Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(10): 28320–28340.
DOI: 10.3934/math.20241374
Received: 08 August 2024
Revised: 21 September 2024
Accepted: 26 September 2024
Published: 30 September 2024

Research article

Resonance with Landesman-Lazer conditions for parameter-dependent equations: a multiplicity result via the Poincaré-Birkhoff theorem

Chunlian Liu[1], **Shuang Wang**[2] **and Fanfan Chen**[3,*]

[1] School of Mathematics and Statistics, Nantong University, Nantong 226019, China
[2] School of Mathematics and Statistics, Yancheng Teachers University, Yancheng 224051, China
[3] School of Science, Zhejiang Sci-Tech University, Hangzhou 310018, China

* **Correspondence:** Email: fan7ch@gmail.com.

Abstract: We investigate the resonance problem and prove the existence of multiple periodic solutions to a second order parameter-dependent equation $x'' + f(t, x) = sp(t)$. We weaken the usual requirement on the sublinearity of perturbations when $|x|$ becomes large; and develop a more general method to investigate the rotational characterizations of the Landesman-Lazer conditions. Furthermore, f does not satisfy the common sign condition, and even the global existence of the solution is not guaranteed.

Keywords: periodic solutions; Poincaré-Birkhoff theorem; resonance; Landesman-Lazer conditions; parameter-dependent
Mathematics Subject Classification: 34B15, 34C15, 34C25

1. Introduction

We are interested in the existence of multiple periodic solutions to the following equation:

$$x'' + f(t, x) = sp(t). \tag{1.1}$$

Here, $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is assumed to be a continuous function, T-periodic in the first variable, and locally Lipschitz-continuous in the second variable. Moreover, $p : \mathbb{R} \to \mathbb{R}$ is assumed to be locally integrable and T-periodic, and s is a positive parameter. A similar result could be provided for s being negative. We investigate the case where $xf(t, x)$ is sign-varying.

The study on such kind of parameter-dependent differential equations dates back to the contribution of Berger and Podolak [1], who investigated the following equation:

$$x'' + g(x) = sw(t), \tag{1.2}$$

or a more general elliptic PDE. g is assumed to be of class C^2, for which $g'' > 0$, and $g'(-\infty) < \lambda_1 < g'(+\infty) < \lambda_2$; and $w(t) = \sin(\pi t/T)$, which is the eigenfunction associated with $\lambda_1 = (\pi/T)^2$, the first eigenvalue, for the corresponding Dirichlet problem on $[0, T]$. The research of this type of model has its physical background, which can be seen from the Lazer-McKenna model of oscillations in suspension bridges in [2]. What sets suspension bridges apart is their inherent nonlinearity. This form of nonlinearity, commonly referred to as asymmetric, arises from its differing behavior for positive and negative values of x. The fluctuation of $sp(t)$ represents varying wind conditions, potentially causing different oscillation patterns in the bridge.

There have been significant advances in this field over the years. In particular, Fabry, Mawhin, and Nkashama [3] initiated the investigation of the corresponding problem with periodic boundary conditions. Lazer and McKenna [4] provided a multiplicity result for a periodic problem. Ortega [5] discussed the corresponding periodic problem for a damped Duffing equation from the point of view of the stability of the solutions. For additional contributions, concerning the existence and multiplicity of periodic solutions for second order equations, see for instance [6–10] and the references therein. Additionally, we refer to the works in [11, 12] for equations with a singularity and the nonlinearity sign-varying, respectively; the work in [13] for weakly coupled parameter-dependent equations.

On the other hand, resonance problems are typical models in ordinary differential equations. There have been many interesting results in this field. It is well known that, under some resonance assumptions, the existence of the periodic solution to the considered problem is not guaranteed. In the last years, several conditions were produced in order to overcome this obstacle, such as Landesman-Lazer conditions, which can be traced back to the work of Landesman and Lazer in [14]. The power effects and classical definitions can be seen in [15–17]. We also refer to the works in [18–20] and the references therein. Boscaggin and Garrione [18] investigated the existence of multiple periodic solutions to a planar system under nonresonance conditions near zero, as well as resonance conditions at infinity. Garrione, Margheri, and Rebelo [19] investigated the periodic problem for the equation

$$x'' + f(t, x) = 0 \tag{1.3}$$

under resonance conditions at zero and infinity. Moreover, Landesman-Lazer conditions possess rotational effects on small and large solutions in the phase-plane.

Furthermore, Fonda and Garrione [17] investigated the double resonance problem for a planar system

$$z' = F(t, z), \quad z \in \mathbb{R}^2. \tag{1.4}$$

Their result was later generalized by Liu, Qian and Torres [20].

From the research status of resonance problems described above, one can find that researches on resonance problems associated with parameter-dependent differential equations are relatively sparse. In sight of this, and motivated by the works in [9, 10, 17–20], a natural question arises: Whether second order parameter-dependent equations possess multiple periodic solutions under certain resonance conditions. Moreover, it is observed that there is a common point of the above works concerning resonance problems, which is that the sublinearity of the perturbation is required as $|x|$ becomes large when discussing the rotational characterizations of Landesman-Lazer conditions. If one weakens the usual requirement on the sublinearity of the perturbations, the discussion of this issue will become more complex.

Therefore, we aim to explore a more general method and apply it to investigate the rotational characterizations of the corresponding Landesman-Lazer conditions; and investigate the multiple periodic solutions to (1.1) with nonresonance assumptions at positive infinity and resonance ones at infinity. Additionally, our analysis allows for the nonlinearity f to be sign-varying, and the global existence of the solution to Eq (1.1) may be destroyed within our framework. Our main tool is the Poincaré-Birkhoff theorem, which has broad applications in the multiplicity of periodic solutions (see for instance [8, 10, 19, 21, 22] and the references therein). Another power tool for studying multiple solutions is the variational method, see for instance [23, 24] and the references therein, where Hamiltonian systems coupling resonant linear components with twisting components and first-order Hamiltonian random impulsive differential equations are studied, respectively.

In the following, standard notations are used, such as $x^+ := \max\{x, 0\}$ and $x^- := \max\{-x, 0\}$. Moreover, $\rho(q)$ and $\rho(\phi)$ are used to denote rotation numbers, the exact definitions of which can be seen in Section 2. We introduce the following assumptions.

(H_0) $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is assumed to be continuous, T-periodic in the first variable, and locally Lipschitz-continuous in the second variable. Moreover, $p : \mathbb{R} \to \mathbb{R}$ is assumed to be locally integrable and T-periodic.

(H_1) There is a function $\phi(t) \in L^1([0, T])$ such that

$$\liminf_{x \to -\infty} \frac{f(t, x)}{x} \geqslant \phi(t), \quad \text{uniformly for a.e. } t \in [0, T].$$

(H_2) There is a function $q(t) \in L^1([0, T])$ such that

$$\lim_{x \to +\infty} \frac{f(t, x)}{x} = q(t), \quad \text{uniformly for a.e. } t \in [0, T].$$

(H_3) There exists an integer $m \geqslant 0$ such that

$$m < \rho(q) < m + 1. \tag{1.5}$$

Here, $\rho(q)$ represents the rotation number of $x'' + q(t)x = 0$. Furthermore, the only T-periodic solution of

$$x'' + q(t)x = p(t) \tag{1.6}$$

is strictly positive.

(H_4) There exists an integer $n \geqslant 0$ such that the rotation number $\rho(\phi)$ to the following equation:

$$x'' + q(t)x^+ - \phi(t)x^- = 0 \tag{1.7}$$

satisfies $\rho(\phi) = n$. Moreover, the argument function of every solution to (1.7) is 2π-periodic.

Furthermore, we assume the Landesman-Lazer condition as follows.

(H_5) If u is a nontrivial T-periodic solution to (1.7), it holds

$$\int_{\{u>0\}} \liminf_{x \to +\infty} (f(t, x) - q(t)x) u(t) dt + \int_{\{u<0\}} \limsup_{x \to -\infty} (f(t, x) - \phi(t)x) u(t) dt > 0. \tag{1.8}$$

Then our main result is as follows.

Theorem 1.1. *Suppose that* (H_0)*–*(H_5) *hold and* $n > m$*. Then, there is a* $s_0 \geqslant 0$ *such that Eq (1.1) has at least* $2(n-m)+1$ *distinct* T*-periodic solutions, for every* $s \geqslant s_0$*.*

Remark 1.1. *Regarding the conclusion presented in Theorem 1.1, the rotational characterization of the Landesman-Lazer condition* (H_5) *plays a crucial role. Specifically, the application framework of the Poincaré-Birkhoff theorem requires a twist condition on the inner and outer boundaries of an appropriate annulus. However, the presence of the resonance assumption in* (H_4) *can hinder the occurrence of the twist, so it is necessary to impose extra conditions to get "far away" from the resonance scenario. The Landesman-Lazer condition* (H_5) *happens to be the extra "powerful" condition needed.*

Remark 1.2. *We prove Theorem 1.1 in Section 3. After two times of change of variables in Section 3, Eq* (1.1) *is changed into an equivalent equation. Then, we employ the Poincaré-Birkhoff theorem, and through the estimates of the* T*-rotation numbers (see the definition in Section 2), of the solutions that have a small amplitude and of those having a large amplitude. The rotational characterization of* (H_5) *is used to provide the estimation to the* T*-rotation number of the solutions with a large amplitude.*

Remark 1.3. *In the nonresonance and resonance conditions* (H_1)*–*(H_4)*,* $\phi(t)$ *and* $q(t)$ *are sign-varying, which implies that* $\operatorname{sgn}(x)f(t,x)$ *could be sign-varying. The following is an interesting example regarding this. Two sign-varying functions are defined as follows:*

$$q(t) = \begin{cases} (2m+1)^2, & t \in [0,\pi], \\ -\lambda^2, & t \in [\pi, 2\pi], \end{cases} \qquad \phi(t) = \begin{cases} (2\alpha+\varrho)^2, & t \in [0,\pi], \\ -\mu^2, & t \in [\pi, 2\pi]. \end{cases}$$

Here, $m \in \mathbb{N}^+$*,* $\alpha, \varrho \in \mathbb{R}^+$*,* $\arctan|\lambda| \leqslant \pi/(2(2m+1))$ *and* $\pi - n\pi/(2\alpha+\varrho) - n\pi/(2m+1) < \max\{2\arctan|\lambda|,\ 2\arctan|\mu|\}$*. Furthermore, there is an integer* $n > 0$ *satisfying*

$$\frac{\pi}{m} + \frac{\pi}{\alpha} \leqslant \frac{2\pi}{n}. \tag{1.9}$$

Then, it follows that (see the detailed proof in Section 4)

$$m < \rho(q) < m+1, \quad \rho(\phi) = n. \tag{1.10}$$

The remainder of the paper is structured as follows. In Section 2, we present the rotational characterization of the Landesman-Lazer condition at infinity. In Section 3, we provide the proof of the main result. In Section 4, we prove several technical lemmas and discuss (1.10). Finally, some conclusions are given in Section 5.

2. The rotational characterization of the Landesman-Lazer condition at infinity

We provide the rotational characterization of the Landesman-Lazer condition at infinity in this part. By utilizing the rotational characterizations, we can estimate of T-rotation numbers of the solutions, and then verify that the solution to the system (3.4) in the next section satisfies some twist condition on an appropriate annulus. This verification is a key step in applying the Poincaré-Birkhoff theorem.

First, we present the definitions of T-rotation numbers and rotation numbers. Consider the following system:

$$x' = y, \quad y' = -h(t,x). \tag{2.1}$$

Here, $h : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is assumed to be continuous, T-periodic with respect to the first variable. Denote $z = (x, y) \in \mathbb{R}^2$, a solution to (2.1) satisfying the initial condition $z(0) = z_0$ can be written as $z(t; z_0)$. If $z(t; z_0) \neq 0$, switch to polar coordinates

$$x = r\cos\theta, \quad y = r\sin\theta,$$

it follows that

$$\begin{cases} \theta' = -\sin^2\theta - \dfrac{h(t,x)}{r}\cos\theta, \\ r' = r\sin\theta\cos\theta - h(t,x)\sin\theta. \end{cases} \tag{2.2}$$

In case that $z(t; z_0)$ exists in $[0, T]$, the T-rotation number associated to $z(t; z_0)$ can be defined as

$$\mathrm{Rot}_h(z_0) = \frac{\theta(0; z_0) - \theta(T; z_0)}{2\pi} = \frac{1}{2\pi}\int_0^T \frac{xh(t,x) + y^2}{x^2 + y^2}dt.$$

Here, $\theta(t; z_0)$ represents the argument function of $z(t; z_0)$. Accordingly, $\mathrm{Rot}_h(z_0)$ represents the total algebraic count of the clockwise rotations of the solution $z(t; z_0)$ around the origin during $[0, T]$.

If (2.1) is assumed to be a system of the form

$$x' = y, \quad y' = -a_+(t)x^+ + a_-(t)x^-, \tag{2.3}$$

with $a_\pm(t) \in L^1([0, T])$, and $\theta(t; z_0)$ satisfying

$$\theta' = -a_+(t)\big((\cos\theta)^+\big)^2 - a_-(t)\big((\cos\theta)^-\big)^2 - \sin^2\theta. \tag{2.4}$$

$\theta(t; z_0)$ depends the beginning moment $t = 0$ as well as the initial value $\theta_0 \in \mathbb{S}^1 = \mathbb{R}/(2\pi\mathbb{Z})$.

Moreover, the function

$$a_+(t)((\cos\theta)^+)^2 + a_-(t)((\cos\theta)^-)^2 + \sin^2\theta$$

is T-periodic with respect to t and 2π-periodic with respect to θ. Therefore, Eq (2.4) is a equation on a torus. Hence, the rotation number of (2.4) can be defined as

$$\rho(a) = \lim_{t\to\infty}\frac{\theta_0 - \theta(t; \theta_0)}{t},$$

which exists independently of $t_0 = 0$ and θ_0. Expanding on this, we call $\rho(a)$ the rotation number of (2.3).

Second, we provide rotational characterizations of Landesman-Lazer conditions. We first present the usual versions of Landesman-Lazer conditions at infinity [16, 17, 19]. Assume the validity of the conditions:

(H^l_∞) There exist two functions $a_\pm \in L^1([0, T])$ such that

$$\liminf_{x\to\pm\infty}\frac{h(t,x)}{x} \geqslant a_\pm(t) \quad \text{uniformly a.e. in } t \in [0, T],$$

and

(H_∞^r) There exist two functions $b_\pm \in L^1([0,T])$ such that

$$\limsup_{x\to\pm\infty} \frac{h(t,x)}{x} \leqslant b_\pm(t) \quad \text{uniformly a.e. in } \ t \in [0,T],$$

respectively, Landesman-Lazer conditions at infinity (LL_∞^+) and (LL_∞^-) are stated as follows.

(LL_∞^+) If u is a nonzero T-periodic solution of the following equation:

$$x'' + a_+(t)x^+ - a_-(t)x^- = 0, \tag{2.5}$$

it holds

$$\int_{\{u>0\}} \liminf_{x\to+\infty}(h(t,x) - a_+(t)x)u(t)dt + \int_{\{u<0\}} \limsup_{x\to-\infty}(h(t,x) - a_-(t)x)u(t)dt > 0. \tag{2.6}$$

(LL_∞^-) If u is a nonzero T-periodic solution of the following equation:

$$x'' + b_+(t)x^+ - b_-(t)x^- = 0, \tag{2.7}$$

it holds

$$\int_{\{u>0\}} \limsup_{x\to+\infty}(h(t,x) - b_+(t)x)u(t)dt + \int_{\{u<0\}} \liminf_{x\to-\infty}(h(t,x) - b_-(t)x)u(t)dt < 0. \tag{2.8}$$

Set $H(t,z) = (y, -h(t,x))$, $L_1(t,z) = (y, -a_+(t)x^+ + a_-(t)x^-)$, $L_2(t,z) = (y, -b_+(t)x^+ + b_-(t)x^-)$. Then, (2.1), (2.5) and (2.7) are equivalent to

$$z' = H(t,z),$$

$$z' = L_1(t,z), \tag{2.9}$$

and

$$z' = L_2(t,z), \tag{2.10}$$

respectively.

Next, we list Lemmas 2.1–2.5 and some foundational conclusions in order to show the rotational characterization of the Landesman-Lazer conditions ($LL_\infty^\pm$). However, we choose to omit the proofs of them in this part, so that we can arrive quickly at the proof of the main result (Theorem 1.1) in the next section. Lemma 2.1 is about the generalized polar coordinates based on solutions of the system $z' = L(t,z)$, it comes from Lemma 5.2 in [20]. Lemma 2.5 will be proved similarly to the proof of Lemma 2.4. The missing proofs of Lemmas 2.2–2.4 are thus provided in the final section.

Lemma 2.1. *Suppose that $L : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}^2$ is an L^1-Carathéodory function, and is T-periodic in the first variable and positive homogeneous with degree one with respect to the second vector variable. Assume that $z_\omega(t)$ is a solution of the system*

$$z' = L(t,z) \tag{2.11}$$

with $z_\omega(t_0) = \omega \in \mathbb{S}^1$. Then any continuous function $z(t)$ can be expressed as $z(t) = r(t)z_{\omega(t)}(t)$, with $r(t)$ and $\omega(t)$ being continuous and $r(t) \geqslant 0$.

Lemma 2.2. *(i) For every* $\nu \in \mathbb{S}^1$*, let* $z_\nu(t) = (x_\nu(t), y_\nu(t)) \not\equiv 0$ *be a solution to* (2.9) *that satisfies* $z_\nu(0) = \nu$*. Then for every* $\mu \in \mathbb{S}^1$*, condition* (2.6) *is equivalent to*

$$\int_0^T \liminf_{(\lambda,\nu)\to(+\infty,\mu)} \Big(\frac{\langle JH(t, \lambda z_\nu(t)), z_\nu(t)\rangle}{|z_\nu(t)|^2} - \lambda \frac{\langle JL_1(t, z_\mu(t)), z_\mu(t)\rangle}{|z_\mu(t)|^2} \Big) dt > 0. \tag{2.12}$$

(ii) For every $\nu \in \mathbb{S}^1$*, let* $z_\nu(t) = (x_\nu(t), y_\nu(t)) \not\equiv 0$ *be a solution to* (2.10) *that satisfies* $z_\nu(0) = \nu$*. Then for every* $\mu \in \mathbb{S}^1$*, condition* (2.8) *is equivalent to*

$$\int_0^T \limsup_{(\lambda,\nu)\to(+\infty,\mu)} \Big(\frac{\langle JH(t, \lambda z_\nu(t)), z_\nu(t)\rangle}{|z_\nu(t)|^2} - \lambda \frac{\langle JL_2(t, z_\mu(t)), z_\mu(t)\rangle}{|z_\mu(t)|^2} \Big) dt < 0. \tag{2.13}$$

Here, $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ *represents a standard symplectic matrix.*

Next, we give a lemma, aiming to find a truncated function of h, which plays a crucial role in the proof of Lemma 2.4.

Lemma 2.3. *Suppose that* $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is continuous, T-periodic in the first variable. Moreover, assume the validity* (H_∞^l)*. Then there is a function* $\hat{h}$*, which can be expressed as*

$$\hat{h}(t, x) = a_+(t)x^+ - a_-(t)x^- + \hat{r}(t, x), \tag{2.14}$$

such that $\hat{r}(t, x)$ *satisfies*

$$\lim_{|x|\to+\infty} \frac{\hat{r}(t, x)}{x} = 0, \tag{2.15}$$

and $\hat{h}$ *satisfies*

$$x\hat{h}(t, x) \leqslant xh(t, x), \quad \textit{for every } x \in \mathbb{R} \textit{ and a.e. } t \in [0, T]. \tag{2.16}$$

Lemma 2.4. *Suppose that* $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is continuous, T-periodic in the first variable, and* (H_∞^l) *and* (LL_∞^+) *hold. Furthermore, let the rotation number of Eq* (2.5) *satisfy* $\rho(a) = k$*, and the argument function of every solution to Eq* (2.5) *is* 2π*-periodic. Then, there exists* $\tilde{R} > 0$ *such that*

$$\mathrm{Rot}_h(z_0) > k$$

is valid for each solution to (2.1) that satisfies $|z(t; z_0)| \geqslant \tilde{R}, \ \forall\, t \in [0, T]$.

Furthermore, symmetrical to Lemma 2.4, replace (H_∞^l) and (LL_∞^+) by (H_∞^r) and (LL_∞^-), respectively, we get a result as follows.

Lemma 2.5. *Suppose that* $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is continuous, T-periodic in the first variable, and* (H_∞^r) *and* (LL_∞^-) *hold. Furthermore, let the rotation number of Eq* (2.7) *satisfy* $\rho(b) = k$*, and the argument function of every solution to Eq* (2.7) *is* 2π*-periodic. Then, there exists* $\tilde{R} > 0$ *such that*

$$\mathrm{Rot}_h(z_0) < k$$

is valid for each solution to (2.1) that satisfies $|z(t; z_0)| \geqslant \tilde{R}, \ \forall\, t \in [0, T]$.

3. The existence of multiple periodic solutions

In this part, we provide the proof of the main result by applying the Poincaré-Birkhoff theorem. We now make a change of variables. In (1.1), let

$$u(t) = \frac{1}{s}x(t).$$

Then, Eq (1.1) is changed into

$$u'' + \frac{f(t, su)}{s} = p(t). \tag{3.1}$$

Similar to Lemmas 3.3 in [12], we present an existence result as follows.

Lemma 3.1. *Assume the validity of* (H_0)–(H_3)*. Then, there is a* $s_1 \geqslant 1$*, such that, for every* $s \geqslant s_1$*, Eq* (3.1) *has a* T*-periodic solution* $\tilde{u}_s$ *that satisfies*

$$c_0 \leqslant \tilde{u}_s(t) \leqslant C_0,$$

for each $t \in [0, T]$*, where* c_0 *and* C_0 *are two positive constants.*

Next, we make the second change of variables. In (3.1), let

$$v(t) = u(t) - \tilde{u}_s(t),$$

Eq (3.1) is thus changed into

$$v'' + \frac{f(t, s(v + \tilde{u}_s(t))) - f(t, s\tilde{u}_s(t))}{s} = 0. \tag{3.2}$$

We can see that $v = 0$ is a solution of (3.2). Similar to Lemmas 3.4 in [12], we have the following lemma.

Lemma 3.2. *Assume* (H_0), (H_2) *and* (H_3)*, then*

$$\lim_{s\to+\infty} \frac{f(t, s(v + \tilde{u}_s(t))) - f(t, s\tilde{u}_s(t))}{s} = q(t)v$$

holds uniformly for almost every $t \in [0, T]$ *and every* $v \in [-\frac{1}{2}c_0, \frac{1}{2}c_0]$.

Set

$$\tilde{f}_s(t, v) = \frac{f(t, s(v + \tilde{u}_s(t))) - f(t, s\tilde{u}_s(t))}{s},$$

then Eq (3.2) is changed into

$$v'' + \tilde{f}_s(t, v) = 0. \tag{3.3}$$

Consider the first order planar system

$$v' = w, \quad w' = -\tilde{f}_s(t, v), \tag{3.4}$$

associated to Eq (3.3), where $\tilde{f}_s(t, v)$ satisfies the following conditions, which can be deduced from (H_0), (H_1), (H_2), and (H_5) by some simple computations.

$(H_0)'$ $\tilde{f}_s : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is assumed to be continuous, T-periodic in the first variable, and locally Lipschitz-continuous in the second variable. Moreover, $\tilde{f}_s(t,0)=0$.

$(H_1)'$ For $\phi(t) \in L^1([0,T],\mathbb{R})$ in (H_1), we have

$$\liminf_{v\to-\infty} \frac{\tilde{f}_s(t,v)}{v} \geqslant \phi(t), \quad \text{uniformly for a.e. } t\in[0,T]. \tag{3.5}$$

$(H_2)'$ For $q(t) \in L^1([0,T],\mathbb{R})$ in (H_2), we have

$$\lim_{v\to+\infty} \frac{\tilde{f}_s(t,v)}{v} = q(t), \quad \text{uniformly for a.e. } t\in[0,T].$$

$(H_5)'$ If u is a nonzero T-periodic solution to (1.7), it holds

$$\int_{\{u>0\}} \liminf_{v\to+\infty}(\tilde{f}_s(t,v)-q(t)v)u(t)dt + \int_{\{u<0\}} \limsup_{v\to-\infty}(\tilde{f}_s(t,v)-\phi(t)v)u(t)dt > 0. \tag{3.6}$$

3.1. Spiral property

It is noted that the global existence of solutions for system (3.4) may be destroyed under the conditions $(H_0)'$, $(H_1)'$, and $(H_2)'$. Global existence is essential in the application of the Poincaré-Birkhoff theorem. In this part, we provide a spiral property of the solutions for Eq (3.3) under the conditions $(H_0)'$, $(H_1)'$, and $(H_2)'$, and modify system (3.4) using the spiral property below. We then obtain the global existence of solutions to the modified system.

Denote by $\tilde{z}_s(t) := (v_s(t), w_s(t))$ a solution to (3.4) that satisfies $\tilde{z}_0 := \tilde{z}_s(0) = (v_s(0), w_s(0))$. By $(H_0)'$, every solution to the initial value problem corresponding to (3.4) is unique. Because $\tilde{z}_s(t) = 0$ is a solution to (3.4), by uniqueness, one has $\tilde{z}_s(t) \neq 0$ when $\tilde{z}_0 \neq 0$. Switch to polar coordinates

$$v = r\cos\theta, \quad w = r\sin\theta,$$

then it follows that

$$\begin{cases} \theta' = -\sin^2\theta - \dfrac{\tilde{f}_s(t,v)}{r}\cos\theta, \\ r' = r\sin\theta\cos\theta - \tilde{f}_s(t,v)\sin\theta. \end{cases} \tag{3.7}$$

Let $(\tilde{\theta}_s(t), \tilde{r}_s(t)) := (\tilde{\theta}_s(t;\tilde{z}_0), \tilde{r}_s(t;\tilde{z}_0))$ denote a solution to (3.7) with $(\tilde{\theta}_s(0), \tilde{r}_s(0)) = (\theta_0, r_0)$. Then, we have the spiral property of the solution for the system (3.4) as follows. Its proof is analogous to that of Lemma 4.1 in [12], and is therefore omitted here.

Lemma 3.3. *Suppose that $(H_0)'$, $(H_1)'$ and $(H_2)'$ hold, $s \geqslant s_1$. For every fixed l, $N_0 \in \mathbb{N}$ and a r^* large enough, there exist two strictly increasing functions $\xi^-_{N_0}$, $\xi^+_{N_0}$: $[r^*, +\infty) \to \mathbb{R}$, such that*

$$\xi^{\pm}_{N_0}(r) \to +\infty \iff r \to +\infty.$$

Furthermore, given any $r_0 \geqslant r^*$, then the solution $(\tilde{\theta}_s(t), \tilde{r}_s(t))$ of (3.4) satisfies that either

$$\xi^-_{N_0}(r_0) \leqslant \tilde{r}_s(t) \leqslant \xi^+_{N_0}(r_0), \quad t\in[0, lT],$$

or there exists $t_{N_0} \in (0,T)$ such that

$$\theta_0 - \tilde{\theta}_s(t_{N_0};\tilde{z}_0) = 2N_0\pi,$$

and

$$\xi^-_{N_0}(r_0) \leqslant \tilde{r}_s(t) \leqslant \xi^+_{N_0}(r_0), \quad t\in[0, t_{N_0}].$$

3.2. The modified system

We now introduce a modified system. To ensure the existence of global solutions to the associated initial value problems, we propose a truncated function as follows:

$$\tilde{g}_s(t,v) = \begin{cases} \tilde{f}_s(t,-R), & v < -R, \\ \tilde{f}_s(t,v), & |v| \leqslant R, \\ \tilde{f}_s(t,R), & v > R, \end{cases}$$

where the positive parameter R satisfies $R > c_0/2$, and its exact value will be provided when proving Theorem 1.1. Then the Hamiltonian system associated to $\tilde{g}_s(t,v)$ is as follows:

$$v' = w, \quad w' = -\tilde{g}_s(t,v). \tag{3.8}$$

We now give some basic properties of the modified system (3.8), containing uniqueness, global existence and rotational property. For convenience, we denote by $\tilde{z}_s(t)$ a solution to (3.8) with an initial value $\tilde{z}_0$, and $\tilde{\theta}_s(t)$ the argument function associated to $\tilde{z}_s(t)$ satisfying $\tilde{\theta}_s(0) = \theta_0$.

Lemma 3.4. *Assume* $(H_0)'$*, then every solution to the initial value problem associated to* (3.8) *is unique and exists globally. If* $|\tilde{z}_s(t)| \leqslant R$*,* $\tilde{z}_s(t)$ *is also a solution of system* (3.4)*.*

Lemma 3.5. *Nonzero solutions satisfy the rotational property. More precisely, if* $\tilde{z}_s(t)$ *is a nontrivial solution of* (3.8)*, then,*

$$\tilde{\theta}_s(t_2) - \tilde{\theta}_s(t_1) < \pi, \quad \text{for any } t_2 > t_1.$$

Lemma 3.5 can be proven similarly to Lemma 4.1 in [25], so we omit it here. Before proving the main result, we present a lemma to determine the inner boundary of an appropriate annulus for applying the Poincaré-Birkhoff theorem, as follows [12].

Lemma 3.6. *There exist three positive constants* δ*,* $\tilde{r}$ *and* s_2 *that satisfy* $\delta < \tilde{r} < \frac{1}{2}c_0$ *and* $s_2 \geqslant s_1$*, such that, for every* $s \geqslant s_2$*, if* $|\tilde{z}_0| = \tilde{r}$*, then we have*

$$\delta < |\tilde{z}_s(t)| < \frac{1}{2}c_0,$$

for every $t \in [0,T]$*.*

3.3. Proof of the main result

The proof will be divided into the following steps.

Step 1. Define a set

$$\Omega := \{z \in \mathbb{R}^2 : \delta < |z| < \frac{1}{2}c_0\},$$

and let

$$\Gamma_- := \{z : |z| = \tilde{r}\}.$$

Consider the solution $\tilde{z}_s(t)$ of (3.8) with $\tilde{z}_0 \in \Gamma_-$. By Lemma 3.6, there is a s_2 with $s_2 \geqslant s_1$, for which $\tilde{z}_s(t) \in \Omega$ when $s \geqslant s_2$, that is

$$\delta < |\tilde{z}_s(t)| < \frac{1}{2}c_0, \quad t \in [0,T].$$

Since $\frac{1}{2}c_0 < R$, right now, $\tilde{z}_s(t)$ is a solution to (3.4). By Lemma 3.2, it follows that

$$\lim_{s\to+\infty} \tilde{f}_s(t, v) = q(t)v$$

holds uniformly for almost every $t \in [0, T]$ and $z = (v, w) \in \Omega$. Then we obtain

$$\lim_{s\to+\infty} \frac{\tilde{f}_s(t, v)}{v} = q(t) \tag{3.9}$$

holds uniformly for almost every $t \in [0, T]$ and any $z = (v, w) \in \Omega$. Additionally, it is observed that

$$s \to +\infty \iff s(v + \tilde{u}_s) \to +\infty, \quad \text{for } v + \tilde{u}_s \in [\frac{1}{2}c_0, \frac{1}{2}c_0 + C_0].$$

Thus, from (H_3) and Lemma 2.3 in [12], we can obtain

$$m < \mathrm{Rot}_{\tilde{f}_s}(\tilde{z}_0) < m + 1.$$

Thus,

$$m < \mathrm{Rot}_{\tilde{g}_s}(\tilde{z}_0) < m + 1, \quad \text{for } \tilde{z}_0 \in \Gamma_-. \tag{3.10}$$

Step 2. By $(H_1)'$, $(H_2)'$, we can deduce that (H^l_∞) in Lemma 2.4 holds. Then by (H_4) and $(H_5)'$, and Lemma 2.4, for $s \geqslant s_2$, there is a $\tilde{R} > 0$, so that for each solution $\tilde{z}_s(t)$ of (3.4) that satisfies $|\tilde{z}_s(t)| \geqslant \tilde{R}$, $t \in [0, T]$, we have

$$\mathrm{Rot}_{\tilde{f}_s}(\tilde{z}_0) > n. \tag{3.11}$$

Define

$$\Gamma_+ := \{z : |z| = R_\infty\},$$

and select $R = R'_\infty$, where

$$R_\infty > (\xi^-_{n+1})^{-1}(\tilde{R}), \quad R'_\infty > \xi^+_{n+1}(R_\infty),$$

as a result, (3.8) is equivalent to (3.4) when $|\tilde{z}_s(t)| \leqslant R$.

Next, Let us focus on the solution to system (3.8) starting from $\tilde{z}_0 \in \Gamma_+$. If $\tilde{R} \leqslant |\tilde{z}_s(t)| \leqslant R'_\infty$, for every $t \in [0, T]$, then by (3.11) we have

$$\mathrm{Rot}_{\tilde{g}_s}(\tilde{z}_0) > n, \quad \text{for } \tilde{z}_0 \in \Gamma_+. \tag{3.12}$$

If for some $t_* \in [0, T]$ it holds $|\tilde{z}_s(t_*)| \geqslant R'_\infty > \xi^+_{n+1}(R_\infty)$, it can be seen that

$$\xi^-_{n+1}(|\tilde{z}_0|) \leqslant |\tilde{z}_s(t)| \leqslant \xi^+_{n+1}(|\tilde{z}_0|)$$

is not valid for every $t \in (0, T)$. Thus, applying Lemma 3.3, there exists $\bar{t}_* \in (0, t_*]$ for which

$$\theta_0 - \tilde{\theta}_s(t'_*) = 2(n + 1)\pi.$$

Furthermore, applying Lemma 3.5, it follows that

$$\theta_0 - \tilde{\theta}_s(T) = \theta_0 - \tilde{\theta}_s(\bar{t}_*) + \tilde{\theta}_s(\bar{t}_*) - \tilde{\theta}_s(T) \geqslant 2(n + 1)\pi - \pi > 2n\pi.$$

Then,

$$\mathrm{Rot}_{\tilde{g}_s}(\tilde{z}_0) > n. \tag{3.13}$$

Last, if for some $t'_* \in (0, T)$ it holds that $|z(t'_*)| \leqslant \tilde{R} < \xi^-_{n+1}(R_\infty)$, then

$$\xi^-_{n+1}(|z_0|) \leqslant |z(t)| \leqslant \xi^+_{n+1}(|z_0|)$$

is not valid for every $t \in (0, T)$. By the same discussion as above we have

$$\mathrm{Rot}_{\tilde{g}_s}(\tilde{z}_0) > n. \tag{3.14}$$

Combined (3.12), (3.13) with (3.14), we have, if the solution of (3.8) begins from $\tilde{z}_0 \in \Gamma_+$,

$$\mathrm{Rot}_{\tilde{g}_s}(\tilde{z}_0) > n. \tag{3.15}$$

Step 3. Give the definition of the Poincaré map

$$\begin{aligned} \mathcal{P} : \ \mathbb{R}^2 &\rightarrow \mathbb{R}^2, \\ \tilde{z}_0 &\mapsto \tilde{z}_s(T). \end{aligned}$$

Lemma 3.4 confirms the global existence of solutions, which implies that $\mathcal{P}$ is well-defined. Furthermore, the fact that solutions are unique guarantees that $\mathcal{P}$ is a homeomorphism, and (3.8) has a Hamiltonian structure, $\mathcal{P}$ is therefore an area-preserving homeomorphism.

Let $k = m + 1,\ m + 2, \cdots, n$, then from (3.10) and (3.15) we have

$$\mathrm{Rot}_{\tilde{g}_s}(\tilde{z}_0) < k, \quad \text{for } \tilde{z}_0 \in \Gamma_-,$$

$$\mathrm{Rot}_{\tilde{g}_s}(\tilde{z}_0) > k, \quad \text{for } \tilde{z}_0 \in \Gamma_+.$$

Thus, by the Poincaré-Birkhoff theorem (see [21, 26]), $\mathcal{P}$ possesses no fewer than $n - m$ pairs of geometrically distinct fixed points $\tilde{z}_{i,j},\ i = 1, \cdots, n - m,\ j = 1, 2$, which associate to $n - m$ pairs of T-periodic solutions

$$\tilde{z}_s(t; \tilde{z}_{i,j}), \quad i = 1, \cdots, n - m,\ j = 1, 2$$

of (3.8) with

$$\mathrm{Rot}_{\tilde{g}_s}(\tilde{z}_{i,j}) = k, \quad i = 1, \cdots, n - m,\ j = 1, 2. \tag{3.16}$$

Step 4. We aim to show that $\tilde{z}_s(t; \tilde{z}_{i,j}),\ i = 1, \cdots, n - m,\ j = 1, 2$ are in fact T-periodic solutions to (3.4). Specifically, we will show that $|\tilde{z}_s(t; \tilde{z}_{i,j})| \leqslant R$, for every $t \in [0, T]$, $i = 1, \cdots, n - m,\ j = 1, 2$. Note that

$$0 < |\tilde{z}_{i,j}| < R_\infty,\ \ i = 1, \cdots, n - m,\ \ j = 1, 2.$$

Take $\tilde{z}_s(t; \tilde{z}_{1,1})$ as an example. Suppose by contradiction that there is $t_1 \in (0, T)$ for which $|\tilde{z}_s(t_1; \tilde{z}_{1,1})| > R = R'_\infty$, as well as

$$|\tilde{z}_s(t; \tilde{z}_{1,1})| \leqslant R'_\infty, \quad \text{for } t \in [0, t_1].$$

Applying Lemma 3.3, then it holds

$$\tilde{\theta}_s(0; \tilde{z}_{1,1}) - \tilde{\theta}_s(t_1; \tilde{z}_{1,1}) = 2(k + 1)\pi.$$

Moreover, by Lemma 3.5, we have

$$\begin{aligned}&\tilde{\theta}_s(0;\tilde{z}_{1,1})-\tilde{\theta}_s(T;\tilde{z}_{1,1})\\&=\tilde{\theta}_s(0;\tilde{z}_{1,1})-\tilde{\theta}_s(t_1;\tilde{z}_{1,1})+\tilde{\theta}_s(t_1;\tilde{z}_{1,1})-\tilde{\theta}_s(T;\tilde{z}_{1,1})\\&\geqslant 2(k+1)\pi-\pi>2k\pi.\end{aligned}$$

Hence,

$$\mathrm{Rot}_{\tilde{g}_s}(\tilde{z}_{1,1})>k,$$

contradicting (3.16). Hence, we can obtain

$$|\tilde{z}_s(t;\tilde{z}_{1,1})|\leqslant R,\quad \text{for } t\in[0,T],$$

so $\tilde{z}_s(t;\tilde{z}_{1,1})$ is indeed a T-periodic solutions of (3.4). The same discussion is valid for other solutions.

Recalling the zero solution to (3.3) which corresponding to $\tilde{u}_s(t)$ of (3.1), we get $2(n-m)+1$ distinct T-periodic solutions for (3.1), which means that Eq (1.1) has $2(n-m)+1$ distinct T-periodic solutions. Therefore, the proof is completed.

Remark 3.1. *To demonstrate the application of the main result, we give an example as follows. Consider a function $f(t,x)$ defined by*

$$f(t,x)=\begin{cases}q(t)x, & x>0,\\ |\phi(t)|x^3, & x\leq 0\end{cases}$$

with $q(t)$ and $\phi(t)$ being defined as that in Remark 1.3. Then we have

$$\liminf_{x\to-\infty}\frac{f(t,x)}{x}\geq\phi(t)\quad \textit{uniformly for a.e. } t\in[0,2\pi];$$

and

$$\lim_{x\to+\infty}\frac{f(t,x)}{x}=q(t)\quad \textit{uniformly for a.e. } t\in[0,2\pi].$$

Therefore, (H_1) and (H_2) hold. By (1.10) in Remark 1.3 and a similar discuss to that in Remark 3.1 in [12], we can verify (H_3). Then, (H_4) can be verified by applying Remark 1.3, and (H_5) can be verified by a simple computation. Furthermore, let $n>m$. Then, by applying Theorem 1.1, there is a $s_0\geq 0$ such that, for every $s\geq s_0$, the equation $x''+f(t,x)=sp(t)$ with $f(t,x)$ defined as above, has at least $2(n-m)+1$ distinct 2π-periodic solutions.

4. Proofs of some technical lemmas

This section is dedicated to proving Lemmas 2.2–2.4 and discussing (1.10).

Proof of Lemma 2.2. We present the proof of the first statement. The second one can be proved similarly. For every $\nu\in\mathbb{S}^1$, since $z_\nu(t)=(x_\nu(t),y_\nu(t))\not\equiv 0$ is a solution to (2.9) that satisfies $z_\nu(0)=\nu$, from the continuous dependence of the solution on the initial value, it holds $\lim_{\nu\to\mu}x_\nu(t)=x_\mu(t)$ for $t\in[0,T]$. Therefore, (2.12) is equivalent to

$$\int_0^T\liminf_{(\lambda,\nu)\to(+\infty,\mu)}\Big(\frac{x_\nu(t)h(t,\lambda x_\nu(t))}{|z_\nu(t)|^2}-\lambda\frac{a_+(t)(x_\nu^+(t))^2+a_-(t)(x_\nu^-(t))^2}{|z_\nu(t)|^2}\Big)dt>0.\tag{4.1}$$

Then, we will prove that (2.6) is equivalent to (4.1).

In another perspective, it holds that

$$\begin{aligned}&\int_0^T \liminf_{(\lambda,\nu)\to(+\infty,\mu)}\Big(\frac{x_\nu(t)h(t,\lambda x_\nu(t))}{|z_\nu(t)|^2}-\lambda\frac{a_+(t)(x_\nu^+(t))^2+a_-(t)(x_\nu^-(t))^2}{|z_\nu(t)|^2}\Big)dt\\ =&\int_0^T \liminf_{(\lambda,\nu)\to(+\infty,\mu)}\frac{1}{|z_\nu(t)|^2}x_\nu(t)\Big(h(t,\lambda x_\nu(t))-\lambda(a_+(t)x_\nu^+(t)-a_-(t)x_\nu^-(t))\Big)dt.\end{aligned}\tag{4.2}$$

Notice that $\{t\in[0,T]:x(t)=0\}$ is a finite subset of $[0,T]$, let

$$\underline{h}_\infty(t)=\liminf_{x\to+\infty}(h(t,x)-a_+(t)x),\qquad \overline{h}_\infty(t)=\limsup_{x\to-\infty}(h(t,x)-a_-(t)x),$$

and denote by $x_\mu^+(t)=\dfrac{x_\mu(t)+|x_\mu(t)|}{2}$, $x_\mu^-(t)=\dfrac{x_\mu(t)-|x_\mu(t)|}{2}$, we have

$$\liminf_{(\lambda,\nu)\to(+\infty,\mu)} x_\nu(t)\Big(h(t,\lambda x_\nu(t))-\lambda\big(a_+(t)x_\nu^+(t)-a_-(t)x_\nu^-(t)\big)\Big)=\underline{h}_\infty(t)x_\mu^+(t)+\overline{h}_\infty(t)x_\mu^-(t),$$

for almost every $t\in[0,T]$. Therefore, we conclude that

$$\begin{aligned}&\int_0^T\frac{1}{|z_\mu(t)|^2}\liminf_{(\lambda,\nu)\to(+\infty,\mu)} x_\nu(t)\Big(h(t,\lambda x_\nu(t))-\lambda\big(a_+(t)x_\nu^+(t)-a_-(t)x_\nu^-(t)\big)\Big)dt\\ =&\int_{\{u_\mu>0\}}\frac{1}{|z_\mu(t)|^2}\liminf_{x\to+\infty}(h(t,x)-a_+(t)x)u_\mu(t)dt\\ &+\int_{\{u_\mu<0\}}\frac{1}{|z_\mu(t)|^2}\limsup_{x\to-\infty}(h(t,x)-a_-(t)x)u_\mu(t)dt,\end{aligned}\tag{4.3}$$

this is because, under (H_∞^l), every integral on the right hand side must be either finite or $+\infty$. Furthermore, (2.6) is equivalent to that the right hand side of (4.3) is greater than zero. By (4.2), (2.6) is equivalent to (4.1). Therefore, the proof is completed. □

Proof of Lemma 2.3. By (H_∞^l), for arbitrary $\varepsilon>0$, there is a positive constant $d>0$ for which

$$\frac{h(t,x)}{x}\geqslant a_+(t)-\varepsilon,\quad \text{for}\quad x\geqslant d,\tag{4.4}$$

and

$$\frac{h(t,x)}{x}\geqslant a_-(t)-\varepsilon,\quad \text{for}\quad x\leqslant -d.\tag{4.5}$$

Focus on a certain $t\in[0,T]$, it is noted that $h(t,x)/x$ will fall into one of two cases as follows.

(i) $h(t,x)/x\in[a_+(t)+\varepsilon,+\infty)$ for $x\geqslant d$, $h(t,x)/x\in[a_-(t)+\varepsilon,+\infty)$ for $x\leqslant -d$;

(ii) $h(t,x)/x\in[a_+(t)-\varepsilon,a_+(t)+\varepsilon]$ for $x\geqslant d$, $h(t,x)/x\in[a_-(t)-\varepsilon,a_-(t)+\varepsilon]$ for $x\leqslant -d$.

Case 1. $h(t,x)/x\in[a_+(t)+\varepsilon,+\infty)$ for $x\geqslant d$, and $h(t,x)/x\in[a_-(t)+\varepsilon,+\infty)$ for $x\leqslant -d$. In such a case, we see that

$$\frac{h(t,x)-a_\pm(t)x}{x}\geqslant\varepsilon>0,\quad \text{for}\quad |x|\geqslant d.$$

Then, it holds

$$h(t,d) - a_+(t)d > 0, \quad h(t,-d) + a_-(t)d < 0. \tag{4.6}$$

Moreover, we have

$$\lim_{x\to+\infty} \frac{h(t,d) - a_+(t)d}{x} = 0, \quad \lim_{x\to-\infty} \frac{h(t,-d) + a_-(t)d}{x} = 0.$$

Thus, for the above $\varepsilon > 0$, there is a positive constant d (which, for simplicity, we can take to be the same as before) such that

$$0 < \frac{h(t,d) - a_+(t)d}{x} \leqslant \varepsilon, \quad \text{for } x \geqslant d, \tag{4.7}$$

and

$$0 < \frac{h(t,-d) + a_-(t)d}{x} \leqslant \varepsilon, \quad \text{for } x \leqslant -d. \tag{4.8}$$

By (4.7), it holds

$$\frac{h(t,x)}{x} \geqslant a_+(t) + \frac{h(t,d) - a_+(t)d}{x}, \quad \text{for } x \geqslant d, \tag{4.9}$$

and by (4.8), it holds

$$\frac{h(t,x)}{x} \geqslant a_-(t) + \frac{h(t,-d) + a_-(t)d}{x}, \quad \text{for } x \leqslant -d. \tag{4.10}$$

We proceed to define a truncated function $\hat{h}$ as follows:

$$\hat{h}(t,x) = \begin{cases} a_+(t)x + h(t,d) - a_+(t)d, & x \geqslant d, \\ h(t,x), & |x| < d, \\ a_-(t)x + h(t,-d) + a_-(t)d, & x \leqslant -d. \end{cases}$$

We can conclude that: focusing on a certain $t \in [0,T]$, if $h(t,x)/x$ falls into Case 1, from (4.9) and (4.10), it follows that

$$\frac{h(t,x)}{x} \geqslant \frac{\hat{h}(t,x)}{x}, \quad \text{for } |x| \geqslant d. \tag{4.11}$$

Multiplying both sides of (4.11) by x^2 and using the definition of $\hat{h}$ yields

$$xh(t,x) \geqslant x\hat{h}(t,x), \quad \text{for } x \in \mathbb{R} \text{ and } a.e.\ t \in [0,T].$$

Hence, one can deduce the validity of (2.16). Furthermore, it can be obtained

$$\hat{r}(t,x) = \begin{cases} h(t,d) - a_+(t)d, & x \geqslant d, \\ h(t,x) - (a_+(t)x^+ - a_-(t)x^-), & |x| < d, \\ h(t,-d) + a_-(t)d, & x \leqslant -d. \end{cases}$$

As a result, $\hat{r}(t,x)$ is bounded: a T-periodic function $\eta(t)$, which belongs to $L^1([0,T])$, exists such that for a.e. $t \in [0,T]$ and any $x \in \mathbb{R}$,

$$|\hat{r}(t,x)| \leqslant \eta(t). \tag{4.12}$$

Therefore, we can also deduce the validity of (2.15).

Case 2. $h(t,x)/x \in [a_+(t)-\varepsilon, a_+(t)+\varepsilon]$ for $x \geqslant d$, and $h(t,x)/x \in [a_-(t)-\varepsilon, a_-(t)+\varepsilon]$ for $x \leqslant -d$. Now, we see that

$$|h(t,x)/x - a_+(t)| \leqslant \varepsilon, \quad \text{for } x \geqslant d,$$

and

$$|h(t,x)/x - a_-(t)| \leqslant \varepsilon, \quad \text{for } x \leqslant -d.$$

From the arbitrariness of ε, we have

$$\lim_{|x|\to+\infty} \frac{h(t,x) - a_+(t)x^+ + a_-(t)x^-}{x} = 0. \tag{4.13}$$

For a certain time instant $t \in [0,T]$, if $h(t,x)/x$ falls into Case 2, let $\hat{h}(t,x) = h(t,x)$, it is observed that (2.16) holds. Moreover, by (4.13), we can obtain

$$\lim_{|x|\to+\infty} \frac{\hat{r}(t,x)}{x} = 0.$$

Thus, (2.15) holds.

From the discussions in the above two cases, we can conclude that regardless of which case $h(t,x)/x$ falls into, we can find a function $\hat{h}$ that satisfies (2.15) and (2.16). Furthermore, after the modification, it is observed that, $\hat{h}$ lies within the interval $[a_+(t)-\varepsilon, a_+(t)+\varepsilon]$ when $x \geqslant d$, and $\hat{h}$ lies within the interval $[a_-(t)-\varepsilon, a_-(t)+\varepsilon]$ when $x \leqslant -d$. The proof is completed. □

We will prove Lemma 2.4 by contradiction in the next. Before this, we want to give some remarks. Based on the assumptions (H^l_∞) and $\rho(a) = k$, one can conclude that $\mathrm{Rot}_h(z_0) \geqslant k$ by Lemmas 2.1–2.3 in [22]. The Landesman-Lazer condition (LL^+_∞) serves to prevent the case where $\mathrm{Rot}_h(z_0) = k$, ensuring instead that $\mathrm{Rot}_h(z_0) > k$. This ensures that we can obtain some twist condition for an appropriate annulus when applying the Poincaré-Birkhoff theorem. When proved through contradiction, the sublinearity of the perturbation is critical in deriving the corresponding contradiction, which can be seen from (4.17) to (4.18) below. However, the sublinearity of the perturbation is not fully satisfied for the system (2.1). Therefore, we define a truncated system (4.14) based on Lemma 2.3, with its perturbation being sublinear and its solutions have a smaller angular velocity. We use the system (4.14) as a medium to deduce the corresponding contradiction.

Proof of Lemma 2.4. Assume, by way of contradiction, that there is a sequence $(z_n)_n$ of solutions that satisfy $z_n(0) = z_n^0$ for which $\min\{|z_n(t)| : t \in [0,T]\} \to +\infty$ as $n \to \infty$, and $\mathrm{Rot}_h(z_n^0) \leqslant k$.

Applying (H^l_∞) and Lemma 2.3, we can find a truncated function $\hat{h}$ that satisfies (2.15) and (2.16). Next, using this function $\hat{h}$, we define a truncated system

$$x' = y, \;\; y' = -\hat{h}(t,x), \tag{4.14}$$

in which $\hat{h}$ can be expressed as

$$\hat{h}(t,x) = a_+(t)x^+ - a_-(t)x^- + \hat{r}(t,x).$$

Therefore, system (4.14) is equivalent to

$$z' = L_1(t,z) + \hat{R}(t,z), \tag{4.15}$$

where $L_1(t,z) = (y, -a_+(t)x^+ + a_-(t)x^-)$, $\tilde{R}(t,z) = (0, -\hat{r}(t,x))$.

Denote by $(\hat{z}_n)_n$ a sequence of solutions to (4.14) with $\hat{z}_n(0) = z_n^0$. Furthermore, employing the definition of T-rotation numbers, we can obtain

$$\mathrm{Rot}_{\hat{h}}(z_n^0) \leqslant \mathrm{Rot}_h(z_n^0) \leqslant k. \tag{4.16}$$

Setting $\hat{w}_n = \hat{z}_n/\|\hat{z}_n\|_\infty$, and substitute it into (4.15), we have

$$\hat{w}_n' = L_1(t, \hat{w}_n) + \frac{\hat{R}(t,\hat{z}_n)}{\|\hat{z}_n\|_\infty}. \tag{4.17}$$

By the definition of $\hat{h}$, we can deduce that every solution to (4.14) exists globally. Consequently, the elastic property of solutions ensues. Furthermore, since $\min\{|z_n(t)| : t \in [0,T]\} \to +\infty$ as $n \to \infty$, it can be concluded that $\|z_n^0\|_\infty \to +\infty$ as $n \to \infty$. As a result, it holds that $\|\hat{z}_n\|_\infty \to +\infty$ as $n \to \infty$.

Due to the boundedness of $(\hat{w}_n)_n$ in $L^2(0,T)$, (4.17) yields the boundedness of $(\hat{w}_n)_n$ in $H^1([0,T])$. Thus, there is a $\hat{w} \in H^1([0,T])$ for which, up to subsequence, $\hat{w}_n \to \hat{w}$ uniformly, as well as $\hat{w}_n \rightharpoonup \hat{w}$ weakly in $H^1([0,T])$. Then, $\|\hat{w}\|_\infty = 1$. Taking the weak limit in (4.17) and noting that the last term disappears due to the sublinearity of $\hat{R}(t,z)$, we can obtain

$$\hat{w}' = L_1(t, \hat{w}). \tag{4.18}$$

Moreover, since the argument function of $\hat{w}(t)$ is 2π-periodic, so $\hat{w}(t)$ performs k clockwise rotations around the origin in $[0,T]$. Thus, $\hat{w}(t) = R_0 v_\mu(t)$ for proper $R_0 > 0$ and $v_\mu(t)$ is a solution to (4.18) that satisfies $v_\mu(0) = \mu \in \mathbb{S}^1$. Thus, we obtain

$$\int_0^T \frac{\langle JL_1(t, v_\mu(t)), v_\mu(t)\rangle}{|v_\mu(t)|^2} dt = \int_0^T \frac{\langle JL_1(t, R_0 v_\mu(t)), R_0 v_\mu(t)\rangle}{|R_0 v_\mu(t)|^2} dt = k. \tag{4.19}$$

Applying the generalized polar coordinates presented from Lemma 2.1, we have $\hat{z}_n(t) = r_n(t) v_{\mu_n(t)}(t)$, where $\mu_n(t) \in \mathbb{S}^1$ for each n. Therefore, by $\hat{w}_n \to \hat{w}(t)$, it holds that

$$\hat{z}_n(t)/\|\hat{z}_n\|_\infty \to R_0 v_\mu(t) \quad \text{uniformly as } n \to \infty.$$

Moreover, by $\hat{z}_n(t) = r_n(t) v_{\mu_n(t)}(t)$, it holds that

$$r_n(t)/\|\hat{z}_n\|_\infty \to R_0, \quad \mu_n(t) \to \mu \quad \text{uniformly when } n \to \infty.$$

Let $\hat{H} = (y, -\hat{h}(t,x))$. By (4.16) and (4.19), it holds

$$\int_0^T \frac{\langle J\hat{H}(t,\hat{z}_n), \hat{z}_n(t)\rangle}{|\hat{z}_n(t)|^2} dt \leqslant \int_0^T \frac{\langle JL_1(t, v_\mu(t)), v_\mu(t)\rangle}{|v_\mu(t)|^2} dt.$$

This implies

$$\int_0^T \frac{R_0\|\hat{z}_n\|_\infty}{r_n(t)} \Big(\frac{\langle J\hat{H}(t,\hat{z}_n(t)), \frac{\hat{z}_n(t)}{r_n(t)}\rangle}{|\hat{z}_n(t)|^2/(r_n(t))^2} - \frac{\langle JL_1(t, r_n(t) v_\mu(t)), v_\mu(t)\rangle}{|v_\mu(t)|^2} \Big) dt \leqslant 0.$$

Formulas (2.14) and (4.12) enable us to use the Fatou's lemma now, so we have

$$\int_0^T \liminf_{n\to+\infty} \frac{R_0\|\hat{z}_n\|_\infty}{r_n(t)} \Big(\frac{\langle J\hat{H}(t,\hat{z}_n(t)), \frac{\hat{z}_n(t)}{r_n(t)}\rangle}{|\hat{z}_n(t)|^2/(r_n(t))^2} - \frac{\langle JL_1(t, r_n(t) v_\mu(t)), v_\mu(t)\rangle}{|v_\mu(t)|^2} \Big) dt \leqslant 0.$$

Applying the usual properties of the inferior limit and keeping in mind that

$$\hat{z}_n(t)/(R_0\|\hat{z}_n\|_\infty) \to v_\mu(t), \quad \text{and} \quad r_n(t)/(R_0\|\hat{z}_n\|_\infty) \to 1,$$

uniformly as $n \to \infty$, it can be assumed, without loss of generality, that $\mu_n(t) \to \mu$ uniformly. If needed, it can be passed to a subsequence. Therefore, for any given $t \in [0, T]$, we are calculating the inferior limit appearing in (2.12) for H replaced by $\hat{H}$, along the specific subsequence $(r_n(t), \mu_n(t))$, for that $\mu_n(t) \to \mu$ and $r_n(t) \to +\infty$. Then, it can be concluded that

$$\int_0^T \liminf_{(\lambda,\nu)\to(+\infty,\mu)} \Big(\frac{\langle J\hat{H}(t, \lambda v_\nu(t)), v_\nu(t)\rangle}{|v_\nu(t)|^2} - \lambda \frac{\langle JL_1(t, v_\mu(t)), v_\mu(t)\rangle}{|v_\mu(t)|^2} \Big) dt \leqslant 0. \tag{4.20}$$

Furthermore, by Lemma 2.2 we can conclude that (4.20) is equivalent to

$$\int_{\{u>0\}} \liminf_{x\to+\infty} (\hat{h}(t,x) - a_+(t)x) u(t) dt + \int_{\{u<0\}} \limsup_{x\to-\infty} (\hat{h}(t,x) - a_-(t)x) u(t) dt \leqslant 0. \tag{4.21}$$

When $\hat{h}(t, x) = h(t, x)$, (4.21) contradicts with the hypothesis (2.6). When $\hat{h}(t, x) = a_+(t)x + h(t, d) - a_+(t)d$ for $x \geqslant d$, and $\hat{h}(t, x) = a_-(t)x + h(t, -d) + a_-(t)d$ for $x \leqslant -d$, (4.21) is changed into

$$\int_{\{u>0\}} (h(t, d) - a_+(t)d) u(t) dt + \int_{\{u<0\}} (h(t, -d) + a_-(t)d) u(t) dt \leqslant 0.$$

This is a contradiction. In fact, by the discussions in Lemma 2.3, we can conclude that

$$\int_{\{u>0\}} (h(t, d) - a_+(t)d) u(t) dt + \int_{\{u<0\}} (h(t, -d) + a_-(t)d) u(t) dt > 0.$$

Therefore, the proof is completed. □

Finally, we discuss in detail on $m < \rho(q) < m + 1$ and $\rho(\phi) = n$ in (1.10). First of all, similar to the discuss of (1.15) in [12], we can prove $m < \rho(q) < m + 1$. Therefore, we mainly focus on the proof of $\rho(\phi) = n$ in the following.

By (ii) of Lemma 5.1 in [12] and (1.9), we have $\rho(\phi) \geqslant n$. Consider the system

$$x' = y, \quad y' = -q(t)x^+ + \phi(t)x^-, \tag{4.22}$$

associated to equations (1.7). If the following statement hold, we can conclude $\rho(\phi) = n$ by Lemma 2.1 in [12].

($*$) There is a solution $z(t; z_0)$ to (4.22) with $z(0; z_0) = z_0$ such that

$$\theta(2\pi) - \theta_0 = -2n\pi, \tag{4.23}$$

with $\theta(t)$ being the argument function corresponding to $z(t; z_0)$ that satisfies $\theta(0) = \theta_0$.

Denote by $\theta(t)$ a nonzero 2π-periodic solution to the following equation:

$$\theta' = -q(t)((\cos\theta)^+)^2 - \phi(t)((\cos\theta)^-)^2 - \sin^2\theta$$

associated to system (4.22), with $\theta(0) = \theta_0$. Applying a simple calculation, we can obtain

$$\theta(0) - \theta\Big(\frac{n\pi}{2m+1} + \frac{n\pi}{2\alpha+\varrho}\Big) = 2n\pi. \tag{4.24}$$

Then, $\theta(2\pi) - \theta\Big(\frac{n\pi}{2m+1} + \frac{n\pi}{2\alpha+\varrho}\Big) = 0$. Using a analogous approach to Lemma 4.5 [20], it follows that

$$\arctan|\lambda| + \arctan|\mu| - \pi < \theta(2\pi) - \theta(\pi) < \max\{2\arctan|\lambda|,\ 2\arctan|\mu|\}.$$

Furthermore, since

$$\theta' = -\sin^2\theta - (2m+1)^2\cos^2\theta < -1, \quad \text{for} \quad t \in \Big(\frac{n\pi}{2m+1} + \frac{n\pi}{2\alpha+\varrho}, \pi\Big),$$

then

$$\theta(\pi) - \theta\Big(\frac{n\pi}{2m+1} + \frac{n\pi}{2\alpha+\varrho}\Big) < \frac{n\pi}{2m+1} + \frac{n\pi}{2\alpha+\varrho} - \pi.$$

By the definitions of λ and μ, we can find a nontrivial solution $\hat{\theta}(t)$ with $\hat{\theta}(0) = \hat{\theta}_0$ such that $\hat{\theta}(\pi) - \hat{\theta}\Big(\frac{n\pi}{2m+1} + \frac{n\pi}{2\alpha+\varrho}\Big) = \hat{\theta}(\pi) - \hat{\theta}(2\pi)$. Thus, by (4.24), we have

$$\hat{\theta}(2\pi) - \hat{\theta}_0 = \hat{\theta}(2\pi) - \hat{\theta}\Big(\frac{n\pi}{2m+1} + \frac{n\pi}{2\alpha+\varrho}\Big) + \hat{\theta}\Big(\frac{n\pi}{2m+1} + \frac{n\pi}{2\alpha+\varrho}\Big) - \hat{\theta}_0 = -2n\pi.$$

Therefore, $\rho(\phi) = n$.

5. Conclusions

In this paper, we studied the resonance problems for the parameter-dependent equation (1.1) under Landesman-Lazer conditions and obtained the multiplicity of periodic solutions. It is formulated in an original way, relying on sufficiently general assumptions.

We weakened the usual requirement on the sublinearity of the perturbations, and developed a more general method to investigate rotational characterizations of the Landesman-Lazer conditions. Moreover, we address the challenges arising from the sign-changing nature of the nonlinearity and the lack of global existence of solutions.

Author contributions

Chunlian Liu: Investigation, Writing–original draft; Shuang Wang and Fanfan Chen: Writing–review and editing; Chunlian Liu, Shuang Wang and Fanfan Chen: Conceptualization, Methodology, Validation. All authors have read and agreed to the published version of the manuscript.

Acknowledgments

This study was supported by the National Natural Science Foundation of China (Grant Nos. 12101337, 11901507, and 12301213).

Conflict of interest

The authors declare that there is no conflicts of interest.

References

1. M. S. Berger, E. Podolak, On the solutions of a nonlinear Dirichlet problem, *Indiana Univ. Math. J.*, **24** (1975), 837–846.
2. A. C. Lazer, P. J. McKenna, Large-amplitude periodic oscillations in suspension bridges: some new connections with nonlinear analysis, *SIAM Rev.*, **32** (1990), 537–578. https://doi.org/10.1137/1032120
3. C. Fabry, J. Mawhin, M. N. Nkashama, A multiplicity result for periodic solutions of forced nonlinear second order ordinary differential equations, *Bull. London Math. Soc.*, **18** (1986), 173–180. https://doi.org/10.1112/blms/18.2.173
4. A. C. Lazer, P. J. McKenna, Large scale oscillatory behaviour in loaded asymmetric systems, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **4** (1987), 243–274. https://doi.org/10.1016/S0294-1449(16)30368-7
5. R. Ortega, Stability of a periodic problem of Ambrosetti-Prodi type, *Differ. Integral Equ.*, **3** (1990), 275–284. https://doi.org/10.57262/die/1371586143
6. M. A. Del Pino, R. F. Manásevich, A. Murua, On the number of 2π periodic solutions for $u'' + g(u) = s(1 + h(t))$ using the Poincaré-Birkhoff theorem, *J. Differ. Equ.*, **95** (1992), 240–258. https://doi.org/10.1016/0022-0396(92)90031-h
7. B. Zinner, Multiplicity of solutions for a class of superlinear Sturm-Liouville problems, *J. Math. Anal. Appl.*, **176** (1993), 282–291. https://doi.org/10.1006/jmaa.1993.1213
8. C. Rebelo, F. Zanolin, Multiplicity results for periodic solutions of second order odes with asymmetric nonlinearities, *Trans. Amer. Math. Soc.*, **348** (1996), 2349–2389. https://doi.org/10.1090/S0002-9947-96-01580-2
9. C. Zanini, F. Zanolin, A multiplicity result of periodic solutions for parameter dependent asymmetric non-autonomous equations, *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.*, **12** (2005), 343–361.
10. A. Fonda, L. Ghirardelli, Multiple periodic solutions of scalar second order differential equations, *Nonlinear Anal.*, **72** (2010), 4005–4015. https://doi.org/10.1016/j.na.2010.01.032
11. A. Boscaggin, A. Fonda, M. Garrione, A multiplicity result for periodic solutions of second order differential equations with a singularity, *Nonlinear Anal.*, **75** (2012), 4457–4470. https://doi.org/10.1016/j.na.2011.10.025
12. C. L. Liu, S. Wang, Multiple periodic solutions of second order parameter-dependent equations via rotation numbers, *AIMS Math.*, **8** (2023), 25195–25219. https://doi.org/ 10.3934/math.20231285
13. A. Calamai, A. Sfecci, Multiplicity of periodic solutions for systems of weakly coupled parametrized second order differential equations, *Nonlinear Differ. Equ. Appl.*, **24** (2017), 1–17. https://doi.org/10.1007/s00030-016-0427-5
14. E. M. Landesman, A. C. Lazer, Nonlinear perturbations of linear elliptic boundary value problems at resonance, *J. Math. Mech.*, **19** (1970), 609–623.

15. C. Fabry, A. Fonda, Periodic solutions of nonlinear differential equations with double resonance, *Ann. Mat. Pura Appl.*, **157** (1990), 99–116. https://doi.org/10.1007/BF01765314
16. C. Fabry, Landesman-Lazer conditions for periodic boundary value problems with asymmetric nonlinearities, *J. Differ. Equ.*, **116** (1995), 405–418. https://doi.org/ 10.1006/jdeq.1995.1040
17. A. Fonda, M. Garrione, Double resonance with Landesman-Lazer conditions for planar systems of ordinary differential equations, *J. Differ. Equ.*, **250** (2011), 1052–1082. https://doi.org/10.1016/j.jde.2010.08.006
18. A. Boscaggin, M. Garrione, Resonance and rotation numbers for planar Hamiltonian systems: multiplicity results via the Poincaré-Birkhoff theorem, *Nonlinear Anal.*, **74** (2011), 4166–4185. https://doi.org/10.1016/j.na.2011.03.051
19. M. Garrione, A. Margheri, C. Rebelo, Nonautonomous nonlinear ODEs: nonresonance conditions and rotation numbers, *J. Math. Anal. Appl.*, **473** (2019), 490–509. https://doi.org/ 10.1016/j.jmaa.2018.12.063
20. C. L. Liu, D. B. Qian, P. J. Torres, Non-resonance and double resonance for a planar system via rotation numbers, *Results Math.*, **76** (2021), 1–23. https://doi.org/10.1007/s00025-021-01401-w
21. A. Fonda, A. J. Ureña, A higher dimensional Poincaré-Birkhoff theorem for Hamiltonian flows, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **34** (2017), 679–698. https://doi.org/ 10.1016/j.anihpc.2016.04.002
22. D. B. Qian, P. J. Torres, P. Y. Wang, Periodic solutions of second order equations via rotation numbers, *J. Differ. Equ.*, **266** (2019), 4746–4768. https://doi.org/10.1016/j.jde.2018.10.010
23. F. F. Chen, D. B. Qian, An extension of the Poincaré-Birkhoff theorem for Hamiltonian systems coupling resonant linear components with twisting components, *J. Differ. Equ.*, **321** (2022), 415–448. https://doi.org/10.1016/j.jde.2022.03.016
24. Q. B. Yin, Y. Guo, D. Wu, X. B. Shu, Existence and multiplicity of mild solutions for first-order Hamilton random impulsive differential equations with Dirichlet boundary conditions, *Qual. Theory Dyn. Syst.*, **22** (2023), 47. https://doi.org/10.1007/s12346-023-00748-5
25. S. Wang, Periodic solutions of weakly coupled superlinear systems with indefinite terms, *Nonlinear Differ. Equ. Appl.*, **29** (2022), 36. https://doi.org/10.1007/s00030-022-00768-1
26. C. Rebelo, A note on the Poincaré-Birkhoff fixed point theorem and periodic solutions of planar systems, *Nonlinear Anal.*, **29** (1997), 291–311. https://doi.org/10.1016/S0362-546X(96)00065-X