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Research article

Decision methods based on Bonferroni mean operators and EDAS for the classifications of circular pythagorean fuzzy Meta-analysis

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Abstract: Meta-analysis is a statistical technique used to process an overall summary estimation, and the technique of meta-analysis is mostly used in medicine, social science, and psychology. In this manuscript, we aimed to combine the techniques of the Bonferroni mean (BM) operator based on circular Pythagorean fuzzy (CPF) sets, called the CPF Bonferroni mean (CPFBM) operator, and CPF weighted Bonferroni mean (CPFWBM) operator and described their special cases with the help of two parameters, "s" and "t", and some describable properties of them are also proposed. Further, we present the evaluation technique based on distance from average solution (EDAS) technique and the proposed operators. Moreover, we use some examples to show the flexibility and dominance of the proposed operators by comparing the proposed methods with some existing techniques.

Keywords: Bonferroni mean operators; Circular Pythagorean fuzzy meta-analysis; EDAS techniques; decision-making analysis
Mathematics Subject Classification: 03B52, 03E72, 03E73, 28E10, 94D05

1. Introduction

One of the most preferable and valuable techniques is called meta-analysis [1], which is a statistical tool used to combine and evaluate the results from various independent studies on a particular topic or research question. The technique of meta-analysis [2] is a very suitable procedure for depicting vague and problematic information, but utilizing the technique of meta-analysis under the consideration of the classical set theory is very complex and vague because the range of crisp set is very limited. Further, the decision-making procedure is also used for finding the best optimal among the collection of alternatives, where the MADM technique is an important part of the decision-making process, and is also used for evaluating the required results under the concern criteria [3]. During the decision-making process, many experts have lost a lot of information because of limited information, such as zero and one. For this, Zadeh [4] proposed the fuzzy sets (FSs) with a truth function, such as $\mathbb{F}_{BM}(\mathcal{L}j) \in [0,1]$, where $\mathbb{F}_{BM}: X \to [0,1]$. Moreover, in dealing with uncertain and unreliable information, the technique of truth function is not enough to resolve some complex problems. Further, the falsity function also plays an essential role in many complex problems because of their features. For this, Atanassov [5,6] derived the intuitionistic FSs (IFSs), which is a very flexible and reliable theory for managing vague and unreliable information, where the truth function and falsity function are a major part of the IFSs with a condition that the sum of the duplet will be contained in the unit interval. Further, the idea of FSs is a special case of the IFSs, and due to these features, many applications have been designed, such as aggregation operators [7,8], hybrid operators [9,10], and decision-making problems [11,12].

The function of truth grade and the function of falsity grade have a lot of potential to cope with vague and uncertain information with the condition that the sum of the duplet is contained in the unit interval; however, in the presence of the following kinds of pair, $(0.6,0.7)$, the IFSs have not worked feasibly because $0.6 + 0.7 = 1.3 \notin [0,1]$. For this reason, in 2013, Yager [13] proposed the Pythagorean FSs (PFSs), where the structure of PFSs is the same as the structure of IFSs, but the condition of both techniques are different such as the sum of the square of the duplet will be contained in unit interval: $0.6^2 + 0.7^2 = 0.36 + 0.49 = 0.85 \in [0,1]$. The technique of PFSs is very wide due to their condition, where the IFSs and FSs are the special cases of the PFSs. Further, Deveci et al. [14] discussed the survey on recent applications of PFSs. Moreover, Mandel and Ranadive [15] exposed the decision-theoretic rough set based on PFSs. Additionally, Perez-Dominguez et al. [16] evaluated the CODAS technique for PFSs and their applications. Further, Alkan and Kahraman [17] presented the CODAS technique for PFSs with application in supply chain management. Moreover, Sun and Wang [18] exposed the distance measures for Pythagorean fuzzy information processing. Additionally, Calik [19] presented the AHP and TOPSIS techniques based on PFSs and their application in Industry 4.0.

IFSs contained the truth function and falsity function with a condition that the sum of the duplet will be contained in the unit interval, but it is also possible to involve a new function, called the radius function between truth and falsity grades. Therefore, Atanassov [20] presented the technique of circular IFSs (CIFSs) with three different functions with the same range, called the truth function, falsity function, and radius function, with a condition that the sum of the duplet will be contained in the unit interval. Many applications have been proposed by different scholars: example, decision-making problems for CIFSs [21], divergence measures for CIFSs [22], distance measures for CIFSs [23], and similarity/entropy measures for CIFSs [24]. Further, Bozyigit et al. [25] presented the technique of circular PFSs (CPFSs) by modifying the condition of the CIFSs such as that the sum of the square of the duplet will be contained in the unit interval. Further, the CPFSs contain the technique of truth function, falsity function, and radius function, which is wider and more reliable than the existing techniques, such as FSs, IFSs, PFSs, and CIFSs, which can cope with vague and unreliable information in genuine life problems. Further, Ali and Yang [26] derived the technique of Hamacher operators for CPFSs.

Keshavarz Ghorabaee et al. [27] proposed a new technique, called the EDAS method, known as evaluation based on the distance from average solution, which is mostly used for evaluating the best optimal among the collection of information. Further, Klement et al. [28] proposed the idea of triangular norms, which contained different types of norms and their geometrical representations. Additionally, the Bonferroni mean (BM) operators [29] were proposed based on algebraic norms, which are used for aggregating the collection of information into a singleton set. Furthermore, Xu and Yager [30] proposed the BM operators for IFSs. Moreover, Xia et al. [31] evaluated the BM operators for generalized IFSs. Additionally, Liang et al. [32] derived the BM operators for PFSs and their applications. Further, Yang et al. [33] exposed the BM operators for PFSs based on triangular norms.

After the overall discussion, we observed that the model of circular Pythagorean fuzzy sets is very reliable and dominant because of the: The model of FSs to the model of CIFSs is the part of the CPF sets. Further, we also noticed that the model of Bonferroni mean operators and weighted Bonferroni mean operators are not proposed yet based on CPFSs, which are used for the aggregation of a finite number of information into a singleton set. The major problem is that up to date no one has derived the BM operators for CPFSs. Further, we also noticed that the EDAS method has not been proposed, which is a very reliable technique for evaluating some complicated and vague information. The major themes of this manuscript are listed below:

1) To propose the technique of the CPFBM, CPFWBM and describes their special cases using two parameters, “s” and “t”. Some describable properties are also proposed for the above techniques.
2) To present the technique of the EDAS technique for the CPFSs.
3) To show the flexibility and dominance of the proposed operators by comparing the proposed methods with some techniques.

This manuscript is arranged as follows: In Section 2, we describe the prevailing notion of CPFSs and their operational laws. Moreover, we discuss the technique of the BM operator for any collection of non-negative integers. In Section 3, we propose the CPFBM operator, and CPFWBM operator and describe their special cases using two parameters, “s” and “t”. Some describable properties are also proposed for the above techniques. In Section 4, we present the technique of the EDAS technique for the CPFSs. In Section 5, we use some examples to show the flexibility and dominance of the proposed operators. In Section 6, we compare the proposed method with some existing techniques to enhance the worth of the proposed theory. Some remarkable statements are given in Section 7.

In this section, we describe the prevailing notion of CPFSs and their operational laws. Moreover, we discuss the technique of the BM operator for any collection of non-negative integers.

2. Preliminaries

In this section, we describe the prevailing notion of CPFSs and their operational laws. Moreover, we discuss the technique of the BM operator for any collection of non-negative integers.

Definition 1. [25] A CPFS P on fixed set X is explained below:

$$P=\left\{\left(x,\mathbb{F}_{BM}(\mathcal{L}j),\mathbb{G}_{BM}(\mathcal{L}j),\mathbb{H}_{BM}(\mathcal{L}j)\right)|\mathcal{L}j\in X\right\}. \tag{1}$$

Here, the representation of the truth grade and falsity grade with radius are follows, such as $\mathbb{F}_{BM}(\mathcal{L}j),\mathbb{G}_{BM}(\mathcal{L}j)$, and $\mathbb{H}_{BM}(\mathcal{L}j)$, satisfying $1\leq\mathbb{F}_{BM}(\mathcal{L}j)\leq 1$, $0\leq\mathbb{G}_{BM}(\mathcal{L}j)\leq 1$ with $\left(\mathbb{F}_{BM}(\mathcal{L}j)\right)^2+\left(\mathbb{G}_{BM}(\mathcal{L}j)\right)^2\leq 1$. Further, for convenience, the simple name of the pair

$\left(\mathbb{F}_{BM}^{P}(\mathcal{L}j), \mathbb{G}_{BM}^{P}(\mathcal{L}j), \mathbb{H}_{BM}^{P}(\mathcal{L}j)\right)$ is called CPFN, such as $P_{\sigma} = \left(\mathbb{F}_{BM}^{\sigma}(\mathcal{L}j), \mathbb{G}_{BM}^{\sigma}(\mathcal{L}j), \mathbb{H}_{BM}^{\sigma}(\mathcal{L}j)\right), \sigma = 1,2, \dots, \varphi$.

Definition 2. [26] Consider three CPFNs, $P = \left(\mathbb{F}_{BM}(\mathcal{L}j), \mathbb{G}_{BM}(\mathcal{L}j), \mathbb{H}_{BM}(\mathcal{L}j)\right)$, $P_1 = \left(\mathbb{F}_{BM}^{1}(\mathcal{L}j), \mathbb{G}_{BM}^{1}(\mathcal{L}j), \mathbb{H}_{BM}^{1}(\mathcal{L}j)\right)$, and $P_2 = \left(\mathbb{F}_{BM}^{2}(\mathcal{L}j), \mathbb{G}_{BM}^{2}(\mathcal{L}j), \mathbb{H}_{BM}^{2}(\mathcal{L}j)\right)$ with λ, thus

$$P_1 \oplus_{TN} P_2 = \begin{pmatrix} \sqrt{\left(\mathbb{F}_{BM}^{1}(\mathcal{L}j)\right)^2 + \left(\mathbb{F}_{BM}^{2}(\mathcal{L}j)\right)^2 - \left(\mathbb{F}_{BM}^{1}(\mathcal{L}j)\right)^2 * \left(\mathbb{F}_{BM}^{2}(\mathcal{L}j)\right)^2}, \\ \mathbb{G}_{BM}^{1}(\mathcal{L}j) * \mathbb{G}_{BM}^{2}(\mathcal{L}j), \\ \sqrt{\left(\mathbb{H}_{BM}^{1}(\mathcal{L}j)\right)^2 + \left(\mathbb{H}_{BM}^{2}(\mathcal{L}j)\right)^2 - \left(\mathbb{H}_{BM}^{1}(\mathcal{L}j)\right)^2 * \left(\mathbb{H}_{BM}^{2}(\mathcal{L}j)\right)^2} \end{pmatrix}, \quad (2)$$

$$P_1 \oplus_{TCN} P_2 = \begin{pmatrix} \sqrt{\left(\mathbb{F}_{BM}^{1}(\mathcal{L}j)\right)^2 + \left(\mathbb{F}_{BM}^{2}(\mathcal{L}j)\right)^2 - \left(\mathbb{F}_{BM}^{1}(\mathcal{L}j)\right)^2 * \left(\mathbb{F}_{BM}^{2}(\mathcal{L}j)\right)^2}, \\ \mathbb{G}_{BM}^{1}(\mathcal{L}j) * \mathbb{G}_{BM}^{2}(\mathcal{L}j), \\ \mathbb{H}_{BM}^{1}(\mathcal{L}j) * \mathbb{H}_{BM}^{2}(\mathcal{L}j) \end{pmatrix}, \quad (3)$$

$$P_1 \otimes_{TN} P_2 = \begin{pmatrix} \mathbb{F}_{BM}^{1}(\mathcal{L}j) * \mathbb{F}_{BM}^{2}(\mathcal{L}j), \\ \sqrt{\left(\mathbb{G}_{BM}^{1}(\mathcal{L}j)\right)^2 + \left(\mathbb{G}_{BM}^{2}(\mathcal{L}j)\right)^2 - \left(\mathbb{G}_{BM}^{1}(\mathcal{L}j)\right)^2 * \left(\mathbb{G}_{BM}^{2}(\mathcal{L}j)\right)^2}, \\ \mathbb{H}_{BM}^{1}(\mathcal{L}j) * \mathbb{H}_{BM}^{2}(\mathcal{L}j) \end{pmatrix}, \quad (4)$$

$$P_1 \otimes_{TCN} P_2 = \begin{pmatrix} \mathbb{F}_{BM}^{1}(\mathcal{L}j) * \mathbb{F}_{BM}^{2}(\mathcal{L}j), \\ \sqrt{\left(\mathbb{G}_{BM}^{1}(\mathcal{L}j)\right)^2 + \left(\mathbb{G}_{BM}^{2}(\mathcal{L}j)\right)^2 - \left(\mathbb{G}_{BM}^{1}(\mathcal{L}j)\right)^2 * \left(\mathbb{G}_{BM}^{2}(\mathcal{L}j)\right)^2}, \\ \sqrt{\left(\mathbb{H}_{BM}^{1}(\mathcal{L}j)\right)^2 + \left(\mathbb{H}_{BM}^{2}(\mathcal{L}j)\right)^2 - \left(\mathbb{H}_{BM}^{1}(\mathcal{L}j)\right)^2 * \left(\mathbb{H}_{BM}^{2}(\mathcal{L}j)\right)^2} \end{pmatrix}, \quad (5)$$

$$(\lambda P)_{TN} = \left(\sqrt{1 - \left(1 - \left(\mathbb{F}_{BM}(\mathcal{L}j)\right)^2\right)^{\lambda}}, \left(\mathbb{G}_{BM}(\mathcal{L}j)\right)^{\lambda}, \sqrt{1 - \left(1 - \left(\mathbb{H}_{BM}(\mathcal{L}j)\right)^2\right)^{\lambda}}\right), \quad (6)$$

$$(\lambda P)_{TCN} = \left(\sqrt{1 - \left(1 - \left(\mathbb{F}_{BM}(\mathcal{L}j)\right)^2\right)^{\lambda}}, \left(\mathbb{G}_{BM}(\mathcal{L}j)\right)^{\lambda}, \left(\mathbb{H}_{BM}(\mathcal{L}j)\right)^{\lambda}\right), \quad (7)$$

$$\left(P^{\lambda}\right)_{TN} = \left(\left(\mathbb{F}_{BM}(\mathcal{L}j)\right)^{\lambda}, \sqrt{1 - \left(1 - \left(\mathbb{G}_{BM}(\mathcal{L}j)\right)^2\right)^{\lambda}}, \left(\mathbb{F}_{BM}(\mathcal{L}j)\right)^{\lambda}\right), \quad (8)$$

$$\left(P^{\lambda}\right)_{TCN} = \left(\left(\mathbb{F}_{BM}(\mathcal{L}j)\right)^{\lambda}, \sqrt{1 - \left(1 - \left(\mathbb{G}_{BM}(\mathcal{L}j)\right)^2\right)^{\lambda}}, \sqrt{1 - \left(1 - \left(\mathbb{H}_{BM}(\mathcal{L}j)\right)^2\right)^{\lambda}}\right). \quad (9)$$

Definition 3. [26] Consider a CPFN, such as $P = \left(\mathbb{F}_{BM}(\mathcal{L}j), \mathbb{G}_{BM}(\mathcal{L}j), \mathbb{H}_{BM}(\mathcal{L}j)\right)$, we have

$$S(P) = \left(\left(\mathbb{F}_{BM}(\mathcal{L}j)\right)^2 - \left(\mathbb{G}_{BM}(\mathcal{L}j)\right)^2\right) * \left(\mathbb{H}_{BM}(\mathcal{L}j)\right)^2 \in [-1,1], \quad (10)$$

$$H(P) = \left(\left(\mathbb{F}_{BM}(\mathcal{L}j)\right)^2 + \left(\mathbb{G}_{BM}(\mathcal{L}j)\right)^2\right) * \left(\mathbb{H}_{BM}(\mathcal{L}j)\right)^2. \quad (11)$$

Called score value and accuracy value with a condition that is if $S(P_1) > S(P_1)$, then $P_1 > P_2$; if $S(P_1) = S(P_1)$, then $P_1 = P_2$, then if $H(P_1) > H(P_1)$, then $P_1 > P_2$.

Definition 4. [29] Consider $P, Q \geq 0$ with a collection of non-negative integers $a_\sigma(\sigma = 1,2, \dots, \varphi)$, thus

$$BM^{P,Q}(a_1, a_2, \dots, a_\varphi) = \left[\frac{1}{\varphi(\varphi-1)} \sum_{\substack{\sigma,\theta=1 \\ \sigma \neq \theta}}^{\varphi} a_\sigma^P \, a_\theta^Q\right]^{\frac{1}{P+Q}}. \tag{12}$$

Called BM operator.

3. CPF bonferroni mean operator

In this section, we compute the technique of CPFBM operator and CPFWBM operator. Further, we discuss some basic properties of the above-proposed operators, called idempotency, monotonicity, and boundedness.

Definition 5. Consider $s, t > 0$ with a collection of CPFNs,

$$P_\sigma = \left(\mathbb{F}_{BM}^{\sigma}(\mathcal{L}j), \mathbb{G}_{BM}^{\sigma}(\mathcal{L}j), \mathbb{H}_{BM}^{\sigma}(\mathcal{L}j)\right)(\sigma = 1,2, \dots, \varphi).$$

Thus,

$$CPFBM^{s,t}(P_1, P_2, \dots, P_\varphi)_{TN} = \left[\frac{1}{\varphi(\varphi-1)}\left[\bigoplus_{\substack{\sigma,\theta=1 \\ \sigma \neq \theta}}^{\varphi} (P_\sigma^s \otimes P_\theta^t)\right]\right]^{\frac{1}{s+t}}, \tag{13}$$

$$CPFBM^{s,t}(P_1, P_2, \dots, P_\varphi)_{TCN} = \left[\frac{1}{\varphi(\varphi-1)}\left[\bigoplus_{\substack{\sigma,\theta=1 \\ \sigma \neq \theta}}^{\varphi} (P_\sigma^s \otimes P_\theta^t)\right]\right]^{\frac{1}{s+t}}. \tag{14}$$

Called $CPFBM^{s,t}$ Operator for t-norm and t-conorm.

Theorem 1. Consider $s, t > 0$ with a collection of CPFNs,

$$P_\sigma = \left(\mathbb{F}_{BM}^{\sigma}(\mathcal{L}j), \mathbb{G}_{BM}^{\sigma}(\mathcal{L}j), \mathbb{H}_{BM}^{\sigma}(\mathcal{L}j)\right)(\sigma = 1,2, \dots, \varphi),$$

thus, we prove that the aggregated theory of the above operators is again CPFN, such as

$$CPFBM^{s,t}(P_1, P_2, \dots, P_\varphi)_{TN} = \begin{pmatrix} \left(1 - \prod_{\substack{\sigma,\theta=1 \\ \sigma \neq \theta}}^{\varphi}\left(1 - \left(\mathbb{F}_{BM}^{\sigma}(\mathcal{L}j)\right)^{2s}\left(\mathbb{F}_{BM}^{\theta}(\mathcal{L}j)\right)^{2s}\right)^{\frac{\varphi}{\varphi(\varphi-1)}}\right)^{\frac{t}{2(s+t)}}, \\ \sqrt{1 - \left(1 - \prod_{\substack{\sigma,\theta=1 \\ \sigma \neq \theta}}^{\varphi}\left(1 - \left(1 - \left(\mathbb{G}_{BM}^{\sigma}(\mathcal{L}j)\right)^2\right)^s\left(1 - \left(\mathbb{G}_{BM}^{\theta}(\mathcal{L}j)\right)^2\right)^t\right)^{\frac{1}{\varphi(\varphi-1)}}\right)^{\frac{t}{2(s+t)}}}, \\ \left(1 - \prod_{\substack{\sigma,\theta=1 \\ \sigma \neq \theta}}^{\varphi}\left(1 - \left(\mathbb{H}_{BM}^{\sigma}(\mathcal{L}j)\right)^{2s}\left(\mathbb{H}_{BM}^{\theta}(\mathcal{L}j)\right)^{2s}\right)^{\frac{\varphi}{\varphi(\varphi-1)}}\right)^{\frac{t}{2(s+t)}} \end{pmatrix} \tag{15}$$

$$CPFBM^{s,t}(P_1,P_2,\dots,P_\varphi)_{TCN}=\begin{pmatrix}\left(1-\prod_{\substack{\sigma,\theta=1\\ \sigma\neq\theta}}^{\varphi}\left(1-\left(\mathbb{F}_{BM}^{\sigma}(\mathcal{L}j)\right)^{2s}\left(\mathbb{F}_{BM}^{\theta}(\mathcal{L}j)\right)^{2s}\right)^{\frac{\varphi}{\varphi(\varphi-1)}}\right)^{\frac{t}{2(s+t)}},\\ \sqrt{1-\left(1-\prod_{\substack{\sigma,\theta=1\\ \sigma\neq\theta}}^{\varphi}\left(1-\left(1-\left(\mathbb{G}_{BM}^{\sigma}(\mathcal{L}j)\right)^2\right)^{s}\left(1-\left(\mathbb{G}_{BM}^{\theta}(\mathcal{L}j)\right)^2\right)^{t}\right)^{\frac{1}{\varphi(\varphi-1)}}\right)^{\frac{t}{2(s+t)}}},\\ \sqrt{1-\left(1-\prod_{\substack{\sigma,\theta=1\\ \sigma\neq\theta}}^{\varphi}\left(1-\left(1-\left(\mathbb{H}_{BM}^{\sigma}(\mathcal{L}j)\right)^2\right)^{s}\left(1-\left(\mathbb{H}_{BM}^{\theta}(\mathcal{L}j)\right)^2\right)^{t}\right)^{\frac{1}{\varphi(\varphi-1)}}\right)^{\frac{t}{2(s+t)}}}\end{pmatrix}. \tag{16}$$

Proof. Using mathematical induction, we prove the above theory, such as if

$$P_\sigma^s=\left(\left(\mathbb{F}_{BM}^{\sigma}(\mathcal{L}j)\right)^{s},\sqrt{1-\left(1-\left(\mathbb{G}_{BM}^{\sigma}(\mathcal{L}j)\right)^2\right)^{s}},\left(\mathbb{H}_{BM}^{\sigma}(\mathcal{L}j)\right)^{s}\right),$$

$$P_\theta^t=\left(\left(\mathbb{F}_{BM}^{\theta}(\mathcal{L}j)\right)^{t},\sqrt{1-\left(1-\left(\mathbb{G}_{BM}^{\theta}(\mathcal{L}j)\right)^2\right)^{t}},\left(\mathbb{H}_{BM}^{\theta}(\mathcal{L}j)\right)^{t}\right).$$

Thus, we combine the above two information with the help of product rules, such as

$$P_\sigma^s\otimes P_\theta^t=\begin{pmatrix}\left(\mathbb{F}_{BM}^{\sigma}(\mathcal{L}j)\right)^{s}*\left(\mathbb{F}_{BM}^{\theta}(\mathcal{L}j)\right)^{t},\sqrt{1-\left(1-\left(\mathbb{G}_{BM}^{\sigma}(\mathcal{L}j)\right)^2\right)^{s}*\left(1-\left(\mathbb{G}_{BM}^{\theta}(\mathcal{L}j)\right)^2\right)^{t}},\\ \left(\mathbb{H}_{BM}^{\sigma}(\mathcal{L}j)\right)^{s}*\left(\mathbb{H}_{BM}^{\theta}(\mathcal{L}j)\right)^{t}\end{pmatrix}.$$

Further, we use the addition technique, we have

$$\bigoplus_{\substack{\sigma,\theta=1\\ \sigma\neq\theta}}^{\varphi}(P_\sigma^s\otimes P_\theta^t)=\begin{pmatrix}\sqrt{1-\prod_{\substack{\sigma,\theta=1\\ \sigma\neq\theta}}^{\varphi}\left(1-\left(\mathbb{F}_{BM}^{\sigma}(\mathcal{L}j)\right)^{2s}*\left(\mathbb{F}_{BM}^{\theta}(\mathcal{L}j)\right)^{2t}\right)},\\ \sqrt{\prod_{\substack{\sigma,\theta=1\\ \sigma\neq\theta}}^{\varphi}\left(1-\left(1-\left(\mathbb{G}_{BM}^{\sigma}(\mathcal{L}j)\right)^2\right)^{s}*\left(1-\left(\mathbb{G}_{BM}^{\theta}(\mathcal{L}j)\right)^2\right)^{t}\right)}\\ \sqrt{1-\prod_{\substack{\sigma,\theta=1\\ \sigma\neq\theta}}^{\varphi}\left(1-\left(\mathbb{H}_{BM}^{\sigma}(\mathcal{L}j)\right)^{2s}*\left(\mathbb{H}_{BM}^{\theta}(\mathcal{L}j)\right)^{2t}\right)}\end{pmatrix}.$$

Finally, by mathematical induction, we have, if $\varphi=2$, such as

$$\bigoplus_{\substack{\sigma,\theta=1\\ \sigma\neq\theta}}^{2}(P_\sigma^s\otimes P_\theta^t)=(P_\sigma^s\otimes P_\theta^t)\oplus(P_\sigma^s\otimes P_\theta^t)$$

$$=\begin{pmatrix}\sqrt{1-\left(1-\left(\mathbb{F}_{BM}^{1}(\mathcal{L}j)\right)^{2s}*\left(\mathbb{F}_{BM}^{2}(\mathcal{L}j)\right)^{2t}\right)*\left(1-\left(\mathbb{F}_{BM}^{2}(\mathcal{L}j)\right)^{2s}*\left(\mathbb{F}_{BM}^{1}(\mathcal{L}j)\right)^{2t}\right)},\\ \sqrt{\left(1-\left(1-\left(\mathbb{G}_{BM}^{1}(\mathcal{L}j)\right)^2\right)^{s}*\left(1-\left(\mathbb{G}_{BM}^{2}(\mathcal{L}j)\right)^2\right)^{t}\right)*\left(1-\left(1-\left(\mathbb{G}_{BM}^{2}(\mathcal{L}j)\right)^2\right)^{s}*\left(1-\left(\mathbb{G}_{BM}^{1}(\mathcal{L}j)\right)^2\right)^{t}\right)},\\ \sqrt{1-\left(1-\left(\mathbb{H}_{BM}^{1}(\mathcal{L}j)\right)^{2s}*\left(\mathbb{H}_{BM}^{2}(\mathcal{L}j)\right)^{2t}\right)*\left(1-\left(\mathbb{H}_{BM}^{2}(\mathcal{L}j)\right)^{2s}*\left(\mathbb{H}_{BM}^{1}(\mathcal{L}j)\right)^{2t}\right)}\end{pmatrix}$$

The proposed theory holds for $\varphi = 2$, if we have done the above theory for $\varphi = \omega$, such as

$$\bigoplus_{\substack{\sigma,\theta=1\\ \sigma\neq\theta}}^{\omega} (P_\sigma^s \otimes P_\theta^t) = \begin{pmatrix} \sqrt{1-\prod_{\substack{\sigma,\theta=1\\ \sigma\neq\theta}}^{\omega}\left(1-\left(\mathbb{F}_{BM}^{\sigma}(\mathcal{L}j)\right)^{2s} * \left(\mathbb{F}_{BM}^{\theta}(\mathcal{L}j)\right)^{2t}\right)}, \\ \sqrt{\prod_{\substack{\sigma,\theta=1\\ \sigma\neq\theta}}^{\omega}\left(1-\left(1-\left(\mathbb{G}_{BM}^{\sigma}(\mathcal{L}j)\right)^{2}\right)^{s} * \left(1-\left(\mathbb{G}_{BM}^{\theta}(\mathcal{L}j)\right)^{2}\right)^{t}\right)}, \\ \sqrt{1-\prod_{\substack{\sigma,\theta=1\\ \sigma\neq\theta}}^{\omega}\left(1-\left(\mathbb{H}_{BM}^{\sigma}(\mathcal{L}j)\right)^{2s} * \left(\mathbb{H}_{BM}^{\theta}(\mathcal{L}j)\right)^{2t}\right)} \end{pmatrix}.$$

Then, we prove it for $\varphi = \omega + 1$, such as

$$\bigoplus_{\substack{\sigma,\theta=1\\ \sigma\neq\theta}}^{\omega+1} (P_\sigma^s \otimes P_\theta^t) = \left(\bigoplus_{\substack{\sigma,\theta=1\\ \sigma\neq\theta}}^{\omega} (P_\sigma^s \otimes P_\theta^t)\right) \oplus \left(\bigoplus_{\sigma=1}^{\omega} (P_\sigma^s \otimes P_{\omega+1}^t)\right) \oplus \left(\bigoplus_{\sigma=1}^{\omega} (P_{\omega+1}^s \otimes P_\theta^t)\right).$$

Where,

$$\bigoplus_{\substack{\sigma,\theta=1\\ \sigma\neq\theta}}^{\omega} (P_\sigma^s \otimes P_{\omega+1}^t) = \begin{pmatrix} \sqrt{1-\prod_{\substack{\sigma,\theta=1\\ \sigma\neq\theta}}^{\omega}\left(1-\left(\mathbb{F}_{BM}^{\sigma}(\mathcal{L}j)\right)^{2s} * \left(\mathbb{F}_{BM}^{\omega+1}(\mathcal{L}j)\right)^{2t}\right)}, \\ \sqrt{\prod_{\substack{\sigma,\theta=1\\ \sigma\neq\theta}}^{\omega}\left(1-\left(1-\left(\mathbb{G}_{BM}^{\sigma}(\mathcal{L}j)\right)^{2}\right)^{s} * \left(1-\left(\mathbb{G}_{BM}^{\omega+1}(\mathcal{L}j)\right)^{2}\right)^{t}\right)}, \\ \sqrt{1-\prod_{\substack{\sigma,\theta=1\\ \sigma\neq\theta}}^{\omega}\left(1-\left(\mathbb{H}_{BM}^{\sigma}(\mathcal{L}j)\right)^{2s} * \left(\mathbb{H}_{BM}^{\omega+1}(\mathcal{L}j)\right)^{2t}\right)} \end{pmatrix}$$

and

$$\bigoplus_{\substack{\sigma,\theta=1\\ \sigma\neq\theta}}^{\omega} (P_{\omega+1}^s \otimes P_\theta^t) = \begin{pmatrix} \sqrt{1-\prod_{\substack{\sigma,\theta=1\\ \sigma\neq\theta}}^{\omega}\left(1-\left(\mathbb{F}_{BM}^{\omega+1}(\mathcal{L}j)\right)^{2s} * \left(\mathbb{F}_{BM}^{\theta}(\mathcal{L}j)\right)^{2t}\right)}, \\ \sqrt{\prod_{\substack{\sigma,\theta=1\\ \sigma\neq\theta}}^{\omega}\left(1-\left(1-\left(\mathbb{G}_{BM}^{\omega+1}(\mathcal{L}j)\right)^{2}\right)^{s} * \left(1-\left(\mathbb{G}_{BM}^{\theta}(\mathcal{L}j)\right)^{2}\right)^{t}\right)}, \\ \sqrt{1-\prod_{\substack{\sigma,\theta=1\\ \sigma\neq\theta}}^{\omega}\left(1-\left(\mathbb{H}_{BM}^{\omega+1}(\mathcal{L}j)\right)^{2s} * \left(\mathbb{H}_{BM}^{\theta}(\mathcal{L}j)\right)^{2t}\right)} \end{pmatrix}.$$

Then,

$$\bigoplus_{\substack{\sigma,\theta=1\\ \sigma\neq\theta}}^{\omega+1}\left(P_\sigma^s\otimes P_\theta^t\right)$$

$$=\left(\begin{array}{c}\sqrt{1-\prod_{\substack{\sigma,\theta=1\\ \sigma\neq\theta}}^{\omega}\left(1-\left(\mathbb{F}_{BM}^{\omega+1}(\mathcal{L}j)\right)^{2s}*\left(\mathbb{F}_{BM}^{\theta}(\mathcal{L}j)\right)^{2t}\right)},\\ \sqrt{\prod_{\substack{\sigma,\theta=1\\ \sigma\neq\theta}}^{\omega}\left(1-\left(1-\left(\mathbb{G}_{BM}^{\sigma}(\mathcal{L}j)\right)^{2}\right)^{s}*\left(1-\left(\mathbb{G}_{BM}^{\omega+1}(\mathcal{L}j)\right)^{2}\right)^{t}\right)},\\ \sqrt{1-\prod_{\substack{\sigma,\theta=1\\ \sigma\neq\theta}}^{\omega}\left(1-\left(\mathbb{H}_{BM}^{\sigma}(\mathcal{L}j)\right)^{2s}*\left(\mathbb{H}_{BM}^{\omega+1}(\mathcal{L}j)\right)^{2t}\right)}\end{array}\right)$$

$$\oplus\left(\begin{array}{c}\sqrt{1-\prod_{\substack{\sigma,\theta=1\\ \sigma\neq\theta}}^{\omega}\left(1-\left(\mathbb{F}_{BM}^{\sigma}(\mathcal{L}j)\right)^{2s}*\left(\mathbb{F}_{BM}^{\omega+1}(\mathcal{L}j)\right)^{2t}\right)},\\ \sqrt{\prod_{\substack{\sigma,\theta=1\\ \sigma\neq\theta}}^{\omega}\left(1-\left(1-\left(\mathbb{G}_{BM}^{\sigma}(\mathcal{L}j)\right)^{2}\right)^{s}*\left(1-\left(\mathbb{G}_{BM}^{\omega+1}(\mathcal{L}j)\right)^{2}\right)^{t}\right)},\\ \sqrt{1-\prod_{\substack{\sigma,\theta=1\\ \sigma\neq\theta}}^{\omega}\left(1-\left(\mathbb{H}_{BM}^{\sigma}(\mathcal{L}j)\right)^{2s}*\left(\mathbb{H}_{BM}^{\omega+1}(\mathcal{L}j)\right)^{2t}\right)}\end{array}\right)$$

$$\oplus\left(\begin{array}{c}\sqrt{1-\prod_{\substack{\sigma,\theta=1\\ \sigma\neq\theta}}^{\omega}\left(1-\left(\mathbb{F}_{BM}^{\omega+1}(\mathcal{L}j)\right)^{2s}*\left(\mathbb{F}_{BM}^{\theta}(\mathcal{L}j)\right)^{2t}\right)},\\ \sqrt{\prod_{\substack{\sigma,\theta=1\\ \sigma\neq\theta}}^{\omega}\left(1-\left(1-\left(\mathbb{G}_{BM}^{\omega+1}(\mathcal{L}j)\right)^{2}\right)^{s}*\left(1-\left(\mathbb{G}_{BM}^{\theta}(\mathcal{L}j)\right)^{2}\right)^{t}\right)},\\ \sqrt{1-\prod_{\substack{\sigma,\theta=1\\ \sigma\neq\theta}}^{\omega}\left(1-\left(\mathbb{H}_{BM}^{\omega+1}(\mathcal{L}j)\right)^{2s}*\left(\mathbb{H}_{BM}^{\theta}(\mathcal{L}j)\right)^{2t}\right)}\end{array}\right)$$

$$=\left(\begin{array}{c}\sqrt{1-\prod_{\substack{\sigma,\theta=1\\ \sigma\neq\theta}}^{\omega+1}\left(1-\left(\mathbb{F}_{BM}^{\omega+1}(\mathcal{L}j)\right)^{2s}*\left(\mathbb{F}_{BM}^{\theta}(\mathcal{L}j)\right)^{2t}\right)},\\ \sqrt{\prod_{\substack{\sigma,\theta=1\\ \sigma\neq\theta}}^{\omega+1}\left(1-\left(1-\left(\mathbb{G}_{BM}^{\sigma}(\mathcal{L}j)\right)^{2}\right)^{s}*\left(1-\left(\mathbb{G}_{BM}^{\omega+1}(\mathcal{L}j)\right)^{2}\right)^{t}\right)},\\ \sqrt{1-\prod_{\substack{\sigma,\theta=1\\ \sigma\neq\theta}}^{\omega+1}\left(1-\left(\mathbb{H}_{BM}^{\sigma}(\mathcal{L}j)\right)^{2s}*\left(\mathbb{H}_{BM}^{\omega+1}(\mathcal{L}j)\right)^{2t}\right)}\end{array}\right)$$

The proposed theory is held for $\varphi=\omega+1$. Then,

$$\frac{1}{\varphi(\varphi-1)}\left(\bigoplus_{\substack{\sigma,\theta=1\\ \sigma\neq\theta}}^{\varphi}\left(P_\sigma^s\otimes P_\theta^t\right)\right)=\left(\begin{array}{c}\sqrt{1-\left(\prod_{\substack{\sigma,\theta=1\\ \sigma\neq\theta}}^{\varphi}\left(1-\left(\mathbb{F}_{BM}^{\sigma}(\mathcal{L}j)\right)^{2s}*\left(\mathbb{F}_{BM}^{\theta}(\mathcal{L}j)\right)^{2s}\right)\right)^{\frac{1}{\varphi(\varphi-1)}}},\\ \prod_{\substack{\sigma,\theta=1\\ \sigma\neq\theta}}^{\varphi}\left(1-\left(1-\left(\mathbb{G}_{BM}^{\sigma}(\mathcal{L}j)\right)^{2}\right)^{s}\left(1-\left(\mathbb{G}_{BM}^{\theta}(\mathcal{L}j)\right)^{2}\right)^{t}\right)^{\frac{1}{2\varphi(\varphi-1)}},\\ \sqrt{1-\left(\prod_{\substack{\sigma,\theta=1\\ \sigma\neq\theta}}^{\varphi}\left(1-\left(\mathbb{H}_{BM}^{\sigma}(\mathcal{L}j)\right)^{2s}*\left(\mathbb{H}_{BM}^{\theta}(\mathcal{L}j)\right)^{2s}\right)\right)^{\frac{1}{\varphi(\varphi-1)}}}\end{array}\right).$$

Hence,

$$CPFBM^{P,Q}\left(P_1,P_2,\dots,P_\varphi\right)=\left[\frac{1}{\varphi(\varphi-1)}\left(\bigoplus_{\substack{\sigma,\theta=1\\ \sigma\neq\theta}}^{\varphi}\left(P_\sigma^s\otimes P_\theta^t\right)\right)\right]^{\frac{1}{s+t}}=$$

$$\left(\begin{array}{c}\left(1-\prod_{\substack{\sigma,\theta=1\\ \sigma\neq\theta}}^{\varphi}\left(1-\left(\mathbb{F}_{BM}^{\sigma}(\mathcal{L}j)\right)^{2s}\left(\mathbb{F}_{BM}^{\theta}(\mathcal{L}j)\right)^{2s}\right)^{\frac{\varphi}{2(s+t)}}\right)^{\frac{1}{2(s+t)}},\\ \sqrt{1-\left(1-\prod_{\substack{\sigma,\theta=1\\ \sigma\neq\theta}}^{\varphi}\left(1-\left(1-\left(\mathbb{G}_{BM}^{\sigma}(\mathcal{L}j)\right)^{2}\right)^{s}\left(1-\left(\mathbb{G}_{BM}^{\theta}(\mathcal{L}j)\right)^{2}\right)^{t}\right)^{\frac{1}{\varphi(\varphi-1)}}\right)^{\frac{1}{s+t}}},\\ \left(1-\prod_{\substack{\sigma,\theta=1\\ \sigma\neq\theta}}^{\varphi}\left(1-\left(\mathbb{H}_{BM}^{\sigma}(\mathcal{L}j)\right)^{2s}\left(\mathbb{H}_{BM}^{\theta}(\mathcal{L}j)\right)^{2s}\right)^{\frac{\varphi}{\varphi(\varphi-1)}}\right)^{\frac{1}{2(s+t)}}\end{array}\right).$$

The proposed theory holds for all values of φ. Similarly, we evaluate the remaining part using the same procedures, such as

$$CPFBM^{s,t}\left(P_1,P_2,\dots,P_\varphi\right)_{TCN}=\left(\begin{array}{c}\left(1-\prod_{\substack{\sigma,\theta=1\\ \sigma\neq\theta}}^{\varphi}\left(1-\left(\mathbb{F}_{BM}^{\sigma}(\mathcal{L}j)\right)^{2s}\left(\mathbb{F}_{BM}^{\theta}(\mathcal{L}j)\right)^{2s}\right)^{\frac{\varphi}{\varphi(\varphi-1)}}\right)^{\frac{t}{2(s+t)}},\\ \sqrt{1-\left(1-\prod_{\substack{\sigma,\theta=1\\ \sigma\neq\theta}}^{\varphi}\left(1-\left(1-\left(\mathbb{G}_{BM}^{\sigma}(\mathcal{L}j)\right)^{2}\right)^{s}\left(1-\left(\mathbb{G}_{BM}^{\theta}(\mathcal{L}j)\right)^{2}\right)^{t}\right)^{\frac{1}{\varphi(\varphi-1)}}\right)^{\frac{t}{2(s+t)}}},\\ \sqrt{1-\left(1-\prod_{\substack{\sigma,\theta=1\\ \sigma\neq\theta}}^{\varphi}\left(1-\left(1-\left(\mathbb{H}_{BM}^{\sigma}(\mathcal{L}j)\right)^{2}\right)^{s}\left(1-\left(\mathbb{H}_{BM}^{\theta}(\mathcal{L}j)\right)^{2}\right)^{t}\right)^{\frac{1}{\varphi(\varphi-1)}}\right)^{\frac{t}{2(s+t)}}}\end{array}\right).$$

Theorem 2. Consider $s,t>0$ with a collection of CPFNs $P_\sigma=\left(\mathbb{F}_{BM}^{\sigma}(\mathcal{L}j),\mathbb{G}_{BM}^{\sigma}(\mathcal{L}j),\mathbb{H}_{BM}^{\sigma}(\mathcal{L}j)\right)(\sigma=1,2,\dots,\varphi)$, then

1) Idempotency: If $P_\sigma=P=\left(\mathbb{F}_{BM}(\mathcal{L}j),\mathbb{G}_{BM}(\mathcal{L}j),\mathbb{H}_{BM}(\mathcal{L}j)\right)$, then

$$CPFBM^{s,t}\left(P_1,P_2,\dots,P_\varphi\right)_{TN}=P=\left(\mathbb{F}_{BM}(\mathcal{L}j),\mathbb{G}_{BM}(\mathcal{L}j),\mathbb{H}_{BM}(\mathcal{L}j)\right).$$

$$CPFBM^{s,t}\left(P_1,P_2,\dots,P_\varphi\right)_{TCN}=P=\left(\mathbb{F}_{BM}(\mathcal{L}j),\mathbb{G}_{BM}(\mathcal{L}j),\mathbb{H}_{BM}(\mathcal{L}j)\right).$$

2) Monotonicity: If $P_\sigma\le P_\sigma^*$ that is $\mathbb{F}_{BM}^{\sigma}(\mathcal{L}j)\le\mathbb{F}^{*\sigma}_{BM}(\mathcal{L}j)$, $\mathbb{G}_{BM}^{\sigma}(\mathcal{L}j)\ge\mathbb{G}_{BM}^{\sigma*}(\mathcal{L}j)$, and $\mathbb{H}_{BM}(\mathcal{L}j)\le\mathbb{H}_{BM}^{*}(\mathcal{L}j)$, thus

$$CPFBM^{s,t}\left(P_1,P_2,\dots,P_\varphi\right)_{TN}\le CPFBM^{s,t}\left(P_1^*,P_2^*,\dots,P_\varphi^*\right)_{TN}.$$

$$CPFBM^{s,t}(P_1, P_2, \dots, P_\varphi)_{TCN} \leq CPFBM^{s,t}(P_1^*, P_2^*, \dots, P_\varphi^*)_{TCN}.$$

3) Commutativity: Consider ${P_\sigma}^*(\sigma = 1,2, \dots, \varphi)$ be the permutation of $P_\sigma(\sigma = 1,2, \dots, \varphi)$, thus

$$CPFBM^{s,t}(P_1, P_2, \dots, P_\varphi)_{TN} = CPFBM^{s,t}({P_1}^*, {P_2}^*, \dots, {P_\varphi}^*)_{TN}.$$

$$CPFBM^{s,t}(P_1, P_2, \dots, P_\varphi)_{TCN} = CPFBM^{s,t}({P_1}^*, {P_2}^*, \dots, {P_\varphi}^*)_{TCN}.$$

4) Boundedness: Consider $P^- = \left(m\sigma\varphi_\sigma\left(\mathbb{F}_{BM}^\sigma(\mathcal{L}j)\right), max_\sigma\left(\mathbb{G}_{BM}^\sigma(\mathcal{L}j)\right), m\sigma\varphi_\sigma\left(\mathbb{H}_{BM}^\sigma(\mathcal{L}j)\right)\right)$ and $P^+ = \left(max_\sigma\left(\mathbb{F}_{BM}^\sigma(\mathcal{L}j)\right), m\sigma\varphi_\sigma\left(\mathbb{G}_{BM}^\sigma(\mathcal{L}j)\right), max_\sigma\left(\mathbb{H}_{BM}^\sigma(\mathcal{L}j)\right)\right)$, thus

$$P^- \leq CPFBM^{s,t}(P_1, P_2, \dots, P_\varphi)_{TN} \leq P^+.$$

$$P^- \leq CPFBM^{s,t}(P_1, P_2, \dots, P_\varphi)_{TCN} \leq P^+.$$

Further, we simplify the supremacy and validity of the proposed theory by discussing their special cases with the help of parameters.

Case 1: When $t \to 0$, then

$$l\sigma m_{t\to 0} CPFBM^{s,t}(P_1, P_2, \dots, P_\varphi)_{TN} = \begin{pmatrix} \left(1 - \prod_{\sigma=1}^{\varphi}\left(1 - \left(\mathbb{F}_{BM}^\sigma(\mathcal{L}j)\right)^{2s}\right)^{\frac{1}{\varphi}}\right)^{\frac{1}{2s}}, \\ \sqrt{1 - \left(1 - \prod_{\sigma=1}^{\varphi}\left(1 - \left(1 - \left(\mathbb{G}_{BM}^\sigma(\mathcal{L}j)\right)^2\right)^s\right)^{\frac{1}{\varphi}}\right)^{\frac{1}{s}}}, \\ \left(1 - \prod_{\sigma=1}^{\varphi}\left(1 - \left(\mathbb{H}_{BM}^\sigma(\mathcal{L}j)\right)^{2s}\right)^{\frac{1}{\varphi}}\right)^{\frac{1}{2s}} \end{pmatrix} = \left[\frac{1}{\varphi}\left(\oplus_{\sigma=1}^{\varphi} P_\sigma^s\right)\right]^{\frac{1}{s}}.$$

$$l\sigma m_{t\to 0} CPFBM^{s,t}(P_1, P_2, \dots, P_\varphi)_{TCN} = \begin{pmatrix} \left(1 - \prod_{\sigma=1}^{\varphi}\left(1 - \left(\mathbb{F}_{BM}^\sigma(\mathcal{L}j)\right)^{2s}\right)^{\frac{1}{\varphi}}\right)^{\frac{1}{2s}}, \\ \sqrt{1 - \left(1 - \prod_{\sigma=1}^{\varphi}\left(1 - \left(1 - \left(\mathbb{G}_{BM}^\sigma(\mathcal{L}j)\right)^2\right)^s\right)^{\frac{1}{\varphi}}\right)^{\frac{1}{s}}}, \\ \sqrt{1 - \left(1 - \prod_{\sigma=1}^{\varphi}\left(1 - \left(1 - \left(\mathbb{H}_{BM}^\sigma(\mathcal{L}j)\right)^2\right)^s\right)^{\frac{1}{\varphi}}\right)^{\frac{1}{s}}} \end{pmatrix} = \left[\frac{1}{\varphi}\left(\oplus_{\sigma=1}^{\varphi} P_\sigma^s\right)\right]^{\frac{1}{s}}.$$

Called the generalized CPF averaging (GCPFA) operator.

Case 2: When $s = 1$ and $t \to 0$, thus

$$l\sigma m_{t\to 0} CPFBM^{1,t}(P_1, P_2, \dots, P_\varphi)_{TN} = \begin{pmatrix} \sqrt{1 - \prod_{\sigma=1}^{\varphi}\left(1 - \left(\mathbb{F}_{BM}^\sigma(\mathcal{L}j)\right)^2\right)^{\frac{1}{\varphi}}}, \\ \left(\prod_{\sigma=1}^{\varphi} \mathbb{G}_{BM}^\sigma(\mathcal{L}j)\right)^{\frac{1}{\varphi}}, \\ \sqrt{1 - \prod_{\sigma=1}^{\varphi}\left(1 - \left(\mathbb{H}_{BM}^\sigma(\mathcal{L}j)\right)^2\right)^{\frac{1}{\varphi}}} \end{pmatrix} = \frac{1}{\varphi}\left(\oplus_{\sigma=1}^{\varphi} P_\sigma\right).$$

$$l\sigma m_{t\to 0}CPFBM^{1,t}(P_1,P_2,\dots,P_\varphi)_{TCN}=\begin{pmatrix}\sqrt{1-\prod_{\sigma=1}^{\varphi}\left(1-\left(\mathbb{F}_{BM}^{\sigma}(\mathcal{L}j)\right)^2\right)^{\frac{1}{\varphi}}},\\ \left(\prod_{\sigma=1}^{\varphi}\mathbb{G}_{BM}^{\sigma}(\mathcal{L}j)\right)^{\frac{1}{\varphi}},\\ \left(\prod_{\sigma=1}^{\varphi}\mathbb{H}_{BM}^{\sigma}(\mathcal{L}j)\right)^{\frac{1}{\varphi}}\end{pmatrix}=\frac{1}{\varphi}\left(\oplus_{\sigma=1}^{\varphi}P_\sigma\right).$$

This is called the CPF averaging (CPFA) operator.

Case 3: When $s=2$ and $t\to 0$, thus

$$l\sigma m_{t\to 0}CPFBM^{2,t}(P_1,P_2,\dots,P_\varphi)_{TN}=\begin{pmatrix}\left(1-\prod_{\sigma=1}^{\varphi}\left(1-\left(\mathbb{F}_{BM}^{\sigma}(\mathcal{L}j)\right)^4\right)^{\frac{1}{\varphi}}\right)^{\frac{1}{4}},\\ \sqrt{1-\left(1-\prod_{\sigma=1}^{\varphi}\left(1-\left(1-\left(\mathbb{G}_{BM}^{\sigma}(\mathcal{L}j)\right)^2\right)^2\right)^{\frac{1}{\varphi}}\right)^{\frac{1}{2}}},\\ \left(1-\prod_{\sigma=1}^{\varphi}\left(1-\left(\mathbb{H}_{BM}^{\sigma}(\mathcal{L}j)\right)^4\right)^{\frac{1}{\varphi}}\right)^{\frac{1}{4}}\end{pmatrix}=\left[\frac{1}{\varphi}\left(\oplus_{\sigma=1}^{\varphi}(P_\sigma)^2\right)\right]^{\frac{1}{2}}.$$

$$l\sigma m_{t\to 0}CPFBM^{2,t}(P_1,P_2,\dots,P_\varphi)_{TCN}=\begin{pmatrix}\left(1-\prod_{\sigma=1}^{\varphi}\left(1-\left(\mathbb{F}_{BM}^{\sigma}(\mathcal{L}j)\right)^4\right)^{\frac{1}{\varphi}}\right)^{\frac{1}{4}},\\ \sqrt{1-\left(1-\prod_{\sigma=1}^{\varphi}\left(1-\left(1-\left(\mathbb{G}_{BM}^{\sigma}(\mathcal{L}j)\right)^2\right)^2\right)^{\frac{1}{\varphi}}\right)^{\frac{1}{2}}},\\ \sqrt{1-\left(1-\prod_{\sigma=1}^{\varphi}\left(1-\left(1-\left(\mathbb{H}_{BM}^{\sigma}(\mathcal{L}j)\right)^2\right)^2\right)^{\frac{1}{\varphi}}\right)^{\frac{1}{2}}}\end{pmatrix}=\left[\frac{1}{\varphi}\left(\oplus_{\sigma=1}^{\varphi}(P_\sigma)^2\right)\right]^{\frac{1}{2}}.$$

Called the CPF square mean (CPFSM) operator.

Definition 6. Consider $s,t>0$ with a collection of CPFNs,

$$P_\sigma=\left(\mathbb{F}_{BM}^{\sigma}(\mathcal{L}j),\mathbb{G}_{BM}^{\sigma}(\mathcal{L}j),\mathbb{H}_{BM}^{\sigma}(\mathcal{L}j)\right)(\sigma=1,2,\dots,\varphi),$$

thus

$$CPFWBM^{s,t}(P_1,P_2,\dots,P_\varphi)_{TN}=\left[\frac{1}{\varphi(\varphi-1)}\left(\oplus_{\substack{\sigma,\theta=1\\ \sigma\neq\theta}}^{\varphi}\left((w_\sigma P_\sigma)^s\otimes(w_\theta P_\theta)^t\right)\right)\right]^{\frac{1}{s+t}} \tag{17}$$

$$CPFWBM^{s,t}(P_1,P_2,\dots,P_\varphi)_{TCN}=\left[\frac{1}{\varphi(\varphi-1)}\left(\oplus_{\substack{\sigma,\theta=1\\ \sigma\neq\theta}}^{\varphi}\left((w_\sigma P_\sigma)^s\otimes(w_\theta P_\theta)^t\right)\right)\right]^{\frac{1}{s+t}}. \tag{18}$$

This is called the $CPFWBM^{s,t}$ operator with weight vector $w_\sigma\in[0,1]$ and $\sum_{\sigma=1}^{\varphi}w_\sigma=1$.

Theorem 3. Consider $s,t>0$ with a collection of CPFNs, $P_\sigma=\left(\mathbb{F}_{BM}^{\sigma}(\mathcal{L}j),\mathbb{G}_{BM}^{\sigma}(\mathcal{L}j),\mathbb{H}_{BM}^{\sigma}(\mathcal{L}j)\right)(\sigma=1,2,\dots,\varphi)$, thus we prove that the aggregated theory of the above operators is again CPFN, such as

$CPFWBM^{s,t}\left(P_1,P_2,\dots,P_\varphi\right)_{TN}=$

$$\left(\begin{array}{c} 1-\left(1-\prod_{\substack{\sigma,\theta=1\\ \sigma\neq\theta}}^{\varphi}\left(1-\left(1-\left(1-\left(\mathbb{G}_{BM}^{\sigma}(\mathcal{L}j)\right)^{2}\right)^{w_\sigma}\right)^{s}\left(1-\left(1-\left(\mathbb{G}_{BM}^{\sigma}(\mathcal{L}j)\right)^{2}\right)^{w_\theta}\right)^{t}\right)^{\frac{1}{2(s+t)}}\right)^{\frac{1}{2(s+t)}}, \\ \sqrt{1-\left(1-\prod_{\substack{\sigma,\theta=1\\ \sigma\neq\theta}}^{\varphi}\left(1-\left(1-\left(\mathbb{G}_{BM}^{\sigma}(\mathcal{L}j)\right)^{2w_\sigma}\right)^{s}\left(1-\left(\mathbb{G}_{BM}^{\theta}(\mathcal{L}j)\right)^{2w_\theta}\right)^{t}\right)^{\frac{1}{\varphi(\varphi-1)}}\right)^{\frac{1}{s+t}}}, \\ 1-\left(1-\prod_{\substack{\sigma,\theta=1\\ \sigma\neq\theta}}^{\varphi}\left(1-\left(1-\left(1-\left(\mathbb{H}_{BM}^{\sigma}(\mathcal{L}j)\right)^{2}\right)^{w_\sigma}\right)^{s}\left(1-\left(1-\left(\mathbb{H}_{BM}^{\sigma}(\mathcal{L}j)\right)^{2}\right)^{w_\theta}\right)^{t}\right)^{\frac{1}{2(s+t)}}\right)^{\frac{1}{2(s+t)}} \end{array}\right). \tag{19}$$

$CPFWBM^{s,t}\left(P_1,P_2,\dots,P_\varphi\right)_{TCN}=$

$$\left(\begin{array}{c} 1-\left(1-\prod_{\substack{\sigma,\theta=1\\ \sigma\neq\theta}}^{\varphi}\left(1-\left(1-\left(1-\left(\mathbb{G}_{BM}^{\sigma}(\mathcal{L}j)\right)^{2}\right)^{w_\sigma}\right)^{s}\left(1-\left(1-\left(\mathbb{G}_{BM}^{\sigma}(\mathcal{L}j)\right)^{2}\right)^{w_\theta}\right)^{t}\right)^{\frac{1}{2(s+t)}}\right)^{\frac{1}{2(s+t)}}, \\ \sqrt{1-\left(1-\prod_{\substack{\sigma,\theta=1\\ \sigma\neq\theta}}^{\varphi}\left(1-\left(1-\left(\mathbb{G}_{BM}^{\sigma}(\mathcal{L}j)\right)^{2w_\sigma}\right)^{s}\left(1-\left(\mathbb{G}_{BM}^{\theta}(\mathcal{L}j)\right)^{2w_\theta}\right)^{t}\right)^{\frac{1}{\varphi(\varphi-1)}}\right)^{\frac{1}{s+t}}}, \\ \sqrt{1-\left(1-\prod_{\substack{\sigma,\theta=1\\ \sigma\neq\theta}}^{\varphi}\left(1-\left(1-\left(\mathbb{H}_{BM}^{\sigma}(\mathcal{L}j)\right)^{2w_\sigma}\right)^{s}\left(1-\left(\mathbb{H}_{BM}^{\theta}(\mathcal{L}j)\right)^{2w_\theta}\right)^{t}\right)^{\frac{1}{\varphi(\varphi-1)}}\right)^{\frac{1}{2(s+t)}}} \end{array}\right). \tag{20}$$

Proof. Omitted.

Theorem 4: Consider $s,t>0$ with a collection of CPFNs $P_\sigma=\left(\mathbb{F}_{BM}^{\sigma}(\mathcal{L}j),\mathbb{G}_{BM}^{\sigma}(\mathcal{L}j),\mathbb{H}_{BM}^{\sigma}(\mathcal{L}j)\right)(\sigma=1,2,\dots,\varphi)$, then

1) Idempotency: If $P_\sigma=P=\left(\mathbb{F}_{BM}(\mathcal{L}j),\mathbb{G}_{BM}(\mathcal{L}j),\mathbb{H}_{BM}(\mathcal{L}j)\right)$, then

$$CPFWBM^{s,t}\left(P_1,P_2,\dots,P_\varphi\right)_{TN}=P=\left(\mathbb{F}_{BM}(\mathcal{L}j),\mathbb{G}_{BM}(\mathcal{L}j),\mathbb{H}_{BM}(\mathcal{L}j)\right)$$

$$CPFWBM^{s,t}\left(P_1,P_2,\dots,P_\varphi\right)_{TCN}=P=\left(\mathbb{F}_{BM}(\mathcal{L}j),\mathbb{G}_{BM}(\mathcal{L}j),\mathbb{H}_{BM}(\mathcal{L}j)\right).$$

2) Monotonicity: If $P_\sigma\leq P_\sigma^{*}$ that is $\mathbb{F}_{BM}^{\sigma}(\mathcal{L}j)\leq \mathbb{F}^{*}{}_{BM}^{\sigma}(\mathcal{L}j)$, $\mathbb{G}_{BM}^{\sigma}(\mathcal{L}j)\geq \mathbb{G}_{BM}^{\sigma*}(\mathcal{L}j)$, and $\mathbb{H}_{BM}(\mathcal{L}j)\leq \mathbb{H}_{BM}^{*}(\mathcal{L}j)$, thus

$$CPFWBM^{s,t}\left(P_1,P_2,\dots,P_\varphi\right)_{TN}\leq CPFWBM^{s,t}\left(P_1^{*},P_2^{*},\dots,P_\varphi^{*}\right)_{TN}$$

$$CPFWBM^{s,t}\left(P_1,P_2,\dots,P_\varphi\right)_{TCN}\leq CPFWBM^{s,t}\left(P_1^{*},P_2^{*},\dots,P_\varphi^{*}\right)_{TCN}.$$

3) Commutativity: Consider ${P_\sigma}^{*}(\sigma=1,2,\dots,\varphi)$ be the permutation of $P_\sigma(\sigma=1,2,\dots,\varphi)$, thus

$$CPFWBM^{s,t}\left(P_1, P_2, \dots, P_\varphi\right)_{TN} = CPFWBM^{s,t}\left({P_1}^*, {P_2}^*, \dots, {P_\varphi}^*\right)_{TN}$$

$$CPFWBM^{s,t}\left(P_1, P_2, \dots, P_\varphi\right)_{TCN} = CPFWBM^{s,t}\left({P_1}^*, {P_2}^*, \dots, {P_\varphi}^*\right)_{TCN}.$$

4) Boundedness: Consider $P^- = \left(m\sigma\varphi_\sigma\left(\mathbb{F}^\sigma_{BM}(\mathcal{L}j)\right), max_\sigma\left(\mathbb{G}^\sigma_{BM}(\mathcal{L}j)\right), m\sigma\varphi_\sigma\left(\mathbb{H}^\sigma_{BM}(\mathcal{L}j)\right)\right)$ and $P^+ = \left(max_\sigma\left(\mathbb{F}^\sigma_{BM}(\mathcal{L}j)\right), m\sigma\varphi_\sigma\left(\mathbb{G}^\sigma_{BM}(\mathcal{L}j)\right), max_\sigma\left(\mathbb{H}^\sigma_{BM}(\mathcal{L}j)\right)\right)$, thus

$$P^- \leq CPFWBM^{s,t}\left(P_1, P_2, \dots, P_\varphi\right)_{TN} \leq P^+$$

$$P^- \leq CPFWBM^{s,t}\left(P_1, P_2, \dots, P_\varphi\right)_{TCN} \leq P^+.$$

Proof. Omitted.

4. Evaluation based on distance from average solution for proposed theory

The major theme of this section is to expose or construct the technique of the EDAS procedure based on the initiated operators. For this, we compute the following procedure, such as

Step 1: Design the matrix by including the terms of CPFNS by assigning to each attribute θ^{th} in every alternative σ^{th}, such as

$$DM = \left[P_{\mathrm{ij}}\right]_{\mathrm{n}\times\mathrm{m}} = \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1m} \\ P_{21} & P_{22} & \cdots & P_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ P_{\varphi 1} & P_{\varphi 2} & \cdots & P_{\varphi m} \end{bmatrix}. \tag{21}$$

The representation of the weight vector is as follows, such as $W = (\mathrm{w}_1, \mathrm{w}_2, \dots, \mathrm{w}_\mathrm{m})^{\mathrm{T}}$ with $\sum_{\theta=1}^{\varphi} \mathrm{w}_\mathrm{j} = 1$.

Step 2: Consider the technique of the CPFBM operator, we expose the average solution, such as

$$CPFBM^{s,t}\left(P_1, P_2, \dots, P_\varphi\right)_{TN} = \left(\begin{array}{c} \left(1 - \prod_{\substack{\sigma,\theta=1 \\ \sigma\neq\theta}}^{\varphi}\left(1 - \left(\mathbb{F}^\sigma_{BM}(\mathcal{L}j)\right)^{2s}\left(\mathbb{F}^\theta_{BM}(\mathcal{L}j)\right)^{2s}\right)^{\frac{\varphi}{\varphi(\varphi-1)}}\right)^{\frac{t}{2(s+t)}}, \\ \sqrt{1 - \left(1 - \prod_{\substack{\sigma,\theta=1 \\ \sigma\neq\theta}}^{\varphi}\left(1 - \left(1 - \left(\mathbb{G}^\sigma_{BM}(\mathcal{L}j)\right)^2\right)^s\left(1 - \left(\mathbb{G}^\theta_{BM}(\mathcal{L}j)\right)^2\right)^t\right)^{\frac{1}{\varphi(\varphi-1)}}\right)^{\frac{t}{2(s+t)}}}, \\ \left(1 - \prod_{\substack{\sigma,\theta=1 \\ \sigma\neq\theta}}^{\varphi}\left(1 - \left(\mathbb{H}^\sigma_{BM}(\mathcal{L}j)\right)^{2s}\left(\mathbb{H}^\theta_{BM}(\mathcal{L}j)\right)^{2s}\right)^{\frac{\varphi}{\varphi(\varphi-1)}}\right)^{\frac{t}{2(s+t)}} \end{array}\right). \tag{22}$$

Step 3: Compute the value of PDA and the value of NDA in the shape of a matrix, such as

$$\mathcal{PDA}_{\sigma\theta} = \frac{\max\left(0, (\mathrm{x_{ij}} - \mathcal{AV}_\theta)\right)}{\mathcal{AV}_\theta}, \tag{23}$$

$$\mathcal{NDA}_{\sigma\theta} = \frac{\max\left(0,\left(\mathcal{AV}_{\theta} - \mathrm{x_{ij}}\right)\right)}{\mathcal{AV}_{\theta}}. \tag{24}$$

These techniques will be used for truth, falsity, and radius functions.

Step 4: Derive the technique of weighted summation according to the technique of PDA and NDA, such as

$$\mathcal{SP}_{\mathrm{i}} = \sum_{\mathrm{j}=1}^{\mathrm{m}} \mathrm{w_j}\, \mathcal{PDA}_{\sigma\theta}, \tag{25}$$

$$\mathcal{SN}_{\mathrm{i}} = \sum_{\mathrm{j}=1}^{\mathrm{m}} \mathrm{w_j}\, \mathcal{NDA}_{\sigma\theta}. \tag{26}$$

Step 5: Compute the normalized values according to the values of $\mathrm{SP_i}$ and $\mathrm{SN_i}$, such as

$$\mathcal{NSP}_{\mathrm{i}} = \frac{\mathcal{SP}_{\mathrm{i}}}{\max\left(\mathcal{SN}_{\mathrm{i}}\right)}, \tag{27}$$

$$\mathcal{NSN}_{\mathrm{i}} = 1 - \frac{\mathcal{SN}_{\mathrm{i}}}{\max\left(\mathcal{SN}_{\mathrm{i}}\right)}. \tag{28}$$

Step 6: Derive the appraisal values based on the above information, such as

$$\mathcal{AS}_{\mathrm{i}} = \frac{1}{2}\left(\mathcal{NSP}_{\mathrm{i}} + \mathcal{NSN}_{\mathrm{i}}\right). \tag{29}$$

Step 7: Rank all alternatives, to examine the best optimal.

Moreover, we discuss the supremacy of the above technique by evaluating some examples with the help of the above technique. For this, we discuss the problem of the classification of the CPF meta-analysis and its application.

5. Classification of the Circular Pythagorean fuzzy Meta-analysis and its applications

In this section, we describe one of the most preferable and dominant techniques or applications called meta-analysis, which is a statistical technique used to evaluate or analyze the data from multiple independent studies for a particular topic or research question. In this application, we concentrate on using the initiated operators and methods to evaluate the above problems. For this, we consider the problem of meta-analysis and find the major key steps involved in meta-analysis, such as

1) Data Extraction “A1”.
2) Pooling of Effect Sizes “A2”.
3) Assessment of Heterogeneity “A3”.
4) Publication Bias Assessment “A4”.
5) Subgroup Analysis and Sensitive Analysis “A5”.

We have some attributes, such as growth analysis, social impact, political impact, environmental impact, and internet resources. For this, we compute the following procedure, such as

Step 1: Design the matrix by including the terms of CPFNS by assigning to each attribute θ^{th} in every alternative σ^{th}, see Table 1.

Table 1. Representation of the CPF matrix.

	G1	G2	G3	G4	G5
A1	(0.8,0.5,0.3)	(0.81,0.51,0.31)	(0.82,0.52,0.32)	(0.83,0.53,0.33)	(0.84,0.54,0.34)
A2	(0.4,0.3,0.1)	(0.41,0.31,0.11)	(0.42,0.32,0.12)	(0.43,0.33,0.13)	(0.44,0.34,0.14)
A3	(0.7,0.6,0.3)	(0.71,0.61,0.31)	(0.72,0.62,0.32)	(0.73,0.63,0.33)	(0.74,0.64,0.34)
A4	(0.6,0.5,0.4)	(0.61,0.51,0.41)	(0.62,0.52,0.42)	(0.63,0.53,0.43)	(0.64,0.54,0.44)
A5	(0.5,0.4,0.2)	(0.51,0.41,0.21)	(0.52,0.42,0.22)	(0.53,0.43,0.23)	(0.54,0.44,0.24)

Step 2: Consider the technique of the CPFBM operator, we expose the average solution, such as

$$\mathcal{AV}_1 = (0.9945, 0.4674, 0.8303),$$
$$\mathcal{AV}_2 = (0.9952, 0.4775, 0.8439),$$
$$\mathcal{AV}_3 = (0.9959, 0.4876, 0.8565),$$
$$\mathcal{AV}_4 = (0.9964, 0.4977, 0.8681),$$
$$\mathcal{AV}_5 = (0.9969, 0.5078, 0.8788).$$

Step 3: Compute the value of PDA and the value of NDA in the shape of a matrix, see Tables 2 and 3.

Table 2. Positive distance matrix.

(0,0.0695,0)	(0,0.0679,0)	(0,0.0663,0)	(0,0.0647,0)	(0,0.0632,0)
(0,0,0)	(0,0,0)	(0,0,0)	(0,0,0)	(0,0,0)
(0,0.2834,0)	(0,0.2773,0)	(0,0.2713,0)	(0,0.2656,0)	(0,0.2601,0)
(0,0.0695,0)	(0,0.0679,0)	(0,0.0663,0)	(0,0.0647,0)	(0,0.0632)
(0,0,0)	(0,0,0)	(0,0,0)	(0,0,0)	(0,0,0)

Table 3. Negative distance matrix.

(0.1956,0, 0.6387)	(0.1861, 0,0.63269)	(0.1766, 0,0.6264)	(0.1670,0, 0.6198)	(0.1574,0, 0.6131)
(0.5978, 0.3582, 0.8795)	(0.5880, 0.3508, 0.8696)	(0.5782,0.3437, 0.8599)	(0.5684, 0.3370, 0.8502)	(0.5586, 0.3305, 0.8407)
(0.2961,0, 0.6387)	(0.2866,0, 0.6326)	(0.2770, 0,0.6264)	(0.26742,0, 0.6198)	(0.2577,0, 0.6131)
(0.3967,0, 0.5182)	(0.3871,0, 0.5142)	(0.3774, 0,0.5096)	(0.367,0, 0.5046)	(0.3580,0, 0.4993)
(0.4972, 0.1443, 0.7591)	(0.4875, 0.1414, 0.7511)	(0.4778,0.1387, 0.7431)	(0.4681, 0.1361, 0.7350)	(0.4583, 0.1336, 0.7269)

Step 4: Derive the technique of weighted summation according to the technique of PDA and NDA based on the weight vector $(0.2, 0.2, 0.2, 0.2, 0.2)$. Tables 4 and 5.

Table 4. Weighted information based on the Positive distance matrix.

$\mathcal{SP}_1$	$(0,0.0845,0)$
$\mathcal{SP}_2$	$(0,0.082,0)$
$\mathcal{SP}_3$	$(0,0.080,0)$
$\mathcal{SP}_4$	$(0,0.0790,0)$
$\mathcal{SP}_5$	$(0,0.0773,0)$

Table 5. Weighted information based on the Negative distance matrix.

$\mathcal{NP}_1$	$(0.3967,0.1005,0.6868)$
$\mathcal{NP}_2$	$(0.3871,0.0984,0.6800)$
$\mathcal{NP}_3$	$(0.3774,0.0965,0.6731)$
$\mathcal{NP}_4$	$(0.367,0.0946,0.6659)$
$\mathcal{NP}_5$	$(0.3580,0.0928,0.6586)$

Step 5: Compute the normalized values according to the values of SP_i and SN_i, see Table 6.

Table 6. Normalized matrix of information.

$\mathcal{NSP}_1\mathcal{NSN}_1$	$(0,0,0)$
$\mathcal{NSP}_2\mathcal{NSN}_2$	$(0,0.0204,0.0098)$
$\mathcal{NSP}_3\mathcal{NSN}_3$	$(0.9514,0.0399,0.0200)$
$\mathcal{NSP}_4\mathcal{NSN}_4$	$(0.9269,0.0586,0.0304)$
$\mathcal{NSP}_5\mathcal{NSN}_5$	$(0.9025,0.0765,0.411)$

Step 6: Derive the appraisal values based on the above information, such as

$$\mathcal{SV}_1 = 0.0, \mathcal{SV}_2 = 0.0, \mathcal{SV}_3 = 0.0181, \mathcal{SV}_4 = 0.0260, \mathcal{SV}_5 = 0.0303.$$

Step 7: Rank all alternatives, to examine the best optimal, such as

$$\mathcal{SV}_5 > \mathcal{SV}_4 > \mathcal{SV}_3 > \mathcal{SV}_2 = \mathcal{SV}_1.$$

The most prominent decision is $\mathcal{SV}_5$, according to the technique of the EDAS method, which is represented by the "Subgroup analysis and Sensitive Analysis". Further, we show the flexibility of the proposed theory by using the proposed operators without the EDAS method to evaluate the dominancy and flexibility of the proposed operators. For this, we consider the information in Table 7.

Table 7. CPF matrix of information.

	G1	G2	G3	G4	G5
A1	(0.8,0.5,0.3)	(0.81,0.51,0.31)	(0.82,0.52,0.32)	(0.83,0.53,0.33)	(0.84,0.54,0.34)
A2	(0.4,0.3,0.1)	(0.41,0.31,0.11)	(0.42,0.32,0.12)	(0.43,0.33,0.13)	(0.44,0.34,0.14)
A3	(0.7,0.6,0.3)	(0.71,0.61,0.31)	(0.72,0.62,0.32)	(0.73,0.63,0.33)	(0.74,0.64,0.34)
A4	(0.6,0.5,0.4)	(0.61,0.51,0.41)	(0.62,0.52,0.42)	(0.63,0.53,0.43)	(0.64,0.54,0.44)
A5	(0.5,0.4,0.2)	(0.51,0.41,0.21)	(0.52,0.42,0.22)	(0.53,0.43,0.23)	(0.54,0.44,0.24)

Further, we evaluate the support of any two CPF values based on distance measures. Thus, we have

$S_{12}^{1}=S_{12}^{1}=0.7333$, $S_{13}^{1}=S_{31}^{1}=0.9333$, $S_{14}^{1}=S_{41}^{1}=0.9$, $S_{15}^{1}=S_{51}^{1}=0.833$, $S_{23}^{1}=S_{32}^{1}=0.8333$, $S_{24}^{1}=S_{42}^{1}=0.7666$, $S_{25}^{1}=S_{52}^{1}=0.9$, $S_{34}^{1}=S_{43}^{1}=0.9$, $S_{35}^{1}=S_{53}^{1}=0.8333$, $S_{45}^{1}=S_{54}^{1}=0.8666$.

$S_{12}^{2}=S_{12}^{2}=0.7333$, $S_{13}^{2}=S_{31}^{2}=0.9333$, $S_{14}^{2}=S_{41}^{2}=0.9$, $S_{15}^{2}=S_{51}^{2}=0.833$, $S_{23}^{2}=S_{32}^{2}=0.8333$, $S_{24}^{2}=S_{42}^{2}=0.7666$, $S_{25}^{2}=S_{52}^{2}=0.9$, $S_{34}^{2}=S_{43}^{2}=0.9$, $S_{35}^{2}=S_{53}^{2}=0.8333$, $S_{45}^{2}=S_{54}^{2}=0.8666$.

$S_{12}^{3}=S_{12}^{3}=0.7333$, $S_{13}^{3}=S_{31}^{3}=0.9333$, $S_{14}^{3}=S_{41}^{3}=0.9$, $S_{15}^{3}=S_{51}^{3}=0.833$, $S_{23}^{3}=S_{32}^{3}=0.8333$, $S_{24}^{3}=S_{42}^{3}=0.7666$, $S_{25}^{3}=S_{52}^{3}=0.9$, $S_{34}^{3}=S_{43}^{3}=0.9$, $S_{35}^{3}=S_{53}^{3}=0.8333$, $S_{45}^{3}=S_{54}^{3}=0.8666$.

$S_{12}^{4}=S_{12}^{4}=0.7333$, $S_{13}^{4}=S_{31}^{4}=0.9333$, $S_{14}^{4}=S_{41}^{4}=0.9$, $S_{15}^{4}=S_{51}^{4}=0.833$, $S_{23}^{4}=S_{32}^{4}=0.8333$, $S_{24}^{4}=S_{42}^{4}=0.7666$, $S_{25}^{4}=S_{52}^{4}=0.9$, $S_{34}^{4}=S_{43}^{4}=0.9$, $S_{35}^{4}=S_{53}^{4}=0.8333$, $S_{45}^{4}=S_{54}^{4}=0.8666$.

$S_{12}^{5}=S_{12}^{5}=0.7333$, $S_{13}^{5}=S_{31}^{5}=0.9333$, $S_{14}^{5}=S_{41}^{5}=0.9$, $S_{15}^{5}=S_{51}^{5}=0.833$, $S_{23}^{5}=S_{32}^{5}=0.8333$, $S_{24}^{5}=S_{42}^{5}=0.7666$, $S_{25}^{5}=S_{52}^{5}=0.9$, $S_{34}^{5}=S_{43}^{5}=0.9$, $S_{35}^{5}=S_{53}^{5}=0.8333$, $S_{45}^{5}=S_{54}^{5}=0.8666$.

Thus, by using the above values, we find the following information, such as

$$\tau_{11}=0.2018, \tau_{12}=0.1896, \tau_{13}=0.2018, \tau_{14}=0.2033, \tau_{15}=0.2033.$$

$$\tau_{21}=0.2018, \tau_{22}=0.1896, \tau_{23}=0.2018, \tau_{24}=0.2033, \tau_{25}=0.2033.$$

$$\tau_{31}=0.2018, \tau_{32}=0.1896, \tau_{33}=0.2018, \tau_{34}=0.2033, \tau_{35}=0.2033.$$

$$\tau_{41}=0.2018, \tau_{42}=0.1896, \tau_{43}=0.2018, \tau_{44}=0.2033, \tau_{45}=0.2033.$$

$$\tau_{51}=0.2018, \tau_{52}=0.1896, \tau_{53}=0.2018, \tau_{54}=0.2033, \tau_{55}=0.2033.$$

Moreover, using the technique of the CPFBM operator, we have the following aggregated values, such as

$z_1=(0.9945,0.46748,0.8303)$, $z_2=(0.9952,0.477,0.8439)$, $z_3=(0.9959,0.4876,0.856)$, $z_4=(0.9964,0.4977,0.8681)$, $z_5=(0.9969,0.5078,0.8788)$.

Thus, we consider the technique of score values, such as

$$S(z_1)=0.6399, S(z_2)=0.6435, S(z_3)=0.6458, S(z_4)=0.6469, S(z_5)=0.6468.$$

Finally, we have the following ranking values, such as

$$S(z_4)>S(z_5)>S(z_3)>S(z_2)>S(z_1).$$

The most prominent decision is $\mathcal{SV}_4$ according to the technique of the CPFBM operator, which is represented by the "Publication Bias Assessment". Further, we compare the proposed theory with some existing techniques for evaluating the supremacy and validity of the proposed theory.

6. Comparative analysis

The main theme of this article is to evaluate the supremacy and flexibility of the proposed method and proposed operators by comparing their ranking values with the obtained values of existing techniques to enhance the worth of the initiated theory. For this, we consider some existing techniques: Ali and Yang [26] derived the technique of Hamacher operators for CPFSs. Furthermore, Xu and Yager [30] proposed the theory of BM operators for IFSs. Moreover, Xia et al. [31] evaluated the technique of BM operators for generalized IFSs. Additionally, Liang et al. [32] derived the technique of BM operators for PFSs and their applications. Further, Yang et al. [33] exposed the BM operators for PFSs based on triangular norms. Finally, using the information in Table 7, the comparative analysis is listed in Table 8.

Table 8. Representation of the comparative analysis.

Methods	Score values	Best optimal
Ali and Yang [26]	$S(z_4) > S(z_5) > S(z_3) > S(z_2) > S(z_1)$	$S(z_4)$
Xu and Yager [30]	No	No
Xia et al. [31]	No	No
Liang et al. [32]	No	No
Yang et al. [33]	No	No
CPFBM operator	$S(z_4) > S(z_5) > S(z_3) > S(z_2) > S(z_1)$	$S(z_4)$

The most prominent decision is $\mathcal{SV}_4$, according to the technique of the CPFBM operator, which is represented by the "Publication Bias Assessment". Further, the existing techniques have failed because of ambiguity and limitations, where the proposed theory of Xu and Yager [30], Xia et al. [31], Liang et al. [32], and Yang et al. [33] are the special cases of the proposed theory. The limitations of the existing models are described below:

1) Ali and Yang [26] derived the technique of Hamacher operators for CPFSs. The models for the Hamacher operators are very flexible and dominant because of their features. According to the theory of Ali and Yang [26], the ranking values are listed as follows: $S(z_4) > S(z_5) > S(z_3) > S(z_2) > S(z_1)$. Once more, the proposed model is different from the existing technique because it is computed based on the Bonferroni operators.
2) Xu and Yager [30] proposed the theory of BM operators for IFSs, which is a special part of the proposed theory. The technique of BM operators for IFSs cannot evaluate the CPF kind of information, because the condition of CPF sets is more general. The three functions are the part of CPFSs, but in the case of IFSs, we have just two functions, which is the special case of the derived theory.
3) Xia et al. [31] evaluated the technique of BM operators for generalized IFSs, which is a special part of the proposed theory. The technique of BM operators for IFSs cannot evaluate the CPF kind of information, because the condition of CPF sets is more general, where the three functions are the part of CPFSs, but in the case of IFSs, we have just two functions, which is the special case of the derived theory.
4) Liang et al. [32] derived the technique of BM operators for PFSs and their applications as part of initiated operators, since this technique contains only two functions, but the proposed model is the modified version. This is because they contained three functions, and if we ignored the

radius function form the CPFSs, then the proposed theory will be converted to the existing models.

5) Further, Yang et al. [33] exposed the BM operators for PFSs based on triangular norms as part of initiated operators, since this technique contains only two functions, but the proposed model is the modified version. This is because they contained three functions, and if we ignored the radius function form the CPFSs, then the proposed theory will be converted to the existing models.

Thus, after our long analysis, we concluded that the proposed model is superior and more general than the existing technique in coping with vague and uncertain information. Hence, the proposed model is more reliable and more general than the existing information.

7. Conclusions

The major themes of this manuscript are listed below:

1) We propose the technique of the CPFBM operator, and CPFWBM operator and describe their special cases with the help of two parameters, "s" and "t". Some describable properties are also proposed for the above techniques.
2) We present the technique of the EDAS technique for the CPFSs.
3) We show the flexibility and dominance of the proposed operators by comparing the proposed methods with some existing techniques.

In the future, we will discuss the novel technique of circular picture fuzzy sets and their extensions. Further, we will develop some new techniques for aggregation operators [34], similarity measures [35], and different kinds of methods [36]. Finally, we will discuss their application in artificial intelligence [37], data science [38], and decision-making theory [39] to enhance the worth of the proposed methods.

Author contributions

Weiwei Jiang: Conceptualization, Methodology, Formal analysis, Writing–review and editing, Visualization; Zeeshan Ali: Conceptualization, Software, Validation, Formal analysis, Investigation, Data Curation, Writing–review and editing; Muhammad Waqas: Conceptualization, Software, Validation, Formal analysis, Investigation, Data Curation, Writing–review and editing; Peide Liu: Software, Validation, Investigation, Resources, Data Curation, Writing–review and editing, Visualization Supervision, Project administration, Funding acquisition. All authors have read and approved the final version of the manuscript for publication.

Conflict of interest

The authors declare there is no conflict of interest.

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