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Research article

Global asymptotic stability and trajectory structure rules of high-order nonlinear difference equation

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Abstract: In this article, global asymptotic stability and trajectory structure of the following high-order nonlinear difference equation

$$z_{n+1} = \frac{z_{n-1}z_{n-2}z_{n-4} + z_{n-1} + z_{n-2} + z_{n-4} + b}{z_{n-1}z_{n-2} + z_{n-1}z_{n-4} + z_{n-2}z_{n-4} + 1 + b}, \quad n \in N,$$

are studied, where $b \in [0, \infty)$ and the initial conditions $z_i \in (0, \infty), i = 0, -1, -2, -3, -4$. Using the semi-cycle analysis method, in a prime period, a continuous length of positive and negative semi-cycles of any nontrivial solution appears periodically: 2,3,4,6,12. Moreover, two examples are given to illustrate the effectiveness of theoretic analysis.

Keywords: global asymptotic stability; trajectory structure; difference equation; nontrivial solution
Mathematics Subject Classification: 39A10

1. Introduction

Difference equations and differential equations are equally important mathematical theories in mathematics. In many fields such as chemistry, engineering, physics, medicine, and many practical problems, real-world data is statistically analyzed at equal intervals and time periods. Mathematical models that describe the relationships between discrete variables are called discrete models, and these discrete variables are usually referred to as discrete variables. In physics, this model is often used to simulate and predict discontinuous phenomena in continuous processes, such as shock waves in fluid dynamics. Difference equations are an effective method for studying discrete models. They reflect the variation law of the values of discrete variables. When seeking numerical solutions to differential equations, the differentiation is often approximated using corresponding differences, and the resulting equation is called a difference equation. Finding an approximate solution to a differential equation by solving a difference equation is an example of discretizing continuous problems. Studying high-order rational difference equations is not only necessary for the development of mathematics itself, but also for the development of multiple fields. It can help us to understand and to predict the dynamic behavior of various systems, providing an important theoretical basis for scientific decision-making. In this context, the study of the dynamical behavior of these mathematical models has practical significance.

In recent years, dynamical behavior of nonlinear difference equations or discrete dynamical systems has become a popular research topic since it is widely applied in many fields, such as biology, economics, chemistry, physics, and so on. Many scholars are interested in studying the qualitative property of the solution for these models and have obtained many very important achievements (see [1–5]).

In 1995, Ladas [6] proposed an open problem that is to find some sufficient conditions on the global stability of the rational difference equation

$$r_{n+1} = \frac{r_n + r_{n-1}r_{n-2}}{r_n r_{n-1} + r_{n-2}}, \quad n \in N, \tag{1.1}$$

with initial conditions $r_i \in (0, \infty), i = 0, -1, -2$.

In 2001, Tim Nesemann [7], using the strong negative feedback property [8], studied the rational difference equation

$$r_{n+1} = \frac{r_{n-1} + r_n r_{n-2}}{r_n r_{n-1} + v_{n-2}}, \qquad n \in N,$$

with initial conditions $r_i \in (0, \infty), i = 0, -1, -2$.

In 2004, Li and Zhu [9] studied the dynamical behaviors of two recursive difference equations

$$r_{n+1} = \frac{r_n r_{n-1} + r_{n-2} + d}{r_n + r_{n-1}r_{n-2} + d} \quad \text{and} \quad r_{n+1} = \frac{r_{n-1} + r_n r_{n-2} + d}{r_n r_{n-1} + r_{n-2} + d}, \quad n \in N,$$

respectively, where $d \in [0, \infty)$ and with initial conditions $r_i \in (0, \infty), i = 0, -1, -2$.

In 2005, Li [10], utilizing the semi-cycle analysis method, obtained the global asymptotic stability of the equilibrium of the difference equation

$$g_{n+1} = \frac{g_{n-1}g_{n-2}g_{n-3} + g_{n-1} + g_{n-2} + g_{n-3} + d}{g_{n-1}g_{n-2} + g_{n-1}g_{n-3} + g_{n-2}g_{n-3} + 1 + d}, \quad n \in N,$$

in which $d \in [0, \infty)$ and with initial conditions $g_i \in (0, \infty), i = 0, -1, -2, -3$.

In 2008, Chen and Li [11] studied the trajectory structure rules of a high-order rational difference equation

$$g_{n+1} = \frac{g_{n-1}g_{n-2} + g_{n-1}g_{n-4} + g_{n-2}g_{n-4} + 1 + d}{g_{n-1}g_{n-2}g_{n-4} + g_{n-1} + g_{n-2} + g_{n-4} + d}, \quad n \in N,$$

in which $d \in [0, \infty)$ and with initial conditions $g_i \in (0, \infty), i = 0, -1, -2, -3, -4$.

In 2012, Elsayed and El-Dessoky [12] obtained a global convergence result, the boundedness, and the periodicity of the solutions to the difference equation

$$g_{n+1} = a g_{n-s} + \frac{b g_{n-l} + c g_{n-k}}{d g_{n-l} + e g_{n-k}}, \quad n \in N,$$

with parameters $a, b, c, d, e \in R^+$ and initial conditions $g_i \in R^+, i = 0, -1, ..., -t, t = \max\{s, l, k\}$.

In 2018, Ibrahim [13] explored the bifurcation and periodically semi-cycles of fifth- order rational difference equation

$$y_{m+1} = \frac{y_m y_{m-2}^{\alpha} y_{m-4}^{\beta} + y_m + y_{m-2}^{\alpha} + y_{m-4}^{\beta} + \gamma}{y_m y_{m-2}^{\alpha} + y_m y_{m-4}^{\beta} + \gamma + 1}, \quad m \in N,$$

where $\gamma \in [0, \infty), \alpha, \beta \in Z^+$, and $y_{-i} \in (0, \infty), i = 0, 1, 2, 3, 4$.

In 2019, Chatzarakis et al. [14] studied the dynamics of solutions of the following difference equation

$$g_{n+1} = \alpha + \frac{\beta g_n^2}{(\gamma + g_n) g_{n-1}}, \quad n \in N,$$

in which $\alpha, \gamma \in [0, \infty)$ and $\beta \in (0, \infty)$, and with initial conditions $g_i \in (0, \infty), i = 0, -1$. The authors obtained the periodicity character, stability, and boundedness of its solutions.

In 2021, Khan and El-Metwally [15] studied the second-order nonlinear difference equation

$$g_n = a_n + \frac{g_n^p}{g_{n-1}^p}, \quad n \in N,$$

in which $p \in N$, with initial conditions $g_{-1}, g_0 \in R^+$, and where $\{a_n\}$ is a nonnegative periodic sequence; when n is even, $a_n = \alpha$, and when n is odd, $a_n = \beta$. The authors explored the local stability, attractor, periodicity character, and boundedness of solutions.

In 2024, Liu and Xu et al. [16] studied the solutions of several systems of the first-order partial differential difference equations

$$\begin{cases} u(z + d)[a_1 u(z) + b_1 u_{z_1} + c_1 u_{z_2} + a_2 v(z) + b_2 v_{z_1} + c_2 v_{z_2}] = 1, \\ v(z + d)[a_1 v(z) + b_1 v_{z_1} + c_1 v_{z_2} + a_2 u(z) + b_2 u_{z_1} + c_2 u_{z_2}] = 1, \end{cases}$$

where $d = (d_1, d_2) \in C^2$ (two-dimensional complex number space) and $a_i, b_i, c_i \in C$ (complex number), $i = 1, 2$. They discussed the forms of the transcendental solutions of the system in respect to the previous existing results.

Inspired by the above works, in this article, we will study the dynamical behaviors of the high-order rational difference equation

$$z_{n+1} = \frac{z_{n-1} z_{n-2} z_{n-4} + z_{n-1} + z_{n-2} + z_{n-4} + b}{z_{n-1} z_{n-2} + z_{n-1} z_{n-4} + z_{n-2} z_{n-4} + 1 + b}, \quad n \in N, \tag{1.2}$$

where $z_i \in (0, \infty), i = 0, -1, -2, -3, -4$, and $b \in [0, \infty)$.

The aim of this article is to obtain some results, such as the global asymptotic stability and the trajectory structure of the positive and negative semi-cycles of any nontrivial solution of Eq (1.2) by utilizing the semi-cycle analysis method.

The rest of this article is arranged as follows. Section 2 introduces some lemmas, definitions, and the trajectory structures of the solution to Eq (1.2). The global asymptotical stability of the unique positive equilibrium of the system is derived by mathematical analysis in Section 3. Two examples demonstrate the effectiveness of theoretic analysis in Section 4. A general conclusion and discussion are drawn in Section 5.

The contribution of this study lies in the following aspects.

• The trajectory structure rules of the solution to the nonlinear difference equation are derived extensively using the semi-cycle method. It is beneficial to fully understand asymptotic behavior of the solution.

• This study provides an effective method to explore the qualitative dynamical behaviors of some high-order rational difference equations.

2. Definitions and some lemmas

For the convenience of readers, we provide definitions and some lemmas used in what follows.

Let K be an interval of a real number g: $K \times K \times K \to K, g \in C^1$ (continuously differentiable function). Then,

$$z_{n+1} = g(z_{n-1}, z_{n-2}, z_{n-4}), \quad n \in N, \tag{2.1}$$

has a unique positive solution $\{z_n\}_{n=-4}^{\infty}$ for the initial conditions $z_j \in K, j = 0, -1, -2, -3, -4$.

Definition 2.1. [10] *$\bar{z}$ is said to be an equilibrium of (2.1) if $\bar{z} = g(\bar{z}, \bar{z}, \bar{z})$. In other words, for $n \geq 0$, $z_n = \bar{z}$ is a solution of (2.1), that is to say, $\bar{z}$ is also a fixed point of g.*

It is clear that $\bar{z} = 1$ is a unique positive equilibrium of Eq (1.2) satisfying

$$\bar{z} = \frac{\bar{z}^3 + 3\bar{z} + b}{3\bar{z}^2 + 1 + b}. \tag{2.2}$$

Definition 2.2. [13] *Let $\bar{z}$ be an equilibrium of (2.1).*

(i) *The equilibrium point $\bar{z}$ is stable if, for any $\varepsilon > 0$ with the initial values $z_j \in K, j = 0, -1, -2, -3, -4$, there exists $\delta > 0$, $\sum_{j=-4}^{0} |z_j - \bar{z}| < \delta$ implies $|z_n - \bar{z}| < \varepsilon$, for $n \geq 1$.*

(ii) *The equilibrium point $\bar{z}$ is locally asymptotically stable (LAS) if it is stable and there exists $\gamma > 0$, for any initial conditions $z_j \in K, j = 0, -1, -2, -3, -4$, $\sum_{j=-4}^{0} |z_j - \bar{z}| < \gamma$, and then $\lim_{n\to\infty} z_n = \bar{z}$.*

(iii) *The equilibrium point $\bar{z}$ is a global attractor if $\lim_{n\to\infty} z_n = \bar{z}$ for any initial conditions $z_j \in K, j = 0, -1, -2, -3, -4$.*

(iv) *The equilibrium point $\bar{z}$ is globally asymptotically stable (GAS) if it is stable and a global attractor.*

Definition 2.3. [10] *A positive semi-cycle of $\{z_n\}_{n=-4}^{\infty}$ of (1.2) is a "string" of terms $\{z_m, z_{m+1}, \cdots, z_l\}$ which is greater than or equal to the equilibrium $\bar{z}$ for $m \geq -4$ and $l \leq \infty$ such that*

$$z_{m-1} < \bar{z}, \quad \textit{either} \quad m = -4 \quad \textit{or} \quad m > -4,$$

and

$$z_{l+1} < \bar{z}, \quad \textit{either} \quad l = \infty \quad \textit{or} \quad l < \infty.$$

A negative semi-cycle of $\{z_n\}_{n=-4}^{\infty}$ of (1.2) is a "string" of terms $\{z_m, z_{m+1}, \cdots, z_l\}$ which is less than the equilibrium $\bar{z}$ for $m \geq -4$ and $l \leq \infty$ such that

$$z_{m-1} \geq \bar{z}, \quad \textit{either} \quad m = -4 \quad \textit{or} \quad m > -4,$$

and

$$z_{l+1} \geq \bar{z}, \quad \textit{either} \quad l = \infty \quad \textit{or} \quad l < \infty.$$

The length of a positive or negative semi-cycle with p terms is denoted by p^+ (or p^-), and the number of terms is the length of semi-cycle.

Definition 2.4. [9] *A solution* $\{z_n\}_{n=-4}^{\infty}$ *of (1.2) is eventually positive or negative if there exists a positive integer* M *for* $n > M$ *such that* $z_n - \bar{z} > 0$ *or* $z_n - \bar{z} < 0$.

Definition 2.5. [13] *If* $\{z_n\}$ *is eventually equal to* $\bar{z}$*, then the solution* $\{z_n\}_{n=-4}^{\infty}$ *of (1.2) is eventually trivial, otherwise the solution is nontrivial.*

Lemma 2.1. *A positive solution* $\{z_n\}_{n=-4}^{\infty}$ *of (1.2) is equal to 1 if and only if*

$$(z_0 - 1)(z_{-1} - 1)(z_{-2} - 1)(z_{-3} - 1)(z_{-4} - 1) = 0. \tag{2.3}$$

Proof. Suppose condition (2.3) holds. Then, the following conclusions are true.

(1) If $z_{-4} = 1$, then $z_n = 1$, for $n \geq 3$.
(2) If $z_{-3} = 1$, then $z_n = 1$, for $n \geq 4$.
(3) If $z_{-2} = 1$, then $z_n = 1$, for $n \geq 3$.
(4) If $z_{-1} = 1$, then $z_n = 1$, for $n \geq 1$.
(5) If $z_0 = 1$, then $z_n = 1$, for $n \geq 2$.

Suppose

$$(z_0 - 1)(z_{-1} - 1)(z_{-2} - 1)(z_{-3} - 1)(z_{-4} - 1) \neq 0, \tag{2.4}$$

which claims that

$$z_n \neq 1, \quad for \quad n \geq 1.$$

Suppose the opposite is true for a positive integer $N \geq 1$:

$$z_N = 1, \quad \text{and} \quad z_n \neq 1 \quad \text{for} \quad -4 \leq n \leq N - 1. \tag{2.5}$$

Obviously,

$$1 = z_N = \frac{z_{N-2}z_{N-3}z_{N-5} + z_{N-2} + z_{N-3} + z_{N-5} + b}{z_{N-2}z_{N-3} + z_{N-2}z_{N-5} + z_{N-3}z_{N-5} + 1 + b},$$

implies that

$$(z_{N-2} - 1)(z_{N-3} - 1)(z_{N-5} - 1) = 0.$$

This contradicts (2.5).

Remark 1. *A positive solution* $\{z_n\}_{n=-4}^{\infty}$ *of Eq (1.2) is eventually nontrivial if and only if* $(z_{-4} - 1)(z_{-3} - 1)(z_{-2} - 1)(z_{-1} - 1)(z_0 - 1) \neq 0$. *If* $z_n \neq 1$ *for* $n \geq -4$, *then* $\{z_n\}_{n=-4}^{\infty}$ *is nontrivial.*

Lemma 2.2. *For any nontrivial positive solution* $\{z_n\}_{n=-4}^{\infty}$ *of (1.2), these assertions are true for* $n \geq 0$.

(i) $(z_{n+1} - 1)(z_{n-1} - 1)(z_{n-2} - 1)(z_{n-4} - 1) > 0$.
(ii) $(z_{n+1} - z_{n-1})(z_{n-1} - 1) < 0$.
(iii) $(z_{n+1} - z_{n-2})(z_{n-2} - 1) < 0$.
(iv) $(z_{n+1} - z_{n-4})(z_{n-4} - 1) < 0$.

Proof. From Eq (1.2), it follows that

$$z_{n+1} - 1 = \frac{(z_{n-1} - 1)(z_{n-2} - 1)(z_{n-4} - 1)}{z_{n-1}z_{n-2} + z_{n-1}z_{n-4} + z_{n-2}z_{n-4} + 1 + b}.$$

$$z_{n+1} - z_{n-1} = \frac{(1 - z_{n-1})[z_{n-2}(1 + z_{n-1}) + z_{n-4}(1 + z_{n-1}) + b]}{z_{n-1}z_{n-2} + z_{n-1}z_{n-4} + z_{n-2}z_{n-4} + 1 + b}.$$

$$z_{n+1} - z_{n-2} = \frac{(1 - z_{n-2})[z_{n-1}(1 + z_{n-2}) + z_{n-4}(1 + z_{n-2}) + b]}{z_{n-1}z_{n-2} + z_{n-1}z_{n-4} + z_{n-2}z_{n-4} + 1 + b}.$$

$$z_{n+1} - z_{n-4} = \frac{(1 - z_{n-4})[z_{n-1}(1 + z_{n-4}) + z_{n-2}(1 + z_{n-4}) + b]}{z_{n-1}z_{n-2} + z_{n-1}z_{n-4} + z_{n-2}z_{n-4} + 1 + b}.$$

Remark 2. *If for Eq (1.2) there exists a non-oscillatory solution with initial values* $z_i \in (0,1), i = 0,-1,-2,-3,-4,$ *then the solution is eventually negative. Otherwise, it is eventually positive for initial values* $z_i \in (1,\infty)$*,* $i = 0,-1,-2,-3,-4.$

Noting Remark 2, for the solution of Eq (1.2), if $z_i > 1, i = 0,-1,-2,-3,-4,$ from the first inequality of Lemma 2.2 it follows that $z_n > 1$ for $n \geq 4$, so the solution is eventually positive. If $z_i \in (0,1), i = 0,-1,-2,-3,-4,$ then $z_n < 1$ for $n \geq 4$, so the solution is eventually negative.

Theorem 2.1. *If* $\{z_n\}_{n=-4}^{\infty}$ *is an oscillatory solution of (1.2), on account of the disturbance of initial conditions, a continuous length of positive and negative semi-cycles appears periodically in different prime periods:2,3,4,6,12. For period 2, the rule is* $1^+, 1^-$ *in a period. For period 3, the rule is* $2^-, 1^+$ *or* $2^+, 1^-$ *in a period. For period 4, the rule is* $2^+, 2^-$ *in a period. For period 6, the rule is* $3^+, 3^-$ *in a period. For period 12, the rule is* $4^+, 1^-, 1^+, 4^-, 1^+, 1^-$ *in a period.*

Proof. From the first inequality of Lemma 2.2, the length of the positive or negative semi-cycle does not exceed 4. According to Remark 1, one has that

$$(z_{-4} - 1)(z_{-3} - 1)(z_{-2} - 1)(z_{-1} - 1)(z_0 - 1) \neq 0.$$

Assuming $q \in N$, at least one of the following eight cases occurs.

Case (i): $z_{q-4} < 1, z_{q-3} > 1, z_{q-2} > 1, z_{q-1} > 1, z_q > 1.$

Case (ii): $z_{q-4} < 1, z_{q-3} > 1, z_{q-2} > 1, z_{q-1} > 1, z_q < 1.$

Case (iii): $z_{q-4} < 1, z_{q-3} > 1, z_{q-2} > 1, z_{q-1} < 1, z_q > 1.$

Case (iv): $z_{q-4} < 1, z_{q-3} > 1, z_{q-2} > 1, z_{q-1} < 1, z_q < 1.$

Case (v): $z_{q-4} < 1, z_{q-3} > 1, z_{q-2} < 1, z_{q-1} > 1, z_q > 1.$

Case (vi): $z_{q-4} < 1, z_{q-3} > 1, z_{q-2} < 1, z_{q-1} > 1, z_q < 1.$

Case (vii): $z_{q-4} < 1, z_{q-3} > 1, z_{q-2} < 1, z_{q-1} < 1, z_q < 1.$

Case (viii): $z_{q-4} < 1, z_{q-3} > 1, z_{q-2} < 1, z_{q-1} < 1, z_q > 1.$

If Case (i) takes place, from the first inequality of Lemma 2.2 it follows that $z_{q+1} < 1, z_{q+2} > 1, z_{q+3} < 1, z_{q+4} < 1, z_{q+5} < 1, z_{q+6} < 1, z_{q+7} > 1, z_{q+8} < 1, z_{q+9} > 1, z_{q+10} > 1, z_{q+11} > 1, z_{q+12} > 1, z_{q+13} < 1, z_{q+14} > 1, z_{q+15} < 1, z_{q+16} < 1, z_{q+17} < 1, z_{q+18} < 1, z_{q+19} > 1, z_{q+20} < 1, \cdots$. So, the positive and negative semi-cycles appear continuously, and the rule is $\cdots, 4^+, 1^-, 1^+, 4^-, 1^+, 1^-, 4^+, 1^-, 1^+, 4^-, 1^+, 1^-, \cdots$.

If Case (ii) takes place, from the first inequality of Lemma 2.2, $z_{q+1} < 1, z_{q+2} < 1, z_{q+3} > 1, z_{q+4} > 1, z_{q+5} > 1, z_{q+6} < 1, z_{q+7} < 1, z_{q+8} < 1, z_{q+9} > 1, z_{q+10} > 1, z_{q+11} > 1, z_{q+12} < 1, z_{q+13} < 1, z_{q+14} < 1, z_{q+15} > 1, z_{q+16} > 1, z_{q+17} > 1, \cdots$. So, the positive and negative semi-cycles appear continuously, and the rule is $\cdots, 3^+, 3^-, 3^+, 3^-, \cdots$.

If Case (iii) takes place, from the first inequality of Lemma 2.2, $z_{q+1} > 1, z_{q+2} < 1, z_{q+3} > 1, z_{q+4} > 1, z_{q+5} < 1, z_{q+6} > 1, z_{q+7} > 1, z_{q+8} < 1, z_{q+9} > 1, z_{q+10} > 1, z_{q+11} < 1, z_{q+12} > 1, z_{q+13} > 1, z_{q+14} <$

$1, z_{q+15} > 1, z_{q+16} > 1, z_{q+17} < 1, \cdots$. So, the positive and negative semi-cycles appear continuously, and the rule is $\cdots, 2^+, 1^-, 2^+, 1^-, \cdots$.

If Case (iv) takes place, from the first inequality of Lemma 2.2, $z_{q+1} > 1, z_{q+2} > 1, z_{q+3} < 1, z_{q+4} < 1, z_{q+5} > 1, z_{q+6} > 1, z_{q+7} < 1, z_{q+8} < 1, z_{q+9} > 1, z_{q+10} > 1, z_{q+11} < 1, z_{q+12} < 1, z_{q+13} > 1, z_{q+14} > 1, z_{q+15} < 1, z_{q+16} < 1, z_{q+17} > 1, z_{q+18} > 1, \cdots$. So, the positive and negative semi-cycles appear continuously, and the rule is $\cdots, 2^+, 2^-, 2^+, 2^-, \cdots$.

If Case (v) takes place, from the first inequality of Lemma 2.2, $z_{q+1} > 1, z_{q+2} > 1, z_{q+3} < 1, z_{q+4} > 1, z_{q+5} < 1, z_{q+6} < 1, z_{q+7} < 1, z_{q+8} < 1, z_{q+9} > 1, z_{q+10} < 1, z_{q+11} > 1, z_{q+12} > 1, z_{q+13} > 1, z_{q+14} > 1, z_{q+15} < 1, z_{q+16} > 1, z_{q+17} < 1, z_{q+18} < 1, z_{q+19} < 1, z_{q+20} < 1, \cdots$. So, the positive and negative semi-cycles appear continuously, and the rule is $\cdots, 4^+, 1^-, 1^+, 4^-, 1^+, 1^-, 4^+, 1^-, 1^+, 4^-, 1^+, 1^-, \cdots$.

If Case (vi) takes place, from the first inequality of Lemma 2.2, $z_{q+1} > 1, z_{q+2} < 1, z_{q+3} > 1, z_{q+4} < 1, z_{q+5} > 1, z_{q+6} < 1, z_{q+7} > 1, z_{q+8} < 1, z_{q+9} > 1, z_{q+10} < 1, z_{q+11} > 1, z_{q+12} < 1, z_{q+13} > 1, z_{q+14} < 1, z_{q+15} > 1, z_{q+16} < 1, z_{q+17} > 1, z_{q+18} < 1, z_{q+19} > 1, z_{q+20} < 1, \cdots$. So, the positive and negative semi-cycles appear continuously, and the rule is $\cdots, 1^+, 1^-, 1^+, 1^-, \cdots$.

If Case (vii) takes place, from the first inequality of Lemma 2.2, $z_{q+1} < 1, z_{q+2} > 1, z_{q+3} < 1, z_{q+4} > 1, z_{q+5} > 1, z_{q+6} > 1, z_{q+7} > 1, z_{q+8} < 1, z_{q+9} > 1, z_{q+10} < 1, z_{q+11} < 1, z_{q+12} < 1, z_{q+13} < 1, z_{q+14} > 1, z_{q+15} < 1, z_{q+16} > 1, z_{q+17} > 1, z_{q+18} > 1, z_{q+19} > 1, \cdots$. So, the positive and negative semi-cycles appear continuously, and the rule is $\cdots, 4^+, 1^-, 1^+, 4^-, 1^+, 1^-, 4^+, 1^-, 1^+, 4^-, 1^+, 1^-, \cdots$.

If Case (viii) takes place, from the first inequality of Lemma 2.2, $z_{q+1} < 1, z_{q+2} < 1, z_{q+3} > 1, z_{q+4} < 1, z_{q+5} < 1, z_{q+6} > 1, z_{q+7} < 1, z_{q+8} < 1, z_{q+9} > 1, z_{q+10} < 1, z_{q+11} < 1, z_{q+12} > 1, z_{q+13} < 1, z_{q+14} < 1, z_{q+15} > 1, z_{q+16} < 1, z_{q+17} < 1, z_{q+18} > 1, \cdots$. So, the positive and negative semi-cycles appear continuously, and the rule is $\cdots, 2^-, 1^+, 2^-, 1^+, \cdots$.

3. Main result

In this section, we derive the global asymptotical stability of the unique positive equilibrium point.

Theorem 3.1. *The equilibrium point $\bar{z} = 1$ of (1.2) is GAS.*

Proof. In fact, it need to show that the equilibrium point $\bar{z} = 1$ of (1.2) is LAS and a global attractor. The linearized equation of (1.2) at the equilibrium point $\bar{z} = 1$ is

$$z_{n+1} = 0 \times z_n + 0 \times z_{n-1} + 0 \times z_{n-2} + 0 \times z_{n-3} + 0 \times z_{n-4}, \qquad n \in N,$$

so the equilibrium point $\bar{z} = 1$ is stable. Now, we need to prove that

$$\lim_{n\to\infty} z_n = \bar{z} = 1. \tag{3.1}$$

If the solution $\{z_n\}_{n=-4}^{\infty}$ of (1.2) is a trivial solution, by Definition 2.5 it is clear that $\lim_{n\to\infty} z_n = \bar{z} = 1$. If the solution $\{z_n\}_{n=-4}^{\infty}$ of (1.2) is a nontrivial solution, then there are two cases.

Case (i): Non-oscillatory solution.

Case (ii): Oscillatory solution.

If Case (i) occurs, noting Remark 2, for the initial conditions $z_i \in (0, 1), i = 0, -1, -2, -3, -4$, the solution $\{z_n\}_{n=-4}^{\infty}$ must be eventually negative. Namely, there exists an integer $N \geq 1$ such that $z_n < 1$ for $n \geq N$.

From the second inequality of Lemma 2.2, the solution $\{z_n\}_{n=-4}^{\infty}$ has two subsequences $\{z_{2n+1}\}$ and $\{z_{2n+2}\}, n \in N$, which are increasing and have upper bound 1, so $\lim_{n\to\infty} z_{2n+1}$ and $\lim_{n\to\infty} z_{2n+2}$ exist.

Suppose that $\lim_{n\to\infty} z_{2n+1} = M$ and $\lim_{n\to\infty} z_{2n+2} = H$. According to Eq (1.2),

$$z_{2n+1} = \frac{z_{2n-1}z_{2n-2}z_{2n-4} + z_{2n-1} + z_{2n-2} + z_{2n-4} + b}{z_{2n-1}z_{2n-2} + z_{2n-1}z_{2n-4} + z_{2n-2}z_{2n-4} + 1 + b}, \tag{3.2}$$

$$z_{2n+2} = \frac{z_{2n}z_{2n-1}z_{2n-3} + z_{2n} + z_{2n-1} + z_{2n-3} + b}{z_{2n}z_{2n-1} + z_{2n}z_{2n-3} + z_{2n}z_{2n-3} + 1 + b}. \tag{3.3}$$

Taking the limit on both sides of (3.2) and (3.3), we have

$$M = \frac{MH^2 + M + H + H + b}{MH + MH + H^2 + 1 + b} \quad \text{and} \quad H = \frac{HM^2 + H + M + M + b}{HM + HM + M^2 + 1 + b}.$$

From which it follows that

$$M = 1 \quad \text{and} \quad H = 1,$$

so $\lim_{n\to\infty} z_n = \bar{z} = 1$.

For the initial conditions $z_i \in (1, \infty), i = 0, -1, -2, -3, -4$, the solution $\{z_n\}_{n=-4}^{\infty}$ is eventually positive. Namely, there exists an integer $N \geq 1$ such that $z_n > 1$ for $n \geq N$, so the solution $\{z_n\}_{n=-4}^{\infty}$ has two subsequences $\{z_{2n+1}\}$ and $\{z_{2n+2}\}$ which are decreasing and have lower bound 1, hence $\lim_{n\to\infty} z_{2n+1}$ and $\lim_{n\to\infty} z_{2n+2}$ exist.

Suppose that $\lim_{n\to\infty} z_{2n+1} = G$ and $\lim_{n\to\infty} z_{2n+2} = T$. Then,

$$z_{2n+1} = \frac{z_{2n-1}z_{2n-2}z_{2n-4} + z_{2n-1} + z_{2n-2} + z_{2n-4} + b}{z_{2n-1}z_{2n-2} + z_{2n-1}z_{2n-4} + z_{2n-2}z_{2n-4} + 1 + b}, \tag{3.4}$$

$$z_{2n+2} = \frac{z_{2n}z_{2n-1}z_{2n-3} + z_{2n} + z_{2n-1} + z_{2n-3} + b}{z_{2n}z_{2n-1} + z_{2n}z_{2n-3} + z_{2n-1}z_{2n-3} + 1 + b}. \tag{3.5}$$

Taking the limit on both sides of (3.4) and (3.5), we have

$$G = \frac{GT^2 + G + T + T + b}{GT + GT + T^2 + 1 + b} \quad \text{and} \quad T = \frac{TG^2 + T + G + G + b}{TG + TG + G^2 + 1 + b}.$$

From which it follows that

$$G = 1 \quad \text{and} \quad T = 1,$$

so $\lim_{n\to\infty} z_n = \bar{z} = 1$.

If Case (ii) occurs and $\{z_n\}$ is an oscillatory solution, then the continuous length of positive and negative semi-cycles appears periodically in different prime periods: 2,3,4,6,12. The rule is

Subcase (i) $\cdots, 1^+, 1^-, 1^+, 1^-, \cdots$.
Subcase (ii) $\cdots, 2^+, 1^-, 2^+, 1^-, \cdots$.
Subcase (iii) $\cdots, 2^-, 1^+, 2^-, 1^+, \cdots$.
Subcase (iv) $\cdots, 4^+, 1^-, 1^+, 4^-, 1^+, 1^-, 4^+, 1^-, 1^+, 4^-, 1^+, 1^-, \cdots$.
Subcase (v) $\cdots, 2^+, 2^-, 2^+, 2^-, \cdots$.
Subcase (vi) $\cdots, 3^+, 3^-, 3^+, 3^-, \cdots$.

Considering Subcase (i), for integer $q \geq 0$, the terms of a positive semi-cycle of length one is denoted by $\{z_q\}^+$ and the negative semi-cycle of length one is denoted by $\{z_{q+1}\}^-$. Since the positive and negative semi-cycles appear continuously, the rule is $\{z_{q+2n}\}^+, \{z_{q+2n+1}\}^-, n \in N$.

By the second inequality of Lemma 2.2, there is $z_{q+2n+2} < z_{q+2n}$, $z_{q+2n+1} < z_{q+2n+3}$, from which it can be inferred that the sequence $\{z_{q+2n}\}_{n=0}^{\infty}$ is decreasing and has lower bound 1 and the sequence $\{z_{q+2n+1}\}_{n=0}^{\infty}$ is increasing and has upper bound 1. Therefore, the limits of two sequences exist.

Suppose that $\lim_{n\to\infty} z_{q+2n} = W$ and $\lim_{n\to\infty} z_{q+2n+1} = G$. Then,

$$z_{q+2n} = \frac{z_{q+2n-2}z_{q+2n-3}z_{q+2n-5} + z_{q+2n-2} + z_{q+2n-3} + z_{q+2n-5} + b}{z_{q+2n-2}z_{q+2n-3} + z_{q+2n-2}z_{q+2n-5} + z_{q+2n-3}z_{q+2n-5} + 1 + b}, \tag{3.6}$$

$$z_{q+2n+1} = \frac{z_{q+2n-1}z_{q+2n-2}z_{q+2n-4} + z_{q+2n-1} + z_{q+2n-2} + z_{q+2n-4} + b}{z_{q+2n-1}z_{q+2n-2} + z_{q+2n-1}z_{q+2n-4} + z_{q+2n-2}z_{q+2n-4} + 1 + b}. \tag{3.7}$$

Taking the limit on both sides of (3.6) and (3.7), one has

$$W = \frac{WG^2 + W + G + G + b}{WG + WG + G^2 + 1 + b} \quad \text{and} \quad G = \frac{GW^2 + G + W + W + b}{GW + GW + W^2 + 1 + b}.$$

From which it follows that

$$W = 1 \quad \text{and} \quad G = 1,$$

so $\lim_{n\to\infty} z_n = \bar{z} = 1$.

Considering Subcase (ii), for integer $q \geq 0$, the terms of a positive semi-cycle of length two are denoted by $\{z_q, z_{q+1}\}^+$ and the negative semi-cycle of length one is denoted by $\{z_{q+2}\}^-$. Due to the positive and negative semi-cycles appearing continuously, the rule is $\{z_{q+3n}, z_{q+3n+1}\}^+, \{z_{q+3n+2}\}^-, n \in N$.

From the third inequality of Lemma 2.2, it follows that $z_{q+3n+3} < z_{q+3n}$, $z_{q+3n+4} < z_{q+3n+1}$, $z_{q+3n+2} < z_{q+3n+5}$, so the sequences $\{z_{q+3n}\}_{n=0}^{\infty}$ and $\{z_{q+3n+1}\}_{n=0}^{\infty}$ are decreasing and have lower bound 1 and the sequence $\{z_{q+3n+2}\}_{n=0}^{\infty}$ is increasing and has upper bound 1. Then, the limits of the three sequences exist.

Suppose that $\lim_{n\to\infty} z_{q+3n} = W$, $\lim_{n\to\infty} z_{q+3n+1} = E$, and $\lim_{n\to\infty} z_{q+3n+2} = D$. Then,

$$z_{q+3n} = \frac{z_{q+3n-2}z_{q+2n-3}z_{q+2n-5} + z_{q+2n-2} + z_{q+2n-3} + z_{q+2n-5} + b}{z_{q+2n-2}z_{q+2n-3} + z_{q+2n-2}z_{q+2n-5} + z_{q+2n-3}z_{q+2n-5} + 1 + b}, \tag{3.8}$$

$$z_{q+3n+1} = \frac{z_{q+3n-1}z_{q+3n-2}z_{q+3n-4} + z_{q+3n-1} + z_{q+3n-2} + z_{q+3n-4} + b}{z_{q+3n-1}z_{q+3n-2} + z_{q+3n-1}z_{q+3n-4} + z_{q+3n-2}z_{q+3n-4} + 1 + b}, \tag{3.9}$$

$$z_{q+3n+2} = \frac{z_{q+3n}z_{q+3n-1}z_{q+3n-3} + z_{q+3n} + z_{q+3n-1} + z_{q+3n-3} + b}{z_{q+3n}z_{q+3n-1} + z_{q+3n}z_{q+3n-3} + z_{q+3n-1}z_{q+3n-3} + 1 + b}. \tag{3.10}$$

Taking the limit on the both sides of (3.8)–(3.10), we have

$$W = \frac{EWE + E + W + E + b}{EW + E^2 + WE + 1 + b}, \qquad E = \frac{DED + D + E + D + b}{DE + D^2 + ED + 1 + b},$$

$$D = \frac{WDW + W + D + W + b}{WD + W^2 + DW + 1 + b}.$$

From which it follows that

$$W = 1, \quad E = 1 \quad \text{and} \quad D = 1,$$

so $\lim_{n\to\infty} z_n = \bar{z} = 1$. The proof of Subcase (iii) is similar to that of Subcase (ii).

Considering Subcase (iv), for integer $q \geq 0$, the terms of a positive semi-cycle of length four are denoted by $\{z_q, z_{q+1}, z_{q+2}, z_{q+3}\}^+$, the negative semi-cycle of length one by $\{z_{q+4}\}^-$, the term of a positive semi-cycle of length one by $\{z_{q+5}\}^+$, a negative semi-cycle of length four is denoted by $\{z_{q+6}, z_{q+7}, z_{q+8}, z_{q+9}\}^-$, the terms of a positive semi-cycle of length one by $\{z_{q+10}\}^+$, and a negative semi-cycle of length one by $\{z_{q+11}\}^-$. With the positive and negative semi-cycles appearing continuously, we can see the rule is as follows: $\{z_{q+12n}, z_{q+12n+1}, z_{q+12n+2}, z_{q+12n+3}\}^+, \{z_{q+12n+4}\}^-, \{z_{q+12n+5}\}^+, \{z_{q+12n+6}, z_{q+12n+7}, z_{q+12n+8}, z_{q+12n+9}\}^-, \{z_{q+12n+10}\}^+, \{z_{q+12n+11}\}^-, n \in N$.

By the second, third, and fourth inequalities of Lemma 2.2, the following inequalities are true.

(1b) $z_{q+12n} > z_{q+12n+2} > z_{q+12n+5} > z_{q+12n+10} > z_{q+12n+12} > z_{q+12n+14} > z_{q+12n+17} > z_{q+12n+22}$.

(2b) $z_{q+12n+1} > z_{q+12n+3} > z_{q+12n+5} > z_{q+12n+10} > z_{q+12n+13} > z_{q+12n+15} > z_{q+12n+17} > z_{q+12n+22}$.

(3b) $z_{q+12n+4} < z_{q+12n+6} < z_{q+12n+8} < z_{q+12n+11} < z_{q+12n+16} < z_{q+12n+18} < z_{q+12n+20} < z_{q+12n+23}$.

(4b) $z_{q+12n+7} < z_{q+12n+9} < z_{q+12n+11} < z_{q+12n+16} < z_{q+12n+19} < z_{q+12n+21}$.

According to (1b) and (2b), it follows that the subsequences

$$\{z_{q+12n}\}_{n=0}^{\infty}, \{z_{q+12n+2}\}_{n=0}^{\infty}, \{z_{q+12n+5}\}_{n=0}^{\infty}, \{z_{q+12n+10}\}_{n=0}^{\infty},$$

and

$$\{z_{q+12n+1}\}_{n=0}^{\infty}, \{z_{q+12n+3}\}_{n=0}^{\infty}, \{z_{q+12n+5}\}_{n=0}^{\infty}, \{z_{q+12n+10}\}_{n=0}^{\infty},$$

are monotonically decreasing and have lower bound 1, so the limits exist.

Let

$$\lim_{n\to\infty} z_{q+12n} = \lim_{n\to\infty} z_{q+12n+2} = \lim_{n\to\infty} z_{q+12n+5} = \lim_{n\to\infty} z_{q+12n+10} = E, \tag{3.11}$$

$$\lim_{n\to\infty} z_{q+12n+1} = \lim_{n\to\infty} z_{q+12n+3} = \lim_{n\to\infty} z_{q+12n+5} = \lim_{n\to\infty} z_{q+12n+10} = Q. \tag{3.12}$$

It follows that $E = Q$.

Since

$$z_{q+12n+5} = \frac{z_{q+12n+3} z_{q+12n+2} z_{q+12n} + z_{q+12n+3} + z_{q+12n+2} + z_{q+12n} + b}{z_{q+12n+3} z_{q+12n+2} + z_{q+12n+3} z_{q+12n} + z_{q+12n+2} z_{q+12n} + 1 + b},$$

from (3.11) and (3.12), we have

$$E = \frac{QEE + Q + E + E + b}{QE + QE + EE + 1 + b}.$$

It follows that $E = Q = 1$.

From (3b) and (4b), one has that the subsequences

$$\{z_{q+12n+4}\}_{n=0}^{\infty}, \{z_{q+12n+6}\}_{n=0}^{\infty}, \{z_{q+12n+8}\}_{n=0}^{\infty}, \{z_{q+12n+11}\}_{n=0}^{\infty},$$

and

$$\{z_{q+12n+7}\}_{n=0}^{\infty}, \{z_{q+12n+9}\}_{n=0}^{\infty}, \{z_{q+12n+11}\}_{n=0}^{\infty},$$

are monotonically increasing and have upper bound 1, hence the limits exist.

Let

$$\lim_{n\to\infty} z_{q+12n+4} = \lim_{n\to\infty} z_{q+12n+6} = \lim_{n\to\infty} z_{q+12n+8} = \lim_{n\to\infty} z_{q+12n+11} = D, \tag{3.13}$$

$$\lim_{n\to\infty} z_{q+12n+7} = \lim_{n\to\infty} z_{q+12n+9} = \lim_{n\to\infty} z_{q+12n+11} = P. \tag{3.14}$$

It follows that $D = P$.

Since

$$z_{q+12n+7} = \frac{z_{q+12n+5}z_{q+12n+4}z_{q+12n+2} + z_{q+12n+5} + z_{q+12n+4} + z_{q+12n+2} + b}{z_{q+12n+5}z_{q+12n+4} + z_{q+12n+5}z_{q+12n+2} + z_{q+12n+4}z_{q+12n+2} + 1 + b},$$

from (3.13) and (3.14), we have

$$P = \frac{EDE + E + D + E + b}{ED + EE + DE + 1 + b}.$$

Since $E = Q = 1$, $P = D = 1$. Namely, $E = Q = P = D = 1$, so $\lim_{n\to\infty} z_n = \bar{z} = 1$.

Considering Subcase (v), for integer $q \geq 0$, the terms of a positive semi-cycle of length two are denoted by $\{z_q, z_{q+1}\}^+$, and the negative semi-cycle of length two by $\{z_{q+2}, z_{q+3}\}^-$. As the positive and negative semi-cycles appear continuously, the rule is $\{z_{q+4n}, z_{q+4n+1}\}^+, \{z_{q+4n+2}, z_{q+4n+3}\}^-,\ n \in N$.

From the third and fourth inequalities of Lemma 2.2, for positive integers k, g, h, and p, one has:

(1c)

$$\begin{aligned} z_{q+4n} &> z_{q+4n+5} > z_{q+4n+8\times1+0} > z_{q+4n+8\times1+5} > z_{q+4n+8\times2+0} \\ &> z_{q+4n+8\times2+5} > \cdots > z_{q+4n+8\times k+0} > z_{q+4n+8\times k+5} > \cdots > 1. \end{aligned}$$

(2c)

$$\begin{aligned} z_{q+4n+1} &> z_{q+4n+4} > z_{q+4n+8\times1+1} > z_{q+4n+8\times1+4} > z_{q+4n+8\times2+1} \\ &> z_{q+4n+8\times2+4} > \cdots > z_{q+4n+8\times g+1} > z_{q+4n+8\times g+4} > \cdots > 1. \end{aligned}$$

(3c)

$$\begin{aligned} z_{q+4n+2} &< z_{q+4n+7} < z_{q+4n+8\times1+2} < z_{q+4n+8\times1+7} < z_{q+4n+8\times2+2} \\ &< z_{q+4n+8\times2+7} < \cdots < z_{q+4n+8\times h+2} < z_{q+4n+8\times h+7} < \cdots < 1. \end{aligned}$$

(4c)

$$\begin{aligned} z_{q+4n+3} &< z_{q+4n+6} < z_{q+4n+8\times1+3} < z_{q+4n+8\times1+6} < z_{q+4n+8\times2+3} \\ &< z_{q+4n+8\times2+6} < \cdots < z_{q+4n+8\times p+3} < z_{q+4n+8\times p+6} < \cdots < 1. \end{aligned}$$

From (1c) and (2c), we have that the two subsequences of the positive semi-cycles are monotonically decreasing and have lower bound 1, so the limits of two subsequences exist, denoted as M and W. Similarly, from (3c) and (4c), the two subsequences of the negative semi-cycles are monotonically increasing and have upper bound 1, so the limits of two subsequences exist, denoted as Q and L. In other words, the following limits can be obtained:

$$\lim_{n\to\infty} z_{q+4n} = \lim_{n\to\infty} z_{q+4n+5} = \lim_{n\to\infty} z_{q+4n+8} = \lim_{n\to\infty} z_{q+4n+13} = \lim_{n\to\infty} z_{q+4n+16} = M. \tag{3.15}$$

$$\lim_{n\to\infty} z_{q+4n+1} = \lim_{n\to\infty} z_{q+4n+4} = \lim_{n\to\infty} z_{q+4n+9} = \lim_{n\to\infty} z_{q+4n+12} = \lim_{n\to\infty} z_{q+4n+17} = W. \tag{3.16}$$

$$\lim_{n\to\infty} z_{q+4n+2} = \lim_{n\to\infty} z_{q+4n+7} = \lim_{n\to\infty} z_{q+4n+10} = \lim_{n\to\infty} z_{q+4n+15} = \lim_{n\to\infty} z_{q+4n+18} = Q. \tag{3.17}$$

$$\lim_{n\to\infty} z_{q+4n+3} = \lim_{n\to\infty} z_{q+4n+6} = \lim_{n\to\infty} z_{q+4n+11} = \lim_{n\to\infty} z_{q+4n+14} = \lim_{n\to\infty} z_{q+4n+19} = L. \tag{3.18}$$

Utilizing Eq (1.2), we have that

$$z_{q+4n+16} = \frac{z_{q+4n+14}z_{q+4n+13}z_{q+4n+11} + z_{q+4n+14} + z_{q+4n+13} + z_{q+4n+11} + b}{z_{q+4n+14}z_{q+4n+13} + z_{q+4n+14}z_{q+4n+11} + z_{q+4n+13}z_{q+4n+11} + 1 + b},$$

$$z_{q+4n+12} = \frac{z_{q+4n+10}z_{q+4n+9}z_{q+4n+7} + z_{q+4n+10} + z_{q+4n+9} + z_{q+4n+7} + b}{z_{q+4n+10}z_{q+4n+19} + z_{q+4n+10}z_{q+4n+7} + z_{q+4n+9}z_{q+4n+7} + 1 + b},$$

$$z_{q+4n+10} = \frac{z_{q+4n+8}z_{q+4n+7}z_{q+4n+5} + z_{q+4n+8} + z_{q+4n+7} + z_{q+4n+5} + b}{z_{q+4n+8}z_{q+4n+7} + z_{q+4n+8}z_{q+4n+5} + z_{q+4n+7}z_{q+4n+5} + 1 + b},$$

$$z_{q+4n+11} = \frac{z_{q+4n+9}z_{q+4n+8}z_{q+4n+6} + z_{q+4n+9} + z_{q+4n+8} + z_{q+4n+6} + b}{z_{q+4n+9}z_{q+4n+8} + z_{q+4n+9}z_{q+4n+6} + z_{q+4n+8}z_{q+4n+6} + 1 + b}.$$

From (3.15)–(3.18), we have

$$M = \frac{LML + L + M + L + b}{LM + LL + ML + 1 + b}, \qquad W = \frac{QWQ + Q + W + Q + b}{QW + QQ + WQ + 1 + b},$$

$$Q = \frac{MQM + M + Q + M + b}{MQ + MM + QM + 1 + b}, \qquad L = \frac{WML + W + M + L + b}{WM + WL + ML + 1 + b}.$$

By simple calculations, we have that

$$Q = L = M = W = 1.$$

So, $\lim_{n\to\infty} z_n = \bar{z} = 1$.

Considering Subcase (vi), for integer $q \geq 0$, the terms of a positive semi-cycle of length three are denoted by $\{z_q, z_{q+1}, z_{q+2}\}^+$, and the negative semi-cycle of length three by $\{z_{q+3}, z_{q+4}, z_{q+5}\}^-$. Because the positive and negative semi-cycles appear continuously, the rule is $\{z_{q+6n}, z_{q+6n+1}, z_{q+6n+2}\}^+$, $\{z_{q+6n+3}, z_{q+6n+4}, z_{q+6n+5}\}^-$.

From the second, third, and fourth inequalities of Lemma 2.2, for some nonnegative integers k, g, h, p, t, and m, we have

(1d)

$$\begin{aligned} z_{q+6n} &> z_{q+6n+2} > z_{q+6n+7} > z_{q+6n+12\times1+0} > z_{q+6n+12\times1+2} > z_{q+6n+12\times1+7} \\ &> z_{q+6n+12\times2+0} > z_{q+6n+12\times2+2} > z_{q+6n+12\times2+7} > \cdots > z_{q+6n+12\times k+0} \\ &> z_{q+6n+12\times k+2} > z_{q+6n+12\times k+7} > \cdots > 1. \end{aligned}$$

(2d)

$$\begin{aligned} z_{q+6n+1} &> z_{q+6n+6} > z_{q+6n+8} > z_{q+6n+12\times1+1} > z_{q+6n+12\times1+6} > z_{q+6n+12\times1+8} \\ &> z_{q+6n+12\times2+1} > z_{q+6n+12\times2+6} > z_{q+6n+12\times2+8} > \cdots > z_{q+6n+12\times g+1} \\ &> z_{q+6n+12\times g+6} > z_{q+6n+12\times g+8} > \cdots > 1. \end{aligned}$$

(3d)

$$z_{q+6n+2} > z_{q+6n+7} > z_{q+6n+12} > z_{q+6n+12\times1+2} > z_{q+6n+12\times1+7} > z_{q+6n+12\times1+12}$$

$$> z_{q+6n+12\times2+2} > z_{q+6n+12\times2+7} > z_{q+6n+12\times2+12} > \cdots > z_{q+6n+12\times h+2}$$
$$> z_{q+6n+12\times h+7} > z_{q+6n+12\times h+12} > \cdots > 1.$$

(4d)

$$\begin{aligned} z_{q+6n+3} &< z_{q+6n+5} < z_{q+6n+10} < z_{q+6n+12\times1+3} < z_{q+6n+12\times1+5} < z_{q+6n+12\times1+10} \\ &< z_{q+6n+12\times2+3} < z_{q+6n+12\times2+5} < z_{q+6n+12\times2+10} < \cdots < z_{q+6n+12\times p+3} \\ &< z_{q+6n+12\times p+5} < z_{q+6n+12\times p+10} < \cdots < 1. \end{aligned}$$

(5d)

$$\begin{aligned} z_{q+6n+4} &< z_{q+6n+9} < z_{q+6n+11} < z_{q+6n+12\times1+4} < z_{q+6n+12\times1+9} < z_{q+6n+12\times1+11} \\ &< z_{q+6n+12\times2+4} < z_{q+6n+12\times2+9} < z_{q+6n+12\times2+11} < \cdots < z_{q+6n+12\times t+4} \\ &< z_{q+6n+12\times t+9} < z_{q+6n+12\times t+11} < \cdots < 1. \end{aligned}$$

(6d)

$$\begin{aligned} z_{q+6n+5} &< z_{q+6n+10} < z_{q+6n+15} < z_{q+6n+12\times1+5} < z_{q+6n+12\times1+10} < z_{q+6n+12\times1+15} \\ &< z_{q+6n+12\times2+5} < z_{q+6n+12\times2+10} < z_{q+6n+12\times2+15} < \cdots < z_{q+6n+12\times m+5} \\ &< z_{q+6n+12\times m+10} < z_{q+6n+12\times m+15} < \cdots < 1. \end{aligned}$$

According to (1d)–(3d), the three subsequences of the positive semi-cycles are monotonically decreasing and have lower bound 1, hence the limits of the three subsequences exist, denoted by M,W, and B, repectively. Similarly, from (4d)–(6d), we have that three subsequences of the negative semi-cycles are monotonically increasing and have upper bound 1, hence the limits of the three subsequences exist, denoted by H, L, and V, repectively. That is,

$$\lim_{n\to\infty} z_{q+6n} = \lim_{n\to\infty} z_{q+6n+2} = \lim_{n\to\infty} z_{q+6n+7} = \lim_{n\to\infty} z_{q+6n+12} = \lim_{n\to\infty} z_{q+6n+14} = M. \tag{3.19}$$

$$\lim_{n\to\infty} z_{q+6n+1} = \lim_{n\to\infty} z_{q+6n+6} = \lim_{n\to\infty} z_{q+6n+8} = \lim_{n\to\infty} z_{q+6n+13} = \lim_{n\to\infty} z_{q+6n+18} = W. \tag{3.20}$$

$$\lim_{n\to\infty} z_{q+6n+2} = \lim_{n\to\infty} z_{q+6n+7} = \lim_{n\to\infty} z_{q+6n+12} = \lim_{n\to\infty} z_{q+6n+14} = \lim_{n\to\infty} z_{q+6n+19} = B. \tag{3.21}$$

$$\lim_{n\to\infty} z_{q+6n+3} = \lim_{n\to\infty} z_{q+6n+5} = \lim_{n\to\infty} z_{q+6n+10} = \lim_{n\to\infty} z_{q+6n+15} = \lim_{n\to\infty} z_{q+6n+17} = H. \tag{3.22}$$

$$\lim_{n\to\infty} z_{q+6n+4} = \lim_{n\to\infty} z_{q+6n+9} = \lim_{n\to\infty} z_{q+6n+11} = \lim_{n\to\infty} z_{q+6n+16} = \lim_{n\to\infty} z_{q+6n+21} = L. \tag{3.23}$$

$$\lim_{n\to\infty} z_{q+6n+5} = \lim_{n\to\infty} z_{q+6n+10} = \lim_{n\to\infty} z_{q+6n+15} = \lim_{n\to\infty} z_{q+6n+17} = \lim_{n\to\infty} z_{q+6n+22} = V. \tag{3.24}$$

From Eq (1.2), we have

$$z_{q+6n+12} = \frac{z_{q+6n+10}z_{q+6n+9}z_{q+6n+7} + z_{q+6n+10} + z_{q+6n+9} + z_{q+6n+7} + b}{z_{q+6n+10}z_{q+6n+9} + z_{q+6n+10}z_{q+6n+7} + z_{q+6n+9}z_{q+6n+7} + 1 + b}.$$

$$z_{q+6n+13} = \frac{z_{q+6n+11}z_{q+6n+10}z_{q+6n+8} + z_{q+6n+11} + z_{q+6n+10} + z_{q+6n+8} + b}{z_{q+6n+11}z_{q+6n+10} + z_{q+6n+11}z_{q+6n+8} + z_{q+6n+10}z_{q+6n+8} + 1 + b}.$$

$$z_{q+6n+14}=\frac{z_{q+6n+12}z_{q+6n+11}z_{q+6n+9}+z_{q+6n+12}+z_{q+6n+11}+z_{q+6n+9}+b}{z_{q+6n+12}z_{q+6n+11}+z_{q+6n+12}z_{q+6n+9}+z_{q+6n+11}z_{q+6n+9}+1+b}.$$

$$z_{q+6n+15}=\frac{z_{q+6n+13}z_{q+6n+12}z_{q+6n+10}+z_{q+6n+13}+z_{q+6n+12}+z_{q+6n+10}+b}{z_{q+6n+13}z_{q+6n+12}+z_{q+6n+13}z_{q+6n+10}+z_{q+6n+12}z_{q+6n+10}+1+b}.$$

$$z_{q+6n+11}=\frac{z_{q+6n+9}z_{q+6n+8}z_{q+6n+6}+z_{q+6n+9}+z_{q+6n+8}+z_{q+6n+6}+b}{z_{q+6n+9}x_{q+6n+8}+z_{q+6n+9}z_{q+6n+6}+z_{q+6n+8}z_{q+6n+6}+1+b}.$$

$$z_{q+6n+10}=\frac{z_{q+6n+8}z_{q+6n+7}z_{q+6n+5}+z_{q+6n+8}+z_{q+6n+7}+z_{q+6n+5}+b}{z_{q+6n+8}x_{q+6n+7}+z_{q+6n+8}z_{q+6n+5}+z_{q+6n+7}z_{q+6n+5}+1+b}.$$

From (3.19)–(3.24) and the above equations, we have

$$M=\frac{HLM+H+L+M+b}{HL+HM+LM+1+b},\qquad W=\frac{LHW+L+H+W+b}{LH+LW+HW+1+b},$$

$$B=\frac{BLL+B+L+L+b}{BL+BL+LL+1+b},\qquad H=\frac{WBH+W+B+H+b}{WB+WH+BH+1+b},$$

$$L=\frac{LWW+L+W+W+b}{LW+LW+WW+1+b},\qquad V=\frac{WBH+W+B+H+b}{WB+WH+BH+1+b}.$$

By simple calculations, one has $M=W=B=H=L=V=1$, so $\lim_{n\to\infty} z_n=\bar{z}=1$.

Based on the above discussion, it follows that the unique equilibrium point $\bar{z}=1$ of (1.2) is GAS. The proof is complete.

4. Two examples

In this section, we give two numerical examples to demonstrate the effectiveness of theoretic analysis.

Example 4.1. With initial values $z_0=1.12, z_{-1}=0.87, z_{-2}=1.13, z_{-3}=0.9, z_{-4}=1.1$, and $b=13$, it is clear that the equilibrium $\bar{z}=1$ is GAS (See Figure 1).

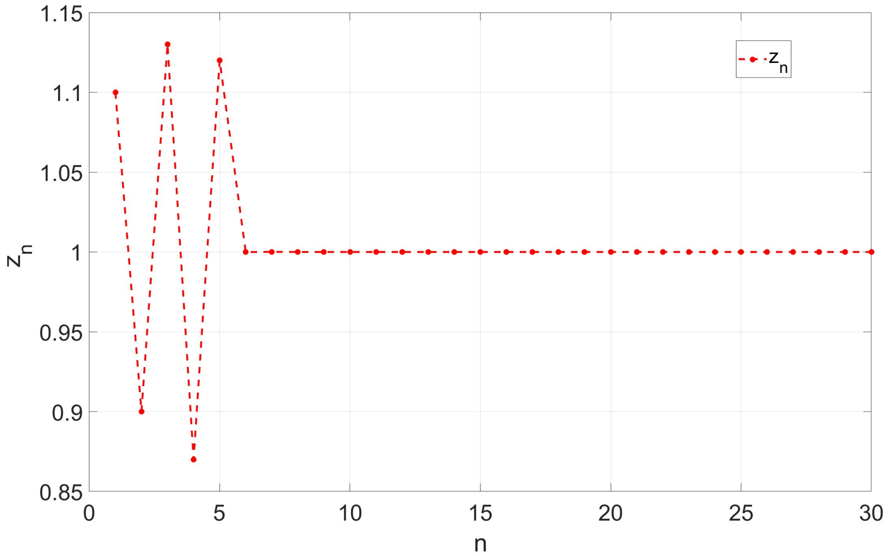

Figure 1. The equilibrium $\bar{z}=1$ is GAS.

Example 4.2. With initial conditions $z_0 = 1.2, z_{-1} = 0.7, z_{-2} = 0.8, z_{-3} = 1.9, z_{-4} = 1.01$, and $b = 8$ for $n \geq 9, z_n = 1$, in other words, $\lim_{n\to\infty} z_n = 1$, i.e., the equilibrium $\bar{z} = 1$ is GAS (See Figure 2).

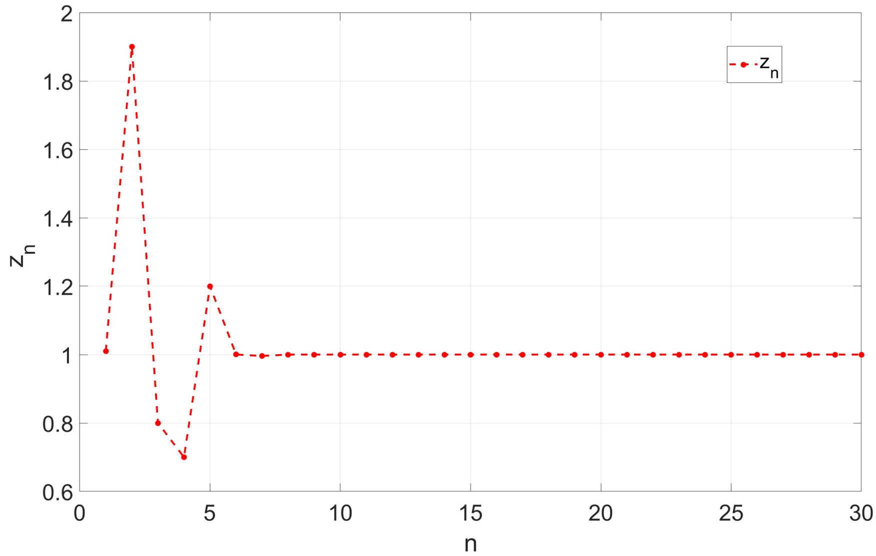

Figure 2. The equilibrium $\bar{z} = 1$ is GAS.

Analyzing the two examples above, we set the model parameters as $z_0 = 1.12, z_{-1} = 0.87, z_{-2} = 1.13, z_{-3} = 0.9, z_{-4} = 1.1$, and $b = 13$ for $n \geq 8$, we can get $z_n = 1$, which implies that $\lim_{n\to\infty} z_n = 1$, so Theorem 3.1 is valid (See Figure 1). Setting the model parameters as $z_0 = 1.2, z_{-1} = 0.7, z_{-2} = 0.8, z_{-3} = 1.9, z_{-4} = 1.01$, and $b = 8$ for $n \geq 8, z_n = 1$, in other words, $\lim_{n\to\infty} z_n = 1$, and Theorem 3.1 is valid (See Figure 2).

Reviewing Figures 1 and 2, for different initial parameters, it is obvious that the solution of Eq (1.2) converges to the equilibrium point at different values. In view of Theorem 3.1, system (1.2) is GAS, and there exists a positive integer m for $n \geq m$, $z_n = 1$. Through these graphics, we have verified the validity of the conclusions drawn in this article.

5. Conclusions

In this article, using the semi-cycle analysis method, we obtain the main results as follows.

(i) If the solution $\{z_n\}_{n=-4}^{\infty}$ of (1.2) is a trivial solution or nontrivial non-oscillatory solution, then the equilibrium $\bar{z} = 1$ of (1.2) is GAS.

(ii) If the solution $\{z_n\}_{n=-4}^{\infty}$ of (1.2) is an oscillatory solution, for a prime period, the successive lengths of positive and negative semi-cycles of any nontrivial solution occur periodically: 2,3,4,6,12. In detail, for period 2, the rule is $1^+, 1^-$; for period 3, the rule is $2^-, 1^+$ or $2^+, 1^-$; for period 4, the rule is $2^+, 2^-$; for period 6, the rule is $3^+, 3^-$; for period 12, the rule is $4^+, 1^-, 1^+, 4^-, 1^+, 1^-$. Moreover, the equilibrium $\bar{z} = 1$ of (1.2) is GAS.

The semi-cycle analysis method is one of most effective approaches for studying the dynamical behaviors of difference equations. However, it also has limitations: For some rational difference equations, using the known semi-cycle analysis method, special semi-cycle rules may appear, but some orbital structures of rational difference equations cannot be obtained using the semi-cycle analysis method. Moreover, there are dozens of possibilities for the distribution of terms in the positive and negative semi-cycle rules of these equations, which can be slightly complex and make it difficult to qualitatively analyze the trajectory structure rules of the solution of the systems. Therefore, other means are needed, such as establishing auxiliary equations and fixed point theorems or using a subsequence analysis method.

Author contributions

Liqin Shen: Writing-original draft, Revising and Editing; Qianhong Zhang: Writing-review, Funding acquisition and Guiding the revision of the paper. All authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

The authors declare that they have no conflict interests.

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