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Research article

Infinitely many solutions for a critical $p(x)$-Kirchhoff equation with Steklov boundary value

Khaled Kefi[1,*], **Abdeljabbar Ghanmi**[2], **Abdelhakim Sahbani**[3] **and Mohammed M. Al-Shomrani**[4]

[1] Faculty of Computing and Information Technology, Northern border University, Arar, Kingdom of Saudi Arabia

[2] ENIT-LAMSIN, BP. 37, 1002 Tunis-Belvédère, Tunis El Manar university, Tunis, Tunisia

[3] Faculté des Sciences de Tunis, Université de Tunis El Manar, Tunis 2092, Tunisia

[4] Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Kingdom of Saudi Arabia

* **Correspondence:** Email: khaled_kefi@yahoo.fr.

Abstract: In this paper, we aim to tackle the questions of existence and multiplicity of solutions of the $p(x)$-Kirchhoff problem involving critical exponent and the Steklov boundary value. Further, we research the results from the theory of variable exponent Sobolev spaces, the concentration-compactness principle, and the symmetric mountain pass theorem.

Keywords: variational methods; critical exponent; the concentration-compactness principle; Kirchhoff equation; Steklov problem
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1. Introduction

In this paper, we focus on a class of critical $\kappa(x)$-Kirchhoff-type problems formulated as follows:

$$\begin{cases} \Im\left(\displaystyle\int_\Omega \frac{1}{\kappa(x)}|\nabla u|^{\kappa(x)}\,dx\right)\Delta_{\kappa(x)}u = |u|^{r(x)-2}u + \mathfrak{g}(x,u), \ \ in\ \Omega, \\ |\nabla u|^{\kappa(x)-2}\frac{\partial u}{\partial \nu} = |u|^{s(x)-2}u \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, with the Lipschitz boundary denoted by $\partial\Omega$. The symbol $\frac{\partial}{\partial\nu}$ represents the outer unit normal derivative and $\Delta_{\kappa(x)}u = div\left(|\nabla u|^{\kappa(x)-2}\ \nabla u\right)$. $\Im(t)$ is a continuous function and the function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function that satisfies appropriate assumptions.

We assume that both κ, r and s are continuous functions in $\overline{\Omega}$, meaning they are defined and continuous on the closure of Ω. Moreover, we consider the condition $1 < \kappa(x) < r(x) \leq \kappa^*(x)$ for all $x \in \Omega$, where $p^*(x)$ represents the critical Sobolev exponent. Additionally, we assume that the set $A = \{x \in \Omega : r(x) = \kappa^*(x)\}$ is non-empty.

Problems with critical growth, the concentration-compactness principle introduced by Lions (see [21]) has been widely recognized as a fundamental tool for establishing the existence of solutions. This principle is particularly crucial when considering equations involving Sobolev embeddings, which capture the critical growth behavior. For a more comprehensive understanding of this topic, we suggest referring to the references [3, 4, 18, 24] and the additional sources mentioned therein.

The study of problems with variable exponents, critical growth, and problems involving fractional p-Laplacian has received significant attention in recent years. These problems have proven to be interesting and relevant in various applications, such as the modeling of electro-rheological fluids [20, 23, 25] and image processing [6]. Additionally, they give rise to challenging mathematical problems that require careful investigation.

In [5], the authors consider the critical variable exponents equation:

$$\begin{cases} (-\Delta)_{p(x)} u = |u|^{r(x)-2} u + a(x)|u|^{q(x)-2} u \quad \text{in } \Omega, \\ \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $1 < p(x) < r(x) \leq p^*(x)$ for all $x \in \Omega$, where $p^*(x)$ denotes the critical Sobolev exponent associated with $p(x)$. The set $A = \{x \in \Omega : r(x) = p^*(x)\}$ is assumed to be non-empty, indicating the presence of critical growth behavior.

To establish the existence of solutions, the authors employ variational methods and make use of the mountain pass theorem. These techniques allow them to construct a suitable functional and apply critical point theory to find nontrivial solutions to the problem (1.2).

Due to their importance, problems involving variable exponents are attracting increasing interest from many researchers. Many authors studied the problem with Dirichlet, Neumann, or Steklov boundary conditions on a bounded domain. In particular, Chammem et al. [7] used the mountain pass theorem combined with Ekland's variational principle to study the following Steklov problem:

$$\begin{cases} (-\Delta)_{p(x)} u + a(x)|u|^{p(x)-2} u = f(x, u) \quad \text{in } \Omega, \\ \\ |\nabla u|^{p(x)-2} \frac{\partial u}{\partial \nu} + b(x)|u|^{q(x)-2} u = g(x, u) \quad \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

After that, Chammem and Sahbani [8] considered the following double-phase problem:

$$\begin{cases} (-\Delta)_{p_1(x)} u + (-\Delta)_{p_2(x)} u + a(x)|u|^{p_1(x)-2} u \\ +b(x)|u|^{p_2(x)-2} u = f(x, u) \quad \text{in } \Omega, \\ \\ |\nabla u|^{p_1(x)-2} \frac{\partial u}{\partial \nu} + |\nabla u|^{p_2(x)-2} \frac{\partial u}{\partial \nu} = g(x, u) \quad \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

By means of variational methods, the mountain pass lemma and its Z_2 symmetric version, the existence and multiplicity of solutions for problem (1.4), was established.

Problems involving Kirchhoff-type with variable exponents are attracting attention and gaining prominence in several research groups for numerous theoretical and practical questions [9, 10, 12, 13] and the references therein. On the other hand, it is also worth mentioning Kirchhoff's problems with fractional operators, which over the years has been increasing exponentially [1, 17, 29]. The $p(x)$-Laplacian possesses more complex nonlinearity which raises some of the essential difficulties. For example, in [30], Z. Yücedag consider the p(x)-Kirchhoff problem with Steklov boundary value conditions:

$$\begin{cases} M\left(\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx\right)\Delta_{p(x)}u = |u|^{q(x)-2}u, \ in\ \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} = \lambda f(x,u) \quad \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where $1 < p(x) \leq p^*(x)$ for all $x \in \Omega$.

Under suitable conditions on the functions q and f, the authors employed variational methods and fountain theorem to establish the existence and multiplicity of solutions for problem (1.5).

Motivated by the results presented in references [5, 7, 8, 30], our paper aims to contribute further by studying the critical case of the aforementioned problem. To this end, we utilize a recent concentration-compactness principle for spaces with variable exponents to investigate the weighted Kirchhoff problem (1.1). Our study provides a generalization, improvement, and extension of the aforementioned references under additional, appropriate conditions. Consequently, this research project holds significant importance and offers valuable insights.

In this paper, we consider problem (1.1), where $g(x,u) = a(x)h(u)$. Under specific hypotheses, we employ the variational method, the mountain pass theorem, and the symmetric mountain pass theorem to establish the existence and multiplicity of nontrivial weak solutions for problem (1.1). This rigorous approach ensures the robustness and reliability of our results.

In summary, our research significantly contributes to the existing literature by exploring the critical case of the Kirchhoff problem with Dirichlet boundary conditions. Through rigorous mathematical techniques and the utilization of recent concentration-compactness principles, we establish the existence and multiplicity of solutions for problem (1.1) under different scenarios, enhancing the overall understanding of this important topic.

In Section 2, we present some necessary preliminary. In Section 3 we give our main results, where we present and prove the existence and multiplicity of solutions for the weighted Kirchhoff problem.

2. Preliminaries

In this section, we provide an overview of some important properties of variable exponent spaces. For more detailed information, we recommend referring to the works [11, 15, 16, 19, 26] and the references therein. We consider the set

$$C_+(\overline{\Omega}) = \{\kappa \in C(\overline{\Omega}), \kappa(\xi) > 1, \forall \xi \in \overline{\Omega}\}.$$

For all $\kappa \in C_+(\Omega)$, consider

$$\kappa^- = \inf_{\overline{\Omega}} \kappa(\xi), \qquad \kappa^+ = \sup_{\overline{\Omega}} \kappa(\xi).$$

Additionally, we define

$$\mathscr{L}^{\kappa(\xi)}(\Omega) = \left\{ u : \Omega \to \mathbb{R}, \text{measurable} : \int_\Omega |u(\xi)|^{\kappa(\xi)} d\xi < \infty \right\},$$

with the norm on $\mathscr{L}^{\kappa(\xi)}(\Omega)$ defined as

$$|u|_{\mathscr{L}^{\kappa(\xi)}(\Omega)} = \inf \left\{ \varpi > 0 : \int_\Omega \left| \frac{u(\xi)}{\varpi} \right|^{\kappa(\xi)} d\xi \leq 1 \right\}.$$

Also, we define

$$\mathscr{L}^{\kappa(\xi)}(\partial\Omega) = \left\{ u : \Omega \to \mathbb{R}, \text{ measurable} : \int_{\partial\Omega} |u(\xi)|^{\kappa(\xi)} d\sigma < \infty \right\},$$

with the norm on $\mathscr{L}^{\kappa(\xi)}(\partial\Omega)$ defined as

$$|u|_{\mathscr{L}^{\kappa(\xi)}(\partial\Omega)} = \inf \left\{ \varpi > 0 : \int_{\partial\Omega} \left| \frac{u(\xi)}{\varpi} \right|^{\kappa(\xi)} d\sigma \leq 1 \right\}.$$

The spaces $\left(\mathscr{L}^{\kappa(\xi)}(\Omega), |\cdot|_{\mathscr{L}^{\kappa(\xi)}(\Omega)}\right)$ and $\left(\mathscr{L}^{\kappa(\xi)}(\partial\Omega), |\cdot|_{\mathscr{L}^{\kappa(\xi)}(\partial\Omega)}\right)$ are a Banach spaces, which we refer to as variable exponent Lebesgue spaces.

The Sobolev space is defined as:

$$W^{1,\kappa(x)}(\Omega) = \left\{ u \in \mathscr{L}^{\kappa(\xi)}(\Omega) : |\nabla u| \in \mathscr{L}^{\kappa(\xi)}(\Omega) \right\}$$

with the norm

$$\|u\| = \|u\|_{W^{1,\kappa(x)}(\Omega)} = \|u\|_{\mathscr{L}^{\kappa(\xi)}(\Omega)} + \|\nabla u\|_{\mathscr{L}^{\kappa(\xi)}(\Omega)}.$$

Denote by $W_0^{1,\kappa(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,\kappa(x)}(\Omega)$.

The following proposition provides important properties of variable exponent spaces.

Proposition 1. [15] *The spaces $\mathscr{L}^{\kappa(\xi)}(\Omega)$ and $W^{1,\kappa(x)}(\Omega)$ are separable and reflexive Banach spaces.*

Proposition 2. [7, 8, 27]

(1) The Hölder inequality holds, that is, for any $u \in \mathscr{L}^{\kappa(\xi)}(\Omega)$ and $v \in \mathscr{L}^{\kappa'(\xi)}(\Omega)$, we have

$$\left| \int_\Omega uv d\xi \right| \leq \left(\frac{1}{\kappa^-} + \frac{1}{(\kappa')^-} \right) |u|_{\kappa(\xi)} |v|_{\kappa'(\xi)}.$$

(2) If $p_1, p_2 \in C_+(\overline{\Omega})$ such that $p_1(\xi) \leq p_2(\xi)$ for all $\xi \in \overline{\Omega}$, then the embedding $\mathscr{L}^{p_2(\xi)}(\Omega) \hookrightarrow \mathscr{L}^{p_1(\xi)}(\Omega)$ is continuous.

Note that $||u||$ and $||\nabla u||_{\mathscr{L}^{\kappa(\xi)}(\Omega)}$ are equivalent in the space $W_0^{1,\kappa(x)}(\Omega)$, so let's use $||u|| = ||\nabla u||_{\mathscr{L}^{\kappa(\xi)}(\Omega)}$, for simplicity [11, 28].

The following proposition highlights the properties of the variable exponent Sobolev spaces.

Proposition 3. [15]

(1) If $q \in C_+(\overline{\Omega})$ with $q(\xi) < \kappa^*(\xi)$ for all $\xi \in \overline{\Omega}$, then the embedding from $W_0^{1,\kappa(x)}(\Omega)$ into $\mathscr{L}^{q(\xi)}(\Omega)$ is compact and continuous. Here, $\kappa^*(\xi)$ is defined as follows:

$$\kappa^*(\xi) = \begin{cases} \frac{N\kappa(\xi)}{N-\kappa(\xi)}, & \text{if } \kappa(\xi) < N, \\ \\ \infty, & \text{if } \kappa(\xi) \geq N. \end{cases}$$

(2) If $q \in C_+(\partial\Omega)$ with $q(\xi) < \kappa_*(\xi)$ for all $\xi \in \partial\Omega$, then the embedding from $W_0^{1,\kappa(x)}(\Omega)$ into $\mathscr{L}^{q(\xi)}(\partial\Omega)$ is compact and continuous. Here, $\kappa_*(\xi)$ is defined as follows:

$$\kappa_*(\xi) = \begin{cases} \frac{(N-1)\kappa(\xi)}{N-\kappa(\xi)}, & \text{if } \kappa(\xi) < N, \\ \\ \infty, & \text{if } \kappa(\xi) \geq N. \end{cases}$$

For simplicity, let us denote

$$\Gamma(u) = \int_\Omega |\nabla u|^{\kappa(\xi)} \, d\xi.$$

The following proposition provides important properties of the functional $\Gamma(u)$:

Proposition 4. [15]

(1) If $\Gamma(u) \geq 1$, then $||u||^{\kappa^-} \leq \Gamma(u) \leq ||u||^{\kappa^+}$.

(2) If $\Gamma(u) \leq 1$, then $||u||^{\kappa^+} \leq \Gamma(u) \leq ||u||^{\kappa^-}$.

(3) $\Gamma(u) \geq 1 (= 1, \leq 1) \Leftrightarrow ||u|| \geq 1 (= 1, \leq 1)$.

Let us define

$$\rho(u) = \int_\Omega |u(\xi)|^{\kappa(\xi)} d\xi.$$

The next proposition provides properties of the functional $\rho(u)$:

Proposition 5. [7, 8] *For all $u \in \mathscr{L}^{\kappa(\xi)}(\Omega)$, we have*

(1) $|u|_{\mathscr{L}^{\kappa(\xi)}(\Omega)} < 1; (resp = 1, > 1) \Leftrightarrow \rho(u) < 1; (resp = 1, > 1)$.

(2) $|u|_{\mathscr{L}^{\kappa(\xi)}(\Omega)} > 1 \Rightarrow |u|^{\kappa^-}_{\mathscr{L}^{\kappa(\xi)}(\Omega)} \leq \rho(u) \leq |u|^{\kappa^+}_{\mathscr{L}^{\kappa(\xi)}(\Omega)}$.

(3) $|u|_{\mathscr{L}^{\kappa(\xi)}(\Omega)} < 1 \Rightarrow |u|^{\kappa^+}_{\mathscr{L}^{\kappa(\xi)}(\Omega)} \leq \rho(u) \leq |u|^{\kappa^-}_{\mathscr{L}^{\kappa(\xi)}(\Omega)}$.

The next proposition relates the norms of a function in variable exponent Lebesgue spaces with its pointwise behavior.

Proposition 6. [7,8] *If p and q are measurable functions such that* $p \in \mathscr{L}^{\infty}(\mathbb{R}^N)$ *and* $1 \leq \kappa(\xi)q(\xi) \leq \infty$ *for all* $\xi \in \mathbb{R}^N$*, then for all* $u \in \mathscr{L}^{q(\xi)}(\mathbb{R}^N)$ *with* $u \neq 0$*, we have*

(1) $|u|_{\mathscr{L}^{\kappa(\xi)q(\xi)}(\Omega)} \leq 1 \Rightarrow |u|^{q^+}_{\mathscr{L}^{\kappa(\xi)q(\xi)}(\Omega)} \leq \||u|^{\kappa(\xi)}|_{\mathscr{L}^{q(\xi)}(\Omega)} \leq |u|^{q^-}_{\mathscr{L}^{\kappa(\xi)q(\xi)}(\Omega)}$.

(2) $|u|_{\mathscr{L}^{\kappa(\xi)q(\xi)}(\Omega)} \geq 1 \Rightarrow |u|^{q^-}_{\mathscr{L}^{\kappa(\xi)q(\xi)}(\Omega)} \leq \||u|^{\kappa(\xi)}|_{\mathscr{L}^{q(\xi)}(\Omega)} \leq |u|^{q^+}_{\mathscr{L}^{\kappa(\xi)q(\xi)}(\Omega)}$.

Denote for $u \in \mathscr{L}^{p(\xi)}(\partial\Omega)$,

$$\rho_{\partial}(u) = \int_{\partial\Omega} |u(\xi)|^{p(\xi)} d\sigma.$$

Proposition 7. [7,8] *For all* $u \in \mathscr{L}^{p(\xi)}(\partial\Omega)$*, we have,*
(1) $|u|_{\mathscr{L}^{p(\xi)}(\partial\Omega)} > 1 \Rightarrow |u|^{p^-}_{\mathscr{L}^{p(\xi)}(\partial\Omega)} \leq \rho_{\partial}(u) \leq |u|^{p^+}_{\mathscr{L}^{p(\xi)}(\partial\Omega)}$,
(2) $|u|_{\mathscr{L}^{p(\xi)}(\partial\Omega)} < 1 \Rightarrow |u|^{p^+}_{\mathscr{L}^{p(\xi)}(\partial\Omega)} \leq \rho_{\partial}(u) \leq |u|^{p^-}_{\mathscr{L}^{p(\xi)}(\partial\Omega)}$.

3. Main results

In this section, we will present our main result of the paper. Firstly, we assume the following hypotheses:

($\mathbf{C_1}$) The function $g(x,u)$ can be expressed as $a(x)h(u)$, where a and h are measurable functions satisfying the following conditions: there exists $c_1 > 0$, $p, q \in C_+(\overline{\Omega})$ such that for all $(x,u) \in \Omega \times \mathbb{R}$, we have

$$a(x) \in \mathscr{L}^{\frac{p(x)}{p(x)-q(x)}}(\Omega),\ h(u) \leq c_1 |u|^{q(x)-1},$$

and

$$\kappa^+ < q(x) < p(x) < \kappa^*(x) \ \ and \ \ \kappa^+ < N. \tag{3.1}$$

($\mathbf{C_2}$) There exists $m_0 > 0$ *such that* $\mathfrak{S}(t) \geq m_0$.

($\mathbf{C_3}$) There exists $0 < \omega < 1$ such that, $1 - \omega \geq \frac{1}{\kappa^+}$ and $\widehat{\mathfrak{S}}(t) \geq (1-\omega)\mathfrak{S}(t)t$, where $\widehat{\mathfrak{S}}(t) = \displaystyle\int_0^t \mathfrak{S}(s)ds$.

($\mathbf{C_4}$) There exist $M_1 > 0$ and $\frac{\kappa^+}{1-\omega} < \theta < \min(r^-, s^-)$, such that for all $x \in \Omega$, we have

$$0 < \theta a(x)H(u) \leq a(x)h(u)u,\ \ |u| \geq M_1,\ \ where\ \ H(t) = \int_0^t h(s)ds.$$

($\mathbf{C_5}$) We have $\kappa^+ \leq s(x) < \kappa_*(x)$.
($\mathbf{C_6}$) For all $x \in \overline{\Omega}$, we have $h(-u) = -h(u)$.

Next, we define a weak solution for the problem (1.1) as follows:

Definition 1. *We say that* $u \in X = W_0^{1,\kappa(x)}(\Omega)$ *is a weak solution for Eq* (1.1) *if, for any* $v \in X$*, we have*

$$\mathfrak{S}\left(\int_{\Omega} \frac{1}{\kappa(x)} |\nabla u|^{\kappa(x)} dx\right) \int_{\Omega} |\nabla u|^{(\kappa(x)-2)} \nabla u \nabla v - \int_{\Omega} |u|^{r(x)-2} uv dx$$
$$- \int_{\Omega} a(x)h(u)v dx - \int_{\partial\Omega} |u|^{s(x)-2} uv dx = 0.$$

Now, we are ready to state and prove the first main results:

Theorem 1. *Under the hypotheses* $(\mathbf{C_1}) - (\mathbf{C_5})$*, problem* (1.1) *has a nontrivial weak solution.*

Theorem 2. *Under the hypotheses* $(\mathbf{C_1}) - (\mathbf{C_6})$*, problem* (1.1) *has infinitely many solutions.*

Now, we introduce the functional $\mathfrak{J}(u)$ associated with problem (1.1), which characterizes the critical points and plays a key role in the existence of solutions.

$$\begin{aligned}\mathfrak{J}(u) &= \widehat{\mathfrak{S}}\left(\int_\Omega \frac{1}{\kappa(x)}|\nabla u|^{\kappa(x)}\,dx\right) - \int_\Omega \frac{|u|^{r(x)}}{r(x)}dx - \int_\Omega a(x)H(u)dx - \int_{\partial\Omega}\frac{|u|^{s(x)}}{s(x)}dx,\\ &= L(u) - I(u) - J(u) - T(u),\end{aligned}$$

where $\widehat{\mathfrak{S}}(t) = \int_0^t \mathfrak{S}(s)ds$, $L(u) = \widehat{\mathfrak{S}}\left(\int_\Omega \frac{1}{\kappa(x)}|\nabla u|^{\kappa(x)}\,dx\right)$,

$$T(u) = \int_{\partial\Omega}\frac{|u|^{s(x)}}{s(x)}dx,\ \ I(u) = \int_\Omega \frac{|u|^{r(x)}}{r(x)}dx \ \ and \ \ J(u) = \int_\Omega a(x)H(u)dx.$$

We recall from [7], that $L \in C^1(X,\mathbb{R})$. Moreover, for all $u, v \in X$, we have

$$< L^{'}(u), v >= \mathfrak{S}\left(\int_\Omega \frac{1}{\kappa(x)}|\nabla u|^{\kappa(x)}\,dx\right)\int_\Omega |\nabla u|^{(\kappa(x)-2)}\nabla u\nabla v dx.$$

The functional L' satisfies the following properties.

Proposition 8. *[7]*
Let $\mathfrak{L}: \ \ \mathcal{H}^{\varpi,\nu;\psi}_{\kappa(x)}(\Omega) \to (\mathcal{H}^{\varpi,\nu;\psi}_{\kappa(x)}(\Omega))^*$,
such that

$$< \mathfrak{L}^{'}(u), v >= \int_\Omega |\nabla u|^{(\kappa(x)-2)}\nabla u\nabla v dx.$$

(1) $\mathfrak{L}^{'}: X \to X^*$ *is a continuous, bounded, and strictly monotone operator.*

(2) $\mathfrak{L}^{'}$ *is a mapping of* (S_+) *type, that is, if* $u_n \rightharpoonup u$ *in* X *and* $\limsup_{n\to\infty} < \mathfrak{L}^{'}(u_n) - \mathfrak{L}^{'}(u), u_n - u > \leq 0$, *then,* $u_n \to u$ *strongly in* X.

Remark 1. *It can be shown, using* (C_1)*, Propositions 4, 6, and the Hölder inequality, that* $J \in C^1(X,\mathbb{R})$*. Furthermore, for all* $u, v \in X$*, we have*

$$< J^{'}(u), v >= \int_\Omega a(x)h(u(x))v(x)dx.$$

From Proposition 8 and Remark 1, it follows that $\mathfrak{J} \in C^1(X,\mathbb{R})$. Moreover, for all $u, v \in X$, we obtain

$$\begin{aligned}< \mathfrak{J}'(u), v > &= \mathfrak{S}\left(\int_\Omega \frac{1}{\kappa(x)}|\nabla u|^{\kappa(x)}\,dx\right)\int_\Omega |\nabla u|^{(\kappa(x)-2)}\nabla u\nabla v dx\\ &- \int_\Omega |u|^{r(x)-2}uv dx - \int_\Omega a(x)h(u(x))v(x)dx - \int_{\partial\Omega}|u|^{s(x)-2}uv dx.\end{aligned}$$

Hence, the weak solutions of problem (1.1) correspond to the critical points of the functional $\mathfrak{J}$.

Now, we establish a key result that provides a lower bound for the functional $\mathfrak{J}(u)$ associated with problem (1.1).

Lemma 1. *Assume that* $(\mathbf{C_1})$–$(\mathbf{C_5})$ *are satisfied. Then, there exist* $m, \eta > 0$ *such that, for* $u \in X$,

$$\text{if} \quad \|u\| = \eta, \ \text{then}, \ \mathfrak{J}(u) \geq m.$$

Proof. Let $u \in X$, with $\|u\| < 1$. Under the hypothesis $(\mathbf{C_1})$, we have for all $x \in \Omega$,

$$H(u) \leq \frac{c_1}{q(x)}|u|^{q(x)}. \tag{3.2}$$

Since $1 < p(x) < \kappa^*(x)$, $1 < r(x) \leq \kappa^*(x)$, $1 < s(x) < \kappa_*(x)$ and according to Proposition 3, we obtain the existence of c_3, c_4, $c_5 > 0$, such that

$$|u|_{\mathscr{L}^{p(x)}(\Omega)} \leq c_3 \|u\|, \ |u|_{\mathscr{L}^{r(x)}(\Omega)} \leq c_4 \|u\|, |u|_{\mathscr{L}^{s(x)}(\partial\Omega)} \leq c_5 \|u\|. \tag{3.3}$$

On the other hand, under hypothesis $(\mathbf{C_2})$ and $(\mathbf{C_3})$ and by Proposition 4, we get,

$$\begin{aligned} L(u) &= \widehat{\mathfrak{S}}\left(\int_\Omega \frac{1}{\kappa(x)} |\nabla u|^{\kappa(x)}\, dx\right) \\ &\geq (1-\omega)\mathfrak{S}\left(\int_\Omega \frac{1}{\kappa(x)} |\nabla u|^{\kappa(x)}\, dx\right)\int_\Omega \frac{1}{\kappa(x)} |\nabla u|^{\kappa(x)}\, dx \\ &\geq \frac{(1-\omega)m_0}{\kappa^+}\int_\Omega |\nabla u|^{\kappa(x)}\, dx \geq \frac{(1-\omega)m_0}{\kappa^+}\|u\|^{\kappa^+}. \end{aligned} \tag{3.4}$$

Now, by (3.2)–(3.4) and using Propositions 2, 6 and 4, we obtain,

$$\begin{aligned} \mathfrak{J}(u) &= \widehat{\mathfrak{S}}\left(\int_\Omega \frac{1}{\kappa(x)} |\nabla u|^{\kappa(x)}\, dx\right) - \int_\Omega a(x)H(u)dx - \int_\Omega \frac{|u|^{r(x)}}{r(x)}dx - \int_{\partial\Omega} \frac{|u|^{s(x)}}{s(x)}dx, \\ &= L(u) - J(u) - I(u) - T(u) \\ &\geq \frac{(1-\omega)m_0}{\kappa^+}\|u\|^{\kappa^+} - \frac{c_3}{q^-}|a|_{\mathscr{L}^{\frac{P(x)}{P(x)-q(x)}}(\Omega)}\|u\|^{q^-} - \frac{c_4}{r^-}\|u\|^{r^-} - \frac{c_5}{s^-}\|u\|^{s^-} \\ &\geq \|u\|^{\kappa^+}\left(\frac{(1-\omega)m_0}{\kappa^+} - \frac{c_3}{q^-}|a|_{\mathscr{L}^{\frac{p(x)}{p(x)-q(x)}}(\Omega)}\|u\|^{q^- - \kappa^+} - \frac{c_4}{r^-}\|u\|^{r^- - \kappa^+} - \frac{c_5}{s^-}\|u\|^{s^- - \kappa^+}\right) \\ &\geq \|u\|^{\kappa^+}\left(\frac{(1-\omega)m_0}{\kappa^+} - t\|u\|^{\min(q^- - \kappa^+, r^- - \kappa^+, s^- - \kappa^+)}\right), \end{aligned}$$

where

$$t = \frac{c_3}{q^-}|a|_{\mathscr{L}^{\frac{p(x)}{p(x)-q(x)}}(\Omega)} + \frac{c_4}{r^-} + \frac{c_5}{s^-}.$$

Since q^-, s^- and r^- are both greater than κ^+, we can choose $\|u\| = \eta$ to be sufficiently small such that

$$\frac{(1-\omega)m_0}{\kappa^+} - t\eta^{\min(q^- - \kappa^+, r^- - \kappa^+, s^- - \kappa^+)} > 0.$$

Finally, we conclude that

$$\mathfrak{J}(u) \geq \eta^{\kappa^+}\Big(\frac{(1-\omega)m_0}{\kappa^+} - t\,\eta^{\min(q^- - \kappa^+, r^- - \kappa^+, s^- - \kappa^+)}\Big) := m > 0.$$

□

Now, define the Palais−Smale (PS) condition at a given level c:

Definition 2. *Let X be a Banach space and $\Im \in C^1(X,\mathbb{R})$, where $c \in \mathbb{R}$. We say that $\Im$ satisfies the Palais−Smale condition at level c if any $\{u_n\} \subset X$, such that*

$$\Im(u_n) \to c, \text{ and } \Im'(u_n) \to 0, \text{ in } X^*, \text{ as } n \to \infty,$$

contains a convergent subsequence.

In the following lemma, we establish a result regarding the boundedness of a Palais−Smale sequence in X.

Lemma 2. *Suppose that conditions* $(\mathbf{C_2}) - (\mathbf{C_5})$ *are satisfied. Let $\{u_n\}$ be a Palais−Smale sequence of $\Im$ in X. Then $\{u_n\}$ is bounded in X.*

Proof. Let $\{u_n\}$ be a sequence in X such that

$$\Im(u_n) \to c, \text{ and } \Im'(u_n) \to 0, \text{ in } X^*, \text{ as } n \to \infty,$$

where c is a positive constant.

Since $\Im(u_n) \to c$, there exists $T_1 > 0$, such that

$$|\Im(u_n)| \leq T_1. \tag{3.5}$$

On the other hand, the fact that $\Im'(u_n) \to 0$ in X^*, implies that $< \Im'(u_n), u_n > \to 0$. In particular, $< \Im'(u_n), u_n >$ is bounded. Thus, there exists $T_2 > 0$, such that

$$\left|< \Im'(u_n), u_n >\right| \leq T_2. \tag{3.6}$$

We claim that the sequence $\{u_n\}$ is bounded. If it is not true, by passing to a sub-sequence if necessary, we may assume that $\|u_n\| \to \infty$. Without loss of generality, we assume that $\|u_n\| \geq 1$.

From (3.5), (3.4) and using $\kappa^+ < \theta < \min(r^-, s^-)$, we obtain

$$\begin{aligned} T_1 \geq \Im(u_n) &= L(u_n) - I(u_n) - J(u_n) - T(u_n) \\ &\geq \frac{(1-\omega)m_0}{\kappa^+}\int_\Omega |\nabla u_n|^{\kappa(x)}\,dx - \frac{1}{r^-}\int_\Omega |u_n|^{r(x)}dx - \frac{1}{s^-}\int_{\partial\Omega} |u_n|^{s(x)}dx - J(u_n) \\ &\geq \frac{(1-\omega)m_0}{\kappa^+}\int_\Omega |\nabla u_n|^{\kappa(x)}\,dx - \frac{1}{\theta}\int_\Omega |u_n|^{r(x)}dx - \frac{1}{\theta}\int_{\partial\Omega} |u_n|^{s(x)}dx - J(u_n). \end{aligned} \tag{3.7}$$

On the other hand by (3.6) and assumption $(\mathbf{C_2})$, we have

$$\begin{aligned} T_2 &\geq -< \Im'(u_n), u_n > \\ &= -\mathfrak{S}\left(\int_\Omega \frac{1}{\kappa(x)}|\nabla u|^{\kappa(x)}\,dx\right)\int_\Omega |\nabla u_n|^{\kappa(x)}\,dx + \int_\Omega |u_n|^{r(x)}dx \\ &+ \int_{\partial\Omega} |u_n|^{s(x)}dx + < J'(u_n), (u_n) > \\ &\geq -m_0\int_\Omega |\nabla u_n|^{\kappa(x)}\,dx + \int_\Omega |u_n|^{r(x)}dx + \int_{\partial\Omega} |u_n|^{s(x)}dx + < J'(u_n), (u_n) > . \end{aligned}$$

By combining the above inequality and (3.7), we obtain

$$\begin{aligned}\theta T_1 + T_2 &\geq (\frac{(1-\omega)\theta}{\kappa^+} - 1)m_0 \int_\Omega |\nabla u_n|^{\kappa(x)}\,dx + < J^{'}(u_n),(u_n) > -\theta J(u_n) \\ &\geq (\frac{(1-\omega)\theta}{\kappa^+} - 1)m_0\,\|u_n\|^{\kappa^-} + \int_\Omega a(x)\Big(h(u_n)u_n - \theta H(u_n)\Big)dx.\end{aligned}$$

Hence, assumption ($\mathbf{C_4}$) implies

$$\theta T_1 + T_2 \geq \Big(\frac{(1-\omega)\theta}{\kappa^+} - 1\Big)m_0\,\|u_n\|^{\kappa^-}. \tag{3.8}$$

By ($\mathbf{C_4}$) and ($\mathbf{C_2}$), $\frac{\kappa^+}{1-\omega} < \theta$ and $m_0 > 0$, then, we have $\Big(\frac{(1-\omega)\theta}{\kappa^+} - 1\Big)m_0 > 0$, so

$$\Big(\frac{(1-\omega)\theta}{\kappa^+} - 1\Big)m_0\,\|u_n\|^{\kappa^-} \to \infty.$$

By (3.8) this is absurder. Then, $\{u_n\}$ is bounded in X. □

Now, we introduce the nonempty set A define by $A = \{x \in \Omega : r(x) = \kappa^*(x)\}$. Also, define the set $A_\delta = \{x \in \Omega : dist((x, A) < \delta\}$ for $\delta > 0$. We note $r_\delta^- = \inf_{\overline{A_\delta}} r(x)$, and $r_A^- = \inf_{\overline{A}} r(x)$.
We will now introduce and recall several important theorem.

Theorem 3. *(Concentration−compactness principle) (see [5]) Let $\kappa(x)$ and $r(x)$ be two continuous functions such that*

$$\kappa^- = \inf_{\overline{\Omega}} \kappa(x) \leq \kappa^+ = \sup_{\overline{\Omega}} \kappa(x) < N \;\; and \;\; 1 < r(x) \leq \kappa^*(x) \quad in \;\; \Omega.$$

Let $\{u_j\}_{j\in\mathbb{N}}$ be a weakly convergent sequence in $W_0^{1,\kappa(x)}(\Omega)$ with weak limit u and such that:
- *$|u_j|^{r(x)} \rightharpoonup \nu$ weakly in the sense of measures.*
- *$|\nabla u_j|^{\kappa(x)} \rightharpoonup \mu$ weakly in the sense of measures.*

Also assume that $A = \{x \in \Omega : r(x) = \kappa^*(x)\}$ is nonempty. Then, for some countable index set I, we have:

$$\nu = |u|^{r(x)} + \sum_{i\in I} \nu_i \delta_{x_i}, \qquad \nu_i > 0.$$

$$\mu \geq |\nabla u|^{\kappa(x)} + \sum_{i\in I} \mu_i \delta_{x_i}, \qquad \mu_i > 0.$$

$$S \nu_i^{\frac{1}{\kappa^*(x_i)}} \leq \mu_i^{\frac{1}{\kappa(x_i)}}.$$

Where $\{x_i\}_{i\in I} \subset A$ and S is the best constant in the Gagliardo-Nirenberg-Sobolev inequality for variable exponents, namely

$$S = S_r(\Omega) = \inf_{\phi \in C_0^\infty(\Omega)} \frac{\|\nabla\phi\|_{\mathscr{L}^{\kappa(x)}}}{\|\phi\|_{\mathscr{L}^{\kappa(x)}}}.$$

If $\{u_n\}$ is a Palais–Smale sequence with energy level c, then according to Theorem 3, we have the following convergence results:

$$|u_n|^{r(x)} \rightharpoonup \nu = |u|^{r(x)} + \sum_{i\in I} \nu_i \delta_{x_i}, \qquad \nu_i > 0. \tag{3.9}$$

$$|\nabla u_n|^{\kappa(x)} \rightharpoonup \phi \geq |\nabla u|^{\kappa(x)} + \sum_{i\in I} \mu_i \delta_{x_i}, \qquad \mu_i > 0. \tag{3.10}$$

$$S\nu_i^{\frac{1}{\kappa^*(x_i)}} \leq \mu_i^{\frac{1}{\kappa(x_i)}}. \tag{3.11}$$

If $I = \emptyset$, then $u_n \to u$ in $\mathscr{L}^{r(x)}(\Omega)$. It should be noted that $\{x_i\}_{i\in A} \subset A$. We aim to demonstrate that if $c < (\frac{1}{\kappa^+} - \frac{1}{r_A^-})S^n$, then $I = \emptyset$, where S is defined in Theorem 3.

The following lemma establishes an important result regarding the behavior of Palais−Smale sequences under certain conditions.

Lemma 3. *If conditions* $(\mathbf{C_1}) - (\mathbf{C_5})$ *are satisfied. Let* $\{u_n\}$ *be a Palais−Smale sequence of* $\mathfrak{J}$ *in* X *with energy level* c. *If* $c < (\frac{1}{\kappa^+} - \frac{1}{r_A^-})S^n$, *then the index set* I *is empty.*

Proof. Suppose that $I \neq \emptyset$ and let $\varphi \in C_0^\infty(\mathbb{R}^n)$ such that $\varphi(0) \neq 0$. Now, we consider the functions $\varphi_{i,\epsilon}(x) = \varphi(\frac{x-x_i}{\epsilon})$. We have $< \mathfrak{J}'(u_n), \varphi_{i,\epsilon}u_n > \to 0$. Thus,

$$\begin{aligned} & < \mathfrak{J}'(u_n), \varphi_{i,\epsilon}u_n > \\ = \ & \mathfrak{S}\Big(\int_\Omega \frac{1}{\kappa(x)}|\nabla u_n|^{\kappa(x)}\,dx\Big)\int_\Omega |\nabla u_n|^{(\kappa(x)-2)}\nabla u_n \nabla(\varphi_{i,\epsilon}u_n)dx \\ - \ & \int_\Omega |u_n|^{r(x)}\varphi_{i,\epsilon}dx - \int_\Omega a(x)h(u_n(x))\varphi_{i,\epsilon}u_n dx - \int_{\partial\Omega} |u_n|^{s(x)}\varphi_{i,\epsilon}dx. \end{aligned}$$

Passing to the limit as $n \to \infty$, we have

$$\begin{aligned} 0 \ = \ & \lim_{n\to\infty}\Big(\mathfrak{S}\Big(\int_\Omega \frac{1}{\kappa(x)}|\nabla u_n|^{\kappa(x)}\,dx\Big)\int_\Omega |\nabla u_n|^{(\kappa(x)-2)}\nabla u_n \nabla(\varphi_{i,\epsilon})u_n dx \\ + \ & \int_\Omega \varphi_{i,\epsilon}d\mu - \int_\Omega \varphi_{i,\epsilon}d\nu - \int_\Omega a(x)h(u_n(x))\varphi_{i,\epsilon}u_n dx - \int_{\partial\Omega} |u_n|^{s(x)}\varphi_{i,\epsilon}dx\Big). \end{aligned}$$

By Hölder's inequality and using hypothesis $(\mathbf{C_2})$ we can show that

$$\lim_{n\to\infty}\int_\Omega |\nabla u_n|^{(\kappa(x)-2)}\nabla u_n \nabla(\varphi_{i,\epsilon})u_n dx = 0.$$

On the other hand, we have

$$\lim_{\epsilon\to 0}\int_\Omega a(x)h(u_n(x))\varphi_{i,\epsilon}u_n dx = 0, \qquad \lim_{\epsilon\to 0}\int_{\partial\Omega} |u_n|^{s(x)}\varphi_{i,\epsilon}dx = 0$$

$$\lim_{\epsilon\to 0}\int_\Omega \varphi_{i,\epsilon}d\mu = \mu_i\varphi(0), \qquad \lim_{\epsilon\to 0}\int_\Omega \varphi_{i,\epsilon}d\nu = \nu_i\varphi(0).$$

Then, $\Big(\mu_i - \nu_i\Big)\varphi(0) = 0$, which implies $\mu_i = \nu_i$. Consequently,

$$S\nu_i^{\frac{1}{\kappa^*(x_i)}} \leq \nu_i^{\frac{1}{\kappa(x_i)}},$$

Thus, we conclude that $\nu_i = 0$ or $S^n \leq \nu_i$.

Now, since $r(x), s(x), \theta > \kappa^+ > 1-\omega$, and by using hypothesis (**C**$_4$), we have

$$\begin{aligned}
c &= \lim_{n\to\infty} \mathfrak{J}(u_n) = \lim_{n\to\infty}\Big(\mathfrak{J}(u_n) - \frac{1}{\kappa^+} < \mathfrak{J}'(u_n), u_n > \Big)\\
&= \lim_{n\to\infty}\Big(\widehat{\mathfrak{S}}\Big(\int_\Omega \frac{1}{\kappa(x)}|\nabla u|^{\kappa(x)}\,dx\Big) - \int_\Omega \frac{|u_n|^{r(x)}}{r(x)}dx - \int_\Omega a(x)H(u_n)dx - \int_\Omega \frac{|u_n|^{s(x)}}{s(x)}dx\\
&- \frac{1}{\kappa^+}\mathfrak{S}\Big(\int_\Omega \frac{1}{\kappa(x)}|\nabla u_n|^{\kappa(x)}\,dx\Big)\int_\Omega \frac{1}{\kappa(x)}|\nabla u_n|^{\kappa(x)}\,dx + \frac{1}{\kappa^+}\int_\Omega a(x)h(u_n)u_n dx\\
&+ \frac{1}{\kappa^+}\int_\Omega |u_n|^{r(x)}dx + \frac{1}{\kappa^+}\int_{\partial\Omega}|u_n|^{s(x)}dx\Big)\\
&\geq \lim_{n\to\infty}\Big((1-\omega)\mathfrak{S}\Big(\int_\Omega \frac{1}{\kappa(x)}|\nabla u_n|^{\kappa(x)}\,dx\Big)\int_\Omega \frac{1}{\kappa(x)}|\nabla u_n|^{\kappa(x)}\\
&- \frac{1}{\kappa^+}\mathfrak{S}\Big(\int_\Omega \frac{1}{\kappa(x)}|\nabla u_n|^{\kappa(x)}\,dx\Big)\int_\Omega \frac{1}{\kappa(x)}|\nabla u_n|^{\kappa(x)}\,dx\\
&+ \int_\Omega (\frac{1}{\kappa(x)} - \frac{1}{r(x)})|u_n|^{r(x)}dx + \int_{\partial\Omega}(\frac{1}{\kappa(x)} - \frac{1}{s(x)})|u_n|^{s(x)}dx\\
&+ \frac{1}{\kappa^+}\int_\Omega a(x)h_1(u_n)u_n dx - \int_\Omega a(x)H(u_n)dx\Big)\\
&\geq \lim_{n\to\infty}\Big((1-\omega-\frac{1}{\kappa^+})\mathfrak{S}\Big(\int_\Omega \frac{1}{\kappa(x)}|\nabla u_n|^{\kappa(x)}\,dx\Big)\int_\Omega \frac{1}{\kappa(x)}|\nabla u|^{\kappa(x)}\,dx\\
&+ \int_\Omega (\frac{1}{\kappa^+} - \frac{1}{r(x)})|u_n|^{r(x)}dx + \int_{\partial\Omega}(\frac{1}{\kappa^+} - \frac{1}{s(x)})|u_n|^{s(x)}dx\\
&+ \frac{1}{\theta}\int_\Omega a(x)h(u_n)u_n dx - \int_\Omega a(x)H(u_n)dx\\
&\geq \lim_{n\to\infty}\int_\Omega (\frac{1}{\kappa^+} - \frac{1}{r(x)})|u_n|^{r(x)}dx\\
&\geq \lim_{n\to\infty}\int_{A_\delta} (\frac{1}{\kappa^+} - \frac{1}{r^-_{A_\delta}})|u_n|^{r(x)}dx.
\end{aligned}$$

On the other hand

$$\begin{aligned}
\lim_{n\to\infty}\int_{A_\delta}(\frac{1}{\kappa^+} - \frac{1}{r^-_{A_\delta}})|u_n|^{r(x)}dx &= (\frac{1}{\kappa^+} - \frac{1}{r^-_{A_\delta}})(\int_{A_\delta}|u|^{r(x)} + \sum_{i\in I}\nu_i)\\
&\geq (\frac{1}{\kappa^+} - \frac{1}{r^-_{A_\delta}})\nu_i\\
&\geq (\frac{1}{\kappa^+} - \frac{1}{r^-_{A_\delta}})S^n. \qquad (3.12)
\end{aligned}$$

Therefore, since δ is positive and arbitrary and r is continuous, we have

$$c \geq (\frac{1}{\kappa^+} - \frac{1}{r^-_A})S^n.$$

Then if $c < (\frac{1}{\kappa^+} - \frac{1}{r^-_A})S^n$, the index set I is empty. □

We now present the following lemma that establishes an important convergence result.

Lemma 4. *If conditions* $(\mathbf{C_1}) - (\mathbf{C_5})$ *are satisfied, let* $\{u_n\}$ *be a Palais–Smale sequence of* $\mathfrak{I}$ *in* X*, with energy level* c*. Then there exists a subsequence of* $\{u_n\}$ *that converges strongly in* X*.*

Proof. Let $\{u_n\}$ be a $(PS)_c$ sequence in X. By Lemma 2, $\{u_n\}$ is bounded in X. Then, there exists a subsequence of $\{u_n\}$, such that $u_n \rightharpoonup u$.

Using Lemma 3, $p(x) < \kappa^*(x)$, $s(x) < \kappa_*(x)$ and Proposition 3, we have

$$\begin{cases} u_n \to u, \text{ strongly in } \mathscr{L}^{p(x)}(\Omega), \\ u_n \to u, \text{ strongly in } \mathscr{L}^{r(x)}(\Omega), \\ u_n \to u, \text{ strongly in } \mathscr{L}^{s(x)}(\partial\Omega). \end{cases}$$

Next, we will show that $u_n \to u$. We start by considering the inner product

$$\begin{aligned} < \mathfrak{I}'(u_n), u_n - u > \;&=\; < L'(u_n), u_n - u > - \int_\Omega |u_n|^{r(x)-2} u_n (u_n - u) dx - \int_{\partial\Omega} |u_n|^{s(x)-2} u_n (u_n - u) dx \\ &\quad - \int_\Omega a(x) h(u_n)(u_n - u) dx. \end{aligned}$$

By applying Hölder's inequality, Propositions 3 and 6, we can estimate the integral term as follows:

$$\begin{aligned} \int_\Omega |u_n|^{r(x)-1} |u_n - u| dx \;&\leq\; |u_n - u|_{\mathscr{L}^{r(x)}} ||u|^{r(x)-1}|_{\mathscr{L}^{\frac{r(x)}{r(x)-1}}} \\ &\leq\; |u_n - u|_{\mathscr{L}^{r(x)}} \max(|u_n|^{r^+-1}_{\mathscr{L}^{r(x)}}, |u_n|^{r^--1}_{\mathscr{L}^{r(x)}}) \\ &\leq\; c_1 |u_n - u|_{\mathscr{L}^{r(x)}} \max(\|u_n\|^{r^+-1}, \|u_n\|^{r^--1}). \end{aligned}$$

This leads to the conclusion

$$\lim_{n\to\infty} \int_\Omega |u_n|^{r(x)-2} u_n (u_n - u) dx = 0. \tag{3.13}$$

Similarly, we get

$$\lim_{n\to\infty} \int_{\partial\Omega} |u_n|^{s(x)-2} u_n (u_n - u) dx = 0. \tag{3.14}$$

Now, by using $(\mathbf{C_1})$, propositions 3 and 6, and Hölder's inequality, we obtain

$$\begin{aligned} \int_\Omega a(x) h(u_n)(u_n - u) dx \;&\leq\; \int_\Omega c_1 |a(x)| |u_n|^{q(x)-1} |u_n - u| dx \\ &\leq\; c_1 |u_n - u|_{\mathscr{L}^{p(x)}} |a(x)|_{\mathscr{L}^{\frac{p(x)}{p(x)-q(x)}}} ||u_n|^{q(x)-1}|_{\mathscr{L}^{\frac{p(x)}{q(x)-1}}} \\ &\leq\; c_1 |u_n - u|_{\mathscr{L}^{p(x)}} |a(x)|_{\mathscr{L}^{\frac{p(x)}{p(x)-q(x)}}} \max(|u_n|^{q^+-1}|_{\mathscr{L}^{p(x)}}, |u_n|^{q^--1}|_{\mathscr{L}^{p(x)}}) \\ &\leq\; c_1 |u_n - u|_{\mathscr{L}^{p(x)}} |a(x)|_{\mathscr{L}^{\frac{p(x)}{p(x)-q(x)}}} \max(\|u_n\|^{q^+-1}|, \|u_n\|^{q^--1}|). \end{aligned}$$

Hence, we have

$$\lim_{n\to\infty} \int_\Omega a(x) h(u_n)(u_n - u) dx = 0. \tag{3.15}$$

By combining (3.13)–(3.15), and using the fact that $< \mathfrak{I}'(u_n), u_n - u > \to 0$, we conclude that

$$< L'(u_n), u_n - u >$$

$$= \quad \mathfrak{S}\Big(\int_\Omega \frac{1}{\kappa(x)}|\nabla u|^{\kappa(x)}\,dx\Big)\int_\Omega |\nabla u_n|^{(\kappa(x)-2)}\nabla u_n\nabla(u_n-u)dx \to 0.$$

Hence by using hypothesis ($\mathbf{C_2}$), we have $\mathfrak{S}\Big(\int_\Omega \frac{1}{\kappa(x)}|\nabla u|^{\kappa(x)}\,dx\Big) \neq 0$, then

$$< \mathfrak{L}'(u_n), u_n - u >= \int_\Omega |\nabla u_n|^{(\kappa(x)-2)}\nabla u_n\nabla(u_n-u)dx \to 0.$$

Since $u_n \rightharpoonup u$, we get $< \mathfrak{L}'(u), u_n - u > \to 0$.
Finally, we get

$$\lim_{n\to\infty} < \mathfrak{L}'(u_n) - \mathfrak{L}'(u), u_n - u >= 0.$$

By Proposition 8, $\mathfrak{L}'$ is of type (S_+), then $u_n \to u$. □

In order to further investigate the properties of the functional $\mathfrak{J}$ and its critical points, we establish the following lemma.

Lemma 5. *If conditions* ($\mathbf{C_2}$) − ($\mathbf{C_4}$) *hold. Then, there exists* $u_0 \in X$ *such that*

$$\|u_0\| > \eta, \quad \textit{and} \quad \mathfrak{J}(u_0) < 0,$$

where η *is defined in Lemma 1.*

Proof. By ($\mathbf{C_4}$) , we get

$$G(x,t) \geq \xi|t|^\theta, \; x \in \Omega, \; |t| \geq M_1. \tag{3.16}$$

By the conditions ($\mathbf{C_2}$) and ($\mathbf{C_3}$) , the function $t \longmapsto \dfrac{\widehat{\mathfrak{S}}(t)}{t^{1/w-1}}$ is decreasing. So for all $t_0 > 0$, when $t > t_0$, yields $\dfrac{\widehat{\mathfrak{S}}(t)}{t^{1/w-1}} \leq \dfrac{\widehat{\mathfrak{S}}(t_0)}{t_0^{1/w-1}}$, then,

$$\widehat{\mathfrak{S}}(t) \leq \frac{\widehat{\mathfrak{S}}(t_0)}{t_0^{1/1-w}} t^{1/1-\omega} \leq ct^{\frac{1}{1-\omega}}, \; for \; t > t_0. \tag{3.17}$$

Let $u \in X$ such that $\int_\Omega |u|^\theta \neq 0$ and let $t > 1$, be sufficiently large. Then, we have by (3.16) and (3.17)

$$\begin{aligned} \mathfrak{J}(tu) &\leq \widehat{\mathfrak{S}}\left(\int_\Omega \frac{1}{\kappa(x)}|\nabla tu|^{\kappa(x)}\,dx\right) - \int_\Omega a(x)H(tu)dx \\ &\leq C\Big(\int_\Omega |\nabla u|^{\kappa(x)}dx\Big)^{1/1-\omega} t^{\frac{\kappa^+}{1-\omega}} - c\xi t^\theta \int_\Omega |u|^\theta dx. \end{aligned}$$

Since $\theta > \frac{\kappa^+}{1-\omega}$, it follows that

$$\mathfrak{J}(tu) \to -\infty, \quad \text{as} \quad t \to \infty.$$

Therefore, we can choose $t_0 > 0$ and set $u_0 = t_0 e$, such that $\|u_0\| > \eta$ and $\mathfrak{J}(u_0) < 0$. This completes the proof. □

Now, we establish the following lemma that provides a key result regarding the boundedness of a set under certain hypotheses.

Lemma 6. *Under hypotheses* $(\mathbf{C_1}) - (\mathbf{C_4})$*, if F is a finite dimensional subspace of X, then the set*

$$T = \{u \in F, \;\; such \;\; that \;\; \Im(u) \geq 0\},$$

is bounded in X.

Proof. Let $u \in T$. By (3.16) and (3.17) , we have:

$$\begin{aligned} \Im(u) &\leq \widehat{\mathfrak{S}}\left(\int_\Omega \frac{1}{\kappa(x)} |\nabla u|^{\kappa(x)}\, dx\right) - \int_\Omega a(x) H(u) dx \\ &\leq C\Big(\int_\Omega |\nabla u|^{\kappa(x)} dx\Big)^{1/1-\omega} - \xi \int_\Omega |u|^\theta dx \\ &\leq C(\|u\|^{\frac{\kappa^+}{1-\omega}} + \|u\|^{\frac{\kappa^-}{1-\omega}}) - \xi |u|^\theta_{\mathscr{L}^\theta}, \end{aligned}$$

where $|.|_{\mathscr{L}^\theta}$ and $\|.\|$ are equivalent norms in F. Thus, there exists a positive constant k such that

$$\|u\|^\theta \leq k |u|^\theta_{\mathscr{L}^\theta}.$$

Therefore, we have

$$\Im(u) \leq \frac{\kappa^+}{1-\omega}(\|u\|^{\frac{\kappa^+}{1-\omega}} + \|u\|^{\frac{\kappa^-}{1-\omega}}) - \frac{\xi}{k}\|u\|^\theta.$$

Hence, since $\frac{\kappa^-}{1-\omega} < \frac{\kappa^+}{1-\omega} < \theta$, we deduce that T is bounded in X. □

In the context of our analysis and proofs, we will now introduce and recall several important theorems: The mountain pass theorem, its symmetric version for even functions. These theorems play a crucial role in establishing our results. Here are the statements of the theorems:

Theorem 4. *(Mountain pass theorem) (see [2]) Let X be a Banach space. Consider a functional* $\Im \in C^1(X, \mathbb{R})$ *satisfying the following conditions:*

(1) $\Im(0) = 0$*;*

(2) $\Im$ *satisfies the (PS) condition;*

(3) There exist positive constants η *and* ρ *such that if* $\|u\| = \eta$*, then* $\Im(u) \geq \rho$*;*

(4) There exists $e \in X$ *with* $\|e\| > \eta$ *such that* $\Im(e) \leq 0$*. Then,* $\Im$ *possesses a critical value* $c \geq \rho$ *which can be characterized as*

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \Im(\gamma(t)),$$

where,

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = e\}.$$

Theorem 5. *(Symmetric mountain pass theorem) (see [2]) Let X be an infinite dimensional real Banach space. Let* $\Im \in C^1(X, \mathbb{R})$*, satisfying the following conditions:*

(1) $\Im$ *is an even functional such that* $\Im(0) = 0$*.*

(2) $\Im$ *satisfies the (PS)-condition.*

(3) There exist positive constants η and ρ, such that if $\|u\| = \eta$, then, $\mathfrak{J}(u) \geq \rho$.

(4) For each finite dimensional subspace $F \subset X$, the set $\{u \in F, \mathfrak{J}(u) \geq 0\}$ is bounded in X. Then $\mathfrak{J}$ has an unbounded sequence of critical values.

Proof of Theorem 1. Lemmas 1, 4, and 5 establish the fulfillment of all the conditions required by Theorem 4 (mountain pass theorem), ensuring the existence of a nontrivial solution to problem (1.1). With this, the proof of Theorem 1 is now concluded. □

Proof of Theorem 2. We observe that $\mathfrak{J}(0) = 0$, and due to $(\mathbf{C_6})$, the functional $\mathfrak{J}$ is even. Furthermore, Lemmas 1, 4, and 6 establish the fulfillment of all the conditions stated in Theorem 5 (symmetric mountain pass theorem). Consequently, we can conclude that problem (1.1) possesses an unbounded sequence of nontrivial solutions. With this, the proof of Theorem 2 is now completed. □

4. Conclusions

In this paper, the existence of a solution and an infinite number of solutions for the Steklov problem have been proven under appropriate conditions on our parameters within variable exponent Sobolev spaces. An interesting perspective is to extend this work to Robin-type problems and to problems involving Leray-Lions type operators.

Author contributions

Khaled Kefi, Abdeljabbar Ghanmi, Abdelhakim Sahbani, and Mohammed M. Al-Shomrani contributed to the conception and design of the study; Abdelhakim Sahbani provided resources and support for the research; Abdeljabbar Ghanmi, Abdelhakim Sahbani, and Mohammed M. Al-Shomrani reviewed and edited the manuscript; Khaled Kefi acquired funding. All authors have read and approved the final version of the manuscript for publication.

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Conflicts of interest

The authors declare no conflict of interest.

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