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Research article

Euclidean hypersurfaces isometric to spheres

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Abstract: Given an immersed hypersurface M^n in the Euclidean space E^{n+1}, the tangential component ω of the position vector field of the hypersurface is called the basic vector field, and the smooth function of the normal component of the position vector field gives a function σ on the hypersurface called the support function of the hypersurface. In the first result, we show that on a complete and simply connected hypersurface M^n in E^{n+1} of positive Ricci curvature with shape operator T invariant under ω and the support function σ satisfies the static perfect fluid equation if and only if the hypersurface is isometric to a sphere. In the second result, we show that a compact hypersurface M^n in E^{n+1} with the gradient of support function σ, an eigenvector of the shape operator T with eigenvalue function the mean curvature H, and the integral of the squared length of the gradient $\nabla\sigma$ has a certain lower bound, giving a characterization of a sphere. In the third result, we show that a compact and simply connected hypersurface M^n of positive Ricci curvature in E^{n+1} has an incompressible basic vector field ω, if and only if M^n is isometric to a sphere.

Keywords: shape operator; n-sphere; Euclidean space; static perfect fluid equation; incompressible vector fields
Mathematics Subject Classification: 53A50, 53C20

1. Introduction

Given an orientable immersed hypersurface M^n in the Euclidean space E^{n+1} with unit normal ξ and shape operator T, we denote by $\psi : M^n \to E^{n+1}$ the immersion, by g the induced metric, and denote the hypersurface by (M^n, ψ, g, ξ, T). The eigenvalues $\mu_1, .., \mu_n$ of the shape operator T are called principal curvatures of the hypersurface (M^n, ψ, g, ξ, T) and play a very important role in the geometry [1, 2] as well as the topology of (M^n, ψ, g, ξ, T) (cf. [3–5]). It is fascinating to see that constraints on principal curvatures also influence the topology of exterior $E^{n+1}\backslash M^n$. An interesting result in [6] proves that if a compact and connected hypersurface (M^n, ψ, g, ξ, T), $n \geq 2$, A^n is the unbounded component of $E^{n+1}\backslash M^n$ and the principal curvatures satisfy $\mu_1 + .. + \mu_n < 0$, then A^n simply connected. There are many important aspects of studying the geometry of hypersurface (M^n, ψ, g, ξ, T) in E^{n+1} and one of them is to study the geometry of (M^n, ψ, g, ξ, T) under the condition $\Delta^2\psi = 0$, and the study of such submanifolds was initiated by Chen [7–9], calling them biharrmonic hypersurfaces. Moreover, Chen conjectured that a biharmonic hypersurface of E^{n+1} is minimal (cf. [10–12]). An interesting result in Euclidean submanifolds is that of Jacobowicz (cf. [6]), which states an n-dimensional Riemannian manifold (M^n, g) with sectional curvatures less than a constant λ^{-2} admits an isometric immersion into the Euclidean space E^{2n-1} can never be contained in a ball of radius λ in E^{2n-1} and this result is the generalization of the nonembeddability result due to Tompkins (cf. [3]). Taking clue from [6, 13, 14], in [2], the author generalized the result to a compact hypersurface (M^n, ψ, g, ξ, T) in E^{n+1}, where it is proved that if the scalar curvature of (M^n, g) is less than a constant $n(n-1)\lambda^{-2}$, then no immersion $\psi : M^n \to E^{n+1}$ is contained in a ball of radius λ in E^{n+1}. It is still open to show that if the Ricci curvatures of a compact Riemannian manifold (M^n, g) are less than a constant $(n-1)\lambda^{-2}$, then no immersion $\psi : M^n \to E^{n+1}$ is contained in a ball of radius λ in E^{n+1}. The geometrical and topological properties [15–17] of hypersurfaces are the branches of differential geometry [18, 19], and a small portion of it is described above [20–22]. The submanifolds theory [23, 24] and soliton theory [25–27], etc., continue to inspire new insights and discoveries to help solve these problems and make it an active area of research in many branches of mathematics and physics.

Inspired by the previous results, in the present paper, we first intend to study the geometrical and topological properties of an orientable hypersurface (M^n, ψ, g, ξ, T) in the Euclidean space E^{n+1}, and we express the vector ψ as

$$\psi = \omega + \sigma\xi, \quad \sigma = \langle \psi, \xi \rangle, \tag{1.1}$$

where ω is tangential projection of ψ to M^n. We call ω the basic vector field and the function σ the support function of the hypersurface (M^n, ψ, g, ξ, T).

Secondly, the paper studied the static perfect fluid equation on a Riemannian manifold (M^n, g), which is given by

$$\sigma Ric - Hes(\sigma) = \frac{1}{n}(\sigma\tau - \Delta\sigma),$$

where Ric is the Ricci tensor, $Hes(\sigma)$ is the Hessian of σ, τ is the scalar curvature and Δ is the Laplace operator on (M^n, g). It is known that the static perfect fluid equation has immense importance in mathematical physics, in particular in fluid dynamics and also in differential geometry (cf [28] and references therein).

Furthermore, we investigate the impacts on the geometry of the complete and simply connected hypersurface (M^n, ψ, g, ξ, T) in the Euclidean space E^{n+1} with support function σ satisfying static perfect fluid equation and find conditions under which this hypersurface (M^n, ψ, g, ξ, T) is isometric to the Euclidean sphere $S^n(c)$ of constant curvature c (see Theorem 3.1).

We also show a compact and simply connected hypersurface (M^n, ψ, g, ξ, T) in E^{n+1} that has support function σ, mean curvature $H = \frac{1}{n} TraceT$, and the shape operator T satisfies

$$T(\nabla\sigma) = H\nabla\sigma, \quad \int_{M^n} \|\nabla\sigma\|^2 \geq \frac{1}{n} \int_{M^n} (\mathrm{div}\omega)^2,$$

where $\nabla\sigma$ is the gradient of the support function σ, if and only if H is a constant and hypersurface (M^n, ψ, g, ξ, T) is isometric to $S^n\left(H^2\right)$ (see Theorem 4.1).

For a vector field ζ on a Riemannian manifold (M^n, g) to be incompressible, it is required that $\mathrm{div}\zeta = 0$. This notion is borrowed from fluid mechanics (cf. [8, 30]). In this paper, further we study compact hypersurface (M^n, ψ, g, ξ, T) in the Euclidean space E^{n+1} of positive Ricci curvature with basic vector field ω incompressible and show that for such hypersurfaces the mean curvature H is a constant and (M^n, ψ, g, ξ, T) is isometric to $S^n\left(H^2\right)$ and also the converse holds (see Theorem 5.1).

2. Preliminaries

Let (M^n, ψ, g, ξ, T) be an orientable hypersurface in the Euclidean space E^{n+1} with immersion $\psi : M^n \to E^{n+1}$. We denote by $\Omega(M^n)$ the space of smooth vector fields on M^n and by ∇_E the covariant derivative in the direction of $E \in \Omega(M^n)$ with respect to the Riemannian connection of the induced metric g. Then, on differentiating Eq (1.1) with respect to $E \in \Omega(M^n)$ and using the Gauss-Weingarten formulae and equating akin parts, we obtain the following for the basic vector field ω [10, 29].

$$\nabla_E \omega = E + \sigma T E, \quad \nabla\sigma = -T\omega, \quad E \in \Omega(M^n). \tag{2.1}$$

The curvature tensor of (M^n, ψ, g, ξ, T) has the following expression:

$$R(E_1, E_2)E_3 = g(TE_2, E_3)\, TE_1 - g(TE_1, E_3)\, TE_2, \quad E_1, E_2, E_3 \in \Omega(M^n), \tag{2.2}$$

and using the following formulas of the Ricci tensor and the mean curvature H of (M^n, ψ, g, ξ, T)

$$Ric(E_1, E_2) = \sum_\alpha g(R(u_\alpha, E_1)E_2, u_\alpha), \quad H = \frac{1}{n} \sum_\alpha g(Tu_\alpha, u_\alpha), \tag{2.3}$$

where $\{u_\alpha\}_1^n$ is a local orthonormal frame on (M^n, ψ, g, ξ, T), we obtain through Eq (2.1)

$$Ric(E_1, E_2) = nHg(TE_1, E_2) - g(TE_1, TE_2), \quad E_1, E_2 \in \Omega(M^n). \tag{2.4}$$

The scalar curvature τ of (M^n, ψ, g, ξ, T) is obtained by taking trace in the above equation, and we have the formula

$$\tau = n^2 H^2 - \|T\|^2, \tag{2.5}$$

where

$$\|T\|^2 = \sum_\alpha g(Tu_\alpha, Tu_\alpha).$$

The shape operator T of the hypersurface (M^n, ψ, g, ξ, T) of the Euclidean space E^{n+1} satisfies the following Codazzi equation:

$$(\nabla T)(E_1, E_2) = (\nabla T)(E_2, E_1), \quad E_1, E_2 \in \Omega(M^n), \tag{2.6}$$

and here $(\nabla T)(E_1, E_2)$ means

$$(\nabla T)(E_1, E_2) = \nabla_{E_1} T E_2 - T(\nabla_{E_1} E_2).$$

On differentiating the expression for H in Eq (2.3) and utilizing Eq (2.6), we confirm

$$nE(H) = g\left(E, \sum_\alpha (\nabla T)(u_\alpha, u_\alpha)\right), \quad E \in \Omega(M^n),$$

and it accounts for the expression for the gradient ∇H of H given by

$$\nabla H = \frac{1}{n} \sum_\alpha (\nabla T)(u_\alpha, u_\alpha). \tag{2.7}$$

For a compact hypersurface (M^n, ψ, g, ξ, T) of the Euclidean space E^{n+1} with support function σ and mean curvature H, Minkowski's integral formula states

$$\int_{M^n} (1 + \sigma H) = 0. \tag{2.8}$$

Recall that for a vector field $\boldsymbol{\omega}$ on a Riemannian manifold (M^n, g), we say that an operator S defined on (M^n, g) is invariant under $\boldsymbol{\omega}$ if the following holds:

$$d\varphi_t \circ S = S \circ d\varphi_i,$$

where $\{\varphi_i\}$ is the local flow of $\boldsymbol{\omega}$. It follows that if S is invariant under $\boldsymbol{\omega}$, then we have

$$\pounds_\omega S = 0,$$

where $\pounds_\omega$ is the Lie derivative with respect to ω.

Next, we discuss the model example of the sphere $S^n(c)$ of constant curvature c as an embedded hypersurface $\psi : S^n(c) \to E^{n+1}$ given by

$$S^n(c) = \left\{ x \in E^{n+1} : \langle x, x \rangle = \frac{1}{c} \right\}, \quad \psi(x) = x.$$

The unit normal ξ to $S^n(c)$ is expressed by $\xi = \sqrt{c}\psi$ and the shape operator $T = -\sqrt{c}I$. Moreover, the support function σ of $S^n(c)$ is given by $\sigma = \frac{1}{\sqrt{c}}$ and the basic vector field $\boldsymbol{\omega}$ of $S^n(c)$ is given by $\boldsymbol{\omega} = 0$. The mean curvature H of $S^n(c)$ is given by $H = -\sqrt{c}$.

3. Hypersurfaces with support function solution of static perfect fluid equation

Let (M^n, ψ, g, ξ, T) be an orientable hypersurface in the Euclidean space E^{n+1} with immersion $\psi : M^n \to E^{n+1}$ support function σ, basic vector field $\boldsymbol{\omega}$, mean curvature H, and scalar curvature τ. We assume that the pressure on the support function σ to satisfy the static perfect fluid equation, namely

$$\sigma Ric - Hes(\sigma) = \frac{1}{n}(\tau\sigma - \Delta\sigma)\, g, \tag{3.1}$$

where $Hes(\sigma)$ is the Hessian, defined by

$$Hes(\sigma)(E_1, E_2) = g(\nabla_{E_1}\nabla\sigma, E_2), \quad E_1, E_2 \in \Omega(M^n),$$

and we also have Hessian operator H^σ and the Ricci operator Q defined by

$$Hes(\sigma)(E_1, E_2) = g(H^\sigma E_1, E_2), \quad Ric(E_1, E_2) = g(QE_1, E_2), \quad E_1, E_2 \in \Omega(M^n).$$

The Laplace operator Δ on (M^n, ψ, g, ξ, T) is defined by $\Delta\sigma = \operatorname{div}(\nabla\sigma)$, and it is also the trace of H^σ. Moreover, we also assume that the hypersurface (M^n, ψ, g, ξ, T) of the Euclidean space E^{n+1} has a basic vector field $\boldsymbol{\omega}$ under which the operator T is invariant, that is,

$$(\pounds_\omega T) = 0,$$

which amounts to the following

$$[\boldsymbol{\omega}, TE] = T[\boldsymbol{\omega}, E], \quad E \in \Omega(M^n). \tag{3.2}$$

We intend to show that a complete and simply connected hypersurface (M^n, ψ, g, ξ, T) of E^{n+1} having positive Ricci curvature, subjected to conditions (3.1) and (3.2), gets ready to acquire the shape of a sphere, as seen in the following:

Theorem 3.1. *A complete and simply connected hypersurface (M^n, ψ, g, ξ, T) of the Euclidean space E^{n+1}, $n > 1$, with positive Ricci curvature satisfies the shape operator T is invariant under the basic vector field $\boldsymbol{\omega}$, and the support function σ satisfies the static perfect fluid equation if and only if the mean curvature H is a constant and (M^n, ψ, g, ξ, T) is isometric to the sphere $S^n(H^2)$.*

Proof. Let (M^n, ψ, g, ξ, T) be a complete and simply connected hypersurface of positive Ricci curvature in the Euclidean space E^{n+1}, $n > 1$, with support function σ and the basic vector field $\boldsymbol{\omega}$ satisfying Eqs (3.1) and (3.2), respectively. On employing Eq (2.1) in Eq (3.2), we extract the following

$$(\nabla T)(\boldsymbol{\omega}, E) = TE + \sigma T^2 E - T(E + \sigma TE) = 0, \quad E \in \Omega(M^n). \tag{3.3}$$

Next, we wish to use Eq (2.1) in order to compute the Hessian operator H^σ as follows:

$$H^\sigma E = \nabla_E \nabla\sigma = -\nabla_E T\boldsymbol{\omega} = -(\nabla T)(E, \boldsymbol{\omega}) - T(\nabla_E \boldsymbol{\omega}), \quad E \in \Omega(M^n),$$

which, in view of Coddazzi Eq (2.6) and Eqs (2.1) and (3.3), yields

$$H^\sigma E = -TE - \sigma T^2 E, \quad E \in \Omega(M^n). \tag{3.4}$$

Equation (2.4) has the form $QE = nHTE - T^2 E$ and employing it in Eq (3.4), it turns out that

$$H^\sigma = -(1 + nH\sigma)T + \sigma Q. \tag{3.5}$$

Defining the second fundamental form Ξ of the hypersurface (M^n, ψ, g, ξ, T), by

$$\Xi(E_1, E_2) = g(TE_1, E_2), \quad E_1, E_2 \in \Omega(M^n).$$

Thus, Eq (3.5) now takes the shape

$$\sigma Ric - Hes(\sigma) = (1 + nH\sigma)\Xi. \tag{3.6}$$

Next, taking trace in the Eq (3.4), we conclude

$$\Delta\sigma = -nH - \sigma \|T\|^2,$$

and using Eq (2.5) in above the equation, we have

$$\Delta\sigma = -nH - n^2\sigma H^2 + \sigma\tau.$$

Inserting this value in Eq (3.1), we confirm

$$\sigma Ric - Hes(\sigma) = H(1 + n\sigma H)\, g. \tag{3.7}$$

Combining Eqs (3.6) and (3.7), we have

$$(1 + n\sigma H)(\Xi - Hg) = 0. \tag{3.8}$$

Assume that $(1 + n\sigma H) = 0$ holds. Then, we conclude $\sigma\nabla H = -H\nabla\sigma$, which, on employing Eq (2.1) implies

$$\sigma\nabla H = HT\boldsymbol{\omega}. \tag{3.9}$$

Now, we use Eqs (2.6) and (3.3) to conclude $(\nabla T)(E, \boldsymbol{\omega}) = 0$, and taking the inner product with E while using the symmetry of T gives

$$g(\boldsymbol{\omega}, (\nabla T)(E, E)) = 0, \quad E \in \Omega(M^n).$$

On taking the sum in the above equation over a local orthonormal frame $\{u_\alpha\}_1^n$ and employing the (2.7), we conclude

$$g(\boldsymbol{\omega}, \nabla H) = 0. \tag{3.10}$$

The Eq (3.9), on taking the inner product with $\boldsymbol{\omega}$ and using Eq (3.10), we get

$$Hg(T\boldsymbol{\omega}, \boldsymbol{\omega}) = 0,$$

and utilizing it with Eq (2.4) in computing $Ric(\boldsymbol{\omega}, \boldsymbol{\omega})$, we get

$$Ric(\boldsymbol{\omega}, \boldsymbol{\omega}) = -\|T\boldsymbol{\omega}\|^2.$$

As $Ric > 0$, the above equation implies the basic vector field $\boldsymbol{\omega} = 0$, that is, $\operatorname{div}(\boldsymbol{\omega}) = n(1 + \sigma H)$ (by virtue of Eq (2.1)) and we get $\sigma H = -1$, and combining it with our assumption $(1 + n\sigma H) = 0$, we get $(n - 1) = 0$, a contradiction as $n > 1$. Hence, we have $(1 + n\sigma H) \neq 0$ and as M^n is simply connected. It is connected and as such, Eq (3.8) now yields

$$\Xi = Hg,$$

that is equivalent to

$$T = HI, \tag{3.11}$$

which implies

$$(\nabla T)(E,E) = E(H)E, \quad E \in \Omega(M^n).$$

Taking $E = u_\alpha$ in the above equation for an orthonormal frame $\{u_\alpha\}_1^n$, and summing while using Eq (2.7), we conclude $n\nabla H = \nabla H$, which, in view of $n > 1$, implies H is a constant. Now, the Eq (2.2), gives

$$R(E_1,E_2)E_3 = H^2\{g(E_2,E_3)E_1 - g(E_1,E_3)E_2\}, \quad E_1,E_2,E_3 \in \Omega(M^n).$$

The above equation confirms that the hypersurface (M^n,ψ,g,ξ,T) has constant curvature H^2. Note that the constant $H^2 > 0$ as $Ric = (n-1)H^2 g > 0$. Hence, the complete and simply connected (M^n,ψ,g,ξ,T) is isometric to $S^n\left(H^2\right)$.

The converse is trivial, as for the support function σ of $S^n\left(H^2\right)$ is a constant $\sigma = -\frac{1}{H}$ that satisfies Eq (3.1) and the basic vector field $\boldsymbol{\omega} = 0$ satisfies (3.2) and also that $S^n\left(H^2\right)$ has positive Ricci curvature. □

4. Hypersurface with gradient of support function an eigenvector of shape operator

Let (M^n,ψ,g,ξ,T) be a complete and simply connected hypersurface in E^{n+1} with basic vector field $\boldsymbol{\omega}$, support function σ, and mean curvature H. At times, simple restrictions lead to very fundamental results. In this section, we are going to witness a similar situation. We are going to show that a simple condition like $T(\nabla\sigma) = H(\nabla\sigma)$, that is, the gradient $\nabla\sigma$ of σ is an eigenvector of T with eigenvalue function H, and an appropriate lower bound on the integral of $\|\nabla\sigma\|^2$ leads to a characterization of the sphere. Indeed, we prove the following:

Theorem 4.1. *A compact and simply connected hypersurface (M^n,ψ,g,ξ,T) of the Euclidean space E^{n+1}, $n > 1$, with support function σ, basic vector field $\boldsymbol{\omega}$ and mean curvature H satisfy*

$$T(\nabla\sigma) = H(\nabla\sigma), \qquad \int_{M^n} \|\nabla\sigma\|^2 \geq \frac{1}{n}\int_{M^n} (div\boldsymbol{\omega})^2;$$

if and only if, H is a constant and (M^n,ψ,g,ξ,T) is isometric to the sphere $S^n\left(H^2\right)$.

Proof. Let (M^n,ψ,g,ξ,T) be a compact and simply connected hypersurface E^{n+1}, $n > 1$, with support function σ, the basic vector field $\boldsymbol{\omega}$ and mean curvature H, satisfying

$$T(\nabla\sigma) = H(\nabla\sigma), \tag{4.1}$$

and

$$\int_{M^n} \|\nabla\sigma\|^2 \geq \frac{1}{n}\int_{M^n} (\mathrm{div}\boldsymbol{\omega})^2. \tag{4.2}$$

Inserting the Eq (2.1) namely $\nabla\sigma = -T\boldsymbol{\omega}$ in Eq (4.1) to reach $T^2\boldsymbol{\omega} = HT\boldsymbol{\omega}$, which by the inner product with $\boldsymbol{\omega}$, gives

$$\|T\boldsymbol{\omega}\|^2 = Hg(T\boldsymbol{\omega},\boldsymbol{\omega}).$$

Employing Eq (2.4) in the above equation, we arrive at

$$Ric\,(\omega, \omega) = (n-1)Hg\,(T\omega, \omega)\,,$$

and combining it with Eq (2.1), confirms

$$Ric\,(\omega, \omega) = -(n-1)H\omega\,(\sigma) = -(n-1)\,(\omega\,(H\sigma) - \sigma\omega\,(H))\,. \tag{4.3}$$

Note that Eq (4.1) also implies $T\,(\nabla\sigma) = \nabla\,(H\sigma) - \sigma\nabla H$, which by the inner product with ω, implies

$$g\,(T\,(\nabla\sigma)\,, \omega) = \omega\,(H\sigma) - \sigma\omega\,(H)\,,$$

and employing Eq (2.1), we conclude

$$-\,\|\nabla\sigma\|^2 = \omega\,(H\sigma) - \sigma\omega\,(H)\,.$$

We insert the above equation in Eq (4.3), yielding

$$Ric\,(\omega, \omega) = (n-1)\,\|\nabla\sigma\|^2\,. \tag{4.4}$$

Next, we use a local frame $\{u_\alpha\}_1^n$ and Eq (2.1), in computing

$$\operatorname{div}(\omega) = n\,(1 + \sigma H)\,, \tag{4.5}$$

$$\|\nabla\omega\|^2 = n + 2n\sigma H + \sigma^2\,\|T\|^2\,, \tag{4.6}$$

and

$$|\pounds_{\omega} g|^2 = 4\left(n + 2n\sigma H + \sigma^2\,\|T\|^2\right), \tag{4.7}$$

where the Lie derivative $\pounds_{\omega} g$ is given by

$$(\pounds_{\omega} g)\,(E_1, E_2) = g\,(\nabla_{E_1}\omega, E_2) + g\,(\nabla_{E_2}\omega, E_1)\,, \quad E_1, E_2 \in \Omega\,(M^n)\,.$$

Now, recall the following integral formula from [31], for the compact hypersurface (M^n, ψ, g, ξ, T)

$$\int_{M^n} \left(Ric\,(\omega, \omega) + \frac{1}{2}\,|\pounds_{\omega} g|^2 - \|\nabla\omega\|^2 - (\operatorname{div}(\omega))^2\right) = 0,$$

and inserting Eqs (4.4), (4.6) and (4.7) in the above equation have

$$\int_{M^n} \left((n-1)\,\|\nabla\sigma\|^2 + \sigma^2\,\|T\|^2 + n + 2n\sigma H - (\operatorname{div}(\omega))^2\right) = 0.$$

Rearranging the above equation, we have

$$\int_{M^n} \left\{\sigma^2\left(\|T\|^2 - nH^2\right) + \frac{1}{n}\,(1 + \sigma H)^2 + (n-1)\,\|\nabla\sigma\|^2 - (\operatorname{div}(\omega))^2\right\} = 0,$$

which, on employing Eq (4.5), gives

$$\int_{M^n} \sigma^2 \left(\|T\|^2 - nH^2\right) = (n-1) \int_{M^n} \left(\frac{1}{n} (\operatorname{div}(\omega))^2 - \|\nabla\sigma\|^2\right).$$

Now, employing the inequality (4.2) in the above equation results in

$$\int_{M^n} \sigma^2 \left(\|T\|^2 - nH^2\right) \leq 0.$$

Further, note that it is due to Schwartz's inequality $\|T\|^2 \geq nH^2$, the integrand in above inequality is non-negative, therefore, we conclude

$$\sigma^2 \left(\|T\|^2 - nH^2\right) = 0. \tag{4.8}$$

Note that the implication $\sigma = 0$ of the above equation is forbidden due to Minkowski's formula (2.8). Hence, $\sigma \neq 0$ on the connected M^n forces Eq (4.8) to yield

$$\|T\|^2 = nH^2,$$

which being an equality in Schwartz's inequality, holds if and only if $T = HI$, and it leads to H a constant (see argument after Eq (3.11)). Hence, as seen in the proof of Theorem 1, the curvature tensor hypersurface (M^n, ψ, g, ξ, T) is given by

$$R(E_1, E_2)E_3 = H^2 \{g(E_2, E_3) E_1 - g(E_1, E_3) E_2\}, \quad E_1, E_2, E_3 \in \Omega(M^n).$$

Now, by a global argument that on a compact hypersurface in a Euclidean space, there exists a point where all sectional curvatures are positive, we see that the constant $H^2 > 0$. Hence, the simply connected hypersurface (M^n, ψ, g, ξ, T) being compact is also complete, and thus, the complete and simply connected hypersurface (M^n, ψ, g, ξ, T) has constant positive curvature H^2. Hence, (M^n, ψ, g, ξ, T) is isometric to $S^n\left(H^2\right)$. The converse is trivial. □

5. Hypersurfaces with incompressible basic vector field

In this section, we study the geometry of a compact and simply connected hypersurface (M^n, ψ, g, ξ, T) in a Euclidean space E^{n+1} with basic vector field $\boldsymbol{\omega}$, support function σ and mean curvature H with basic vector field $\boldsymbol{\omega}$ incompressible, that is, satisfying $\operatorname{div}(\omega) = 0$. The notion that $\boldsymbol{\omega}$ is incompressible for the compact hypersurface (M^n, ψ, g, ξ, T) is so strong that it alone suffices in forcing the hypersurface (M^n, ψ, g, ξ, T) of positive Ricci curvature to acquire the shape of a sphere, as seen in the following:

Theorem 5.1. *A compact and simply connected hypersurface (M^n, ψ, g, ξ, T) of the Euclidean space E^{n+1}, $n > 1$, of positive Ricci curvature with support function σ, mean curvature H, has the basic vector field $\boldsymbol{\omega}$ incompressible if and only if H is a constant and (M^n, ψ, g, ξ, T) is isometric to the sphere $S^n\left(H^2\right)$.*

Proof. Suppose (M^n, ψ, g, ξ, T) is a compact and simply connected hypersurface of the Euclidean space E^{n+1}, $n > 1$, with the basic vector field $\boldsymbol{\omega}$ incompressible. Then, as $\operatorname{div}(\boldsymbol{\omega}) = 0$, by Eq (4.5), $\sigma H = -1$ and, therefore, both functions σ and H are nowhere zero on M^n and we have

$$\nabla H = -\frac{1}{\sigma^2}\nabla\sigma, \tag{5.1}$$

and joining it with Eq (2.7), we conclude

$$\sum_\alpha (\nabla T)(u_\alpha, u_\alpha) = -\frac{n}{\sigma^2}\nabla\sigma. \tag{5.2}$$

Now, using $\nabla\sigma = -T\boldsymbol{\omega}$ from Eq (2.1), which on differentiation gives the following expression for H^σ

$$H^\sigma E = -(\nabla T)(E, \boldsymbol{\omega}) - T(E + \sigma T E),$$

and taking trace, while using the symmetry of T, we conclude

$$\Delta\sigma = -g\left(\boldsymbol{\omega}, \sum_\alpha (\nabla T)(u_\alpha, u_\alpha)\right) - nH - \sigma\|T\|^2.$$

Now using the above equation with Eq (4.5) in the form $\sigma H = -1$ and Eq (5.2), we confirm

$$\Delta\sigma = \frac{n}{\sigma^2}\boldsymbol{\omega}(\sigma) + \frac{n}{\sigma} - \sigma\|T\|^2.$$

Multiplying the above equation by σ and then integrating by parts would lead us to

$$-\int_{M^n}\|\nabla\sigma\|^2 = \int_{M^n}\left(\frac{n}{\sigma}\boldsymbol{\omega}(\sigma) + n - \sigma^2\|T\|^2\right),$$

which could be rearranged as

$$\int_{M^n}\sigma^2\left(\|T\|^2 - nH^2\right) = \int_{M^n}\left(\frac{n}{\sigma}\boldsymbol{\omega}(\sigma) + n\left(1 - \sigma^2 H^2\right) + \|\nabla\sigma\|^2\right).$$

Now, using $\sigma H = -1$ and (5.1), in the above equation, it yields

$$\int_{M^n}\sigma^2\left(\|T\|^2 - nH^2\right) = \int_{M^n}\left(\|\nabla\sigma\|^2 - n\sigma\boldsymbol{\omega}(H)\right). \tag{5.3}$$

We observe that

$$\operatorname{div}(H(\sigma\boldsymbol{\omega})) = \sigma\boldsymbol{\omega}(H) + H\operatorname{div}(\sigma\boldsymbol{\omega}),$$

and that as $\boldsymbol{\omega}$ is incompressible implies $\operatorname{div}(\sigma\boldsymbol{\omega}) = \boldsymbol{\omega}(\sigma)$. Thus, we have

$$\operatorname{div}(H(\sigma\boldsymbol{\omega})) = \sigma\boldsymbol{\omega}(H) + H\boldsymbol{\omega}(\sigma).$$

Also, we see by Eq (2.1), $\boldsymbol{\omega}(\sigma) = g(\boldsymbol{\omega}, \nabla\sigma) = -g(T\boldsymbol{\omega}, \boldsymbol{\omega})$ and the above equation becomes

$$\operatorname{div}(H(\sigma\boldsymbol{\omega})) = \sigma\boldsymbol{\omega}(H) - Hg(T\boldsymbol{\omega}, \boldsymbol{\omega}).$$

Next, we plug the above equation with Eq (2.4) and get

$$n\sigma\omega(H) = n\mathrm{div}(H(\sigma\omega)) + nHg(T\omega, \omega) = n\mathrm{div}(H(\sigma\omega)) + \left(Ric(\omega, \omega) + \|T\omega\|^2\right).$$

Plugging the above equation with Eq (5.3), we achieve

$$\int_{M^n} \sigma^2 \left(\|T\|^2 - nH^2\right) = \int_{M^n} \left(\|\nabla\sigma\|^2 - Ric(\omega, \omega) - \|T\omega\|^2\right),$$

and by Eq (2.1), $\nabla\sigma = -T\omega$, the above equation reduces to

$$\int_{M^n} \sigma^2 \left(\|T\|^2 - nH^2\right) = -\int_{M^n} Ric(\omega, \omega).$$

Owing to Schwartz's inequality, the left-hand side in the above equation is non-negative, and since the Ricci curvature is positive, the left-hand side in the above equation is strictly negative. The only possible conclusion is

$$\sigma^2 \left(\|T\|^2 - nH^2\right) = 0, \text{ and } \quad \omega = 0. \tag{5.4}$$

The second equation in Eq (5.4) together with $\nabla\sigma = -T\omega$ implies that σ is a constant. This constant σ has to be non-zero, for otherwise both $\omega = 0$ and $\sigma = 0$ would imply $\psi = \omega + \sigma\xi = 0$ a contradiction. Hence, σ is a non-zero constant and also simultaneously H is a non-zero constant (owing to $\sigma H = -1$) and Eq (5.4) reduces to

$$\|T\|^2 - nH^2 = 0.$$

Then, using the following Eq (4.8) as in the proof of Theorem 4.1, we conclude (M^n, ψ, g, ξ, T) is isometric to $S^n\left(H^2\right)$. The converse follows trivially as the sphere $S^n\left(H^2\right)$ has positive Ricci curvature, and as a hypersurface the sphere has a basic vector field $\omega = 0$, which is automatically incompressible.

□

6. Conclusions

There is a lot to comment on each result in this paper and future scopes of their respective generalizations. However, we shall concentrate on the Theorem 3.1, where it is proved that a complete and simply connected hypersurface (M^n, ψ, g, ξ, T) of the Euclidean space E^{n+1}, $n > 1$, with positive Ricci curvature, satisfies the shape operator T is invariant under the basic vector field ω, and the support function σ satisfies the static perfect fluid equation if and only if the mean curvature H is a constant and (M^n, ψ, g, ξ, T) is isometric to the sphere $S^n\left(H^2\right)$. There are natural questions tagged to this result, namely:

(a) Can we relax the condition that the hypersurface (M^n, ψ, g, ξ, T) has positive Ricci curvature?

(b) Can we replace the condition in Theorem 3.1 that the operator T is invariant under ω by the condition $T(\omega) = \frac{\tau}{n}\omega$?

(c) Note that apart from the support function σ of the hypersurface (M^n, ψ, g, ξ, T), there is yet another function $\delta : M^n \to R$ defined by

$$\delta = \frac{1}{2}\|\psi\|^2,$$

and this function δ satisfies $\nabla\delta = \omega$. A natural question would be to find additional conditions under which the complete and simply connected hypersurface (M^n, ψ, g, ξ, T) with function δ satisfying static perfect fluid equation

$$\delta Ric - Hes(\delta) = \frac{1}{n}(\delta\tau - \Delta\delta)$$

is isometric to the sphere $S^n\left(H^2\right)$?

These questions would be our focus for future studies on the hypersurface (M^n, ψ, g, ξ, T) of the Euclidean space E^{n+1}.

Author contributions

Yanlin Li: Conceptualization, investigation, methodology, writing-review and editing; Nasser Bin Turki: Conceptualization, investigation, methodology, writing-review and editing; Sharief Deshmukh: Conceptualization, investigation, methodology, writing-review and editing; Olga Belova: Conceptualization, investigation, methodology, writing-review and editing. All authors of this article have been contributed equally. All authors have read and agreed to the published version of the manuscript.

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Conflict of interest

The authors declare no conflict of interest.

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