numerical methods [6–10], dynamical system methods [11], Darboux transformations [12], the improved F-expansion method with Riccati equation [13, 14], the sine-Gordon expansion method [15–17], the $(\frac{G'}{G}, \frac{1}{G})$-expansion method [18], the $(m + 1/G)$ expansion method [19], a simplified Hirota method [20], the auxiliary equation method [21], the generalized exponential rational function method [22], a newly extended direct algebraic technique [23], the generalized bilinear form [24], the sine pseudo-spectral method [25], the extended trial equation method [26], the differential transform method [27], the newly extended direct algebraic technique [28], the polynomial expansion method [29], the modified Hirota bilinear method [30], the $\tanh(\frac{\phi(\xi)}{2})$ expansion method [31], the hyperbolic trigonometric functions method [32], integration algorithms [33], Bernoulli sub-equation function method [34], the modified exp $(-\phi(\eta))$-expansion function method [35], new sine-Gordon equation expansion algorithms [36], the assumption of equal background height [37], the extended sinh-Gordon equation expansion method [38], the ansatz method [39–43] and so on [44–47].

In this paper, first of all, by using the powerful sine-Gordon expansion method [6,15,48], we extract some new traveling wave solutions for the third extended fifth-order nonlinear equation [49] given as

$$u_{\zeta\zeta\zeta} - u_{\zeta\epsilon\epsilon\epsilon\epsilon} - u_{\zeta\xi\xi} - 4(u_{\epsilon}u_{\epsilon\zeta})_{\epsilon} = 0, \tag{1.1}$$

in which ζ, ϵ, ξ are independent variables, while the function $u = u(\zeta, \epsilon, \xi)$ is a dependent variable. Seadawy et al. presented multiple soliton solutions in [49]. Equation (1.1) is used to model the complex behaviors of nonlinear dispersive waves in shallow water, also symbolizing complex wave phenomena by balancing nonlinear steepening and dispersion. By using the simplified form of Hirota's direct method, Eq (1.1) was observed by Wazwaz [50].

Second, the equation of motion for compressible fluids in the (1+1)-dimensional Van der Waals gas system is studied. This model may be read as [51]

$$\begin{aligned} v_t + p_x(w) = 0, \\ w_t - v_x = 0, \end{aligned} \tag{1.2}$$

where v represents velocity, and p and w denote pressure and specific volume, respectively [52]. Equation (1.2) was developed to represent the longitudinal isothermal motion in elastic bars or fluids arising in real world problems. The term $p_x(w)$ is used to describe a van der Waals-like structure as $p_x(w) = w - w^3$. The particles in Eq (1.2) have rigid cores and communicate with each other via pair potentials. In particular, this system represents the ideal gas state law equation, which takes into consideration the effects of intermolecular interactions and the fact that molecules have a finite size [53]. Furthermore, this system is significant in chemistry and fluid dynamics, particularly for the study of ideal gases in chemistry and one-dimensional longitudinal isothermal motion in fluids. It extends the classical ideal gas law by considering intermolecular forces and the finite size of gas molecules, providing a more accurate representation of real gases. The model offers insights into the dynamic interactions between pressure, fluid motion, and the internal structure of gases, helping to understand complex behaviors such as wave propagation, phase transitions, and shock waves in compressible fluids influenced by both macroscopic and microscopic properties.

The paper is organized as follows: Section 2 outlines the foundational steps of the SGEM. In Section 3, the SGEM is first applied to obtain the some new travelling wave solutions of the third extended fifth-order nonlinear equation. Then, the (1+1)-dimensional framework of the Van der Waals gas system is studied by using the SGEM. The geometric interpretations of the obtained solutions for the relevant parameter values are provided in various simulations. Section 4 presents the results and discussion. Finally, Section 5 concludes the paper with closing remarks.

2. Overview of the SGEM

This section provides an overview of the general properties of the SGEM. Let us consider the sine-Gordon equation [15, 54, 55] given by

$$v_{xx} - v_{tt} = m^2 \sin(v), \tag{2.1}$$

where $v = v(x,t)$ and $m \in \mathbb{R} \setminus \{0\}$.

Performing the wave transformation $v = V(\eta)$, $\eta = \mu(x - ct)$ upon Eq (2.1) yields the nonlinear ordinary differential equation (NODE)

$$V'' = \frac{m^2}{\mu^2(1-c^2)} \sin(V), \tag{2.2}$$

in which $V = V(\eta)$, η is the amplitude, and c is the speed of the traveling wave. Integrating Eq (2.2), we get

$$\left[\left(\frac{V}{2}\right)'\right]^2 = \frac{m^2}{\mu^2(1-c^2)} \sin^2\left(\frac{V}{2}\right) + K, \tag{2.3}$$

where K is the constant of integration. Setting $K = 0$, $\omega(\eta) = \frac{V}{2}$, and $a^2 = \frac{m^2}{\mu^2(1-c^2)}$ in Eq (2.3) gives

$$\omega' = a \sin(\omega). \tag{2.4}$$

Substituting $a = 1$ in Eq (2.4), we get

$$\omega' = \sin(\omega). \tag{2.5}$$

After simplifying Eq (2.5), we have the following two equations:

$$\sin(\omega) = \sin(w(\eta)) = \left.\frac{2pe^{\eta}}{1 + p^2 e^{2\eta}}\right|_{p=1} = sech(\eta), \tag{2.6}$$

$$\cos(\omega) = \cos(w(\eta)) = \left.\frac{p^2e^{2\eta} - 1}{p^2e^{2\eta} + 1}\right|_{p=1} = \tanh(\eta), \tag{2.7}$$

where p is the constant of integration.

Considering the following properties, we can consider the nonlinear partial differential equations (PDEs)

$$P(u, u_x, u_t, u_{xx}, u_{tt}, u_{xt}, \cdots) = 0. \tag{2.8}$$

If we put the wave transform $u = U(\eta)$, $\eta = \mu(x - ct)$ into Eq (2.8), we get the NODE

$$NODE(U, U', U'', \cdots) = 0,$$

where $U = U(\eta), U' = \frac{dU}{d\eta}, U'' = \frac{d^2U}{d\eta^2}, \cdots$. According to the general properties of the SGEM, in this NODE, we select the trial solution formula given by

$$U(\eta) = \sum_{i=1}^{n} \tanh^{i-1}(\eta)[B_i \operatorname{sech}(\eta) + A_i \tanh(\eta)] + A_0. \tag{2.9}$$

Taking Eqs (2.6) and (2.7) into (2.9), we may rewrite it as

$$U(\omega) = \sum_{i=1}^{n} \cos^{i-1}(\omega)[B_i \sin(\omega) + A_i \cos(\omega)] + A_0. \tag{2.10}$$

We determine the value of n by using the balance principle according to the terms of the NODE (obtained from Eq (2.8)). We derive a set of algebraic equations by summing the coefficients of $\sin^i(\omega)\cos^j(\omega)$ with the same power and setting each sum to zero. We use Wolfram Mathematica 14 to simplify the set of algebraic equations, allowing us to determine the values of the coefficients A_i, B_i, μ, and c. Finally, by substituting the values of A_i, B_i, μ, and c into Eq (2.9), we obtain the new traveling wave solutions to Eq (2.8).

3. Applications

In this part of the paper, we apply SGEM to obtain some new solutions of the third extended fifth-order nonlinear model and the (1+1)-dimensional Van der Waals gas system.

3.1. The third extended fifth-order nonlinear model

This section applies the SGEM to find new solutions to the strain wave equation given in Eq (1.1). Using the wave transformation

$$u(\xi, \epsilon, \zeta) = W(\eta), \eta = k(\xi + l\epsilon - \omega\zeta)$$

yields the following NODE:

$$-\omega^3 W''' + k^2\omega l^4 W^{(5)} + \omega W''' + 4\omega k l^3 (W'W'')' = 0. \tag{3.1}$$

If we assume the integral constant is zero and integrate Eq (3.1) twice, we get

$$k^2 l^4 W''' + 2kl^3 (W')^2 + (1 - \omega^2)W' = 0. \tag{3.2}$$

For simplicity, we take

$$W' = V,$$

and then Eq (3.2) can be written as

$$k^2 l^4 V'' + 2kl^3 V^2 + (1 - \omega^2)V = 0. \tag{3.3}$$

Applying the principle of homogeneous balancing, we get $n = 2$ from Eq (3.3).

Using Eq (2.10) with $n = 2$ yields

$$V(\omega) = B_1 \sin(\omega) + A_1 \cos(\omega) + B_2 \cos(\omega)\sin(\omega) + A_2 \cos^2(\omega) + A_0, \tag{3.4}$$

and differentiating Eq (3.4) twice, we get

$$\begin{aligned} V''(\omega) =& B_1 \cos^2(\omega)\sin(\omega) - B_1 \sin^3(\omega) - 2A \sin^2(\omega)\cos(\omega) + B_2 \cos^3(\omega)\sin(\omega) \\ & - 5B_2 \sin^3(\omega)\cos(\omega) - 4A_2 \cos^2(\omega)\sin^2(\omega) + 2A_2 \sin^4(\omega). \end{aligned} \tag{3.5}$$

Equations (3.4) and (3.5) are inserted into Eq (3.3), resulting in an equation that involves some trigonometric functions, $\sin^i(\omega)\cos^i(\omega)$. So, we get a set of algebraic equations by setting all the trigonometric identities with the same power of coefficients equal to zero. The coefficient values are obtained by solving the set of generated equations. With the help of Wolfram Mathematica, we simplify the extracted set of equations in order to discover new solutions to Eq (1.1). We then insert the values of the coefficients that we have found into Eq (2.9) by taking into consideration the value of $n = 2$.

Case 1. When $A_0 = \frac{3lk}{2}$, $A_1 = 0$, $A_2 = -\frac{3lk}{2}$, $B_1 = 0$, $B_2 = \frac{3ilk}{2}$, and $\omega = \sqrt{1 + l^4k^2}$, we obtain the hyperbolic solution to the Eq (1.1), given as

$$u_1(\xi, \epsilon, \zeta) = -\frac{3}{2} ilk \operatorname{sech}[k(l\epsilon - \zeta\sqrt{1 + l^4k^2} + \xi)] + \frac{3}{2} lk \tanh[k(l\epsilon - \zeta\sqrt{1 + l^4k^2} + \xi)], \tag{3.6}$$

where $l \neq 0, k \neq 0$, and $\zeta \neq 0$ are arbitrary real constants.

Case 2. When $A_0 = -iB_2$, $A_1 = 0$, $A_2 = iB_2$, $B_1 = 0$, $k = -\frac{2iB_2}{3l}$, and $\omega = -\frac{1}{3}\sqrt{9 - 4l^2B_2^2}$, we obtain the following trigonometric solution to Eq (1.1) as

$$u_2(\xi, \epsilon, \zeta) = -\sec[\frac{2B_2(l\epsilon + \xi + \frac{1}{3}\zeta\sqrt{9 - 4l^2B_2^2})}{3l}]B_2 - B_2 \tan[\frac{2B_2(l\epsilon + \xi + \frac{1}{3}\zeta\sqrt{9 - 4l^2B_2^2})}{3l}], \tag{3.7}$$

where $l \neq 0, B_2 \neq 0$ are real constants.

Case 3. When $A_0 = \frac{2iB_2}{3}$, $A_1 = 0$, $A_2 = -iB_2$, $B_1 = 0$, $k = \frac{2iB_2}{3l}$, and $\omega = \frac{1}{3}\sqrt{9 + 4l^2B_2^2}$, we get the trigonometric wave solution to the Eq (1.1) as

$$\begin{aligned} u_3(\xi, \epsilon, \zeta) =& -\sec[\frac{2B_2(l\epsilon + \xi - \frac{1}{3}\zeta\sqrt{9 + 4l^2B_2^2})}{3l}]B_2 + \frac{2B_2^2(l\epsilon + \xi + \frac{1}{3}\zeta\sqrt{9 - 4l^2B_2^2})}{9l} \\ & - B_2 \tan[\frac{2B_2(l\epsilon + \xi + \frac{1}{3}\zeta\sqrt{9 - 4l^2B_2^2})}{3l}], \end{aligned} \tag{3.8}$$

where $B_2 \neq 0, l \neq 0$ as strain conditions for the validity of Eq (3.8).

Case 4. When $A_0 = -A_2$, $A_1 = 0$, $B_1 = 0$, $B_2 = iA_2$, $k = -\frac{2A_2}{3l}$, and $\omega = \frac{1}{3}\sqrt{9 + 4l^2A_2^2}$, we get the hyperbolic function solution to Eq (1.1) as

$$u_4(\xi, \epsilon, \zeta) = -i\,\text{sech}[\frac{2A_2(l\epsilon + \xi - \frac{1}{3}\sqrt{9 + 4l^2A_2^2})}{3l}]A_2 + A_2\tanh[\frac{2A_2(l\epsilon + \xi - \frac{1}{3}\sqrt{9 + 4l^2A_2^2})}{3l}], \quad (3.9)$$

where $l \neq 0$, $A_2 \neq 0$ are arbitrary real non-zero constants.

Case 5. When $A_0 = -\frac{2A_2}{3}$, $A_1 = 0$, $B_1 = 0$, $B_2 = -iA_2$, $k = -\frac{2A_2}{3l}$, and $\omega = \frac{1}{3}\sqrt{9 - 4l^2A_2^2}$, we obtain the hyperbolic function solution to Eq (1.1) as

$$\begin{aligned} u_5(\xi, \epsilon, \zeta) = \frac{2}{3}i\,\text{sech}[&\frac{2A_2\Big(l\epsilon + \xi - \frac{1}{3}\zeta\sqrt{9 - 4l^2A_2^2}\Big)}{3l}]A_2+ \\ &\frac{2}{3}A_2\tanh[\frac{2A_2\Big(l\epsilon + \xi - \frac{1}{3}\zeta\sqrt{9 - 4l^2A_2^2}\Big)}{3l}], \end{aligned} \quad (3.10)$$

where $l \neq 0$, $A_2 \neq 0$ are real constants.

3.2. The (1+1)-dimensional framework of the Van der Waals gas system

In this part of the paper, we study the (1+1)-dimensional framework of the Van der Waals gas system given by Eq (1.2). The viscosity-capillarity regularization for the equation is rewritten as (1.2)

$$\begin{aligned} v_t + p(w)_x &= \mu v_{xx} - \omega\mu^2 w_{xxx}, \\ v_t - w_x &= 0. \end{aligned} \quad (3.11)$$

The structure of the function $p(w)$ is similar to that of van der Waals, as described in [52].

$$p(w) = w - w^3. \quad (3.12)$$

To tackle the system of Eq (3.11), the transformation waves are selected as follows:

$$\begin{aligned} v(x, t) &= V(\xi), \\ w(x, t) &= W(\xi), \\ \xi &= \kappa x - \kappa ct, \end{aligned} \quad (3.13)$$

where κ and c are the wave width and speed, respectively. Using the wave transformation (3.13), we get the following NODE:

$$\begin{aligned} -cV' + (W - W^3)' - \mu\kappa V'' + \omega\mu^2\kappa^2 W''' &= 0, \\ cW' + V' &= 0. \end{aligned} \quad (3.14)$$

Integrating Eq (3.14) yields

$$\begin{aligned} -cV' + \kappa^2\mu^2\omega W'' - \kappa\mu V' - W^3 + W &= 0, \\ cW + V &= 0. \end{aligned} \quad (3.15)$$

The system is reduced to the following nonlinear ordinary differential equation by combining it with $V = -cW$:

$$(c^2+1)W + c\mu\kappa^2 W' + \omega\mu^2\kappa^2 W'' - W^3 = 0. \tag{3.16}$$

Equation (3.16) has a well-known Duffing oscillator with periodic solution, as illustrated in [56]. Using Eq (2.10) with $n = 1$ yields the trial solution formula

$$W(w) = B_1 \sin(w) + A_1 \cos(w) + A_0. \tag{3.17}$$

Equation (3.17) and its necessary derivations are inserted into Eq (3.16), resulting in an equation that involves some trigonometric functions, $\sin^i(w)\cos^i(w)$. So, we get a set of algebraic equations by setting all the trigonometric identities with the same power of coefficients equal to zero. With the help of Wolfram Mathematica, we obtain the values of coefficients which result in the following solutions to the Eq (1.2).

Case 1. After selecting $\kappa = -1$, $A_0 = \frac{1}{\sqrt{3}}$, $A_1 = \frac{1}{\sqrt{3}}$, $B_1 = -\frac{i}{\sqrt{3}}$, $\mu = 2\sqrt{3}$, $c = \frac{1}{\sqrt{3}}$, and $\omega = \frac{1}{18}$, we obtain the complex rational function solution to Eq (3.11) as

$$\begin{aligned} v_1(x,t) &= \frac{1}{\sqrt{3}\left(\frac{1}{2} + \frac{1}{2} i e^{-\frac{t}{\sqrt{3}}+x}\right)}, \\ w_1(x,t) &= -\frac{1}{\sqrt{3}} v_1(x,t). \end{aligned} \tag{3.18}$$

Some important simulations of Eq (3.18) are showed in Section 4.

Case 2. With $\kappa = -1$, $A_0 = -\frac{1}{\sqrt{3}}$, $A_1 = -\frac{1}{\sqrt{3}}$, $B_1 = \frac{i}{\sqrt{3}}$, $\mu = 2\sqrt{3}$, $c = \frac{1}{\sqrt{3}}$, and $\omega = \frac{1}{18}$, we extract another complex rational function solution to Eq (3.11) as

$$\begin{aligned} v_2(x,t) &= \frac{1}{\sqrt{3}\left(-\frac{1}{2} - \frac{1}{2} i e^{-\frac{t}{\sqrt{3}}+x}\right)}, \\ w_2(x,t) &= -\frac{1}{\sqrt{3}} v_2(x,t). \end{aligned} \tag{3.19}$$

Some important simulations of Eq (3.19) are showed in Section 4.

Case 3. With $\kappa = -1$, $A_0 = 1$, $A_1 = 1$, $B_1 = i$, $\mu = 2\sqrt{3}$, $c = \sqrt{3}$, and $\omega = \frac{1}{6}$, we find another complex rational function solution to Eq (3.11) as

$$\begin{aligned} v_3(x,t) &= \frac{2}{1 - i e^{-\sqrt{3}t+x}}, \\ w_3(x,t) &= -\frac{1}{\sqrt{3}} v_3(x,t). \end{aligned} \tag{3.20}$$

Various important figures of Eq (3.20) are showed in Section 4.

Case 4. With $\kappa = -1$, $A_0 = \frac{1}{3}\sqrt{2\mu^2 + \sqrt{\mu^2(-9+4\mu^2)}}$, $A_1 = \frac{1}{3}\sqrt{2\mu^2 + \sqrt{\mu^2(-9+4\mu^2)}}$, $B_1 = 0$, $c = \frac{2\mu^2 + \sqrt{\mu^2(-9+4\mu^2)}}{3\mu}$, and $\omega = \frac{1}{4\mu^2 - 2\sqrt{\mu^2(-9+4\mu^2)}}$, we obtain the hyperbolic traveling wave solutions of Eq (3.11) as

$$
\begin{aligned}
v_4(x,t) &= -\frac{1}{3}\sqrt{2\mu^2+\sqrt{\mu^2(-9+4\mu^2)}}\Big(-1+\tanh\Big[x-\frac{t(2\mu^2+\sqrt{\mu^2(-9+4\mu^2)})}{3\mu}\Big]\Big),\\
w_4(x,t) &= -\frac{1}{\sqrt{3}}v_4(x,t),
\end{aligned}
\tag{3.21}
$$

where $4\mu^2-9>0$ and $\mu\neq 0$ are the strain conditions for the validity of Eq (3.21). Various important figures of Eq (3.21) are showed in Section 4.

Case 5. When we choose $\kappa=-1$, $A_0=-1$ $A_1=-1$, $B_1=-i$, $\mu=2\sqrt{3}$, $c=\sqrt{3}$, and $\omega=\frac{1}{6}$, we obtain another complex solution of Eq (3.11) as

$$
\begin{aligned}
v_5(x,t) &= \frac{1}{-\frac{1}{2}+\frac{1}{2}ie^{-\sqrt{3}t+x}},\\
w_5(x,t) &= -\frac{1}{\sqrt{3}}v_5(x,t).
\end{aligned}
\tag{3.22}
$$

Several important figures of Eq (3.22) are showed in Section 4.

4. Results and discussion

In this paper, the SGEM is employed as a powerful technique to derive new traveling wave solutions for nonlinear partial differential equations. The method is based on the two key properties of the sine-Gordon equation, particularly those expressed in Eqs (2.6) and (2.7). The SGEM utilizes trigonometric functions, which are instrumental in deriving novel solutions as demonstrated in Eq (2.10). The trigonometric properties integral to SGEM facilitate the generation of many new solutions. This characteristic makes the SGEM highly effective, allowing for the incorporation of various coefficients, including complex, exponential, and trigonometric, into the model under consideration.

The results obtained have been visualized through various figures using Mathematica, including 2D, 3D, and contour plots, which provide a comprehensive view of the behavior of the solutions under different parametric conditions. Figures 1–18 illustrate the graphical surfaces of the solutions, highlighting their physical properties and complex structures. These visualizations help in understanding the dynamics and characteristics of the solutions derived using SGEM, offering deeper insights into the nonlinear phenomena modeled by the equations.

It is understood that these solutions have significant physical meanings. For instance, the hyperbolic function solutions are relevant in the study of wave propagation and stability in shallow water, while the complex function solutions can model interactions in nonlinear optics, such as light pulse propagation in optical fibers. The exponential solutions provide insights into the behavior of shock waves and other discontinuities in plasma physics. These physical interpretations highlight the relevance and applicability of the solutions derived using SGEM in various scientific and engineering contexts, such as fluid dynamics, nonlinear optics, and plasma physics.

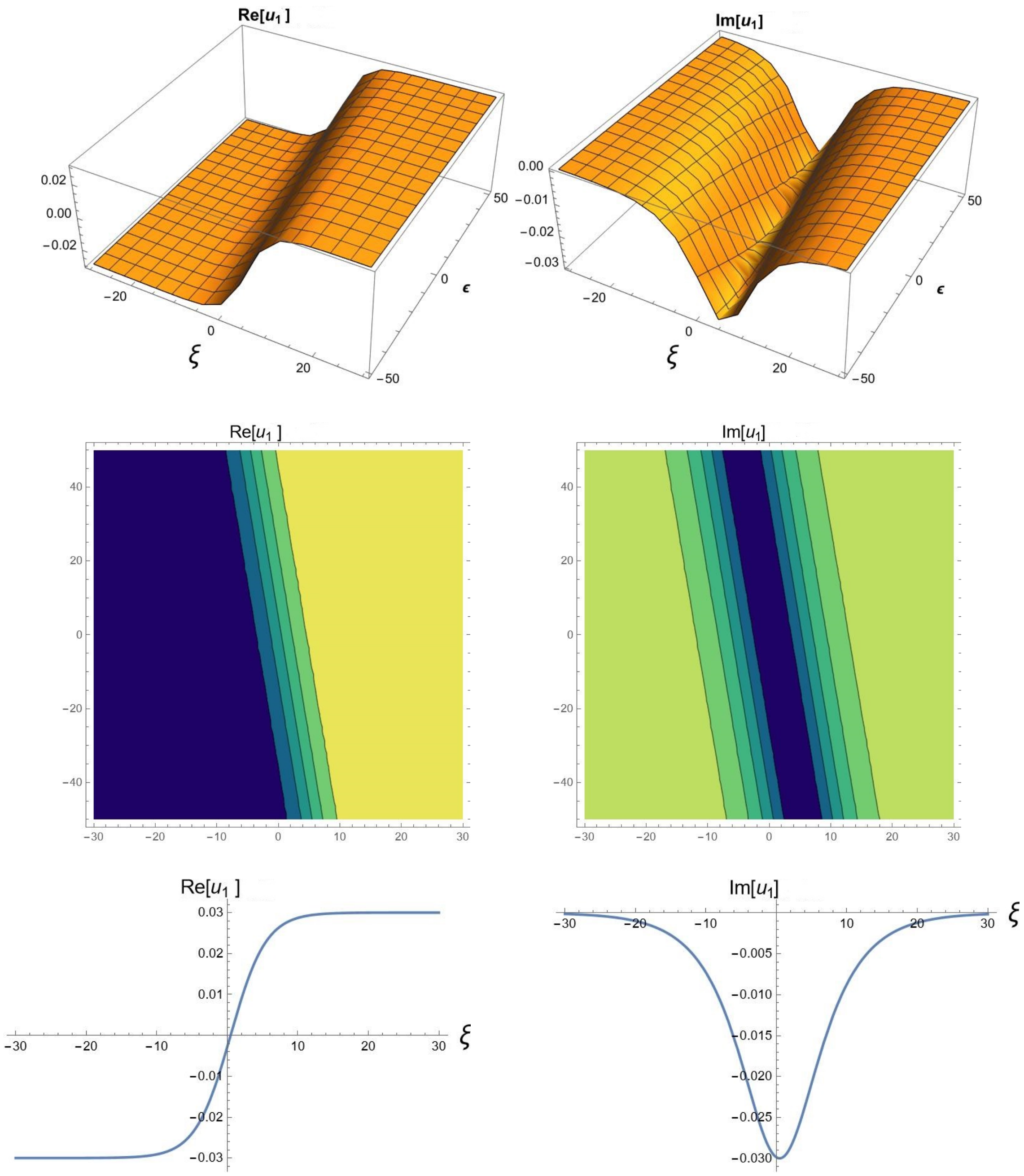

Figure 1. The 3D, contour, and 2D plots for $l = 0.1, \zeta = 0.5, k = 0.2$.

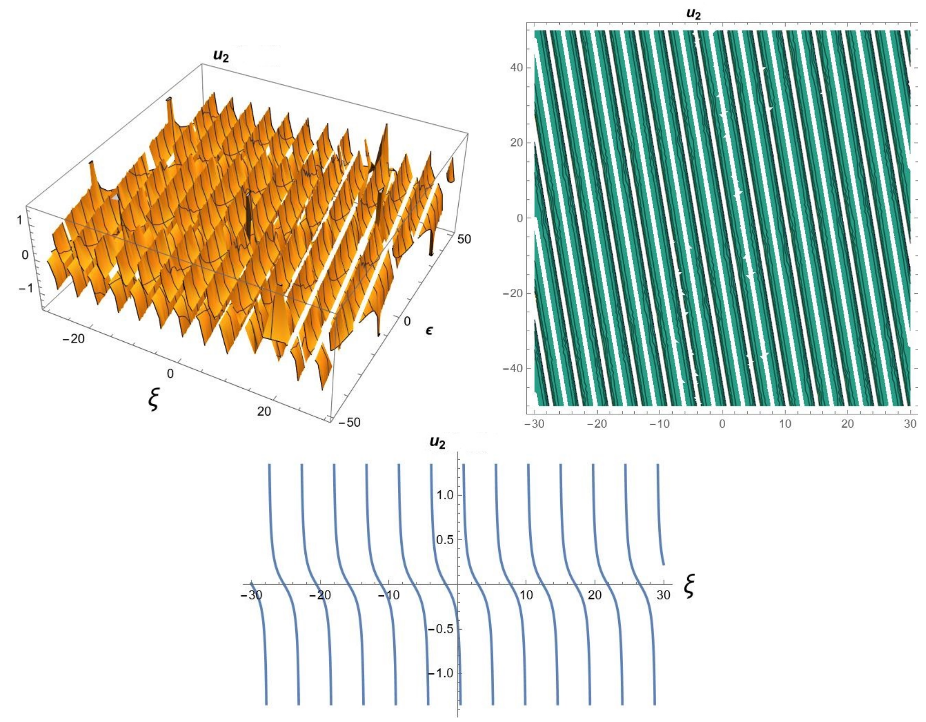

Figure 2. The 3D, contour, and 2D plots for $l = 0.1, \zeta = 0.5, B_2 = 0.2$.

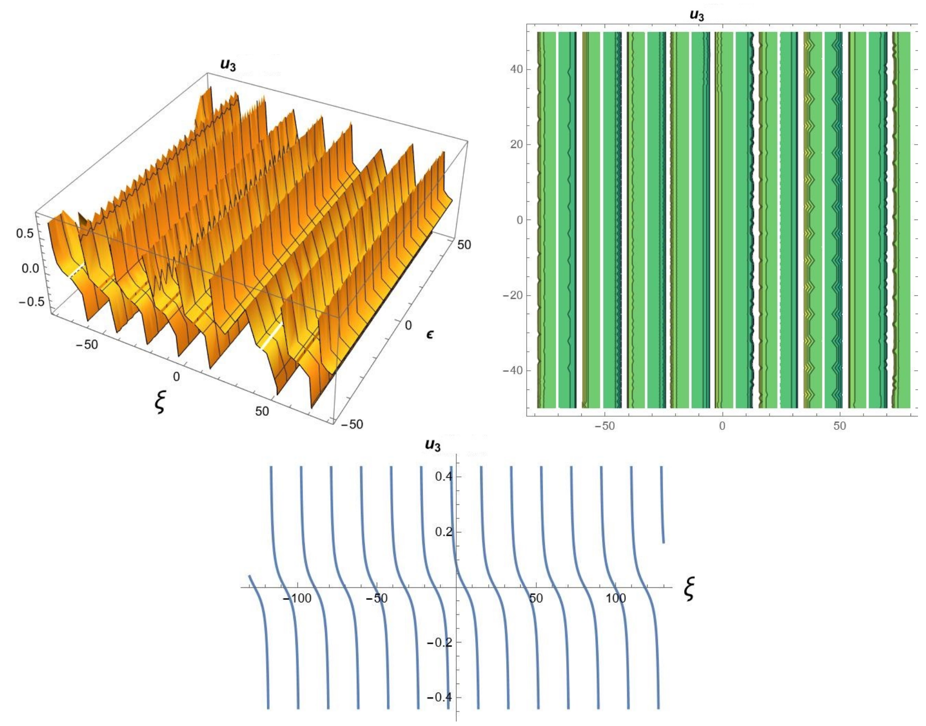

Figure 3. The 3D, contour, and 2D plots for $\epsilon = 0.3$, $l = 0.2, \zeta = 0.5, B_2 = 0.1$.

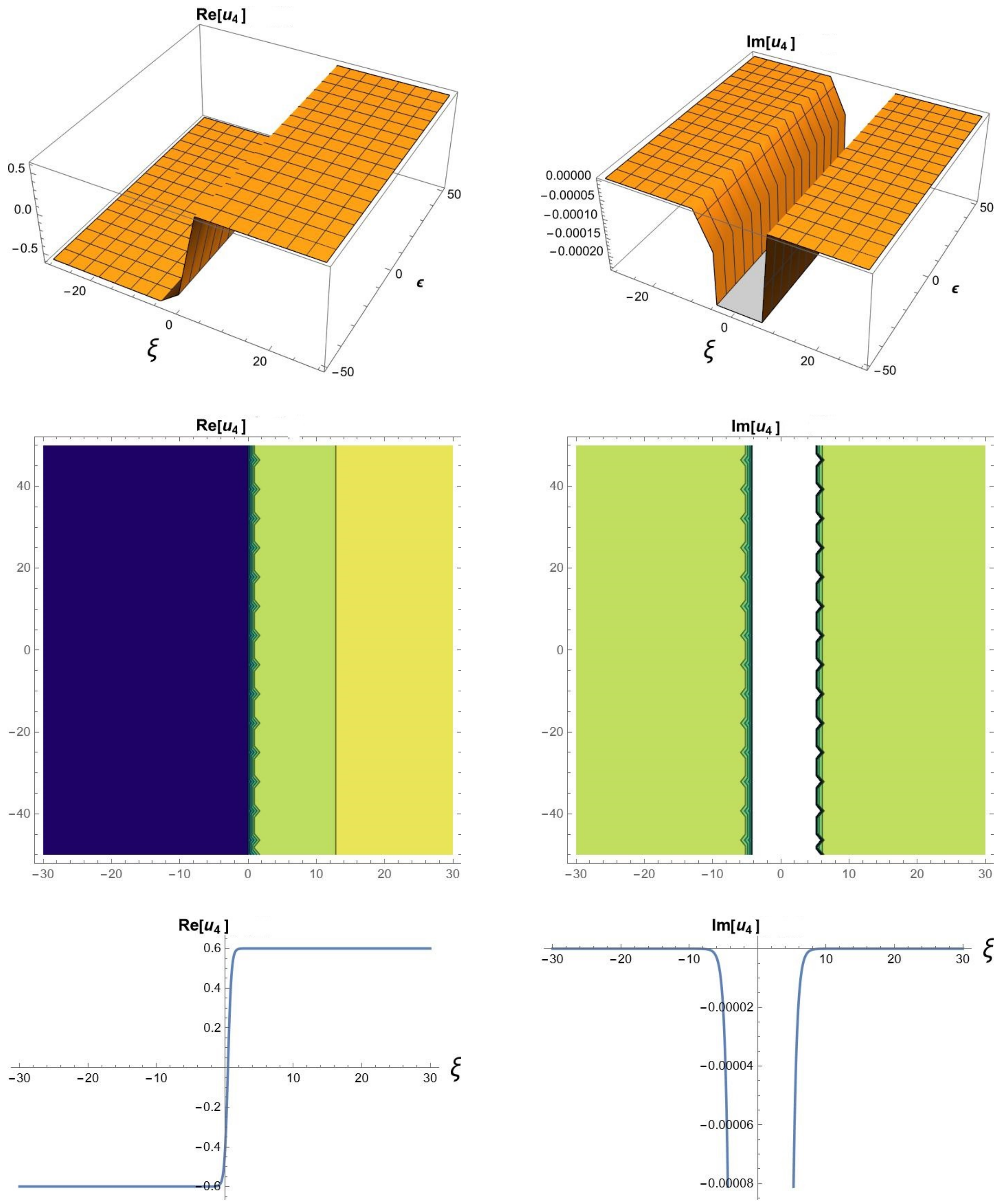

Figure 4. The 3D, contour, and 2D plots for $\epsilon = 0.3$, $l = 0.2$, $\zeta = 0.5$, $A_2 = 0.6$.

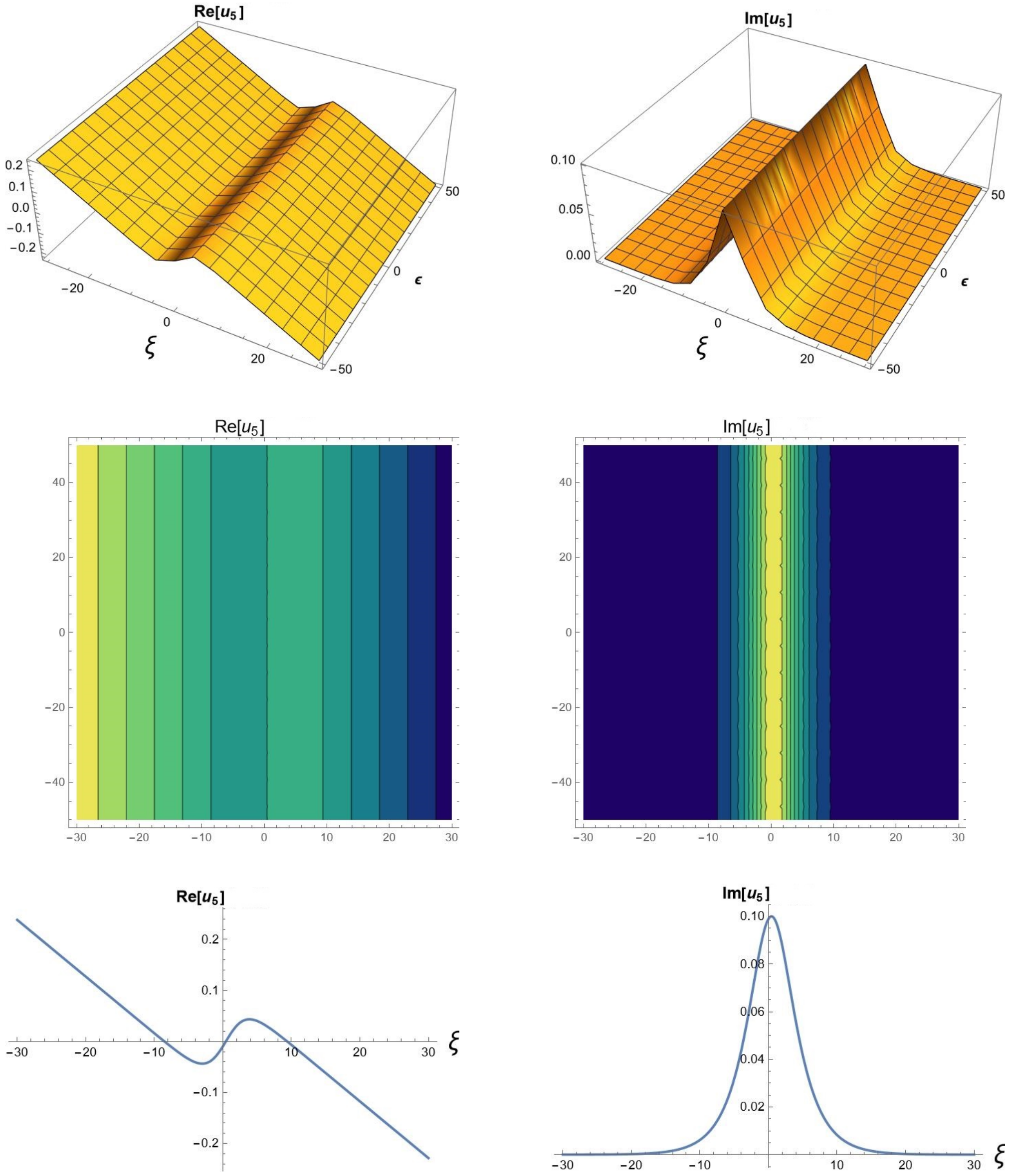

Figure 5. The 3D, contour, and 2D plots for $\epsilon = 0.3$, $l = 0.2$, $\zeta = 0.5$, $A_2 = 0.1$.

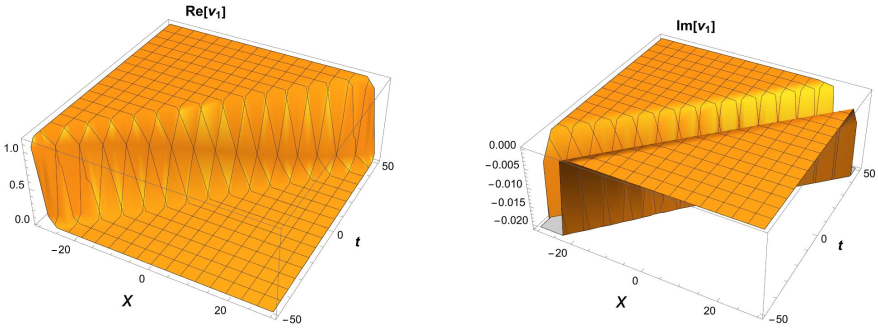

Figure 6. 3D graphs for the real and imaginary parts of Eq (3.18).

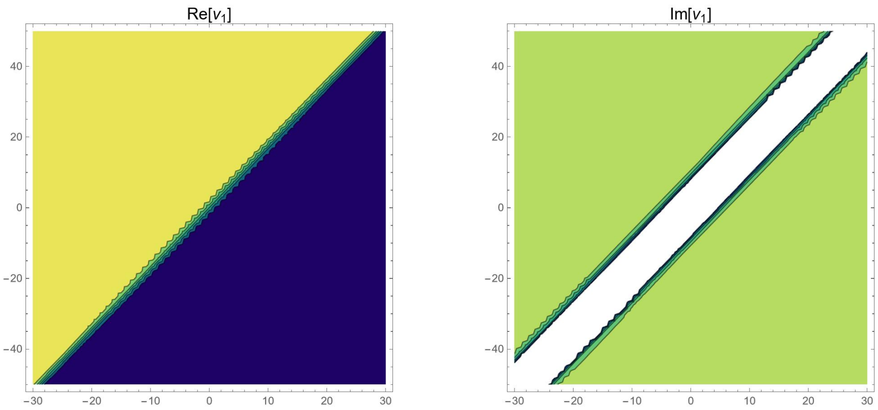

Figure 7. Contour graphs for the real and imaginary parts of Eq (3.18).

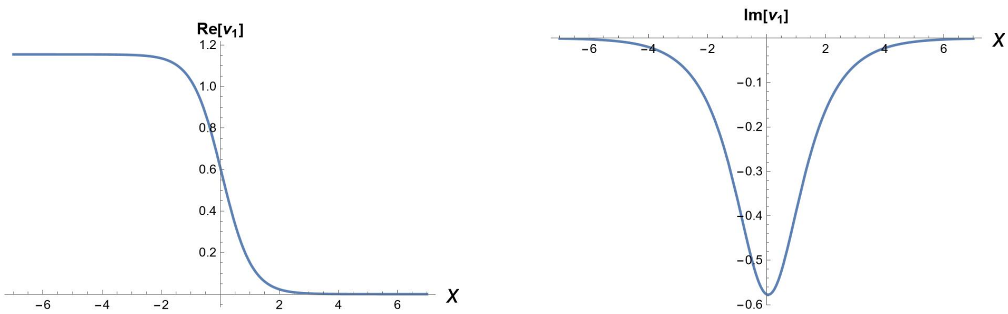

Figure 8. 2D graphs for the real and imaginary parts of Eq (3.18).

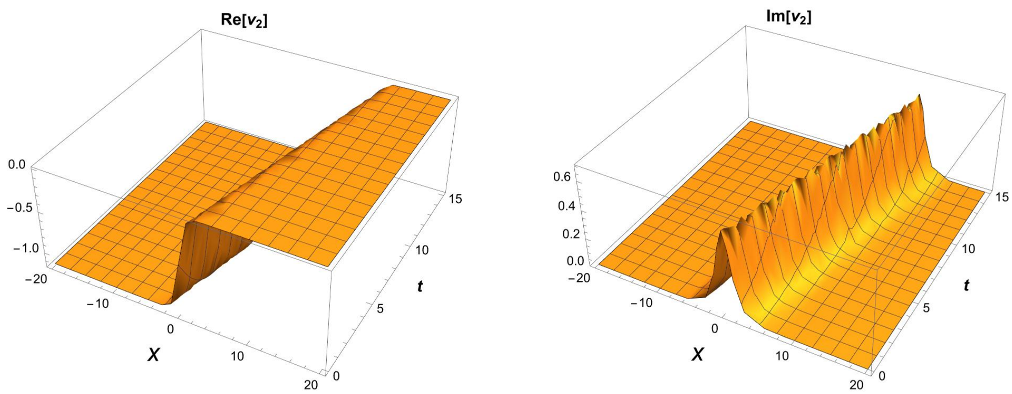

Figure 9. 3D graphs for the real and imaginary parts of Eq (3.19).

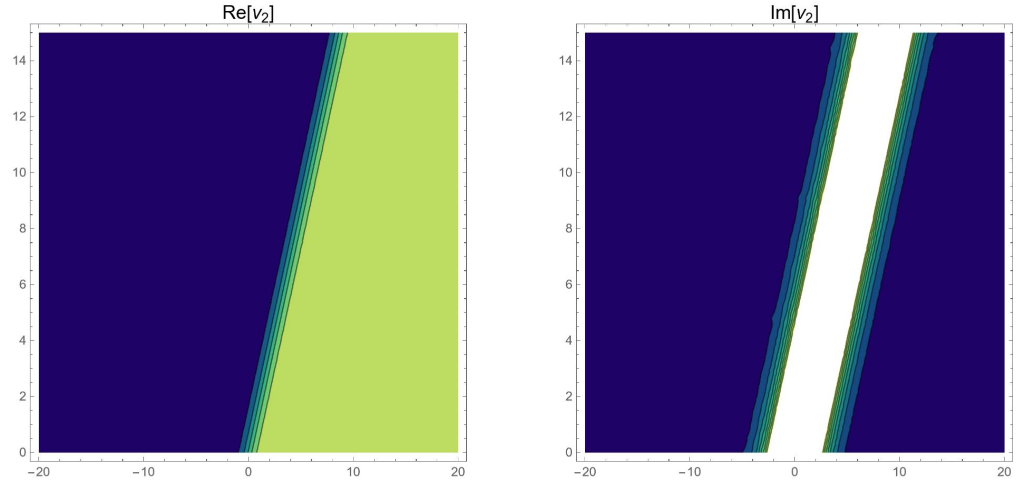

Figure 10. Contour graphs for the real and imaginary parts of equation (3.19).

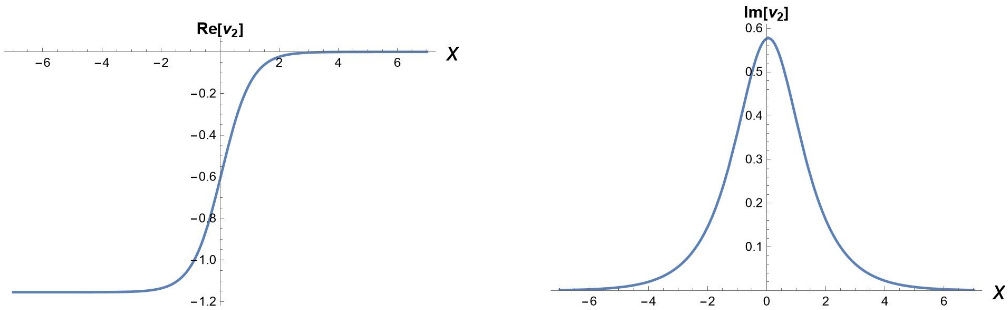

Figure 11. 2D graphs for the real and imaginary parts of Eq (3.19).

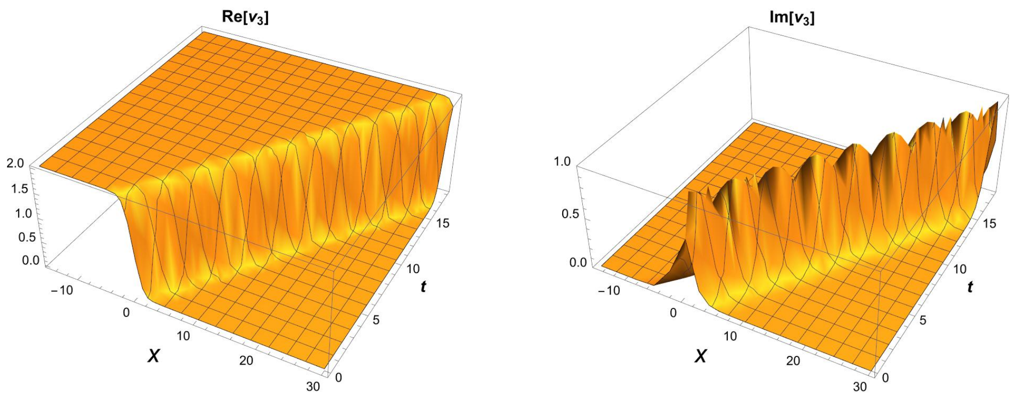

Figure 12. 3D graphs for the real and imaginary parts of Eq (3.20).

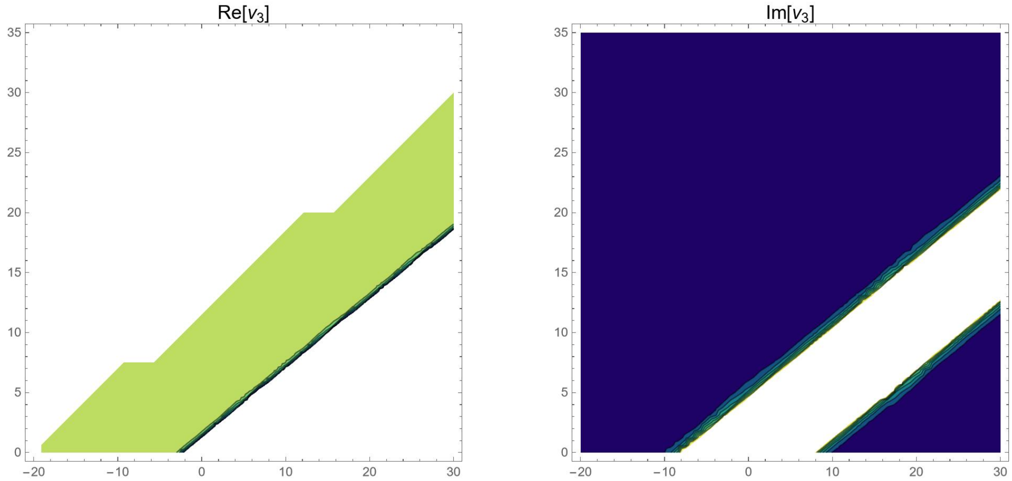

Figure 13. Contour graphs for the real and imaginary parts of Eq (3.20).

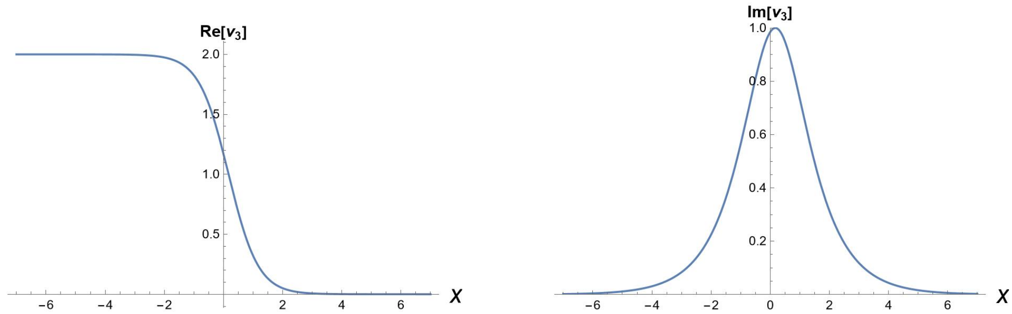

Figure 14. 2D graphs for the real and imaginary parts of Eq (3.20).

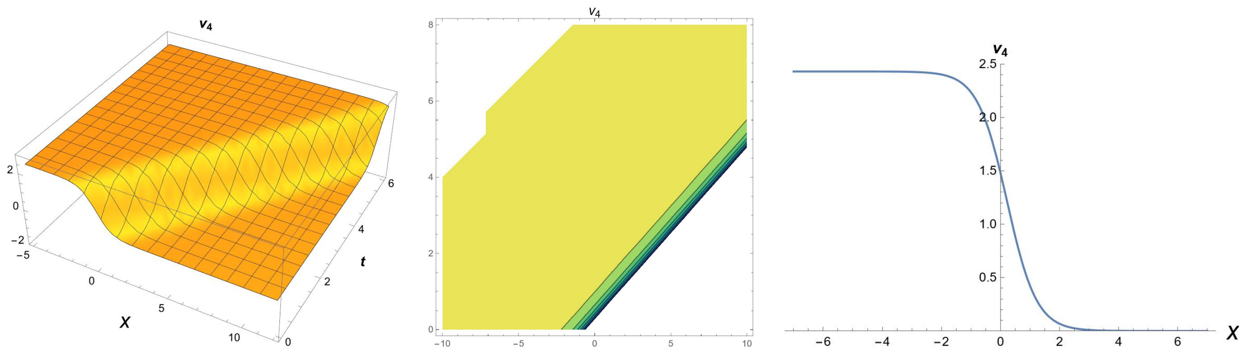

Figure 15. The 3D, contour, and 2D plots of Eq (3.21) for $t = 0.1, \mu = 2$.

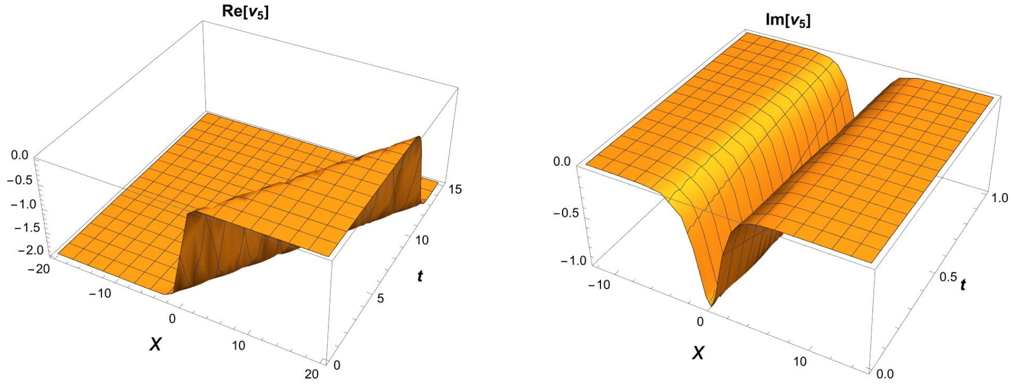

Figure 16. 3D graphs for the real and imaginary parts of Eq (3.22).

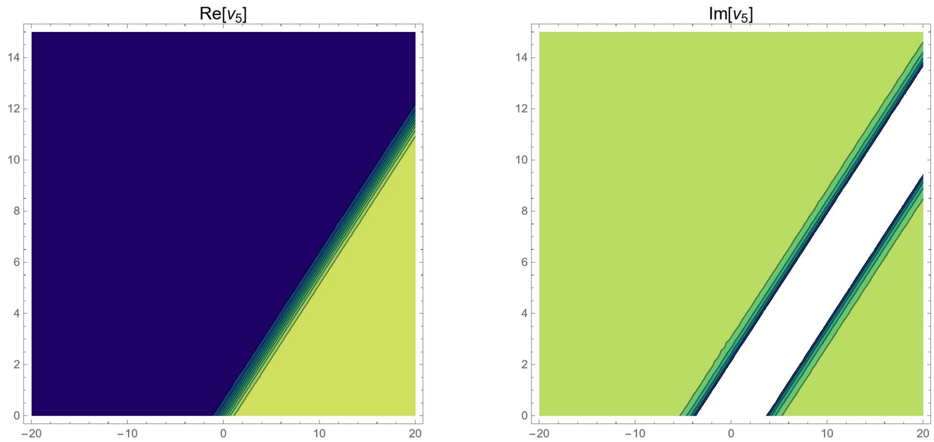

Figure 17. Contour graphs for the real and imaginary parts of Eq (3.22).

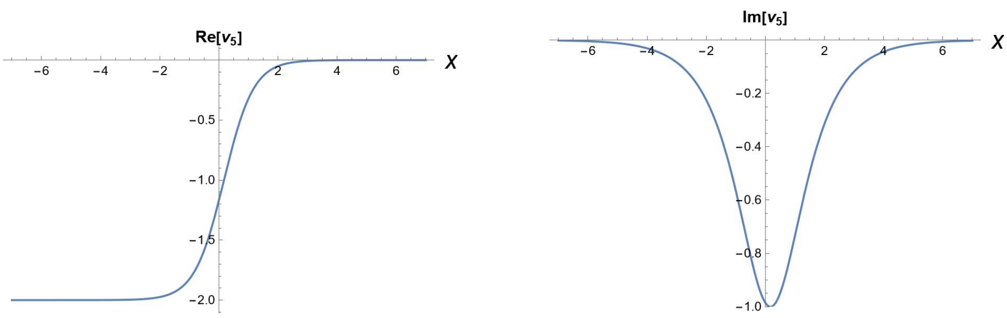

Figure 18. 2D graphs for the real and imaginary parts of Eq (3.22).

5. Conclusions

In this manuscript, we successfully applied the SGEM to find some new solutions for the third extended fifth-order nonlinear equation and the (1+1)-dimensional Van der Waals gas system. By exploring various complex, exponential, and hyperbolic function solutions, we validated these solutions through parametric conditions and visualized them using 2D, 3D, and contour plots (Figures 1–18). Our results revealed new physical properties of Eqs (1.1) and (1.2), demonstrating the effectiveness and simplicity of the SGEM as a powerful and reliable mathematical tool for solving various nonlinear evolution equations. Computational calculations and graphical visualizations were performed using advanced software packages, and to the best of our knowledge, the application of the SGEM to the higher-order Boussinesq dynamical-wave equation and (1+1)-dimensional Van der Waals gas system are novel. These insights contribute significantly to the theoretical understanding and potential practical applications of these nonlinear models, demonstrating the broad applicability of the SGEM across fields such as fluid dynamics, nonlinear optics, plasma physics, and material science. It is estimated that these solutions obtained in this study enhance our understanding of the dynamics of nonlinear dispersive waves and complex wave phenomena, providing a valuable foundation for future research and applications in real-world wave distribution problems. Moreover, this idea in terms of the general properties of SGEM or the models considered in this paper may be also developed to better understand the natural phenomena.

Author contributions

Haci Mehmet Baskonus: conceptualization, methodology; Md Nurul Raihen: software, writing-review editing; Mehmet Kayalar: formal analysis, validation, writing-original draft. The authors read and approved the final submitted version of this manuscript.

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Conflict of interest

The authors have no competing interests to declare that are relevant to the content of this article.

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