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Research article

Locally quasihyperbolic mappings in real Banach spaces

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Abstract: In this paper, we use locally quasihyperbolic mappings to establish a characterization for quasihyperbolic mappings in Banach spaces. More generally, we prove that locally coarsely quasihyperbolic mappings are coarsely quasihyperbolic mappings, while the converse is invalid.

Keywords: quasihyperbolic metric; (locally) quasihyperbolic mapping; (locally) coarsely quasihyperbolic mapping; quasisymmetric mapping
Mathematics Subject Classification: 30C65, 30F45, 30L10

1. Introduction

The quasihyperbolic metric was introduced by Gehring and his students Palka and Osgood in the 1970's in the setting of Euclidean spaces $\mathbb{R}^n$ ($n \geq 2$) [1, 2]. With the aid of the quasihyperbolic metric, from late 1980's onwards, Väisälä developed the theory of (dimension) freely quasiconformal mappings in Banach spaces [7–11].

We assume throughout this paper that E and E' are real Banach spaces with dimension at least 2, and that $G \subsetneq E$ and $G' \subsetneq E'$ are domains, i.e., nonempty connected open sets. A homeomorphism $f\colon G \to G'$ is said to be an M-quasihyperbolic (M-QH) mapping with $M \geq 1$ if

$$\frac{1}{M} k_G(z_1, z_2) \leq k_{G'}(z_1', z_2') \leq M k_G(z_1, z_2)$$

for all $z_1, z_2 \in G$. Here and hereafter, the primes always denote the images in G' of the points in G under the mapping f, and k_G and $k_{G'}$ are the quasihyperbolic metrics of G and G', respectively; see Subsection 2.2 for the precise definitions.

Quasihyperbolic mappings constitute a basic mapping class in the theory of freely quasiconformal mappings, and they are also one of the important tools for the study of the freely quasiconformal mapping theory. Therefore, the study of the properties of quasihyperbolic mappings has attracted extensive attention.

In this paper, we will investigate an open problem raised by Väisälä concerning the quasihyperbolic mappings in the freely quasiconformal mapping theory [11].

Open problem 1.1. *[11, 13.2.13] Suppose that $f\colon G \to G'$ is a homeomorphism and that each point of G has a neighborhood $D \subset G$ such that $f|_D\colon\ D \to f(D)$ is M-QH. Is f M'-QH with $M' = M'(M)$?*

There are some related references that discuss Väisälä's open problem [3,4,12,13]. Here the authors positively answered the problem under certain additional conditions. In [4], Huang et al. considered Open problem 1.1 in the setting of metric spaces and obtained the following:

Theorem 1.1. *[4, Theorem 1.10] Let X be a c_1-quasiconvex and dense metric space, and let Y be a c_2-quasiconvex, dense, and proper metric space. Let $G \subsetneq X$ and $G' \subsetneq Y$ be two domains. Suppose that $f\colon G \to G'$ is a homeomorphism that is both semi-locally M-QH and semi-locally η-QS with $M > 1$ a constant and $\eta\colon [0,\infty) \to [0,\infty)$ a homeomorphism. Then f is M_1-QH with $M_1 = M_1(M, \eta, c_1, c_2)$.*

Here a metric space X is said to be *dense* if for any two points $x, y \in X$ and two positive real numbers r_1, r_2 with $|x-y| < r_1 + r_2$, we have $B(x, r_1) \cap B(y, r_2) \neq \emptyset$, and a homeomorphism $f\colon G \to G'$ is said to be semi-locally M-QH (resp. semi-locally η-QS), if for each point $z \in G$, the homeomorphism $f|_{B(z,d_G(z))}$ is M-QH (resp. η-QS), where $d_G(z) = \operatorname{dist}(z, \partial G)$.

Subsequently, in [12], Zhou considered Open problem 1.1 in the setting of Banach spaces and obtained the following:

Theorem 1.2. *[12, Theorem 1.2] Suppose that $f\colon G \to G'$ is a homeomorphism. If there exists a constant $M > 1$ and a homeomorphism $\eta\colon [0,\infty) \to [0,\infty)$ such that f is semi-locally M-QH and semi-locally η-QS, then f is M_1-QH with $M_1 = M_1(M, \eta)$.*

At the same time, in [13], Zhou et al. considered Open problem 1.1 in the setting of length metric spaces, and weakened the condition "semi-locally η-QS" to "semi-locally relatively η-QS" (see Subsection 2.4 for the precise definition), strengthened the above Theorem 1.2, and obtained the following.

Theorem 1.3. *[13, Theorem 2] Suppose that X is a length metric space and Y is a c-quasiconvex and complete metric space, and that $G \subsetneq X$ and $G' \subsetneq Y$ are two domains. Suppose that $M > 1$ is a constant, $\eta\colon [0,\infty) \to [0,\infty)$ a homeomorphism, and $f\colon G \to G'$ a homeomorphism that is both semi-locally M-QH and semi-locally relatively η-QS. Then f is M_1-QH with $M_1 = M_1(M, \eta, c)$.*

Remark 1.1. *Note that Theorems 1.1–1.3 show that Open problem 1.1 has a positive answer under the condition that f is both semi-locally M-QH and semi-locally (relatively) η-QS. Naturally, one will ask whether the related conditions of Theorems 1.1–1.3 can be removed or not. In [3, Example 2.1], He et al. constructed an example to illustrate that Open problem 1.1 has a negative answer, which shows that these related conditions cannot be removed.*

In [7], Väisälä investigated the properties of QH mappings and proved the following result:

Theorem 1.4. *[7, Theorem 4.7] Suppose that $G \subsetneq E$, $G' \subsetneq E'$, and that $f\colon G \to G'$ is M-QH. Then for every subdomain D of G, the restriction $f|_D$ is $4M^2$-QH.*

Remark 1.2. *Based on Theorem 1.4, we know that the condition "f being semi-locally M-QH" in Theorems 1.1–1.3 is necessary. Naturally, one will ask whether the condition "f being semi-locally (relatively) η-QS" in Theorems 1.1–1.3 is necessary or not. In Section 3, we construct an example to show that it is not a necessary condition; see Example 3.1.*

The main motivation of this paper is to consider Open problem 1.1. We obtain the following necessary and sufficient condition for QH mappings. For $0 < q \le 1$, a homeomorphism $f\colon G \to G'$ is said to be q-locally M-QH, if for each point $z \in G$, the restriction $f|_{B_q^G(z)}\colon B_q^G(z) \to f(B_q^G(z))$ is M-QH, where $B_q^G(z) = B(z, qd_G(z))$. Obviously, 1-locally M-QH means semi-locally M-QH in Theorems 1.1–1.3. Our main result is as follows:

Theorem 1.5. *Suppose that $G \subsetneq E$, $G' \subsetneq E'$, and $f\colon G \to G'$ is a homeomorphism. Then f is M-QH if and only if both f and f^{-1} are q-locally M'-QH such that in the necessity part q is arbitrary and M' depends only on M and such that in the sufficiency part M depends only on (M', q).*

There are several ways to prove Theorem 1.5, such as using [11, Theorem 5.26] or [12, Theorem 1.1], but these proofs may not be so elementary. In this paper, we shall provide an elementary proof of Theorem 1.1. In fact, we obtained as follows a more general result related to (locally) coarsely quasihyperbolic (briefly, CQH) mappings.

Theorem 1.6. *Suppose that $G \subsetneq E$, $G' \subsetneq E'$, and $f\colon G \to G'$ is a homeomorphism. If f and f^{-1} are q-locally (M, C)-CQH, then f is (M', C')-CQH with the constants M' and C' depending only on the constants M, C, and q.*

Remark 1.3. *The converse of Theorem 1.6 is invalid; see Example 3.2.*

The rest of this paper is organized as follows; In Section 2, we recall necessary definitions and preliminary results. In Section 3, we will prove Remarks 1.2 and 1.3 and Theorems 1.5 and 1.6.

2. Preliminaries

2.1. Notation

Throughout this paper, we assume that E and E' are real Banach spaces with dimension at least 2, and that $G \subsetneq E$ and $G' \subsetneq E'$ are domains, i.e., nonempty connected open sets. The norm of a vector z in E is written as $|z|$, and for every pair of points z_1, z_2 in E, the distance between them is denoted by $|z_1 - z_2|$. Let $B(x, r)$ denote the open ball with the center $x \in E$ and radius $r(> 0)$, and $S(x, r)$ and $\overline{B}(x, r)$ denote the boundary and the closure of $B(x, r)$, respectively. In particular, for $0 < q \le 1$ and $x \in G$, let $B_q^G(x) = B(x, qd_G(x))$ and $S_q^G(x) = S(x, qd_G(x))$, where $d_G(x)$ denotes the distance from x to the boundary ∂G of G.

2.2. Quasihyperbolic metric and quasigeodesic

Let $G \subsetneq E$ be a domain. For a rectifiable curve γ in G, that is, of the length $\ell(\gamma) < \infty$, its quasihyperbolic length is defined by

$$\ell_{k_G}(\gamma) = \int_\gamma \frac{|dx|}{d_G(x)}.$$

For each pair of points z_1, z_2 in G, the quasihyperbolic distance $k_G(z_1, z_2)$ between z_1 and z_2 is defined in the usual way:

$$k_G(z_1, z_2) = \inf_\gamma \ell_{k_G}(\gamma)$$

with the infimum taken over all rectifiable curves γ in G joining z_1 to z_2. It is known that k_G is a metric in G, the quasihyperbolic metric. We introduce some estimates on the quasihyperbolic metric.

Theorem 2.1. *[7, Lemma 2.2]*

(1) *For all* $z_1, z_2 \in G$, *we have*

$$k_G(z_1, z_2) \geq \log\Big(1 + \frac{|z_1 - z_2|}{\min\{d_G(z_1), d_G(z_2)\}}\Big) \geq \Big|\log \frac{d_G(z_1)}{d_G(z_2)}\Big|.$$

(2) *For* $w \in G$, $0 < t < 1$, *and* z_1, $z_2 \in \overline{B}(w, td_G(w))$,

$$k_G(z_1, z_2) \leq \frac{1}{1-t}\frac{|z_1 - z_2|}{d_G(w)}.$$

In addition, if $t \leq \frac{1}{2}$, *then*

$$k_G(z_1, z_2) \geq \frac{1}{1+2t}\frac{|z_1 - z_2|}{d_G(w)}.$$

A rectifiable arc γ in G is called a c-quasigeodesic with $c \geq 1$ if

$$\ell_{k_G}(\gamma[z_1, z_2]) \leq ck_G(z_1, z_2)$$

for all $z_1, z_2 \in \gamma$. In particular, γ is a quasihyperbolic geodesic if and only if γ is a 1-quasigeodesic.

In [11], Väisälä proved the existence of quasigeodesics in Banach spaces.

Theorem 2.2. *[11, Theorem 9.4] Suppose that* $G \subsetneq E$, $z_1, z_2 \in G$, *and* $c > 1$. *Then there is a* c-*quasigeodesic from* z_1 *to* z_2 *in* G.

2.3. Quasihyperbolic mappings

Let $M \geq 1$, $C \geq 0$, and $0 < q \leq 1$. Following the notation of [11], we say that a homeomorphism $f\colon G \to G'$ is

(1) C-coarsely M-quasihyperbolic (briefly, (M, C)-CQH) if

$$\frac{k_G(z_1, z_2) - C}{M} \leq k_{G'}(z_1', z_2') \leq Mk_G(z_1, z_2) + C$$

for all $z_1, z_2 \in G$;

(2) q-locally (M, C)-CQH if for each point $z \in G$, the restriction

$$f|_{B_q^G(z)}\colon B_q^G(z) \to f(B_q^G(z))$$

is (M, C)-CQH.

Note that f is M-QH if f is $(M, 0)$-CQH and that f is q-locally M-QH if f is q-locally $(M, 0)$-CQH.

2.4. Quasisymmetric mappings

Let $0 < q \leq 1$ and $\eta\colon [0, \infty) \to [0, \infty)$ be a homeomorphism. We say that a homeomorphism $f\colon G \to G'$ is

(1) η-quasisymmetric (briefly, η-QS) if for each $t > 0$ and for each triple x, a, b of distinct points in G,

$$|x - a| \leq t|x - b| \text{ implies } |x' - a'| \leq \eta(t)|x' - b'|.$$

(2) Relatively η-QS if f has a continuous extension to the boundary ∂G, with the extended mapping on $\overline{G}$ is still being denoted by f, such that

$$|x-a| \leq t|x-b| \text{ implies } |x'-a'| \leq \eta(t)|x'-b'|$$

for each triple x, a, b in $\overline{G}$ with $x \in \partial G$ or $a, b \in \partial G$.

(3) q-locally η-QS if the homeomorphism $f|_{B_q^G(x)}$ is η-QS for each $x \in G$.

(4) Semi-locally relatively η-QS if the homeomorphism $f|_{B(x,d_G(x))}$ is relatively η-QS for each $x \in G$.

Remark 2.1. *It follows from the definition above and [6, Theorem 2.21] that if a homeomorphism $f\colon G \to G'$ is relatively η-QS, then the restriction $f|_{\partial G}$ of f is constant or an η-QS embedding.*

Theorem 2.3. *[6, Corollary 2.6] A quasisymmetric embedding maps every bounded set onto a bounded set.*

3. Proofs of the main results

3.1. The proof of Remark 1.2

In this subsection, we construct an example establishing the correctness of Remark 1.2.

Example 3.1. *Let $\mathbb{C}$ be the complex plane and $G = \{z \in \mathbb{C} | z = x + iy, y > 0\}$. Define f to be the conformal mapping*

$$f(z) = \log z = \log|z| + i\arg z.$$

Thus $f(G) = G'$ with $G' = \{z \in \mathbb{C} | z = x + iy, 0 < y < \pi\}$, see Figure 1. Let $\eta\colon [0,\infty) \to [0,\infty)$ be a homeomorphism.

Then we have the following:

(1) *$f\colon G \to G'$ is M-QH for some $M \geq 1$.*

(2) *$f|_{B_1^G(i)}$ is not η-QS; recall that $B_1^G(i) = B(i, d_G(i))$.*

(3) *$f|_{B_1^G(i)}$ is not relatively η-QS.*

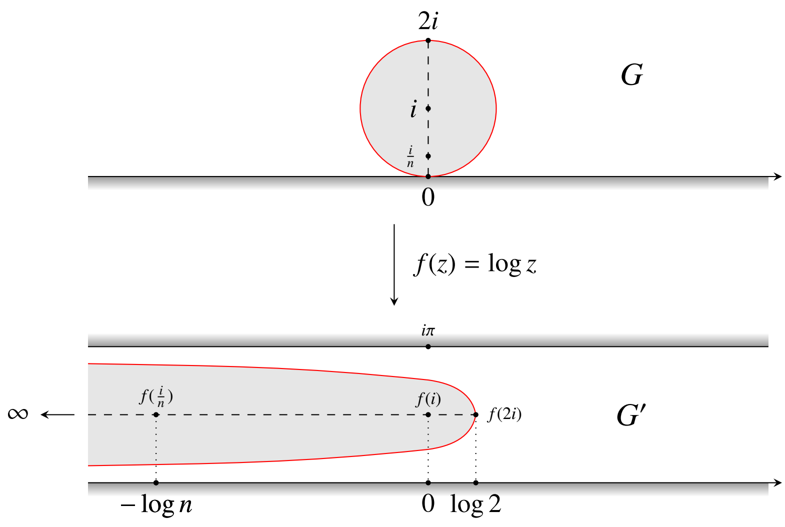

Figure 1. The mapping $f\colon G \to G'$.

Proof. (1) By [5, Proposition 1.6], f is 4-QH. Alternatively, since $G' \subset \mathbb{C}$ is an infinite strip, $G \subset \mathbb{C}$ a half-plane, and $g = f^{-1}\colon G' \to G$ defined by the exponential map $z \mapsto e^z$, [7, 4.11] shows that g is $\frac{\pi}{2}$-QH; thus, f is $\frac{\pi}{2}$-QH.

(2) We first show that $f(B_1^G(i))$ is unbounded. Let $z_n = \frac{i}{n} \in B_1^G(i)$, $n \geq 1$; then

$$f(z_n) = -\log n + i\frac{\pi}{2} \to \infty$$

as $n \to \infty$. Therefore, $f(B_1^G(i))$ is unbounded.

Now, we assume that $f|_{B_1^G(i)}$ is η-QS. Since $B_1^G(i)$ is bounded, by Theorem 2.3 we get that $f(B_1^G(i))$ is also bounded, which is absurd. Hence $f|_{B_1^G(i)}$ is not η-QS.

(3) As $\overline{B_1^G(i)}$ is compact while $f(B_1^G(i))$ is unbounded, $f|_{B_1^G(i)}$ has no extension to a continuous map $\overline{B_1^G(i)} \to \mathbb{C}$. Hence $f|_{B_1^G(i)}$ cannot be relatively η-QS.

3.2. The proof of Theorem 1.6

The purpose of this subsection is to prove Theorem 1.6. Suppose that $G \subsetneq E$, $G' \subsetneq E'$, $f\colon G \to G'$ is a homeomorphism, and f and f^{-1} are q-locally (M, C)-CQH. We must prove that f is (M', C')-CQH, that is, prove that for any $x, y \in G$,

$$\frac{1}{M'}(k_G(x,y) - C') \leq k_{G'}(x', y') \leq M' k_G(x,y) + C'.$$

By the assumption, we only need to prove that

$$k_{G'}(x', y') \leq M' k_G(x,y) + C'. \tag{3.1}$$

Fix $x, y \in G$. In the following, we divide the proof of (3.1) into two cases.

Case 3.1. $|x-y| \leq \frac{q}{2} d_G(x)$.

It follows from Theorem 2.1(2) that

$$\frac{1}{1+q}\frac{|x-y|}{d_G(x)} \leq k_G(x,y) \leq \frac{1}{1-\frac{q}{2}}\frac{|x-y|}{d_G(x)}$$

and hence

$$k_{B_q^G(x)}(x,y) \leq 2\frac{|x-y|}{d_{B_q^G(x)}(x)} = \frac{2}{q}\frac{|x-y|}{d_G(x)}.$$

These imply that

$$k_{B_q^G(x)}(x,y) \leq \frac{2}{q}\frac{|x-y|}{d_G(x)} \leq \frac{2(1+q)}{q} k_G(x,y).$$

Therefore,

$$\begin{aligned} k_{G'}(x', y') &\leq k_{f(B_q^G(x))}(x', y') \leq M k_{B_q^G(x)}(x,y) + C \\ &\leq \frac{2M(1+q)}{q} k_G(x,y) + C. \end{aligned}$$

Case 3.2. $|x-y| > \frac{q}{2}d_G(x)$.

In this case, by Theorem 2.1(1), we obtain

$$k_G(x,y) \geq \log\Big(1+\frac{|x-y|}{\min\{d_G(x),d_G(y)\}}\Big) > \log\Big(1+\frac{q}{2}\Big) =: t_0. \tag{3.2}$$

By Theorem 2.2, we take a 2-quasigeodesic γ joining x and y in G, and $\gamma' = f(\gamma) \subset G'$. Then there exists a unique integer m such that

$$mt_0 < \ell_{k_G}(\gamma) \leq (m+1)t_0.$$

From (3.2), we see that $m \geq 1$, and thus

$$m+1 \leq 2m \leq \frac{2}{t_0}\ell_{k_G}(\gamma). \tag{3.3}$$

Now, we can choose a sequence of successive points $\{x_i\}_{i=0}^{m+1}$ in γ with $x_0 = x$ and $x_{m+1} = y$ such that

$$\ell_{k_G}(\gamma[x_{i-1},x_i]) = t_0 \tag{3.4}$$

for $i \in \{1,2,\ldots,m\}$, and

$$\ell_{k_G}(\gamma[x_m,x_{m+1}]) \leq t_0. \tag{3.5}$$

For all $i \in \{1,2,\ldots,m+1\}$,

$$\frac{|x_{i-1}-x_i|}{d_G(x_{i-1})} \leq \frac{q}{2}, \tag{3.6}$$

since otherwise an estimate similar to (3.2) shows that

$$\ell_{k_G}(\gamma[x_{i-1},x_i]) \geq k_G(x_{i-1},x_i) > t_0,$$

which contradicts (3.4) and (3.5).

Thus, (3.6) and Case 3.1 imply

$$k_{G'}(x'_{i-1},x'_i) \leq \frac{2M(1+q)}{q}k_G(x_{i-1},x_i) + C \tag{3.7}$$

for all $i \in \{1,2,\ldots,m+1\}$.

It follows from γ being a 2-quasigeodesic, (3.3) and (3.7) that

$$\begin{aligned} k_{G'}(x',y') &\leq \sum_{i=1}^{m+1} k_{G'}(x'_{i-1},x'_i) \leq \frac{2M(1+q)}{q}\sum_{i=1}^{m+1} k_G(x_{i-1},x_i) + C(m+1) \\ &\leq \frac{2M(1+q)}{q}\ell_{k_G}(\gamma) + \frac{2C}{t_0}\ell_{k_G}(\gamma) \\ &= \Big(\frac{2M(1+q)}{q} + \frac{2C}{t_0}\Big)\ell_{k_G}(\gamma) \\ &\leq \Big(\frac{4M(1+q)}{q} + \frac{4C}{t_0}\Big)k_G(x,y). \end{aligned}$$

Therefore, from Case 3.1 and Case 3.2, by letting

$$M' = \frac{4M(1+q)}{q} + \frac{4C}{\log(1+\frac{q}{2})} \quad \text{and} \quad C' = C, \tag{3.8}$$

we know that (3.1) holds true. Hence f is (M',C')-CQH. This completes the proof of Theorem 1.6.

3.3. The proof of Theorem 1.5

Sufficiency: If f and f^{-1} are q-locally M'-QH, then f and f^{-1} are q-locally $(M', 0)$-CQH. By Theorem 1.6 (see (3.8)), we know that $f: G \to G'$ is (M, C)-CQH, where $M = \frac{4M'(1+q)}{q}$ and $C = 0$, i.e., $f: G \to G'$ is M-QH.

Necessity: If f is M-QH, then by Theorem 1.4 we see that f and f^{-1} are q-locally M'-QH for any $q \in (0, 1]$ and $M' = 4M^2$. Hence, we complete the proof of Theorem 1.5.

3.4. The proof of Remark 1.3

In this subsection, we construct an example establishing the correctness of Remark 1.3.

Example 3.2. *Let $G = \{z \in \mathbb{C} | z = x + iy, x > -1, |y| < 2\}$ and $f: G \to G$ be a self-homeomorphism of G defined by*

$$f(x+iy) = \begin{cases} x + iy, & -1 < x \le 1; \\ x + i\frac{y}{x}, & x > 1 \text{ and } |y| \le 1; \\ x + i(2 - (2 - |y|)(2 - \frac{1}{x}))\frac{y}{|y|}, & x > 1 \text{ and } 1 < |y| < 2. \end{cases}$$

See Figure 2. Then we have the following:

(1) *$f: G \to G$ is $(1, 2)$-CQH.*

(2) *$f: G \to G$ is not q-locally (M, C)-CQH for any constants $M \ge 1$, $C \ge 0$, and $0 < q \le \frac{1}{\sqrt{3}}$.*

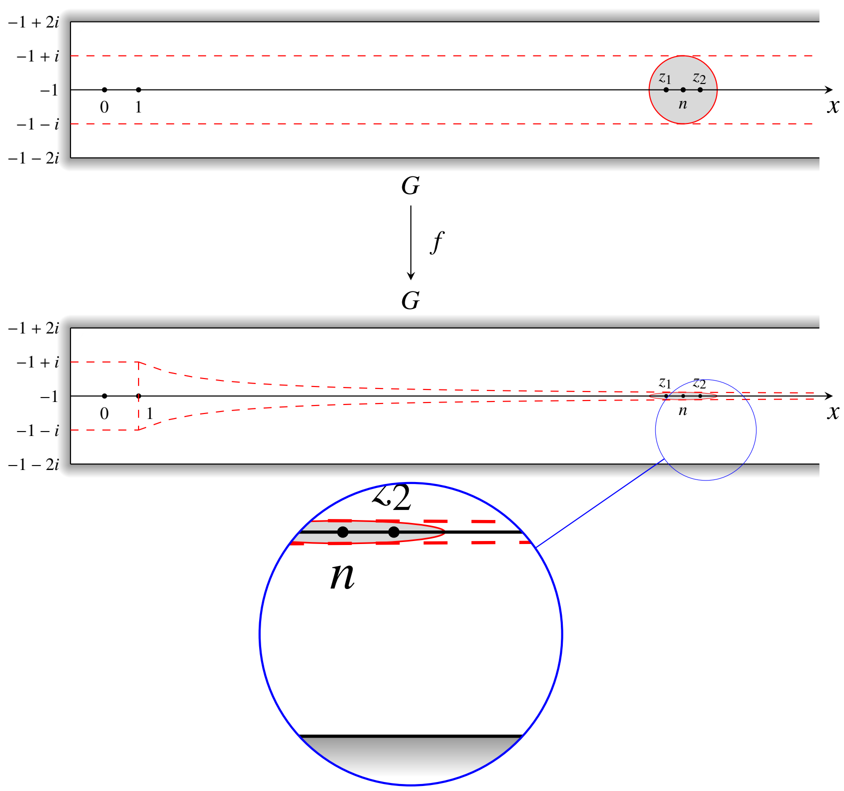

Figure 2. The mapping $f: G \to G$.

Proof. (1) Before proving that $f\colon G \to G$ is $(1,2)$-CQH, we first show that

$$k_G(z, z') \le 1 \tag{3.9}$$

for any $z \in G$.

Fix $z = x + iy \in G$. If $-1 < x \le 1$, then $z' = z$, and so

$$k_G(z, z') = 0. \tag{3.10}$$

If $x > 1$, then we claim that

$$d_G(z) \le d_G(z') \le 2d_G(z). \tag{3.11}$$

Obviously, $d_G(z) = 2 - |y|$. If $|y| \le 1$, then

$$d_G(z') = 2 - |y'| = 2 - \frac{|y|}{x} \ge 2 - |y| = d_G(z)$$

and

$$d_G(z') = 2 - |y'| = 2 - \frac{|y|}{x} \le 2 \le 2(2 - |y|) = 2d_G(z);$$

these imply (3.11). On the other hand, if $1 < |y| < 2$, then

$$d_G(z') = 2 - |y'| = (2 - |y|)(2 - \frac{1}{x}) > 2 - |y| = d_G(z)$$

and

$$d_G(z') = 2 - |y'| = (2 - |y|)(2 - \frac{1}{x}) < 2(2 - |y|) = 2d_G(z).$$

Below, we show that the following equation holds:

$$d_G(z') - d_G(z) = |z - z'|. \tag{3.12}$$

If $-1 < x \le 1$, then $z' = z$ and thus $d_G(z') - d_G(z) = 0 = |z - z'|$.

Suppose $x > 1$. Then $|z - z'| = |y - y'|$ and $d_G(z') - d_G(z) = (2 - |y'|) - (2 - |y|) = |y| - |y'|$. With $\overline{z} = x - iy$ we have $f(\overline{z}) = \overline{f(z)}$. Therefore, we may assume that $0 \le y < 2$, and then it suffices to show that $0 \le y' \le y$. Now, if $0 \le y \le 1$, then $0 \le y' = y/x \le y$, whereas if $1 < y < 2$, then $y' = 2 - (2 - y)(2 - 1/x)$ and thus $y' < 2 - (2 - y) = y$ and $y' > 2 - (2 - y)2 = 2(y - 1) > 0$.

Now, we continue to prove (3.9). It follows from (3.11) and (3.12) that

$$|z - z'| = d_G(z') - d_G(z) \le \frac{1}{2} d_G(z').$$

This, together with Theorem 2.1(2), shows that

$$k_G(z, z') \le \frac{1}{1 - \frac{1}{2}} \frac{|z - z'|}{d_G(z')} \le 1. \tag{3.13}$$

Therefore, the inequality (3.9) follows from (3.10) and (3.13).

We are ready to prove that $f\colon G \to G$ is $(1,2)$-CQH. For any $z_1, z_2 \in G$, it follows from (3.9) that

$$k_G(z_1', z_2') \le k_G(z_1, z_2) + k_G(z_1, z_1') + k_G(z_2, z_2') \le k_G(z_1, z_2) + 2,$$

and

$$k_G(z_1, z_2) \le k_G(z_1', z_2') + k_G(z_1, z_1') + k_G(z_2, z_2') \le k_G(z_1', z_2') + 2.$$

Hence, $f\colon G \to G$ is $(1,2)$-CQH.

(2) Next, we assume that $f\colon G \to G$ is q-locally (M, C)-CQH for some constants $M \ge 1, C \ge 0$, and $0 < q \le \frac{1}{\sqrt{3}}$. Consider the ball $B_q^G(n) = B(n, 2q), n \ge 1$. Then the restriction $f|_{B(n,2q)}$ is (M, C)-CQH. Take $z_1 = n - q$ and $z_2 = n + q$ in $B(n, 2q)$. We obtain $z_1' = z_1, z_2' = z_2$, and with $z = z_2 + iq\sqrt{3} \in S(n, 2q)$ that

$$d_{f(B(n,2q))}(z_2') \le |z_2' - z'| = \frac{q\sqrt{3}}{n+q} < \frac{1}{n}.$$

We obtain from $f|_{B(n,2q)}$ being (M, C)-CQH and Theorem 2.1(2) that

$$k_{f(B(n,2q))}(z_1', z_2') \le M k_{B(n,2q)}(z_1, z_2) + C \le \frac{M}{1 - \frac{1}{2}} \frac{|z_1 - z_2|}{d_{B(n,2q)}(n)} + C = 2M + C. \tag{3.14}$$

On the other hand, by Theorem 2.1(1), we obtain

$$k_{f(B(n,2q))}(z_1', z_2') \ge \log\left(1 + \frac{|z_1' - z_2'|}{d_{f(B(n,2q))}(z_2')}\right) \ge \log\left(1 + \frac{2q}{\frac{1}{n}}\right) = \log(1 + 2qn) \to \infty$$

as $n \to \infty$. This contradicts (3.14). Hence $f\colon G \to G$ is not q-locally (M, C)-CQH for any constants $M \ge 1$, $C \ge 0$, and $0 < q \le \frac{1}{\sqrt{3}}$. The example is proved.

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Conflict of interest

The author declares that there are no conflicts of interest.

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