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Research article

On the study the radius of analyticity for Korteweg-de-Vries type systems with a weakly damping

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Abstract: In the present paper, we considered a Korteweg-de Vries type system with weakly damping terms and initial data in the analytic Gevery spaces. The presence of tow functions $c_1(x), c_2(x)$, called damping coefficients, made the system more interesting from an application point of view due to their great importance in physics. To start, by using the fixed point theorem in Banach space, we investigated the local well-posedness. Additionally, by employing an approximate conservation law, we extended this to be global in time, ensuring that the radius of analyticity of solutions remained uniformly bounded below by a fixed positive number for all time.

Keywords: KdV system; global Well-posedness; analytic spaces; nonlinear equations; approximate conservation law; iterative methods
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1. Introduction

The Korteweg-de Vries (KdV) equation is a fundamental partial differential equation that describes the propagation of solitary waves in shallow water channels. In recent years, there has been many authors who have studied the behavior of the analytic radius for the solution of the KdV equation with analytic initial data. The author in [1] proved the exponential convergence rate for a spectral projection of the periodic initial-value problem for the generalized KdV equation. Based on this convergence result, a new method to determine the radius of analyticity of solutions to the generalized KdV equation is derived. Wang [2] considered the following KdV equation:

$$\partial_t\varphi + \partial_x^3\varphi + \varphi\partial_x\varphi + a(x)\varphi = 0, \qquad \varphi(0,x) = \varphi_0(x),$$

where the author established the local well-posedness solution and studied the long-time behavior of the analytic radius for the solution of the KdV equation with damping term and an analytic initial data on the real line. Boukarou and da Silva [3] considered a KdV-Kawahara equation with a weak damping term

$$\partial_t\varphi + \alpha\partial_x^5\varphi + \beta\partial_x^3\varphi + \mu\partial_x\varphi^2 + \lambda\partial_x\varphi^3 + a(x)\varphi = 0, \qquad \varphi(0,x) = \varphi_0(x).$$

The authors used linear, bilinear, and trilinear estimates in analytic Bourgain spaces, to prove the local well-posedness and the behavior of the analytic radius for this problem. Similar articles have the same problems, but without the weakly damping term we mention [4–7].

T. Oh [8] investigated the local well-posedness of the KdV type systems

$$\begin{cases} \varphi_t + a_{11}\varphi_{xxx} + a_{12}\psi_{xxx} + b_1\varphi\varphi_x + b_2\varphi\psi_x + b_3\varphi_x\psi + b_4\psi\psi_x = 0, \\ \psi_t + a_{21}\varphi_{xxx} + a_{22}\psi_{xxx} + b_5\varphi\varphi_x + b_6\varphi\psi_x + b_7\varphi_x\psi + b_8\psi\psi_x = 0, \\ (\varphi,\psi)|_{t=0} = (\varphi_0,\psi_0), \end{cases}$$

in different cases, both periodic and nonperiodic. Guo et al. [9] presented the following problem:

$$\begin{cases} \varphi_t = \varphi_{xxx} - \varphi\varphi_x + \frac{1}{2}(\varphi\psi)_x, \\ \psi_t = \psi_{xxx} - \psi\psi_x + \frac{1}{2}(\varphi\psi)_x, \\ \varphi(0,x) = \varphi_0(x), \psi(0,x) = \psi_0(x), \end{cases}$$

where the authors investigated the global well-posedness of solutions for the system of KdV equations.

In 2018, Yang and Zhang [10] considered the couple KdV system

$$\begin{cases} \varphi_t + a_1\varphi_{xxx} = c_{11}\varphi\varphi_x + c_{12}\psi\psi_x + d_{11}\varphi_x\psi + d_{12}\varphi\psi_x, \\ \psi_t + a_2\psi_{xxx} = c_{21}\varphi\varphi_x + c_{22}\psi\psi_x + d_{21}\varphi_x\psi + d_{22}\varphi\psi_x, \\ \quad (\varphi,\psi)|_{t=0} = (\varphi_0,\psi_0). \end{cases}$$

By denoting $r = \dfrac{a_2}{a_1}$ with $a_1, a_2 \in \mathbb{R} \setminus \{0\}$, in various cases of a constant r, they established the local well-posedness of the problem.

Then, Carvajal and Panthee [11] introduced the system

$$\begin{cases} \partial_t\varphi + \partial_x^3\varphi + \partial_x\left(\varphi\psi^2\right) = 0, & \varphi(t=0,x) = f(x), \\ \partial_t\psi + \alpha\partial_x^3\psi + \partial_x\left(\varphi^2\psi\right) = 0, & \psi(t=0,x) = g(x). \end{cases} \tag{1.1}$$

By different ways, the authors studied the local well-posedness of solutions for this system. Furthermore, if $\alpha = 1$, the problem (1.1) transformed to the system considered by Ablowiz et al. [12].

For more works in KdV systems, we refer to [13–16]. In this paper, we consider a KdV type system with weakly damping terms of the form:

$$\begin{cases} \varphi_t + \varphi_{xxx} + b_1\varphi\varphi_x + b_2\varphi\psi_x + b_3\varphi_x\psi + b_4\psi\psi_x + c_1(x)\varphi = 0, \\ \psi_t + \alpha\psi_{xxx} + b_5\varphi\varphi_x + b_6\varphi\psi_x + b_7\varphi_x\psi + b_8\psi\psi_x + c_2(x)\psi = 0, \\ (\varphi, \psi)|_{t=0} = (\varphi_0, \psi_0), \end{cases} \tag{1.2}$$

where b_i, $i = 1...8$ are nonegative constants.

Several conservation laws are known for the system

$$I_1 = \int \varphi dx,$$

$$I_2 = \int \psi dx,$$

$$I_3 = \int \varphi^2 + \psi^2 dx,$$

$$I_4 = \int \varphi_x^2 + \alpha\psi^2 - \varphi\psi^2 dx,$$

where I_4 is the Hamiltonian of the system and no other conservation laws seem apparent.

2. Preliminaries

Here, we state certain notations, tools, definitions, and functional spaces which will be used later. Let $f \in L^2(\mathbb{R}^2)$. The spatial Fourier transform is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^2} e^{-ix\xi} f(x)dx,$$

and then the space-time Fourier transform is given by

$$\tilde{u}(\xi, \tau) = \int_{\mathbb{R}^2} e^{-i(x\xi + t\tau)} u(x, t)dxdt.$$

Remark 2.1. *We denote by $\mathcal{F}^{-1}(f)$ for the inverse Fourier transformation of the function f.*

For $\sigma \geq 0, b \in \mathbb{R}$, we denote $G^{\sigma,b}(\mathbb{R}^2)$ the Gevery type Bourgain space defined by the norm

$$\|u\|_{G^{\sigma,b}(\mathbb{R}^2)} := \|e^{\sigma|\xi|}(1 + |\tau - \xi^3|)^b \tilde{u}(\xi, \tau)\|_{L^2(\mathbb{R}^2)}, \tag{2.1}$$

and

$$\|u\|_{G^{\sigma,b,\beta}(\mathbb{R}^2)} := \|e^{\sigma|\xi|}(1 + |\tau - \beta\xi^3|)^b \tilde{u}(\xi, \tau)\|_{L^2(\mathbb{R}^2)}, \tag{2.2}$$

and for $\forall\sigma > 0$, we have $G^{\sigma,b}$ is denoted to be the restrictions of $G^{\sigma,b}(\mathbb{R}^2)$ to $\mathbb{R} \times (-\nu, \nu)$. The spaces $G_\nu^{\sigma,b}$ are defined by:

$$\|u\|_{G_\nu^{\sigma,b}} = \inf \{\|g\|_{G^{\sigma,b}} : g = u \text{ on } \mathbb{R} \times (-\nu, \nu)\}.$$

Remark 2.2. *When we replace $e^{\sigma|\xi|}$ by $(1+|\xi|)^s$ in (2.1), it will be the classical Bourgain spaces $X^{s,b}$ and $X_\nu^{s,b}$.*

Let $e^{\sigma|D_x|}$ be the Fourier multiplier operator with symbol $e^{\sigma|\xi|}$, where

$$\widehat{e^{\sigma|D_x|}f} = e^{\sigma|\xi|}\hat{f}.$$

Then, the norm of G^σ can be expressed as

$$\|\varphi\|_{G^\sigma} = \|e^{\sigma|D_x|}\varphi\|_{L^2}.$$

The interest in these spaces is due to the following fact, for which a discussion can be found in [17].

Theorem 2.1. *(Paley-Wiener theorem) Let $\sigma > 0$, then $f \in G^\sigma$ if, and only if, it is the restriction to the real line of a function F, which is holomorphic in the strip*

$$S_\sigma = \{x + iy : \ x, y \in \mathbb{R}, |y| < \sigma\},$$

and satisfies

$$\sup_{|y|<\sigma} \|F(x+iy)\|_{L^2_x} < \infty.$$

By the Paley-Wiener theorem, every function in G^σ has a uniform analytic radius σ on the real line, see Figure 1.

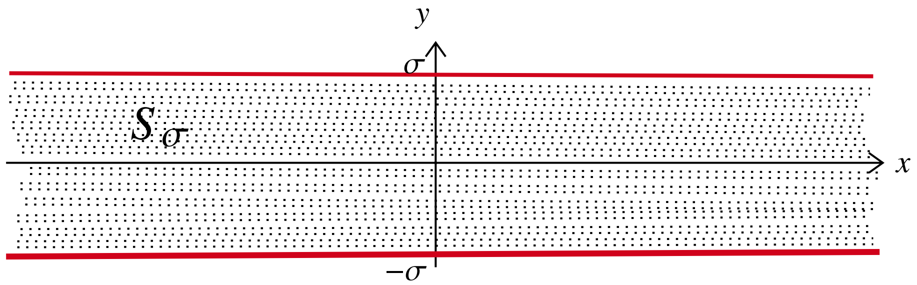

Figure 1. Strip around x-axis.

The question of local existence in G^σ was studied in [6]. These works show the following fact:
If $(\varphi_0, \psi_0) \in G^{\sigma_0} \times G^{\sigma_0}$ with some $\sigma_0 > 0$, then

$$(\varphi(t), \psi(t)) \in G^{\sigma_0} \times G^{\sigma_0}, \tag{2.3}$$

for t small, and $(\varphi(t), \psi(t))$ is a solution of the KdV type system with an appropriate initial data (φ_0, ψ_0). The global well-posedness of the KdV type system in $G^\sigma \times G^\sigma$ is not well treated, since the KdV type system has no conservation law in the analytic space $G^\sigma \times G^\sigma$. Then, we can pose the question

$$\text{whether (2.3) holds } \forall t \in \mathbb{R}? \tag{2.4}$$

Instead of attacking, we can consider the next more suitable problem: If $(\varphi_0, \psi_0) \in G^{\sigma_0} \times G^{\sigma_0}$, then for what kind of $\sigma(t)$ so that

$$(\varphi(t), \psi(t)) \in G^{\sigma(t)} \times G^{\sigma(t)}, \quad \forall t > 0? \tag{2.5}$$

There has been some progress toward answering question (2.5). The main ideas are:

(1) Show the local existence in $G^\sigma \times G^\sigma$ with a lifespan $\nu > 0$;
(2) find an almost conservation law in $G^\sigma \times G^\sigma$, namely, for some $\alpha > 0$,

$$\|\varphi(\nu)\|_{G^\sigma}^2 \le \|\varphi_0\|_{G^\sigma}^2 + C\sigma^\alpha \|\varphi_0\|_{G^\sigma}^3, \tag{2.6}$$

$$\|\psi(\nu)\|_{G^\sigma}^2 \le \|\psi_0\|_{G^\sigma}^2 + C\sigma^\alpha \|\psi_0\|_{G^\sigma}^3; \tag{2.7}$$

(3) by shrinking σ gradually, and using the intervals $[0, \nu]$, $[\nu, 2\nu], \ldots$, we can get a global bound of the solution on $[0, T]$, with $T > 0$ large enough.

As in [6], it is shown that the analytic radius $\sigma(t)$ of solution at t for the system has the lower bound

$$\sigma(t) \ge t^{-\frac{4}{3}-\varepsilon}, \quad t \to \infty,$$

where $\varepsilon > 0$ is a small enough.

Note here that the lower bound does not rule out the possibility that $\sigma(t) \to 0$ as $t \to \infty$. Then, the question (2.4) cannot be answered, as (2.4) is equivalent to see that

$$\sigma(t) \ge \sigma_0, \quad \forall t \ge 0. \tag{2.8}$$

Now, let us consider the analytic radius for a damped KdV type system and prove that

$$\sigma(t) \ge \tilde{\sigma_0}, \quad \forall t \ge 0,$$

for $\tilde{\sigma_0} > 0$. Although, this is still weaker than (2.8), which implies that the analytic radius does not shrink to 0 as $t \longrightarrow \infty$.

We consider the KdV type system on $\mathbb{R}$ with a damping term as

$$\begin{cases} \varphi_t + \varphi_{xxx} + b_1\varphi\varphi_x + b_2\varphi\psi_x + b_3\varphi_x\psi + b_4\psi\psi_x + c_1(x)\varphi = 0, \\ \psi_t + \alpha\psi_{xxx} + b_5\varphi\varphi_x + b_6\varphi\psi_x + b_7\varphi_x\psi + b_8\psi\psi_x + c_2(x)\psi = 0, \\ (\varphi, \psi)|_{t=0} = (\varphi_0, \psi_0). \end{cases}$$

We should impose certain assumptions on the damping coefficients $c_1(\cdot)$ and $c_2(\cdot)$

(H1) Damping effect. There exists $\varepsilon > 0$ such that

$$c_i(x) \ge \varepsilon, \ \ \forall x \in \mathbb{R}, i = 1, 2.$$

(H2) Analyticity. There exist nonegative constants C, M so that

$$\|\partial_x^k c_i\|_{L^\infty(\mathbb{R})} \le CM^k k!, \ \ k \in \mathbb{N}, \ \ i = 1, 2.$$

Proposition 2.1. *[18] Let $\sigma > 0$ and $\nu > 0$.*

(1) If $b > \frac{1}{2}$, then

$$\|f\|_{L_t^\infty G^\sigma} \le C_b \|f\|_{G^{\sigma,b}}.$$

(2) If $-\frac{1}{2} < b < b' < \frac{1}{2}$, then

$$\|f\|_{G_\nu^{\sigma,b}} \le C_{b,b'} \nu^{b'-b} \|f\|_{G_\nu^{\sigma,b'}}.$$

(3) $-\frac{1}{2} < b < \frac{1}{2}$, *then for any interval* $I \subset [-\nu, \nu]$,

$$\|\chi_I f(t)\|_{G^{\sigma,b}} \leq C_b \|f(t)\|_{G_\nu^{\sigma,b}},$$

where χ_I *is the characteristic function of* I.

Proposition 2.2. *[18] (Linear estimates) Let* $\sigma \geq 0$, $\frac{1}{2} < b \leq 1$, *and* $0 < \nu \leq 1$. *Then, for some* $\alpha_1 > 0$, *we have*

$$\begin{aligned}
&\|S(t)\varphi_0\|_{G_\nu^{\sigma,b}} \leq \alpha_1 \|\varphi_0\|_{G^\sigma}, \\
&\|S(t)_\alpha \psi_0\|_{G_\nu^{\sigma,b}} \leq \alpha_1 \|\psi_0\|_{G^\sigma}, \\
&\left\| \int_0^t S(t-s) f(s) ds \right\|_{G_\nu^{\sigma,b}} \leq \|f\|_{G_\nu^{\sigma,b-1}}, \\
&\left\| \int_0^t S_\alpha(t-s) f(s) ds \right\|_{G_\nu^{\sigma,b}} \leq \|f\|_{G_\nu^{\sigma,b-1}}.
\end{aligned}$$

Now, we state the bilinear estimates.

Lemma 2.1. *[18] (Bilinear estimates) Let* $\sigma > 0, b > \frac{1}{2}$ *be sufficiently close to* $\frac{1}{2}$, *and* $b' > \frac{1}{2}$, *then*

$$\begin{aligned}
&\|\varphi\varphi_x\|_{G^{\sigma,b-1}} \leq \|\varphi\|^2_{G^{\sigma,b'}}, \\
&\|\varphi\psi_x\|_{G^{\sigma,b-1}} \leq \|\varphi\|_{G^{\sigma,b'}} \|\psi\|_{G^{\sigma,b'}}, \\
&\|\varphi_x\psi\|_{G^{\sigma,b-1}} \leq \|\varphi\|_{G^{\sigma,b'}} \|\psi\|_{G^{\sigma,b'}}, \\
&\|\psi\psi_x\|_{G^{\sigma,b-1}} \leq \|\psi\|^2_{G^{\sigma,b'}}.
\end{aligned}$$

We shall need to define a special class of functions by A^σ by

$$\|f\|_{A^\sigma} = \sum_{k=0}^{\infty} (k+1)^{\frac{1}{4}} \frac{\sigma^k}{k!} \|\partial_x^k f\|_{L^\infty}.$$

Note that the norms of A^σ and G^σ can be connected. Let us give an equivalent norm of G^σ.

Lemma 2.2. *[2] Let* $\sigma > 0$ *and* $f \in G^\sigma$, *then we have*

$$\|f\|^2_{G^\sigma} \sim \sum_{k \geq 0} \sqrt{k+1} \left(\frac{\sigma^k}{k!} \right) \|\partial_x^k f\|^2_{L^2(\mathbb{R})}.$$

Furthermore, we state the product estimates

Lemma 2.3. *[2] For all* $\sigma > 0, (c, f) \in A^\sigma \times G^\sigma$, *we have*

$$\|cf\|_{G^\sigma} \leq \|c\|_{A^\sigma} \|f\|_{G^\sigma}.$$

Lemma 2.4. *[2] Let* $(c, f) \in A^\sigma \times G^\sigma$ *and* $(\sigma, \nu) \in \mathbb{R}_+^* \times (0, 1], b \geq 0, b' \leq 0$, *then*

$$\|cf\|_{G_\nu^{\sigma,b'}} \leq \|c\|_{A^\sigma} \|f\|_{G_\nu^{\sigma,b}}.$$

These spaces will be essential for constructing analytic solutions to the problem (1.2).

3. Local well-posedness

Let $S(t) = e^{-t\partial_x^3}$ and $S_\alpha(t) = e^{-\alpha t\partial_x^3}$. By Duhamel principle, (φ, ψ) is a solution to system (1.2) if, and only if,

$$\begin{cases} \varphi(t) = S(t)\varphi_0 - \int_0^t S(t-s) F_1(s)\, ds, \\ \psi(t) = S_\alpha(t)\psi_0 - \int_0^t S_\alpha(t-s) F_2(s)\, ds, \end{cases} \tag{3.1}$$

for $-1 \leq t \leq 1$, where

$$F_1(s) = (b_1\varphi\varphi_x + b_2\varphi\psi_x + b_3\varphi_x\psi + b_4\psi\psi_x + c_1(x)\varphi)(s),$$
$$F_2(s) = (b_5\varphi\varphi_x + b_6\varphi\psi_x + b_7\varphi_x\psi + b_8\psi\psi_x + c_2(x)\psi)(s),$$

and we denote $B = \max\{b_i, i = 1, .., 8\}$.

3.1. Existence

Here, we state and prove the local well-posedness theorem.

Theorem 3.1. *(Local well-posedness) Let $b \in \left(\frac{1}{2}, 1\right)$ and $b' \in (b, 1)$ be given by Lemma 2.1. Then, $\forall\sigma \geq 0$ and any $(\varphi_0, \psi_0) \in G^\sigma \times G^\sigma$, and there exists a time $\nu > 0$ given by*

$$\nu \sim_{b,b'} \frac{1}{(2\alpha_2(\|c\|_{A^\sigma} + 4R))^{\frac{1}{b'-b}}}, \tag{3.2}$$

with R being a constant will be define later, and a unique solution (φ, ψ) of (3.1) such that

$$\|(\varphi, \psi)\|_{Y_\nu^{\sigma,b}} \leq 2C_b \|(\varphi_0, \psi_0)\|_{N^\sigma}. \tag{3.3}$$

Proof. Let us consider the mapping

$$\Gamma(\cdot, \cdot) = (\Gamma_1, \Gamma_2), \tag{3.4}$$

where

$$\Gamma_1(\varphi, \psi) = S(t)\varphi_0 + \int_0^t S(t-s)(b_1\varphi\varphi_x + b_2\varphi\psi_x + b_3\varphi_x\psi + b_4\psi\psi_x + c_1(x)\varphi)ds,$$
$$\Gamma_2(\varphi, \psi) = S_\alpha(t)\psi_0 + \int_0^t S_\alpha(t-s)(b_5\varphi\varphi_x + b_6\varphi\psi_x + b_7\varphi_x\psi + b_8\psi\psi_x + c_2(x)\psi)ds,$$

and define the spaces $Y^{\sigma,b}$, $Y_\nu^{\sigma,b}$, and N^σ by

$$N^\sigma = G^\sigma \times G^\sigma,$$
$$Y^{\sigma,b} = G^{\sigma,b} \times G^{\sigma,b,\alpha},$$
$$Y_\nu^{\sigma,b} = G_\nu^{\sigma,b} \times G_\nu^{\sigma,b,\alpha},$$

equipped with norms

$$\|(\varphi, \psi)\|_{Y^{\sigma,b}} = \max\{\|\varphi\|_{G^{\sigma,b}}, \|\psi\|_{G^{\sigma,b,\alpha}}\}, \tag{3.5}$$
$$\|(\varphi, \psi)\|_{Y_\nu^{\sigma,b}} = \max\left\{\|\varphi\|_{G_\nu^{\sigma,b}}, \|\psi\|_{G_\nu^{\sigma,b,\alpha}}\right\}, \tag{3.6}$$

$$\|(\varphi_0,\psi_0)\|_{N^\sigma} = \max\{\|\varphi_0\|_{G^\sigma}, \|\psi_0\|_{G^\sigma}\}, \tag{3.7}$$

and we define the ball $\mathcal{B}$ by

$$\mathcal{B} = \left\{(\varphi,\psi) : \|(\varphi,\psi)\|_{Y_\nu^{\sigma,b}} \le R\right\}. \tag{3.8}$$

where $R = 2\alpha_1\|(\varphi_0,\psi_0)\|_{Y_\nu^{\sigma,b}}$.
The idea of the proof is to show that the functional Γ is contraction in $\mathcal{B}$.

Step1. In this step we show that $\Gamma(\mathcal{B}) \subset \mathcal{B}$. After the Proposition 2.2, we have

$$\|S(t)\varphi_0, S(t)_\alpha\psi_0\|_{Y_\nu^{\sigma,b}} \le \alpha_1\|(\varphi_0,\psi_0)\|_{N^\sigma}. \tag{3.9}$$

Using (2) in Propositions 2.1 and 2.2 and Lemmas 2.1 and 2.4, we find

$$\begin{aligned}
&\left\|\int_0^t S(t-s)(c_1(x)\varphi + b_1\varphi\varphi_x + b_2\varphi\psi_x + b_3\varphi_x\psi + b_4\psi\psi_x)(s)ds\right\|_{G_\nu^{\sigma,b}} \\
\le\ & C_b\nu^{b'-b}\left\|\int_0^t S(t-s)(c_1(x)\varphi + b_1\varphi\varphi_x + b_2\varphi\psi_x + b_3\varphi_x\psi + b_4\psi\psi_x)(s)ds\right\|_{G_\nu^{\sigma,b'}} \\
\le\ & C_{b,b'}\nu^{b'-b}\|c_1(x)\varphi + b_1\varphi\varphi_x + b_2\varphi\psi_x + b_3\varphi_x\psi + b_4\psi\psi_x\|_{G_\nu^{\sigma,b'-1}} \\
\le\ & C_{b,b'}\nu^{b'-b}\left(\|c_1\|_{A^\sigma}\|\varphi\|_{G_\nu^{\sigma,b}} + b_1\|\varphi\|^2_{G_\nu^{\sigma,b}} + (b_2+b_3)\|\varphi\|_{G_\nu^{\sigma,b}}\|\psi\|_{G_\nu^{\sigma,b,\alpha}} + b_4\|\psi\|^2_{G_\nu^{\sigma,b,\alpha}}\right) \\
\le\ & C_{b,b'}\max\{1,b\}\nu^{b'-b}\left(\|c\|_{A^\sigma} + 4\|(\varphi,\psi)\|_{Y_\nu^{\sigma,b}}\right)\|(\varphi,\psi)\|_{Y_\nu^{\sigma,b}},
\end{aligned}$$

where $\|c\|_{A^\sigma} = \max\{\|c_1\|_{A^\sigma}, \|c_2\|_{A^\sigma}\}$. Like the same as before, we get

$$\begin{aligned}
&\left\|\int_0^t S_\alpha(t-s)(c_2(x)\varphi + b_5\varphi\varphi_x + b_6\varphi\psi_x + b_7\varphi_x\psi + b_8\psi\psi_x)(s)ds\right\|_{G_\nu^{\sigma,b,\alpha}} \\
\le\ & C_b\nu^{b'-b}\left\|\int_0^t S_\alpha(t-s)(c_2(x)\varphi + b_5\varphi\varphi_x + b_6\varphi\psi_x + b_7\varphi_x\psi + b_8\psi\psi_x)(s)ds\right\|_{G_\nu^{\sigma,b',\alpha}} \\
\le\ & C_{b,b'}\nu^{b'-b}\|c_2(x)\varphi + b_5\varphi\varphi_x + b_6\varphi\psi_x + b_7\varphi_x\psi + b_8\psi\psi_x\|_{G_\nu^{\sigma,b'-1,\alpha}} \\
\le\ & C_{b,b'}\nu^{b'-b}\left(\|c_2\|_{A^\sigma}\|\varphi\|_{G_\nu^{\sigma,b}} + b_5\|\varphi\|^2_{G_\nu^{\sigma,b}} + (b_6+b_7)\|\varphi\|_{G_\nu^{\sigma,b}}\|\psi\|_{G_\nu^{\sigma,b,\alpha}} + b_8\|\psi\|^2_{G_\nu^{\sigma,b,\alpha}}\right) \\
\le\ & C_{b,b'}\max\{1,b\}\nu^{b'-b}\left(\|c\|_{A^\sigma} + 4\|(\varphi,\psi)\|_{Y_\nu^{\sigma,b}}\right)\|(\varphi,\psi)\|_{Y_\nu^{\sigma,b}}.
\end{aligned}$$

In other words, there exists $\alpha_2 = \alpha_2(b', b, \max\{1,b\}) > 0$ so that

$$\begin{aligned}
&\left\|\int_0^t S(t-s)(c_1(x)\varphi + (b_1\varphi\varphi_x + b_2\varphi\psi_x + b_3\varphi_x\psi + b_4\psi\psi_x)(s)ds\right\|_{Y_\nu^{\sigma,b}} \\
&\le \alpha_2\nu^{b'-b}\left(\|c\|_{A^\sigma} + 4\|(\varphi,\psi)\|_{Y_\nu^{\sigma,b}}\right)\|(\varphi,\psi)\|_{Y_\nu^{\sigma,b}},
\end{aligned} \tag{3.10}$$

$$\begin{aligned}
&\left\|\int_0^t S_\alpha(t-s)(c_2(x)\varphi + (b_5\varphi\varphi_x + b_6\varphi\psi_x + b_7\varphi_x\psi + b_8\psi\psi_x)(s)ds\right\|_{Y_\nu^{\sigma,b}} \\
&\le \alpha_2\nu^{b'-b}\left(\|c\|_{A^\sigma} + 4\|(\varphi,\psi)\|_{Y_\nu^{\sigma,b}}\right)\|(\varphi,\psi)\|_{Y_\nu^{\sigma,b}}.
\end{aligned} \tag{3.11}$$

On one hand, if $(\varphi,\psi)\in\mathcal{B}$, then we deduce from (3.9)–(3.11) such that

$$\begin{aligned}\|\Gamma(\varphi,\psi)\|_{Y_\nu^{\sigma,b}} &\le \alpha_1\|(\varphi_0,\psi_0)\|_{N^\sigma}+\alpha_2\nu^{b'-b}\left(\|c\|_{A^\sigma}+4\|(\varphi,\psi)\|_{Y_\nu^{\sigma,b}}\right)\|(\varphi,\psi)\|_{Y_\nu^{\sigma,b}}\\ &\le \frac{R}{2}+\alpha_2\nu^{b'-b}\left(\|c\|_{A^\sigma}+4R\right)R.\end{aligned}$$

We choose

$$\nu=\frac{1}{\left(2\alpha_2\left(\|c\|_{A^\sigma}+4R\right)\right)^{\frac{1}{b'-b}}}, \tag{3.12}$$

then we get

$$\|\Gamma(\varphi,\psi)\|_{Y_\nu^{\sigma,b}}\le R. \tag{3.13}$$

Step2. Here, we infer that the functional Γ is a contraction in $\mathcal{B}$. Then, for all $[(\varphi,\psi),(z,w)]\in\mathcal{B}\times\mathcal{B}$, similarly we have

$$\|\Gamma(\varphi,\psi)-\Gamma(z,w)\|_{Y_\nu^{\sigma,b}}=\|\Gamma_1(\varphi,\psi)-\Gamma_1(z,w),\Gamma_2(\varphi,\psi)-\Gamma_2(z,w)\|_{Y_\nu^{\sigma,b}}.$$

So, we estimate $\Gamma_1(\varphi,\psi)-\Gamma_1(z,w)$ and $\Gamma_2(\varphi,\psi)-\Gamma_2(z,w)$,

$$\begin{aligned}&\Gamma_1(\varphi,\psi)-\Gamma_1(z,w)\\ =\ &\int_0^t S(t-s)\Big[c_1(x)(\varphi-z)+b_1(\varphi\varphi_x-zz_x)+b_2(\varphi\psi_x-zw_x)+b_3(\varphi_x\psi-z_xw)\\ +\ &b_4(\psi\psi_x-ww_x)\Big]ds.\end{aligned}$$

So, we have

$$\begin{aligned}b_1(\varphi\varphi_x-zz_x)&=b_1[(\varphi-z)\varphi_x+(\varphi_x-z_x)z],\\ b_2(\varphi\psi_x-zw_x)&=b_2[(\varphi-z)\psi_x+(\psi_x-w_x)z],\\ b_3(\varphi_x\psi-z_xw)&=b_3[(\varphi_x-z_x)\psi+(\psi-w)z_x],\\ b_4(\psi\psi_x-ww_x)&=b_4[(\psi-w)\psi_x+(\psi_x-w_x)w],\end{aligned}$$

then, using Lemmas 2.1 and 2.4 and Propositions 2.1 and 2.2,

$$\begin{aligned}&\|\Gamma_1(\varphi,\psi)-\Gamma_1(z,w)\|_{G_\nu^{\sigma,b}}\\ &\le C_{b,b'}\nu^{b'-b}\Big\|\int_0^t S(t-s)\Big[c_1(x)(\varphi-z)+b_1(\varphi\varphi_x-zz_x)+b_2(\varphi\psi_x-zw_x)\\ &+b_3(\varphi_x\psi-z_xw)+b_4(\psi\psi_x-ww_x)\Big]ds\Big\|_{G^{\sigma,b'}}.\end{aligned}$$

It means that there is $\alpha_2(b,b',\max\{1,b\})>0$, such that

$$\begin{aligned}&\|\Gamma_1(\varphi,\psi)-\Gamma_1(z,w)\|_{G_\nu^{\sigma,b}}\\ \le\ &\alpha_2\nu^{b'-b}\Big(\|c\|_{A^\sigma}\|\varphi-z\|_{G^{\sigma,b}}+\|\varphi-z\|_{G^{\sigma,b}}\|\varphi\|_{G^{\sigma,b}}+\|\varphi-z\|_{G^{\sigma,b}}\|z\|_{G^{\sigma,b}}+\|\varphi-z\|_{G^{\sigma,b}}\|\psi\|_{G^{\sigma,b,\alpha}}\\ &+\|\psi-w\|_{G^{\sigma,b,\alpha}}\|z\|_{G^{\sigma,b}}\Big)\\ &+\alpha_2\nu^{b'-b}\Big(\|\varphi-z\|_{G^{\sigma,b}}\|\psi\|_{G^{\sigma,b,\alpha}}+\|\psi-w\|_{G^{\sigma,b,\alpha}}\|z\|_{G^{\sigma,b}}+\|\psi-w\|_{G^{\sigma,b,\alpha}}\|\psi\|_{G^{\sigma,b,\alpha}}\end{aligned}$$

$$+\|\psi - w\|_{G^{\sigma,b,\alpha}}\|w\|_{G^{\sigma,b,\alpha}}\Big)$$
$$\leq \alpha_2 \nu^{b'-b}\left(\|c\|_{A^\sigma} + 4\|(\varphi,\psi)\|_{Y_\nu^{\sigma,b}} + 4\|(z,w)\|_{Y_\nu^{\sigma,b}}\right)\|(\varphi - z, \psi - w)\|_{Y_\nu^{\sigma,b}}.$$

We remind that $[(\varphi,\psi)\times(z,w)] \in \mathcal{B}\times\mathcal{B}$, and we get

$$\|\Gamma_1(\varphi,\psi) - \Gamma_1(z,w)\|_{G_\nu^{\sigma,b}} \leq \alpha_2\nu^{b'-b}\left(\|c\|_{A^\sigma} + 8R\right)\|(\varphi - z,\psi - w)\|_{Y_\nu^{\sigma,b}}. \tag{3.14}$$

Similar to that use as before, we obtain

$$\|\Gamma_2(\varphi,\psi) - \Gamma_2(z,w)\|_{G_\nu^{\sigma,b}} \leq \alpha_2\nu^{b'-b}\left(\|c\|_{A^\sigma} + 8R\right)\|(\varphi - z,\psi - w)\|_{Y_\nu^{\sigma,b}}. \tag{3.15}$$

Inequalities (3.14) and (3.15) lead to

$$\|\Gamma(\varphi,\psi) - \Gamma(z,w)\|_{Y_\nu^{\sigma,b}} \leq \alpha_2\nu^{b'-b}\left(\|c\|_{A^\sigma} + 8R\right)\|(\varphi - z,\psi - w)\|_{Y_\nu^{\sigma,b}}.$$

Then, because $\nu = (2\alpha_2(\|c\|_{A^\sigma} + 4R))^{-1}$, we find

$$\|\Gamma(\varphi,\psi) - \Gamma(z,w)\|_{Y_\nu^{\sigma,b}} < \|(\varphi - z,\psi - w)\|_{Y_\nu^{\sigma,b}}. \tag{3.16}$$

This means that Γ is a contraction in $\mathcal{B}$.

□

3.2. Uniqueness

Uniqueness of the solution in $C([0,\nu],G^\sigma)\times C([0,\nu],G^\sigma)$ can be proved as follows.

Lemma 3.1. *Suppose* (φ,ψ) *and* (φ^*,ψ^*) *are two solutions to (3.1) in* $C([0,\nu],G^\sigma)\times C([0,\nu],G^\sigma)$ *with initial data* $(\varphi_0,\psi_0) = (\varphi_0^*,\psi_0^*)$ *then* $(\varphi,\psi) = (\varphi^*,\psi^*)$.

Proof. Let the conservation law $I(w,w')$ be defined by

$$I(w,w') = \int_{\mathbb{R}}(w^2 + w'^2)dx,$$

and

$$I_1(w) = \int_{\mathbb{R}}(w^2)dx, \qquad I_2(w') = \int_{\mathbb{R}}(w'^2)dx.$$

Suppose (φ,ψ) and (φ^*,ψ^*) are two solutions to (3.1), then

$$\begin{cases} \varphi_t + \varphi_{xxx} + b_1\varphi\varphi_x + b_2\varphi\psi_x + b_3\varphi_x\psi + b_4\psi\psi_x + c_1(x)\varphi = 0, \\ \varphi_t^* + \varphi_{xxx}^* + b_1\varphi^*\varphi_x^* + b_2\varphi^*\psi_x^* + b_3\varphi_x^*\psi^* + b_4\psi^*\psi_x^* + c_1(x)\varphi^* = 0, \end{cases}$$

thus

$$\partial_t(\varphi - \varphi^*) + \partial_x^3(\varphi - \varphi^*) + (b_1\varphi\varphi_x + b_2\varphi\psi_x + b_3\varphi_x\psi + b_4\psi\psi_x + c_1(x)\varphi)$$
$$-(b_1\varphi^*\varphi_x^* + b_2\varphi^*\psi_x^* + b_3\varphi_x^*\psi^* + b_4\psi^*\psi_x^* + c_1(x)\varphi^*) = 0.$$

We have $w = \varphi - \varphi^*$ and $w' = \psi - \psi^*$, then

$$\begin{aligned}\partial_t w + \partial_x^3 w + \Big[c_1(x)(\varphi - \varphi^*) + b_1(\varphi\varphi_x - \varphi^*\varphi_x^*) + b_2(\varphi\psi_x - \varphi^*\psi_x^*)&\\ +b_3(\varphi_x\psi - \varphi_x^*\psi^*) + b_4(\psi\psi_x - \psi^*\psi_x^*)\Big] &= 0.\end{aligned}$$

We have

$$\begin{aligned}b_1(\varphi\varphi_x - \varphi^*\varphi_x^*) &= b_1[w\varphi_x + w_x\varphi^*],\\ b_2(\varphi v_x - \varphi^*\psi_x^*) &= b_2[w\psi_x + w_x'\varphi^*],\\ b_3(\varphi_x\psi - \varphi_x^*\psi^*) &= b_3[w_x\psi + w'\varphi_x^*],\\ b_4(\psi\psi_x - \psi^*\psi_x^*) &= b_4[w'\psi_x + w_x'\psi^*].\end{aligned}$$

So,

$$\begin{aligned}\partial_t w + \partial_x^3 w + \Big[c_1(x)w + b_1[w\varphi_x + w_x\varphi^*] + b_2[w\psi_x + w_x'\varphi^*]&\\ +b_3[w_x\psi + w'\varphi_x^*] + b_4[w'\psi_x + w_x'\psi^*]\Big] &= 0.\end{aligned}$$

Multiplying both sides by w and integrating in space yields

$$\begin{aligned}w\partial_t w + w\partial_x^3 w + w\Big[c_1(x)w + b_1[w\varphi_x + w_x\varphi^*] + b_2[w\psi_x + w_x'\varphi^*]&\\ +b_3[w_x\psi + w'\varphi_x^*] + b_4[w'\psi_x + w_x'\psi^*]\Big] &= 0.\end{aligned}$$

Then, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w\|_{L^2}^2 &= -\int_{\mathbb{R}} w\Big[c_1(x)w + b_1[w\varphi_x + w_x\varphi^*]\\ &\quad +b_2[w\psi_x + w_x'\varphi^*] + b_3[w_x\psi + w'\varphi_x^*] + b_4[w'\psi_x + w_x'\psi^*]\Big]dx\\ \left|\frac{1}{2}\frac{d}{dt}I_1(w)\right| = \left|\frac{1}{2}\frac{d}{dt}\|w\|_{L^2}^2\right| &\le C\|c_1\|_{L^\infty}\|w\|_{L^2}^2 + \|w\|_{L^2}^2\|\varphi_x\|_{L^\infty} + \|w\|_{L^2}^2\|\varphi^*\|_{L^\infty}\\ &\quad +\|w\|_{L^2}^2\|\psi_x\|_{L^\infty} + \|w\|_{L^2}\|w'\|_{L^2}\|\varphi^*\|_{L^\infty}\\ &\quad +\|w\|_{L^2}^2\|\psi\|_{L^\infty} + \|w\|_{L^2}\|w'\|_{L^2}\|\varphi_x^*\|_{L^\infty}\\ &\quad +\|w\|_{L^2}\|w'\|_{L^2}\|\psi_x\|_{L^\infty} + \|w\|_{L^2}\|w_x'\|_{L^2}\|\psi^*\|_{L^\infty}.\end{aligned}$$

We have

$$\int_{\mathbb{R}} w\partial_x^3 w dx = 0.$$

Assume that

$$\|(w, w')\|_{L^2} = max\{\|w\|_{L^2}, \|w'\|_{L^2}\},$$

then

$$\left|\frac{1}{2}\frac{d}{dt}I_1(w)\right| \le C_1\|(w, w')\|_{L^2}^2.$$

By the Gronwall lemma, we get

$$\|w\|_{L^2}^2 \le e^{C_1}\|(w(0), w'(0))\|_{L^2}^2, \quad 0 \le t \le \nu.$$

A similar way for

$$\begin{cases} \psi_t + \alpha\psi_{xxx} + b_5\varphi\varphi_x + b_6\varphi\psi_x + b_7\varphi_x\psi + b_8\psi\psi_x + c_2(x)\psi = 0, \\ \psi_t^* + \alpha\psi_{xxx}^* + b_5\varphi^*\varphi_x^* + b_6\varphi^*\psi_x^* + b_7\varphi_x^*\psi^* + b_8\psi^*\psi_x^* + c_2(x)\psi^* = 0, \end{cases} \tag{3.17}$$

then

$$\left|\frac{1}{2}\frac{d}{dt}I_2(w')\right| \le C_2\|(w, w')\|_{L^2}^2.$$

By the Gronwall lemma, we get

$$\|w'\|_{L^2}^2 \le e^{C_2}\|(w(0), w'(0))\|_{L^2}^2, \quad 0 \le t \le \nu.$$

Then,

$$\begin{aligned} \left|\frac{1}{2}\frac{d}{dt}I(w, w')\right| &= \left|\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}(w^2 + w'^2)dx\right| \\ &\le (e^{C_1} + e^{C_2})\|(w(0), w'(0))\|_{L^2}^2, \quad 0 \le t \le \nu. \end{aligned}$$

Since $\|(w(0), w'(0))\|_{L^2}^2 = 0$, we obtain that $(w, w') = (0, 0)$, $\quad 0 \le t \le \nu$, or $(\varphi, \psi) = (\varphi^*, \psi^*)$.

3.3. Continuous dependence of the initial data

We are going to show that the solution map $\varphi_0 \mapsto \varphi$ is Lipschitz continuous.

Lemma 3.2. *[3] Let $\sigma > 0$ and $b > \frac{1}{2}$, $\nu > 0$, and $(\varphi, \psi), (z, w)$ be solutions of problem* (1.2). *We pertain to initial data (φ_0, ψ_0) and (z_0, w_0), respectively. Then, there exists a constant $C > 0$ such that*

$$\|(\varphi, \psi) - (z, w)\|_{Y_\nu^{\sigma,b}} \le C\|(\varphi_0, \psi_0) - (z_0, w_0)\|_{N^\sigma}.$$

4. Lower bound of analytic radius

4.1. Approximate conservation law

We begin by reminding that, for smooth, compactly supported solutions to (1.2), we have

$$\frac{d}{dt}\|\varphi\|_{L^2}^2 + 2\int_{\mathbb{R}} c_1(x)\varphi^2 dx = 0,$$
$$\frac{d}{dt}\|\psi\|_{L^2}^2 + 2\int_{\mathbb{R}} c_2(x)\psi^2 dx = 0.$$

Under assumption (**H1**), we get

$$\frac{d}{dt}\|\varphi\|_{L^2}^2 \leq -2\int_{\mathbb{R}} \varepsilon\varphi^2 dx,$$
$$\frac{d}{dt}\|\psi\|_{L^2}^2 \leq -2\int_{\mathbb{R}} \varepsilon\psi^2 dx.$$

By applying Grönwall's lemma, we obtain

$$\|\varphi(t)\|_{L^2} \leq \|\varphi_0\|_{L^2} e^{-\varepsilon t}, \tag{4.1}$$
$$\|\psi(t)\|_{L^2} \leq \|\psi_0\|_{L^2} e^{-\varepsilon t}, \tag{4.2}$$

so

$$\|(\varphi(t),\psi(t))\|_{N^0} \leq \|(\varphi_0,\psi_0)\|_{N^0} e^{-\varepsilon t}. \tag{4.3}$$

The aim here is to demonstrate the energy growth bound in N^σ. Then, we state and prove the main theorem.

Theorem 4.1. *Suppose that $c_1, c_2 \in A^{\sigma_0}$ with some $0 < \sigma < \sigma_0$. Let $(\varphi_0,\psi_0) \in N^\sigma$ and $(\varphi,\psi) \in Y_\nu^{\sigma,b}$ be the solution on $[-\nu,\nu]$. Then, we have the following estimate:*

$$\|(\varphi,\psi)(\nu)\|_{Y_\nu^{\sigma,b}} \leq \|(\varphi_0,\psi_0)\|_{N^\sigma}^2 + C_1\left(4\sigma^\eta\|(\varphi_0,\psi_0)\|_{N^\sigma} + \sigma\|c\|_{A_0^\sigma}\right)\|(\varphi_0,\psi_0)\|_{N^\sigma}^2$$
$$+ C_2\|c\|_{L_x^\infty}\|(\varphi_0,\psi_0)\|_{\mathcal{L}^2(\mathbb{R})}\|(\varphi_0,\psi_0)\|_{N^\sigma}, \tag{4.4}$$

where $\|(\varphi_0,\psi_0)\|_{\mathcal{L}^2(\mathbb{R})} = \max\{\|\varphi_0\|_{L^2(\mathbb{R})}, \|\psi_0\|_{L^2(\mathbb{R})}\}$, C_1, C_2 are two constants and η will be defined later.

Proof. We set, for fixed $\sigma > 0$,

$$(\Phi,\Psi) = \left(e^{\sigma|D_x|}\varphi, e^{\sigma|D_x|}\psi\right).$$

We apply $e^{\sigma|D_x|}$ to system (1.2), and we find

$$\partial_t\Phi + \partial_x^3\Phi + b_1\Phi\partial_x\Phi + b_2\Phi\partial_x\Psi + b_3\partial_x\Phi V + b_4\Psi\partial_x\Psi + c_1(x)\Phi$$
$$= I_1 + I_2 + I_3 + I_4 + I_5, \tag{4.5}$$
$$\partial_t\Psi + \partial_x^3\Psi + b_5\Phi\partial_x\Phi + b_6\Phi\partial_x\Psi + b_7\partial_x\Phi V + b_8\Psi\partial_x\Psi + c_2(x)\Phi$$
$$= I'_1 + I'_2 + I'_3 + I'_4 + I'_5, \tag{4.6}$$

where

$$\begin{aligned}
I_1 &= b_1\left(e^{\sigma|D_x|}\varphi\partial_x e^{\sigma|D_x|}\varphi - e^{\sigma|D_x|}(\varphi\partial_x\varphi)\right)\\
I_2 &= b_2\left(e^{\sigma|D_x|}\varphi\partial_x e^{\sigma|D_x|}\psi - e^{\sigma|D_x|}\varphi\partial_x\psi\right)\\
I_3 &= b_3\left(\partial_x e^{\sigma|D_x|}\varphi e^{\sigma|D_x|}\psi - e^{\sigma|D_x|}\psi\partial_x\varphi\right)\\
I_4 &= b_4\left(e^{\sigma|D_x|}\psi\partial_x e^{\sigma|D_x|}\psi - e^{\sigma|D_x|}(\psi\partial_x\psi)\right)\\
I_5 &= c_1(x)e^{\sigma|D_x|}\varphi - e^{\sigma|D_x|}(c_1(x)\varphi),
\end{aligned}$$

and

$$\begin{aligned}
I_1' &= b_5\left(e^{\sigma|D_x|}\varphi\partial_x e^{\sigma|D_x|}\varphi - e^{\sigma|D_x|}(\varphi\partial_x\varphi)\right)\\
I_2' &= b_6\left(e^{\sigma|D_x|}\varphi\partial_x e^{\sigma|D_x|}\psi - e^{\sigma|D_x|}\varphi\partial_x\psi\right)\\
I_3' &= b_7\left(\partial_x e^{\sigma|D_x|}\varphi e^{\sigma|D_x|}\psi - e^{\sigma|D_x|}\psi\partial_x\varphi\right)\\
I_4' &= b_8\left(e^{\sigma|D_x|}\psi\partial_x e^{\sigma|D_x|}\psi - e^{\sigma|D_x|}(\psi\partial_x\psi)\right)\\
I_5' &= c_2(x)e^{\sigma|D_x|}\psi - e^{\sigma|D_x|}(c_1(x)\psi).
\end{aligned}$$

By multiplying (4.5) with Φ and (4.6) by Ψ, and integrating over $\mathbb{R}$ with respect to x, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\Phi^2dx + \int_{\mathbb{R}}c_1(x)\Phi^2dx = \int_{\mathbb{R}}(I_1 + I_2 + I_3 + I_4 + I_5)\,\Phi dx, \tag{4.7}$$

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\psi^2dx + \int_{\mathbb{R}}c_1(x)\psi^2dx = \int_{\mathbb{R}}(I_1' + I_2' + I_3' + I_4' + I_5')\,\Psi dx. \tag{4.8}$$

We remind that $c_i(x) \geq \varepsilon, \forall x \in \mathbb{R}$, and Eqs (4.7) and (4.8) become

$$\frac{d}{dt}\int_{\mathbb{R}}\Phi^2dx + 2\varepsilon\int_{\mathbb{R}}\Phi^2dx \leq \int_{\mathbb{R}}(I_1 + I_2 + I_3 + I_4 + I_5)\,\Phi dx, \tag{4.9}$$

$$\frac{d}{dt}\int_{\mathbb{R}}\psi^2dx + 2\varepsilon\int_{\mathbb{R}}\psi^2dx \leq \int_{\mathbb{R}}(I_1' + I_2' + I_3' + I_4' + I_5')\,\Psi dx. \tag{4.10}$$

By using Grönwall's lemma ror the last inequalities, we get

$$\|\varphi(t)\|^2_{G^\sigma} \leq e^{-2\varepsilon t}\|\varphi_0\|_{G^\sigma} + 2\left|\int_0^t\int_{\mathbb{R}}e^{-2\varepsilon(t-s)}(I_1 + I_2 + I_3 + I_4 + I_5)\Phi dxds\right|, \tag{4.11}$$

$$\|\psi(t)\|^2_{G^\sigma} \leq e^{-2\varepsilon t}\|\psi_0\|_{G^\sigma} + 2\left|\int_0^t\int_{\mathbb{R}}e^{-2\varepsilon(t-s)}(I_1' + I_2' + I_3' + I_4' + I_5')\Psi dxds\right|. \tag{4.12}$$

Now, we need to estimate the second term in the righthand side of inequalities (4.10) and (4.11). We start by

$$\begin{aligned}
&\left|\int_0^t\int_{\mathbb{R}}e^{-2\varepsilon(t-s)}I_1\Phi dxds\right| = \left|\int_{\mathbb{R}}\int_{\mathbb{R}}e^{-2\varepsilon(t-s)}\chi_{[0,t]}(s)I_1\Phi dxds\right|\\
&\leq \|\chi_{[0,t]}(t)I_1\|_{G^{0,b-1}}\|\chi_{[0,t]}(t)e^{-2\varepsilon(t-s)}\varphi\|_{G^{\sigma,b-1}}.
\end{aligned}$$

So, by using the fractional Leibnitz rule (see Theorem 2.8 in [19]), we infer that

$$
\begin{aligned}
&\|\chi_{[0,t]}e^{-2\varepsilon(t-s)}\varphi\|_{G^{\sigma,b-1}} = \left\|\chi_{[0,t]}e^{-2\varepsilon(\nu-t)}S(-t)\varphi\right\|_{H_t^{1-b}G_x^{\sigma}} \\
&\leq C_b \left\|\chi_{[0,t]}e^{-2\varepsilon(\nu-t)}\right\|_{L^\infty(\mathbb{R})} \left\|\chi_{[0,t]}S(-t)\varphi\right\|_{H_t^{1-b}G_x^{\sigma}} \\
&\quad + \left\|\chi_{[0,t]}e^{-2\varepsilon(\nu-t)}\right\|_{H^{1-b}(\mathbb{R})} \left\|\chi_{[0,t]}S(-t)\varphi\right\|_{L_t^\infty G_x^{\sigma}} \\
&\leq C_{b,\varepsilon} \left\|\chi_{[0,t]}\varphi\right\|_{G^{\sigma,1-b}} + \left\|\chi_{[0,t]}\varphi\right\|_{G^{\sigma,b}} \qquad \text{(by Proposition (2.1))} \\
&\leq C_b \left\|\chi_{[0,t]}\varphi\right\|_{G^{\sigma,b}} \leq C_b\|\varphi\|_{G_\nu^{\sigma,b}}, \qquad \text{(by Proposition 2.1)}
\end{aligned}
\tag{4.13}
$$

where in the last line we used $1 - b < b$ and Proposition 2.1.
On other hand, after some calculation there is a constant $\eta > 0$ so that

$$
\|\chi_{[0,t]}I_1\| \leq C_{\varepsilon,b,b_1}\sigma^\eta\|\varphi\|^2_{G_\nu^{\sigma,b}}, \tag{4.14}
$$

for more details, see [3]. Then, combining (4.13) and (4.14), we get

$$
\left|\int_0^t \int_{\mathbb{R}} e^{-2\varepsilon(t-s)}I_1\Phi dxds\right| \leq C_{\varepsilon,b,b_1}\sigma^\eta\|\varphi\|^3_{G_\nu^{\sigma,b}}| \leq C_{\varepsilon,b,b_1}\sigma^\eta\|(\varphi,\psi)\|^3_{Y_\nu^{\sigma,b}}. \tag{4.15}
$$

Like the same as before, we conclude that

$$
\left|\int_0^t \int_{\mathbb{R}} e^{-2\varepsilon(t-s)}I_2\Phi dxds\right| \leq C_{\varepsilon,b,b_2}\sigma^\eta\|\varphi\|^2_{G_\nu^{\sigma,b}}\|\psi\|_{G_\nu^{\sigma,b}} \leq C_{\varepsilon,b,b_2}\sigma^\eta\|(\varphi,\psi)\|^3_{Y_\nu^{\sigma,b}},
$$

$$
\left|\int_0^t \int_{\mathbb{R}} e^{-2\varepsilon(t-s)}I_3\Phi dxds\right| \leq C_{\varepsilon,b,b_3}\sigma^\eta\|\varphi\|^2_{G_\nu^{\sigma,b}}\|\psi\|_{G_\nu^{\sigma,b}} \leq C_{\varepsilon,b,b_3}\sigma^\eta\|(\varphi,\psi)\|^3_{Y_\nu^{\sigma,b}},
$$

and

$$
\left|\int_0^t \int_{\mathbb{R}} e^{-2\varepsilon(t-s)}I_4\Phi dxds\right| \leq C_{\varepsilon,b,b_4}\sigma^\eta\|\psi\|^3_{G_\nu^{\sigma,b}} \leq C_{\varepsilon,b,b_4}\sigma^\eta\|(\varphi,\psi)\|^3_{Y_\nu^{\sigma,b}}.
$$

All that's left is to estimate $\left|\int_0^t \int_{\mathbb{R}} e^{-2\varepsilon(t-s)}I_1\Phi dxds\right|$. By using the inequality of Cauchy Schwarz, we find

$$
\left|\int_0^t \int_{\mathbb{R}} e^{-2\varepsilon(t-s)}I_1\Phi dxds\right| \leq \int_0^t e^{-2\varepsilon(t-s)}\|I_5(s)\|_{L^2(\mathbb{R})}\|\varphi(s)\|_{G^\sigma}ds.
$$

Therefore, using Lemma 3.3 in [2], we have

$$
\|I_5\|_{L^2(\mathbb{R})} \leq 2\frac{\sigma}{\sigma_0}\|c\|_{A^\sigma}\|\varphi\|_{G^\sigma} + 2\|c\|_{L^\infty(\mathbb{R})}\|\varphi\|_{L^2(\mathbb{R})}.
$$

This leads for $0 \leq \sigma \leq \sigma_0$ to become

$$
\begin{aligned}
&\left|\int_0^t \int_{\mathbb{R}} e^{-2\varepsilon(t-s)}I_1\Phi dxds\right| \\
&\leq \int_0^t e^{-2\varepsilon(t-s)} \int_{\mathbb{R}} \left(2\frac{\sigma}{\sigma_0}\|c\|_{A^\sigma}\|\varphi\|_{G^\sigma} + 2\|c\|_{L^\infty(\mathbb{R})}\|\varphi\|_{L^2(\mathbb{R})}\right)\|\varphi\|_{G^\sigma}dxds \\
&\leq \int_{\mathbb{R}^2} \chi_{[0,t]}e^{-2\varepsilon(t-s)} \left(2\frac{\sigma}{\sigma_0}\|c\|_{A^\sigma}\|\varphi\|_{G^\sigma} + 2\|c\|_{L^\infty(\mathbb{R})}\|\varphi\|_{L^2(\mathbb{R})}\right)\|\varphi\|_{G^\sigma}dxds
\end{aligned}
$$

$$\leq \int_{\mathbb{R}^2} \chi_{[0,t]} e^{-2\varepsilon(t-s)} \left(2\frac{\sigma}{\sigma_0} \|c\|_{A^\sigma} \|\varphi\|_{G^\sigma}^2 + 2\|c\|_{L^\infty(\mathbb{R})} \|\varphi\|_{L^2(\mathbb{R})} \|\varphi\|_{G^\sigma} \right) dxds$$
$$\leq \int_{\mathbb{R}^2} \chi_{[0,t]} e^{-2\varepsilon(t-s)} \left(2\frac{\sigma}{\sigma_0} \|c\|_{A^\sigma} \|(\varphi,\psi)\|_{N^\sigma}^2 + 2\|c\|_{L^\infty(\mathbb{R})} \|\varphi\|_{L^2(\mathbb{R})} \|(\varphi,\psi)\|_{N^\sigma} \right) dxds.$$

Combining all the estimations obtained before and using (3.3), we find

$$\|\varphi\|_{G^\sigma} \leq e^{-2\varepsilon t} \|(\varphi_0,\psi_0)\|_{N^\sigma}^2 + C_1 \left(4\sigma^\eta \|(\varphi_0,\psi_0)\|_{N^\sigma} + \sigma \|c\|_{A_0^\sigma} \right) \|(\varphi_0,\psi_0)\|_{N^\sigma}^2$$
$$+ C_2 \|c\|_{L_x^\infty} \|(\varphi_0,\psi_0)\|_{\mathcal{L}^2(\mathbb{R})} \|(\varphi_0,\psi_0)\|_{N^\sigma}, \tag{4.16}$$

where C_1, C_2 are two constants depending on $\varepsilon, b, \sigma_0, \max\{b_i, i = 1, ..., 8\}$. We show the same calculation as before and we infer that

$$\|\psi\|_{G^\sigma} \leq e^{-2\varepsilon t} \|(\varphi_0,\psi_0)\|_{N^\sigma}^2 + C_1 \left(4\sigma^\eta \|(\varphi_0,\psi_0)\|_{N^\sigma} + \sigma \|c\|_{A_0^\sigma} \right) \|(\varphi_0,\psi_0)\|_{N^\sigma}^2$$
$$+ C_2 \|c\|_{L_x^\infty} \|(\varphi_0,\psi_0)\|_{\mathcal{L}^2(\mathbb{R})} \|(\varphi_0,\psi_0)\|_{N^\sigma}. \tag{4.17}$$

Finally, inequalities (4.16) and (4.17) lead to

$$\|(\varphi,\psi)\|_{N^\sigma} \leq \|(\varphi_0,\psi_0)\|_{N^\sigma}^2 + C_1 \left(4\sigma^\eta \|(\varphi_0,\psi_0)\|_{N^\sigma} + \sigma \|c\|_{A_0^\sigma} \right) \|(\varphi_0,\psi_0)\|_{N^\sigma}^2$$
$$+ C_2 \|c\|_{L_x^\infty} \|(\varphi_0,\psi_0)\|_{\mathcal{L}^2(\mathbb{R})} \|(\varphi_0,\psi_0)\|_{N^\sigma},$$

which establishes the proof of Theorem 4.1. □

4.2. Main theorem

In this part, we state and prove the main theorem in this paper.

Theorem 4.2. *Let (**H1**) and (**H2**) hold, and let $(\varphi_0,\psi_0) \in N^{\sigma_0} \times G^{\sigma_0}$ for certain $\sigma_0 > 0$. Then, there is a number $\tilde{\sigma_0}$ such that $\forall T > 0$, and the problem* (1.2) *has a unique solution*

$$(\varphi,\psi) \in C\left([0,T]; G^{\tilde{\sigma_0}}(\mathbb{R})\right)^2.$$

Moreover, we have

$$\|(\varphi,\psi)(t)\|_{N^{\tilde{\sigma_0}}} \leq Ce^{-\frac{\varepsilon t}{2}}.$$

To demonstrate Theorem 4.2, we need the following lemma.

Lemma 4.1. *Let (φ,ψ) be a local solution of system* (1.2)*. Then, there exist some σ_1 such that, for each $n \in \mathbb{N}$, $\exists D_n > 0$, satisfying*

$$\|(\varphi,\psi)(n\nu)\|_{N^{\sigma_1}}^2 \leq \|(\varphi_0,\psi_0)\|_{N^{\sigma_0}}^2 + D_n \|(\varphi_0,\psi_0)\|_{\mathcal{L}^2}^2,$$

where D_n depending on ν, $\|(\varphi_0,\psi_0)\|_{N^\sigma}$, and $\|c\|_{L^\infty}$.

Proof. The proof by induction will be used.
Case1. For $n = 1$, thanks to (4.1), we have

$$\|(\varphi,\psi)(\nu)\|_{N^{\sigma,b}} \leq e^{-2\varepsilon\nu} \|(\varphi_0,\psi_0)\|_{N^\sigma}^2 + C_1 \left(4\sigma^\eta \|(\varphi_0,\psi_0)\|_{N^\sigma} + \sigma \|c\|_{A^{\sigma_0}}\right) \|(\varphi_0,\psi_0)\|_{N^\sigma}^2$$

$$+ C_2\|c\|_{L_x^\infty}\|(\varphi_0,\psi_0)\|_{\mathcal{L}^2(\mathbb{R})}\|(\varphi_0,\psi_0)\|_{N^\sigma}.$$

Applying Young's inequality on the final term on the righthand side of the last inequality, for $\nu > 0$, we get

$$C_2\|c\|_{L_x^\infty}\|(\varphi_0,\psi_0)\|_{\mathcal{L}^2(\mathbb{R})}\|(\varphi_0,\psi_0)\|_{N^\sigma} \leq \frac{1-e^{-2\varepsilon\nu}}{2}\|(\varphi_0,\psi_0)\|_{N^\sigma}^2 + \frac{C_2^2\|c\|_{L_x^\infty}^2}{2(1-e^{-2\varepsilon\nu})}\|(\varphi_0,\psi_0)\|_{\mathcal{L}^2(\mathbb{R})}^2.$$

The case is satisfied if σ is chosen so that

$$\begin{aligned} C_1 4\sigma^\eta\|(\varphi_0,\psi_0)\|_{N^\sigma} &\leq \frac{1-e^{-2\varepsilon\nu}}{4}, \\ C_1\sigma\|c\|_{A^{\sigma_0}} &\leq \frac{1-e^{-2\varepsilon\nu}}{4}, \end{aligned} \tag{4.18}$$

and

$$D_1 = \frac{C_2^2\|c\|_{L_x^\infty}^2}{2(1-e^{-2\varepsilon\nu})}.$$

Since we are using σ to control these quantities, we see that it is thus necessary to restrict η to be strictly greater than zero; otherwise, the necessary σ terms would reduce to 1; see [20].

Case2. Suppose that the result holds for $n = k$ and we prove that it keeps holding for $n+1 = k+1$. We set $(\varphi,\psi)(k\nu)$ as initial data $(\tilde{\varphi}_0,\tilde{\psi}_0)$ and apply the estimate from the base case. We have

$$\begin{aligned} \|(\varphi,\psi)((k+1)\nu)\|_{N^\sigma}^2 &\leq \|(\tilde{\varphi}_0,\tilde{\psi}_0)\|_{N^\sigma}^2 + D_1\|(\tilde{\varphi}_0,\tilde{\psi}_0)\|_{\mathcal{L}^2}^2 \\ &\leq \|(\varphi,\psi)(k\nu)\|_{N^\sigma}^2 + D_1\|(\varphi,\psi)(k\nu)\|_{\mathcal{L}^2}^2 \\ &\leq \|(\varphi_0,\psi_0)\|_{N^\sigma}^2 + D_k\|(\varphi_0,\psi_0)\|_{\mathcal{L}^2}^2 + D_1\|(\varphi,\psi)(k\nu)\|_{\mathcal{L}^2}^2. \text{ (by inductive hypotheses)} \end{aligned}$$

By using Eqs (4.1) and (4.2) with the last inequality, we get

$$\|(\varphi,\psi)((k+1)\nu)\|_{N^\sigma}^2 \leq \|(\varphi_0,\psi_0)\|_{N^\sigma}^2 + D_k\|(\varphi_0,\psi_0)\|_{\mathcal{L}^2}^2 + D_1 e^{-2k\nu\varepsilon}\|(\varphi_0,\psi_0)\|_{\mathcal{L}^2}^2. \tag{4.19}$$

We take

$$D_{k+1} = D_k + D_1 e^{-2k\nu\varepsilon},$$

and the proof is established.

□

Proof of Theorem 4.2. It is worth noting that the estimate (4.19) leads to

$$\|(\varphi,\psi)(k\nu)\|_{N^\sigma} \leq \left(\sqrt{(2+D_1)\sum_{i=0}^{k-1} e^{-2i\nu\varepsilon}}\right)\|(\varphi_0,\psi_0)\|_{N^\sigma},$$

which implies that

$$\|(\varphi,\psi)(k\nu)\|_{N^\sigma} \leq C\|(\varphi_0,\psi_0)\|_{N^\sigma},$$

for some $C > 0$. With the local well-posedness result repeatedly at $t = n\nu$, the last inequality becomes

$$\|(\varphi,\psi)(t)\|_{N^\sigma} \leq C\|(\varphi_0,\psi_0)\|_{N^\sigma}, \tag{4.20}$$

for any σ satisfying inequalities in (4.18) and $\sigma \leq \sigma_0$.
Let σ_1 be any number nonnegative, thanks to Eqs (4.1), (4.2), and (4.18), and we have

$$\|(\varphi, \psi)(t)\|_{N^{\frac{\sigma_1}{2}}} \leq \|(\varphi, \psi)\|_{\mathcal{L}^2}^{\frac{1}{2}} \|(\varphi, \psi)\|_{N^{\sigma_1}}^{\frac{1}{2}} \leq Ce^{-\frac{t\nu}{2}} \|(\varphi_0, \psi_0)\|_{N^{\sigma_0}}.$$

The proof of Theorem 4.2 is established by choosing $\tilde{\sigma_0} = \dfrac{\sigma_1}{2}$.

5. Conclusions

The local well-posedness of the KdV type system with weak damping is investigated in the modified analytic space $Y_\delta^{\sigma,b}$. The local well-posedness is established using the Banach contraction mapping principle, along with bilinear estimates in the Fourier restriction space. The local result, involving the approximate conservation law

$$\begin{aligned} &\frac{d}{dt}\|\varphi\|_{L^2}^2 + 2\int_{\mathbb{R}} c_1(x)\varphi^2 dx, \\ &\frac{d}{dt}\|\psi\|_{L^2}^2 + 2\int_{\mathbb{R}} c_2(x)\psi^2 dx, \end{aligned}$$

is extended to hold globally in time. Additionally, a lower bound for the analytic radius is established. The presence of tow functions $c_1(x), c_2(x)$, called damping coefficients, makes the system more interesting from an application point of view due to their great importance in physics. In the case where $c_1(x) = c_2(x) = 0$, the authors studied a similar model as a single equation in [21], where a KdV type equations in Bourgain type spaces is considered and quantitative results are obtained, while our results in the present paper are qualitative studies related to the behavior of solutions in more suitable analytic spaces.

Author contributions

Sadok Otmani: Conceptualization, formal analysis, writing-original draft preparation; Aissa Boukarou: Investigation, methodology; Keltoum Bouhali: Investigation, methodology; Mohamed Bouye: Writing-review and editing; Abdelkader Moumen: Writing-review and editing; Khaled Zennir: Supervision. All authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

The authors declare that there is no conflict of interest.

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