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Research article

On the optimal second order decrease rate for nonlinear and symmetric control systems

Mauro Costantini and Pierpaolo Soravia*

Dipartimento di Matematica, Università di Padova, via Trieste 63, Padova 35121, Italy

* **Correspondence:** Email: soravia@math.unipd.it; Tel: +39498271496.

Abstract: When a control system has all its vector fields tangent to the level set of a given smooth function u at a point $\hat{x}$, under appropriate assumptions that function can still have a negative rate of decrease with respect to the trajectories of the control system in an appropriate sense. In the case when the system is symmetric and u has a decrease rate of the second order, we characterise this fact and investigate the existence of a best possible rate in the class of piecewise constant controls. The problem turns out to be purely algebraic and depends on the eigenvalues of matrices constructed from a basis matrix whose elements are the second order Lie derivatives of u at $\hat{x}$ with respect to the vector fields of the system.

Keywords: nonlinear mathematical control; controllability conditions; higher order controllability conditions; higher order decrease rate; asymptotic distribution of eigenvalues; control Lyapunov function
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1. Introduction

In this paper, we consider a nonlinear and symmetric control system,

$$\begin{cases} \dot{x}_t = \sigma(x_t)a_t = \sum_{i=1}^m (a_i)_t\ \sigma_i(x_t), \\ x_0 = x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $\sigma : \mathbb{R}^n \to M_{n,m}(\mathbb{R})$ is continuous and matrix valued, the control function $a_\cdot : \mathbb{R}_+ \to B_1$ is measurable, $a = (a_i)_{i=1,\dots,m}$, and $B_1 = \{a \in \mathbb{R}^m : |a| \leq 1\}$ is the control set. The columns σ_i of $\sigma = (\sigma_i)_{i=1,\dots,m}$ are a family of smooth vector fields in $\mathbb{R}^n$. We call x_t a trajectory of the control system (depending on the control and the initial condition). Given a smooth function $u : \mathbb{R}^n \to \mathbb{R}$ and a point $\hat{x} \in \mathbb{R}^n$ such that $\nabla u(\hat{x}) \neq 0$ but $\nabla u\ \sigma(\hat{x}) = 0$, we are interested in the behaviour of the trajectories of (1.1) in the neighborhood of $\hat{x}$ satisfying at their initial point $u(x_0) \geq u(\hat{x})$, in particular the fact that trajectories can nonetheless enter in small time the interior of the sublevel set

$$\mathcal{U} = \{x : u(x) \leq u(\hat{x})\}. \tag{1.2}$$

This property is named "small time local attainability" (STLA) of $\mathcal{U}$ at $\hat{x}$ by the control system, and it is more precisely stated by saying that the minimum time function to reach $\mathcal{U}$ is continuous (and null) at $\hat{x}$.

When we deal with a simple dynamical system

$$\dot{x}_t = f(x_t), \quad x_0 = x,$$

the classical Lyapunov method, see e.g., Lyapunov [16], aims at finding a function u having at the origin a strict minimum, with the additional property that u is strictly decreasing along the trajectory of the system, in order to show that the trajectory reaches the origin (either in finite time or asymptotically). The classical Lyapunov method requires that

$$H_f u(x) := \langle f(x), \nabla u(x) \rangle < 0, \quad x \neq 0$$

in a neighborhood of the origin. The operator $H_f : C^1(\mathbb{R}^n) \to C(\mathbb{R})$ is called the Hamiltonian of the system and $H_f u$ is also called the Lie derivative of u with respect to the vector field f. However, even when $H_f u(x_0) = 0$, at some $x_0 \neq 0$, one can still prove that after a short time $t > 0$, one has $u(x_t) < u(x_0)$ by using the second order Taylor expansion

$$u(x_t) = u(x_0) + \frac{1}{2}\frac{d^2}{dt^2}u(x_t)|_{t=0}t^2 + t^2 o(1)$$

provided

$$\frac{d^2}{dt^2}u(x_t)|_{t=0} = H_f \circ H_f u(x_0) =: H_f^{(2)} u(x_0) < 0.$$

The previous equation defines the second order Lie derivative of u at x_0 and we name the operator $H_f^{(2)} : C^2(\mathbb{R}^n) \to C(\mathbb{R})$ a second order Hamiltonian, see e.g., [25, 27]. We may view the quantity $(1/2)H_f^{(2)} u(x_0)$ as a second order decrease rate of u at x_0 along the trajectory of the system.

For control systems the situation is more varied. One can extend the Lyapunov idea to control systems like (1.1), see e.g., Artstein [1] and Clarke et al. [8], even in much more general cases, by imposing, in a suitable sense if u is nonsmooth,

$$H(x, \nabla u(x)) = |\nabla u \; \sigma(x)| = \max_{a \in B_1}\{-H_{\sigma(x)a} u(x)\} > 0, \quad x \neq 0 \tag{1.3}$$

in a neighborhood of the origin (u is a control Lyapunov function). If u is smooth at x, the negative quantity

$$-H(x, \nabla u(x))$$

can still be seen as the optimal decrease rate of the trajectories of the control system at x, and the control

$$\bar{a} = \frac{{}^t(\nabla u \; \sigma(x))}{H(x, \nabla u(x))} \in B_1,$$

where the maximum is attained in (1.3) as an optimal control in that sense. Control Lyapunov functions will not be smooth in general but allowing the inequality (1.3) to be large, we may sometimes be able to find better behaved functions, see e.g., Motta and Rampazzo [20], when we look for conditions also of a different nature. When at some point $\hat{x} \in \mathbb{R}^n$ we have $\nabla u\ \sigma(\hat{x}) = 0$, i.e., all the vector fields of the control system are tangent to the level set of u at $\hat{x}$, it is well known that if there exists a Lie bracket of the vector fields σ_i pointing inward the sublevel set in (1.2) at $\hat{x}$, we can still reach the interior of $\mathcal{U}$ in small time with the trajectories of the control system starting at $\hat{x}$ or in its neighborhood, although u is no longer strictly decreasing along the trajectories in general. The goal of this paper is to determine, in this case, an optimal second order decrease rate for the trajectories of the system (1.1) in some appropriate sense. We have not seen this optimality question treated before in the literature, but we believe it is interesting in order to investigate the differences between local optimization and global optimization when we want, for instance, to reach a target. We will restrict our investigation to trajectories determined by piecewise constant control functions. Even if the system is nonlinear, the problem will turn out to be purely algebraic for a symmetric system, and we will end up computing the eigenvalues of a sequence of matrices and their asymptotic behaviour. Our method will mix some classical tools of nonlinear control systems as the use of Lie algebra in the problem of small time attainability, some newer Hamilton–Taylor expansions developed by one of the authors [24–27], and linear algebra of matrices.

It is known that in order to have some negative second order decrease rate, it is necessary and sufficient, see e.g., [3], that the system satisfies a second order attainability condition, namely in the neighborhood of $\hat{x}$ the minimum time function T to reach $\mathcal{U}$ satisfies an estimate of the form

$$T(x) \leq C|x - \hat{x}|^{1/2}, \tag{1.4}$$

where $T(x) = \inf\{t \geq 0 : x_t \in \mathcal{U},\ a(\cdot) \in L^\infty(0, +\infty; B_1)\}$. The function u we use in stating our problem is generic, but it can well be, for instance, the smooth distance function $d(x) = \mathrm{dist}(x, \mathcal{T})$ from a closed set $\mathcal{T} \subset \mathbb{R}^n$ either convex or with at least a C^2 boundary, to give the problem a more metric significance.

In mathematical control theory, controllability to a point is well studied, has a long story, and has a huge literature. Symmetric systems are studied for instance in Chow and Rashevski [7], where the famous sufficient condition using the Lie algebra is derived. For general nonlinear systems Petrov [21, 22] introduced the positive basis condition, and Liverowskii [17] extended that result to second order sufficient and necessary conditions. For affine systems, some classical results are, for instance, due to Sussmann [28, 29]. Frankowska [10] and Kawski [11] discussed higher order conditions for affine systems at an equilibrium point. Other results can be found in Bianchini and Stefani [4–6] and Krastanov [13] for dynamics on manifolds. A summary of the main classical results for the point is contained in the chapter on controllability of control systems in the book by Coron [9], where many additional references can be found. For attainability of sets, we recall Bardi and Falcone [2] for necessary and sufficient first order conditions, and our paper with Bardi and Feleqi [3] for necessary and sufficient second order conditions, see also [23]. For affine systems with drift vanishing on the target, sufficient conditions with different generality can be found in Krastanov and Quincampoix [12–14], Marigonda and coauthors [15, 18, 19].

As notations used in this paper, given a square matrix A, we denote its transposed as ${}^tA$, and respectively its symmetric part A^* and its antisymmetric part A^e as

$$A^* = (A + {}^tA)/2, \quad A^e = (A - {}^tA)/2.$$

We also indicate by $M_{n,m}(\mathbb{R})$ the space of $n \times m$ matrices with elements in $\mathbb{R}$, and also $M_n(\mathbb{R}) \equiv M_{n,n}(\mathbb{R})$. We denote by I the identity matrix (of the appropriate dimension).

2. Preliminaries

In the following, we always assume that $u : \mathbb{R}^n \to \mathbb{R}$ is a C^2 function, $\sigma : \mathbb{R}^n \to M_{n,m}(\mathbb{R}^N)$ is C^1, $\hat{x} \in \mathbb{R}^n$ is given, and that $\nabla u\ \sigma(\hat{x}) = 0$. In this case a first order decay rate at $\hat{x}$ for u relative to the trajectories of the system (1.1) is not feasible. One therefore needs to look for higher order decay rates that exploit the nonlinearity of the controlled vector field or that of the level set $\mathcal{U}$. A very general definition to express this is in a quantitative way is the following:

Definition 2.1. We say that a function $u \in C^2(\mathbb{R}^n)$ has a second order decrease rate $v < 0$ for the system (1.1) at $\hat{x}$ if there are sequences $(a[n](\cdot))_{n\geq 1}$ of control functions and $t_n \to 0+$ such that if $x[n](\cdot)$ are the corresponding trajectories in (1.1) with initial point $\hat{x}$ then

$$u(x[n]_{t_n}) = u(\hat{x}) + v\ t_n^2 + t_n^2 o(1), \quad \text{as } n \to +\infty.$$

Sometimes in the literature one may find the terminology *second order variation of* $u(x_t)$ for v in the definition.

Remark 2.2. It is important to stress the fact that in the previous definition we have a sequence of trajectories and that we are checking each one at a specific time that will change with the parameter. We are therefore building a fictitious discrete trajectory $(x[n]_{t_n})_n$ along which we expect the function u to decrease at a determined rate. Our following construction is more specific as the trajectory family parameter is continuous and all trajectories in the family are constructed similarly and consistently.

There is a standard way in the literature where the idea of Definition 2.1 is implemented and applies in the following way. Suppose that we fix a measurable control $\hat{a}(\cdot) \in L^\infty([0,1], B_1)$. We define the following family of control functions parametrized by $t > 0$ (as $t \to 0+$)

$$a[t]_s = a_{s/t}, \quad s \in [0, t].$$

Computing the corresponding family of trajectories $(x[t](\cdot))_t$ of (1.1) at the end time t, in order to check Definition 2.1, we hope to find some second order rate $v < 0$ such that

$$u(x_t) \equiv u(x[t]_t) = u(\hat{x}) + v\ t^2 + t^2 o(1), \quad \text{as } t \to 0+.$$

We will usually avoid showing explicitly the parameter of the family. In order to exploit the best performance of the system relative to the function u, in this paper we will adopt the previous construction, although we further limit ourselves to piecewise constant control functions in the interval $[0, 1]$ and compute the lowest possible decrease rate among them.

Example 2.3. In order to bridge our approach with the literature, the best known example of what we just discussed is when for controls $a_1, a_2 \in B_1$, we take a control function

$$a_s = \begin{cases} a_1, & s \in [0, 1/4[, \\ a_2, & s \in [1/4, 1/2[, \\ -a_1, & s \in [1/2, 3/4[, \\ -a_2, & s \in [3/4, 1], \end{cases}$$

and consider the family of trajectories $(x[t](\cdot))_t$ of (1.1) corresponding to the control functions $a[t](\cdot) \equiv a_{\cdot/t}$ with the same initial point $\hat{x}$. It is then well known that

$$u(x[t]_t) = u(\hat{x}) + \frac{1}{16}\langle[\sigma a_1, \sigma a_2](\hat{x}), \nabla u(\hat{x})\rangle\ t^2 + t^2 o(1), \quad \text{as } t \to 0+, \tag{2.1}$$

where $[\sigma a_1, \sigma a_2] = D(\sigma a_2)\ \sigma a_1 - D(\sigma a_1)\ \sigma a_2$ is the Lie bracket of the two vector fields. Therefore u has a second order decrease rate $\frac{1}{16}\langle[\sigma a_1, \sigma a_2](\hat{x}), \nabla u(\hat{x})\rangle$ when negative.

In order to make things more general, following one of the authors [24, 25, 27], we introduce the matrix valued function

$$S : \mathbb{R}^n \to M_m(\mathbb{R}), \quad {}^tS(x) = D(\nabla u\ \sigma)\sigma(x) = \left(H_{\sigma_j} \circ H_{\sigma_i} u(x)\right)_{i,j=1,\dots,m},$$

which is the matrix of all second order Lie derivatives of u with respect to the vector fields $(\sigma_i)_{i=1,\dots,m}$.

Remark 2.4. Notice that $S(x)$ is not a symmetric matrix in general, as one easily gets that

$$2\ S^e(x) = S(x) - {}^tS(x) = \left(\langle[\sigma_i, \sigma_j], \nabla u(x)\rangle\right)_{i,j=1,\dots,m}.$$

Example 2.5. Suppose that in $\mathbb{R}^3$,

$$\sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and $u(x, y, z) = z - x^2 - y^2$. At the origin we have $\nabla u\ \sigma(0, 0, 0) = 0$ and we compute

$${}^tS(0, 0, 0) = D(-2x, -2y)\ \sigma = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix},$$

which is symmetric. Notice that in this case there is no sensible Lie bracket of the vector fields since they are constant.

Example 2.6. Again, in $\mathbb{R}^3$, suppose that

$$\sigma(x, y, z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ y & -x \end{pmatrix}$$

and $u(x, y, z) = z$, then at any point of the z-axis

$${}^tS(0, 0, z) = D((0, 0, 1)\ \sigma)\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

which is antisymmetric.

We will return to these examples to comment on the results in the next section. To see the role of the matrix valued function $S(x)$, notice that given two controls $a_1, a_2 \in B_1$ and corresponding vector fields $f(x) = \sigma(x)a_1$, $g(x) = \sigma(x)a_2$, we can compute the second order Lie derivative of u with respect to the vector fields f, g as

$$H_f \circ H_g u = \langle f, \nabla\langle g, \nabla u\rangle\rangle = \langle \sigma a_1, \nabla\langle \sigma a_2, \nabla u\rangle\rangle = \langle D(\nabla u\ \sigma)\sigma\ a_1, a_2\rangle = \langle S\ \ a_2, a_1\rangle.$$

That is indeed a bilinear operator on the controls (a_1, a_2). Consider, for instance, a constant control function $a_t \equiv a$, $|a| \le 1$ and $\langle \sigma(\hat{x})a, \nabla u(\hat{x})\rangle = 0$, then the standard Taylor estimate gives

$$u(x_t) = u(\hat{x}) + \frac{t^2}{2!} H^{(2)}_{\sigma a} u(\hat{x}) + t^2 o(1), \quad \text{as } t \to 0+,$$

where we used the notation $H^{(2)}_{\sigma a} = H_{\sigma a} \circ H_{\sigma a}$. In particular, $H^{(2)}_{\sigma a} u(\hat{x}) = \langle S(\hat{x})a, a\rangle = \langle S^*(\hat{x})a, a\rangle$ and, if negative, it is twice the second order decrease rate of u for the trajectory of the control system. Thus, the minimum decrease rate among all constant controls $|a| \le 1$ is $1/2$ the minimum eigenvalue of the symmetric matrix $S^*(\hat{x})$.

As an example of a more general trajectory, consider the following family of control functions parametrized by $t > 0$

$$a[t]_s = \begin{cases} a_1, & \text{if } s \in [0, t/2[, \\ a_2, & \text{if } s \in [t/2, t], \end{cases} \tag{2.2}$$

then the trajectories corresponding to the control functions $a[t](\cdot)$, all starting at $\hat{x}$ satisfy (see [25])

$$u(x_t) \equiv u(x[t]_t) = u(\hat{x}) + \frac{1}{2!}\left(\frac{t}{2}\right)^2 (H_{\sigma a_1} \boxplus H_{\sigma a_2})^2 u(\hat{x}) + t^2 o(1), \quad \text{as } t \to 0+$$

where

$$(H_{\sigma a_1} \boxplus H_{\sigma a_2})^2 u(\hat{x}) := \langle S(\hat{x})a_1, a_1\rangle + \langle S(\hat{x})a_2, a_2\rangle + 2\langle S(\hat{x})a_2, a_1\rangle$$

defines the square of the sum of two Hamiltonians, which is a helpful operator. Indeed, it has been previously proved in [24] that (1.4) for the system (1.1) is equivalent to the following algebraic property of $S(\hat{x})$, see e.g., [3, 25], we can find $a_1, a_2 \in B_1$ such that

$$(H_{\sigma a_1} \boxplus H_{\sigma a_2})^2 u(\hat{x}) < 0. \tag{2.3}$$

Notice that the left hand side in (2.3) identifies a quadratic form in $\mathbb{R}^{2m}$ as

$$(H_{\sigma a_1} \boxplus H_{\sigma a_2})^2 u(\hat{x}) = \langle K_2(S(\hat{x}))\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}\rangle, \quad K_2(S(\hat{x})) = \begin{pmatrix} S^*(\hat{x}) & S(\hat{x}) \\ {}^tS(\hat{x}) & S^*(\hat{x}) \end{pmatrix}.$$

Therefore, if (2.3) holds, u has a second order decrease rate at $\hat{x}$ given by $v = \frac{1}{8}(H_{\sigma a_1} \boxplus H_{\sigma a_2})^2 u(\hat{x})$.

Remark 2.7. Note that, if $f, g : \mathbb{R}^n \to \mathbb{R}^n$ are vector fields, then (see [25])

$$(H_f \boxplus H_g)^2 u = \mathrm{Tr}({}^t(f + g)\ D^2 u\ (f + g)) + \langle [f, g], \nabla u\rangle.$$

Therefore $(H_f \boxplus H_g)^2$ is, as a differential operator, second order degenerate elliptic and has two parts: the second order measures the curvature of u in the direction of the sum of the vector fields, and the first order that compares the direction of the Lie bracket of the vector fields and the normal to the level set of u.

We can more generally consider piecewise constant control functions with an arbitrary number of switches. Namely for any given $t > 0$, a family of k controls $a_i \in B_1,\ i = 1, \dots, k$ and real numbers $\alpha_i \in]0, 1]$, $\sum_{i=1}^{k} \alpha_i = 1$, we indicate $t_i = t\alpha_i$ and define the piecewise constant control function

$$a[t]_s = \begin{cases} a_1, & \text{if } s \in [0, t_1[, \\ a_2, & \text{if } s \in [t_1, t_1 + t_2[, \\ \dots & \\ a_k, & \text{if } s \in [t - t_k, t]. \end{cases} \tag{2.4}$$

From the system (1.1) with initial point $\hat{x}$, we therefore obtain the corresponding family of trajectories indexed by t, that we indicate as $(x[t]_s)_{s\in[0,t]}$. By the results of one of the authors, see [27] and also [17], such a family of trajectories, satisfies the following Hamilton–Taylor expansion

$$u(x_t) \equiv u(x[t]_t) = u(\hat{x}) + \frac{t^2}{2!}(H_{\alpha_1\sigma a_1} \boxplus \cdots \boxplus H_{\alpha_k\sigma a_k})^2 u(\hat{x}) + t^2 o(1) \quad \text{as } t \to 0+, \tag{2.5}$$

where for $k \geq 2$ the coefficient v of the second order term above contains the square of the sum of the k Hamiltonians which is defined as follows:

$$\begin{aligned} 2!\ v &= (H_{\alpha_1\sigma a_1} \boxplus \cdots \boxplus H_{\alpha_k\sigma a_k})^2 u(\hat{x}) \\ &:= \left(\sum_{i=1}^{k} H^{(2)}_{\alpha_i\sigma a_i} u(\hat{x}) + 2\sum_{i<j=1}^{k} H_{\alpha_i\sigma a_i} \circ H_{\alpha_j\sigma a_j} u(\hat{x})\right) \\ &= \left(\sum_{i=1}^{k} \langle S(\hat{x})(\alpha_i a_i), (\alpha_i a_i)\rangle + 2\sum_{i<j=1}^{k} \langle S(\hat{x})(\alpha_j a_j), (\alpha_i a_i)\rangle\right) \\ &= \left(\sum_{i=1}^{k} \alpha_i^2 \langle S(\hat{x}) a_i, a_i\rangle + 2\sum_{i<j;i,j=1}^{k} \alpha_i\alpha_j \langle S(\hat{x}) a_j, a_i\rangle\right). \end{aligned}$$

Also, in this case, v is given by a quadratic form since

$$v = \frac{1}{2} K_k(S(\hat{x})) \begin{pmatrix} \alpha_1 a_1 \\ \vdots \\ \alpha_k a_k \end{pmatrix} \cdot \begin{pmatrix} \alpha_1 a_1 \\ \vdots \\ \alpha_k a_k \end{pmatrix}, \quad |a_1|, \dots, |a_k| \leq 1,\ \sum_{i=1}^{k} \alpha_i = 1, \tag{2.6}$$

where

$$K_k(S(\hat{x})) = \begin{pmatrix} S^*(\hat{x}) & S(\hat{x}) & \cdots & S(\hat{x}) \\ {}^tS(\hat{x}) & S^*(\hat{x}) & S(\hat{x}) & \vdots \\ {}^tS(\hat{x}) & {}^tS(\hat{x}) & S^*(\hat{x}) & \vdots \\ {}^tS(\hat{x}) & \cdots & {}^tS(\hat{x}) & S^*(\hat{x}) \end{pmatrix} \in M_{km}(\mathbb{R}),$$

as it is easily seen by an induction argument. When negative, v is the decrease rate of u with respect to the family of control functions in (2.4). We have therefore defined a sequence of matrices $(K_k(S(\hat{x})))_k$ where the k−th element appears in the Hamilton–Taylor expansion of trajectories of the system constructed as above and having $k-1$ switches.

The main goal of the paper is now to compute by induction on k the (nonpositive, according to the properties of the matrix $S(\hat{x})$) minimum of each quadratic form (2.6) in order to obtain the lowest possible decrease rate of the trajectories of the system (1.1) among all piecewise constant controls and the indicated construction. We also want to understand if the minimum is reached for a specific number of switches among the vector fields or in the limit. We could not find this type of analysis in the literature, but we think it could also help identify time optimal trajectories for the system for appropriate choices of the function u.

We now anticipate the main result of the paper. Its proof can be found in the next section at the end of Subsection 3.2.

Theorem 2.8. Let $u \in C^2(\mathbb{R}^n)$ and $\sigma \in C^1(\mathbb{R}^n; M_{n,m}(\mathbb{R}))$. Assume that at $\hat{x} \in \mathbb{R}^n$ we have $\nabla u\ \sigma(\hat{x}) = 0$ and let $S(\hat{x}) = \left(H_{\sigma_i} \circ H_{\sigma_j} u(\hat{x})\right)_{i,j=1,\ldots,r}$. Then u has at $\hat{x}$ a (negative) second order decrease rate for the control system (1.1) if and only if $S(\hat{x})$ is not symmetric and positive semidefinite. In this case, the highest rate of decrease among all piecewise constant controls is

$$\frac{1}{2}\inf_{k\geq 1}\frac{\lambda_1^{(k)}}{k} \quad \left(= \inf_{k,a_i,\alpha_i}\{\frac{1}{2}(H_{\alpha_1\sigma a_1} \boxplus \cdots \boxplus H_{\alpha_k\sigma a_k})^2 u(\hat{x})\}\right), \tag{2.7}$$

where $\lambda_1^{(k)}$ is the minimal eigenvalue of $K_k(S(\hat{x}))$.

The right hand side of Eq (2.7) shows how the highest decrease rate is computed from a fully nonlinear elliptic operator that plays the role of the Hamilton-Jacobi-Bellman one in the case of higher order conditions.

Notice that (2.6) also contains information about the controls and times that we can use in the system to achieve the decay rates appearing in Theorem 2.8. In particular, if w is an eigenvector of $K_k({}^tS(\hat{x}))$ with eigenvalue $\lambda_1^{(k)}$ and $|w| = 1/\sqrt{k}$, then

$$v = \frac{1}{2}\frac{\lambda_1^{(k)}}{k}.$$

This is the case when $w = (\alpha_i a_i)_{i=1,\ldots,k}$ with $|a_i| = 1$, $\alpha_i = 1/k$ for all $i = 1,\ldots,k$; see Propositions 3.1 and 3.3 on how controls a_i are choosen and the proof of the theorem.

As main examples of the applicability of Theorem 2.8, where calculations can be performed explicitly, we can consider in the next section, three cases when $S(\hat{x})$ is antisymmetric, symmetric or the sum of an antisymmetric and a scalar matrix. This is done below in Theorems 3.14, 3.17, and 3.20, also with examples of control systems where this happens. Furthermore, in addition to showing the calculation of the optimal decreasing rate in practice, we also show how to easily construct the controls that give the optimal rate $\lambda_k/(2k)$ above for a given integer k. This is done in the Propositions 3.3 and 3.1.

3. Results

Let A be in $M_r(\mathbb{R})$. Let us pose $K_1(A) = A^*$, and

$$K_n(A) = \begin{pmatrix} A^* & A & \cdots & A \\ {}^tA & A^* & A & \vdots \\ {}^tA & {}^tA & A^* & \vdots \\ {}^tA & \cdots & {}^tA & A^* \end{pmatrix} = \begin{pmatrix} & & & A \\ & K_{n-1}(A) & & \vdots \\ & & & A \\ {}^tA & \cdots & {}^tA & A^* \end{pmatrix} \in M_{nr}(\mathbb{R})$$

if $n \geq 2$. Let $v = {}^t(v_1,\ldots,v_n) \in \mathbb{R}^{nr}$.

We now study the properties of the eigenvalues and the eigenvectors of $K_n(A)$. We notice that when we will discuss Theorem 2.8 we will choose $A = {}^tS(\hat{x})$.

3.1. Properties of the eigenvectors of $K_n(A)$

Let λ be nonzero real. Since A^e is alternating, $\pm\lambda$ is not eigenvalue of A^e hence, both $\lambda I + A^e$ and $\lambda I - A^e$ are invertible. We define $\Gamma_\lambda = (\lambda I + A^e)^{-1}(\lambda I - A^e)$. Note that in case A^e is non singular, then we may drop the assumption $\lambda \neq 0$: in this case $\Gamma_0 = (A^e)^{-1}(-A^e) = -I$. We always have $\det \Gamma_\lambda = 1$, if $\lambda \neq 0$, see also Remark 3.5.

Proposition 3.1. Let $v = {}^t(v_1, \dots, v_n) \in \mathbb{R}^{nr}$ be an eigenvector of $K_n(A)$ with $v_i \in \mathbb{R}^r$ for $i = 1, \dots, n$ so that

$$K_n(A)v = \lambda v$$

for some $\lambda \in \mathbb{R}$. Then the following holds true:
(i) $A^e(v_i + v_{i+1}) = \lambda(v_i - v_{i+1})$, $i = 1, \dots, n-1$. Assume $\lambda \neq 0$ if A^e is singular.
(ii) Γ_λ is an isometry of $\mathbb{R}^r$.
(iii) $v_{i+1} = \Gamma_\lambda v_i$, $i = 1, \dots, n-1$.
(iv) $|v_i| = |v_j|$ for each i, j.

Proof. Let us prove (i). Consider the i-th and $i+1$-th equations of the linear system $K_n(A)v = \lambda v$ in block form:

$$\begin{aligned} {}^tAv_1 + {}^tAv_2 + \cdots + A^*v_i + Av_{i+1} + \cdots + Av_n &= \lambda v_i, \\ {}^tAv_1 + {}^tAv_2 + \cdots + {}^tAv_i + A^*v_{i+1} + \cdots + Av_n &= \lambda v_{i+1}. \end{aligned}$$

Subtracting gives

$$A^e(v_i + v_{i+1}) = (A^* - {}^tA)v_i + (A - A^*)v_{i+1} = \lambda(v_i - v_{i+1}). \tag{3.1}$$

(ii) Since $\Gamma_\lambda = (\lambda I + A^e)^{-1}(\lambda I - A^e)$ and A^e is alternating, we get $\langle \Gamma_\lambda v, w\rangle = \langle v, \Gamma_\lambda^{-1} w\rangle$ for each v, $w \in \mathbb{R}^r$, so Γ_λ is an isometry. Unless A^e is non singular and $\lambda = 0$ (in which case $\Gamma_0 = -I$), -1 is not an eigenvalue of Γ_λ.
(iii) From $A^e(v_i + v_{i+1}) = \lambda(v_i - v_{i+1})$ it follows $(\lambda I + A^e)v_{i+1} = (\lambda I - A^e)v_i$, hence

$$v_{i+1} = (\lambda I + A^e)^{-1}(\lambda I - A^e)v_i = \Gamma_\lambda v_i.$$

Finally (iv) follows from (ii) and (iii). ◻

Remark 3.2. We note that from $v_{i+1} = \Gamma_\lambda v_i$ it follows that $v_k = \Gamma_\lambda^{k-1} v_1$ and then

$$v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} v_1 \\ \Gamma_\lambda v_1 \\ \vdots \\ \Gamma_\lambda^{n-1} v_1 \end{pmatrix}$$

is the form on an eigenvector of $K_n(A)$. Moreover

$$\frac{1}{2}\langle K_n(A)v, v\rangle = \frac{n}{2}|v_1|^2\lambda.$$

The case A symmetric will be specifically treated in the next section. Otherwise the following necessary condition for eigenvalues holds.

Proposition 3.3. Suppose that the antisymmetric part A^e of A is not singular. Then, λ is an eigenvalue of $K_n(A)$ if and only if

$$\det(A\Gamma_\lambda^n - {}^tA) = 0.$$

Moreover

$$v_1 \in \ker\,(A\Gamma_\lambda^n - {}^tA)(I - \Gamma_\lambda)^{-1},$$

if $v = {}^t(v_1, \dots, v_n) \in \mathbb{R}^{nr}$ is an eigenvector of $K_n(A)$.

Proof. We begin by observing that since A^e is not singular, 1 is not an eigenvalue of Γ_λ, and so $I - \Gamma_\lambda$ is invertible.

Let $v = {}^t(v_1, \dots, v_n) \in \mathbb{R}^{nr}$, $\lambda \in \mathbb{R}$. By the previous proposition, the linear system $K_n(A)v = \lambda v$ is equivalent to

$$\begin{gathered} A^* v_1 + A v_2 + \cdots + A v_n = \lambda v_1, \\ v_{i+1} = \Gamma_\lambda v_i, i = 1, \dots, n-1. \end{gathered}$$

Therefore $v \neq 0$ is an eigenvector of $K_n(A)$ relative to λ if and only if

$$\begin{gathered} A^* v_1 + A\Gamma_\lambda v_1 + \cdots + A\Gamma_\lambda^{n-1} v_1 = \lambda v_1, \\ A(v_1 + \Gamma_\lambda v_1 + \cdots + \Gamma_\lambda^{n-1} v_1) = (\lambda I + A^e) v_1, \\ A(I - \Gamma_\lambda^n)(I - \Gamma_\lambda)^{-1} v_1 = (\lambda I + A^e) v_1 = (\lambda I + A^e)(I - \Gamma_\lambda)(I - \Gamma_\lambda)^{-1} v_1. \end{gathered}$$

Setting $w = (I - \Gamma_\lambda)^{-1} v_1$, we get

$$\begin{gathered} A(I - \Gamma_\lambda^n) w = (\lambda I + A^e) w - (\lambda I - A^e) w = 2A^e w, \\ {}^tA w - A\Gamma_\lambda^n w = 0. \end{gathered} \tag{3.2}$$

Since $w \neq 0$, λ is an eigenvalue of $K_n(A)$ if and only if $\det(A\Gamma_\lambda^n - {}^tA) = 0$. □

Remark 3.4. Observe that if A^e is non singular, the determinant condition of Proposition 3.3 allows us to detect when $\lambda = 0$ is an eigenvalue of $K_n(A)$. This happens if and only if $\det(A(-1)^n - {}^tA) = 0$. If n is even, $\lambda = 0$ is not eigenvalue of $K_n(A)$, since we are assuming $\det A^e \neq 0$. If instead n is odd, then $\lambda = 0$ is an eigenvalue of $K_n(A)$ if and only if $\det A^* = 0$. In this case

$$\ker K_n(A) = \{v = (v_1, -v_1, \dots, v_1, -v_1, v_1) \mid A^* v_1 = 0\} \cong \ker A^* \neq \{0\}.$$

In general, one can easily show that for n even, $\lambda = 0$ is an eigenvalue of $K_n(A)$ if and only if A^e is singular, while for n odd, $\lambda = 0$ is an eigenvalue of $K_n(A)$ either if A^e is non singular and A^* is singular, or if A^e is singular.

Remark 3.5. Let $a \neq 0$ and

$$B = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} \in M_2(\mathbb{R}).$$

Then

$$\Gamma_\lambda = (\lambda I + B)^{-1}(\lambda I - B) = \begin{pmatrix} \frac{\lambda^2 - a^2}{\lambda^2 + a^2} & -\frac{2a\lambda}{\lambda^2 + a^2} \\ \frac{2a\lambda}{\lambda^2 + a^2} & \frac{\lambda^2 - a^2}{\lambda^2 + a^2} \end{pmatrix} = \begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix}$$

is the rotation of angle ϑ, where $\frac{\lambda}{a} = \cot(\vartheta/2)$, $0 < \vartheta < 2\pi$. Observe that posing (e_1, e_2) the canonical basis of $\mathbb{R}^2$, we have $B(e_1 + ie_2) = ia(e_1 + ie_2)$, $B(e_1 - ie_2) = -ia(e_1 - ie_2)$. In addition, if $\lambda \neq 0$,

$$(I + \Gamma_\lambda)^{-1}(I - \Gamma_\lambda) = \frac{1}{\lambda}B.$$

Let $A \in M_r(\mathbb{R})$ be antisymmetric. Then there exists P orthogonal such that $P^{-1}AP = R$,

$$R = \begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & 0 & \vdots \\ 0 & 0 & B_s & \vdots \\ 0 & \cdots & 0 & 0_k \end{pmatrix} \in M_r(\mathbb{R}) \tag{3.3}$$

with

$$B_j = \begin{pmatrix} 0 & a_j \\ -a_j & 0 \end{pmatrix} \in M_2(\mathbb{R}) \tag{3.4}$$

$a_j > 0$ for $j = 1, \ldots, s$, and we can assume $a_1 \geq a_2 \geq \cdots \geq a_s > 0$. Then the eigenvalues of A are 0 with multiplicity k, $k \geq 0$ and $ia_1, -ia_1, \ldots, ia_s, -ia_s$. Let $(v_1, w_1, v_2, w_2, \ldots, v_s, w_s, z_1, \ldots, z_k)$ be the orthonormal basis of $\mathbb{R}^r$ given by the columns of P. If $\lambda \in \mathbb{R}$, $\lambda \neq 0$, then

$$(\lambda I + B_j)^{-1}(\lambda I - B_j) = \begin{pmatrix} \cos\vartheta_j & -\sin\vartheta_j \\ \sin\vartheta_j & \cos\vartheta_j \end{pmatrix}$$

is the rotation of angle ϑ_j in the plane $\langle v_i, w_i \rangle$, where $a_j = \lambda \tan(\vartheta_j/2)$, $-\pi < \vartheta_j < \pi$, $\vartheta_j \neq 0$. Then

$$(\lambda I + R)^{-1}(\lambda I - R) = \begin{pmatrix} \begin{matrix} \cos\vartheta_1 & -\sin\vartheta_1 \\ \sin\vartheta_1 & \cos\vartheta_1 \end{matrix} & 0 & \cdots & 0 \\ 0 & \begin{matrix} \cos\vartheta_2 & -\sin\vartheta_2 \\ \sin\vartheta_2 & \cos\vartheta_2 \end{matrix} & 0 & \vdots \\ 0 & 0 & \begin{matrix} \cos\vartheta_s & -\sin\vartheta_s \\ \sin\vartheta_s & \cos\vartheta_s \end{matrix} & \vdots \\ 0 & \cdots & 0 & 1_k \end{pmatrix}$$

the product of s rotations (this orthogonal matrix does not have -1 as its eigenvalue), in particular

$$\det(\lambda I + R)^{-1}(\lambda I - R) = 1.$$

The vectors v_j, w_j are also constructed from eigenvectors relative to ia_j and $-ia_j$, while the z_k constitutes an orthonormal basis of the core of R. Note that every isometry X of $\mathbb{R}^r$ of this form can be achieved from an appropriate antisymmetrix matrix R: $R = (I + X)^{-1}(I - X)$. □

We can characterize the cases when the matrix $K_2(A)$ has a negative eigenvalue.

Lemma 3.6. Let A be in $M_r(\mathbb{R})$. Then $K_2(A)$ is positive semidefinite if and only if A is symmetric and positive semidefinite.

Proof. If A is symmetric and positive semidefinite, then

$$\langle K_2(A)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\rangle = \langle A(v_1+v_2), (v_1+v_2)\rangle \tag{3.5}$$

and therefore $K_2(A)$ is positive semidefinite as well. We next prove the converse. Notice that choosing $v_1 = -v_2 \in \mathbb{R}^r$ also shows that $K_2(A)$ always has 0 as an eigenvalue.

Suppose first that A is symmetric. By (3.5), if $K_2(A)$ is positive semidefinite and we choose $v_1 = v_2 \in \mathbb{R}^r$, then we get $0 \leq 4\langle Av_1, v_1\rangle$, for all $v_1 \in \mathbb{R}^r$ and A is also positive semidefinite.

Suppose now that A is not symmetric and we show that $K_2(A)$ must have a negative eigenvalue. In particular, A is not the null matrix. Consider thus the positive semidefinite matrix ${}^tAA$; it will have at least one positive eigenvalue λ^2, with $\lambda > 0$, with corresponding unit eigenvector v_2. Thus,

$${}^tAAv_2 = \lambda^2 v_2$$

and then

$$|Av_2|^2 = \langle Av_2, Av_2\rangle = \langle {}^tAAv_2, v_2\rangle = \lambda^2\langle v_2, v_2\rangle = \lambda^2,$$

so that $\lambda = |Av_2| > 0$. Just notice that if $\bar{v}$ is an eigenvector of ${}^tAA$ with null eigenvalue, then the same argument shows that $A\bar{v} = 0$. Let

$$v_1 = -\frac{Av_2}{\lambda},$$

so that $|v_1| = 1$. We can now obtain

$${}^tAv_1 = -\lambda v_2, \quad \langle Av_2, v_1\rangle = -\lambda, \quad \langle Av_1, v_1\rangle = -\lambda\langle v_1, v_2\rangle = \langle Av_2, v_2\rangle.$$

Thus we conclude that

$$\langle K_2(A)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\rangle = -2\lambda(1 + \langle v_1, v_2\rangle)$$

and we reach our conclusion provided $v_1 \neq -v_2$. If instead $v_1 = -v_2$, it then follows

$$Av_1 = \lambda v_1, \quad {}^tAv_1 = \lambda v_1.$$

If this happens for all eigenvectors of ${}^tAA$ with positive eigenvalues, and we consider an orthonormal basis of eigenvectors of ${}^tAA$, then this is also a family of eigenvectors for A which can then be diagonalised by an orthogonal matrix, and it is thus symmetric, which was supposed not to be the case. □

Corollary 3.7. Let A be in $M_r(\mathbb{R})$. Then for all $n \geq 2$, $K_n(A)$ is positive semidefinite if and only if A is symmetric and positive semidefinite.

Proof. If $n = 2$, then the result is the content of Lemma 3.6. Let us assume $n \geq 3$. If $K_n(A)$ is positive semidefinite, then $K_2(A)$ is positive semidefinite, hence A is symmetric positive semidefinite. Assume A symmetric positive semidefinite. Then

$$\langle K_n(A)\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}\rangle = \langle A(v_1 + \cdots + v_n), (v_1 + \cdots + v_n)\rangle \tag{3.6}$$

hence $K_n(A)$ is positive semidefinite. □

Corollary 3.8. Let A be in $M_r(\mathbb{R})$. Then, for all $n \geq 2$, $K_n(A)$ is never positive definite.

Proof. If $K_n(A)$ is positive semidefinite, then A is symmetric. Hence $A^e = 0$ is singular, and therefore $\ker K_n(A)$ is not zero. □

3.2. The minimum of the quadratic form

Given $A \in M_r(\mathbb{R})$ and $n \geq 1$, we consider the quadratic form $Q_n(A) : \mathbb{R}^{nr} \to \mathbb{R}$,

$$Q_n(A) : \; v \mapsto \langle K_n(A)v, v\rangle \; = {}^t v \, K_n(A)\, v, \quad v \in \mathbb{R}^{nr}. \tag{3.7}$$

We want to compute the minimum of $Q_n(A)$ on different compact subsets $D \subset \mathbb{R}^{nr}$.

Remark 3.9. It is well known that on $D_{n,2} = \{v \in \mathbb{R}^{nr} : |v| = 1\}$, then $\min_D Q_n(A)$ is the minimal eigenvalue $\lambda_1^{(n)}$ of $K_n(A)$. If instead $D = B_1$, then $\min_{B_1} Q_n(A) = 0$ if $K_n(A)$ is positive semidefinite, while again $\min_{B_1} Q_n(A) = \lambda_1^{(n)}$ if $K_n(A)$ is not positive semidefinite.

Remark 3.10. Suppose that we have a sequence of domains $D_n \subset \mathbb{R}^{nr}$, $n \geq 2$, with the property that if $v = (v_1, \ldots, v_{n-1}) \in D_{n-1}$ then $\hat{v} = (v_1, \ldots, v_{n-1}, 0) \in D_n$ so that we can identify D_{n-1} with a subset of D_n. Since $\langle K_{n-1}(A)v, v\rangle = \langle K_n(A)\hat{v}, \hat{v}\rangle$, it is then clear that $\min_{D_n} Q_n(A) \leq \min_{D_{n-1}} Q_{n-1}(A)$. Therefore the sequence $(\min_D Q_n(A))_n$ is nonincreasing and $\inf_n \min_{D_n} Q_n(A)$ is either attained at some $\bar{n}$ if the sequence is constant for $n \geq \bar{n}$, or it is attained asymptotically as $n \to +\infty$.

From now on we will suppose that A is not symmetric and positive semidefinite so that $\lambda_1^{(n)}$, the minimal eigenvalue of $K_n(A)$, is negative for $n \geq 2$, as we proved in the previous section. We turn to a more interesting case for us, which is, $n \geq 1$,

$$D_{n,\infty} = \{v = (v_1, \ldots, v_n) \in \mathbb{R}^{nr} : \max_i |v_i| = 1\}.$$

The property of Remark 3.10 holds true for the sequence $(D_{n,\infty})_n$. Notice that if $v \in D_{n,\infty}$ then $|v| \leq \sqrt{n}\max_i |v_i| = \sqrt{n}$ and that the equality is reached only if $|v_i| = 1$, for all $i = 1, \ldots, n$.

Lemma 3.11. If A is not symmetric and positive semidefinite, then

$$\min_{D_{n,\infty}} Q_n(A) = n\lambda_1^{(n)}.$$

Proof. If $n = 1$, the thesis is obvious since $D_{1,2} = D_{1,\infty}$. For $n \geq 2$ and $v \in D_{n,\infty}$, we also have that $v/|v| \in D_{n,2}$. Therefore, as $\lambda_1^{(n)} < 0$,

$$Q_n(A)(v) = |v|^2 Q_n(A)\left(\frac{v}{|v|}\right) \geq |v|^2\lambda_1^{(n)} \geq n\lambda_1^{(n)}$$

and by Proposition 3.1(i), the equalities are actually reached for $v = (v_i)_i$ being an eigenvector for $\lambda_1^{(n)}$ with $|v_i| = 1$ for all $i = 1, \ldots, n$. Thus $\min_{D_{n,\infty}} Q_n(A) = n\lambda_1^{(n)}$. □

We now consider

$$D_{n,1} = \{v = {}^t(v_1, \ldots, v_n) \in \mathbb{R}^{nr} : \sum_{i=1}^{n} |v_i| = 1\}.$$

Remark 3.10 applies to this case as well, and $D_{1,1} = D_{1,2}$.

Proposition 3.12. Suppose that A is not symmetric and positive semidefinite, so that $K_n(A)$ has a negative eigenvalue, for $n \geq 2$. Then

$$\min_{D_{n,1}} Q_n(A) = \min\left\{\lambda_1^{(1)}, \frac{\lambda_1^{(2)}}{2} \dots, \frac{\lambda_1^{(n)}}{n}\right\},$$

where $\lambda_1^{(i)}$ is the minimal eigenvalue of $K_i(A)$.

Proof. We proceed by an induction argument on n. If $n = 1$ then $K_1(A) = A^*$ and $D_{1,1} = \partial B_1 = D_{1,2}$. Therefore, $\min_{D_{1,1}} Q_1(A) = \lambda_1^{(1)}$ is the minimal eigenvalue of A^*.

For $n \geq 2$, we use the Lagrange multipliers necessary condition. Let $v = (v_1, \dots, v_n) \in D_{n,1}$ be a minimum point of $Q_n(A)$. If some $v_i = 0$, then we can reduce the problem to that of the minimum of $Q_m(A)$ with $m < n$ (and v is at some corner of the domain), which we can assume is solved by the inductive hypothesis. So we assume $v_i \neq 0$ for all i in order to get necessary conditions genuinely for the given n. The Lagrangian is

$$L(v, \lambda) = \frac{1}{2}\langle K_n(A)v, v\rangle - \lambda(\sum_{i=1}^{n} |v_i| - 1),$$

so that the Lagrange necessary system is as follows:

$$K_n(A)\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \lambda \begin{pmatrix} v_1/|v_1| \\ \vdots \\ v_n/|v_n| \end{pmatrix}, \quad v \in D_{n,1}. \tag{3.8}$$

Notice that then

$$Q_n(A)(v) = \langle K_n(A)v, v\rangle = \lambda(\sum_{i=1}^{n} |v_i|) = \lambda,$$

therefore, the Lagrange multiplier gives the possible minimum. If $\lambda = 0$, then

$$Q_n(A)(v) \;=\; \langle K_n(A)v, v\rangle \;= 0$$

which is the minimum only if $K_n(A)$ is positive semidefinite, i.e., if A is symmetric and positive semidefinite by Corollary 3.7, which is not the case by assumption. Hence $\lambda \neq 0$. Let us define $\hat{v}_i = \frac{v_i}{|v_i|}$, then $v_i = \gamma_i \hat{v}_i$, with $\gamma_i = |v_i| > 0$, $\gamma_1 + \cdots + \gamma_n = 1$. We will prove that all γ_i are equal (to $1/n$), or that we can find anyway

$$v' = \begin{pmatrix} v'_1 \\ \vdots \\ v'_n \end{pmatrix}$$

with the v'_i all having the same norm equal to $1/n$, such that (3.8) is also satisfied by v' with the same λ. Consider the i-th and $(i+1)$-th equations in (3.8), $i = 1, \dots, n-1$,

$$\begin{aligned} {}^t\!Av_1 + {}^t\!Av_2 + \cdots + A^*v_i + Av_{i+1} + \cdots + Av_n &= \lambda \hat{v}_i, \\ {}^t\!Av_1 + {}^t\!Av_2 + \cdots + {}^t\!Av_i + A^*v_{i+1} + \cdots + Av_n &= \lambda \hat{v}_{i+1}. \end{aligned}$$

Subtracting this gives

$$A^e(v_i + v_{i+1}) = \lambda(\hat{v}_i - \hat{v}_{i+1}).$$

But now A^e is alternating, so

$$\langle v_i + v_{i+1}, \lambda(\hat{v}_i - \hat{v}_{i+1})\rangle = \langle v_i + v_{i+1}, A^e(v_i + v_{i+1})\rangle = 0$$

from which it follows

$$\langle v_i + v_{i+1}, \hat{v}_i - \hat{v}_{i+1}\rangle = 0$$

being $\lambda \neq 0$. Recall that $v_i = \gamma_i \hat{v}_i$, $v_{i+1} = \gamma_{i+1}\hat{v}_{i+1}$, so

$$(\gamma_i - \gamma_{i+1})(1 - \langle \hat{v}_i, \hat{v}_{i+1}\rangle) = 0.$$

We have two possible cases. Suppose first $\hat{v}_i \neq \hat{v}_{i+1}$. Then $\gamma_i = \gamma_{i+1}$ (and thus $|v_i| = |v_{i+1}| = \gamma_i$).

Suppose next $\hat{v}_i = \hat{v}_{i+1}$. The equation

$$A^e(v_i + v_{i+1}) = \lambda(\hat{v}_i - \hat{v}_{i+1})$$

becomes $(\gamma_i + \gamma_{i+1})A^e\hat{v}_i = 0$ and therefore $A^e\hat{v}_i = 0$. Rewrite for convenience

$$K_n(A) = K_n(A^* + A^e) = \begin{pmatrix} A^* & A^* + A^e & \cdots & A^* + A^e \\ A^* - A^e & A^* & A^* + A^e & \vdots \\ A^* - A^e & A^* - A^e & A^* & \vdots \\ A^* - A^e & \cdots & A^* - A^e & A^* \end{pmatrix} \in M_{nr}(\mathbb{R}).$$

It is possible that some other $\hat{v}_{i+2}, \hat{v}_{i+3}, \ldots, \hat{v}_j$ is also equal to $\hat{v}_i$. We then consider the string

$$\hat{v}_i = \hat{v}_{i+1} = \hat{v}_{i+2} = \hat{v}_{i+3} = \ldots = \hat{v}_r$$

where $1 \leq i < r \leq n$. Let us start supposing either $i \geq 2$ or $r \leq n - 1$, and therefore either $\hat{v}_{i-1} \neq \hat{v}_i$ or $\hat{v}_r \neq \hat{v}_{r+1}$. Thus either $\gamma_{i-1} = \gamma_i$ or $\gamma_r = \gamma_{r+1}$ by the discussion above. Recall that $v_s = \gamma_s\hat{v}_s = \gamma_s\hat{v}_i$ and hence $A^e v_s = 0$ for $i \leq s \leq r$. The j-th equation in (3.8) results in

$$A^*(v_1 + \cdots + v_n) + A^e(v_{j+1} + \cdots + v_n) - A^e(v_1 + \cdots + v_{j-1}) = \lambda\hat{v}_j.$$

Let $\delta_1, \ldots, \delta_n$ in $\mathbb{R}$, positive, with $\gamma_i + \cdots + \gamma_r = \delta_i + \cdots + \delta_r$ and $\delta_j = \gamma_j$ if $j < i$ or $j > r$. We consider the vector

$$v' = \begin{pmatrix} v'_1 \\ \vdots \\ v'_n \end{pmatrix} \in M_n(\mathbb{R}^r)$$

where we put $\delta_s\hat{v}_s = v'_s$ for $1 \leq s \leq n$, respectively. Then

$$v_1 + \cdots + v_n = v'_1 + \cdots + v'_n$$

and

$$A^*(v_1 + \cdots + v_n) = A^*(v'_1 + \cdots + v'_n).$$

But also if $j < n$

$$A^e(v_{j+1} + \cdots + v_n) = A^e(v'_{j+1} + \cdots + v'_n)$$

and if $j > 1$

$$A^e(v_1 + \cdots + v_{j-1}) = A^e(v'_1 + \cdots + v'_{j-1}),$$

since at places s where we varied the coefficient, we have $A^e v_s = 0$. We therefore have

$$K_n(A^* + A^e)\begin{pmatrix} v'_1 \\ \vdots \\ v'_n \end{pmatrix} = \lambda \begin{pmatrix} v_1/|v_1| \\ \vdots \\ v_n/|v_n| \end{pmatrix} = \lambda \begin{pmatrix} v'_1/|v'_1| \\ \vdots \\ v'_n/|v'_n| \end{pmatrix},$$

$$\langle K_n(A^* + A^e)v', v'\rangle \ = \ \langle \lambda \begin{pmatrix} \hat{v}'_1 \\ \vdots \\ \hat{v}'_n \end{pmatrix}, \begin{pmatrix} \delta_1 \hat{v}'_1 \\ \vdots \\ \delta_n \hat{v}'_n \end{pmatrix} \rangle \ = \lambda(\delta_1 + \cdots + \delta_n) = \lambda.$$

Since we can modify at will the δ_s, $s = i$ or $s = r$, this contradicts one of the following two

$$\gamma_{i-1} = \gamma_i, \quad \gamma_r = \gamma_{r+1},$$

unless we are really in the extreme case, i.e., $i = 1$, $r = n$, that is, all $\hat{v}_i$ are equal.

Let us then put ourselves in this case. We then have

$$v_i = \gamma_i \hat{v}_1, \quad A^e v_i = 0$$

for every $i = 1, \ldots, n$, and again, as above, we can modify all v_i by choosing $\delta_s = 1/n$ for every $1 \leq s \leq n$. Then

$$v' = \frac{1}{n}\begin{pmatrix} \hat{v}_1 \\ \vdots \\ \hat{v}_1 \end{pmatrix} \in \mathbb{R}^{rn},$$

and this choice satisfied

$$K_n(A)v' = n\lambda v', \quad Q_n(A)(v') = \langle K_n(A)v', v'\rangle \ = \lambda$$

therefore, v' is an eigenvector of $K_n(A)$ with $n\lambda$ as an eigenvalue and $Q_n(A)(v') = \lambda$. Thus the best choice to reach a minimum of $Q_n(A)$ is for the v' eigenvector of $K_n(A)$ with $\lambda_1^{(n)} = n\lambda$ as eigenvalue, and finally $Q_n(A)(v') = \frac{\lambda_1^{(n)}}{n}$ as we intended.

Then, by the induction assumption, the minimum is

$$\min_{D_{n,1}} Q_n(A) = \min\{\min_{D_{n,1}} Q_{n-1}(A), \frac{\lambda_1^{(n)}}{n}\} = \min\{\frac{\lambda_1^{(1)}}{1}, \ldots, \frac{\lambda_1^{(n)}}{n}\} \tag{3.9}$$

as we wanted. □

We are now in the position to prove Theorem 2.8.

Proof of Theorem 2.8. As we saw in Section 2, (2.6) is the expression of the second order coefficient for the family of trajectories corresponding to the family of controls defined in (2.4), where n is the number of controls being used and $\alpha_1, \dots, \alpha_n$ are the percentage of the time being spent on each control respectively. We will use the above with $A = {}^tS(\hat{x})$. We need to compute for each n the minimum of the quadratic form $(1/2)Q_n(A)$ on the following domain

$$\hat{D}_n = \{v = {}^t(\alpha_1 v_1, \dots, \alpha_n v_n) \in \mathbb{R}^{nr} : |v_i| \leq 1,\ \alpha_i \geq 0,\ \sum_i \alpha_i = 1\}.$$

We will show that $\min_{\hat{D}_n} Q_n(A) = \min_{D_{n,1}} Q_n(A)$ reaching our thesis. This fact will be a consequence of Proposition 3.12 and the proof that

$$\hat{D}_n = \{v = {}^t(v_1, \dots, v_n) \in \mathbb{R}^{nr} : \sum_{i=1}^n |v_i| \leq 1\}.$$

Indeed, on one hand, for $v = {}^t(\alpha_1 v_1, \dots, \alpha_n v_n) \in \hat{D}_n$ we have that

$$\sum_{i=1}^n |\alpha_i v_i| \leq \sum_{i=1}^n \alpha_i = 1. \tag{3.10}$$

On the other hand let $v = {}^t(v_1, \dots, v_n)$ and $\sum_{i=1}^n |v_i| \leq 1$. We may clearly suppose $v \neq 0$ and thus $S = \sum_{i=1}^n |v_i| > 0$. Then we put

$$\alpha_i = \frac{|v_i|}{S}, \quad \hat{v}_i = \begin{cases} \frac{v_i}{\alpha_i}, & \text{if } v_i \neq 0, \\ v_i, & \text{if } v_i = 0. \end{cases}$$

Therefore $\sum_{i=1}^n \alpha_i = 1$, $\alpha_i \hat{v}_i = v_i$, and either $\hat{v}_i = 0$ or $|\hat{v}_i| = S \leq 1$ so that $v \in \hat{D}_n$.

Also notice that by choosing ${}^t(v_1, \dots, v_n)$ an eigenvector of $K_n(A)$ with eigenvalue $\lambda_1^{(n)}$, $|v_i| = 1$ and $\alpha_i = 1/n$ for all i, so that ${}^t(\alpha_1 v_1, \dots, \alpha_n v_n) \in \hat{D}_n$, we obtain

$$Q_n(A)(v) = \frac{1}{n^2} K_n(A) \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \frac{1}{n^2} \lambda_1^{(n)} \sum_{i=1}^n |v_i|^2 = \frac{\lambda_1^{(n)}}{n}.$$

□

3.3. *Explicit examples for special classes of the matrix A and applications to control systems*

The first example deals with the case when A is antisymmetric.

Proposition 3.13. Let $A \in M_r(\mathbb{R})$ be antisymmetric, $A \neq 0$. Then the extremal eigenvalues of $K_n(A)$, $n \geq 2$, are

$$\pm \alpha_1 \cot\left(\frac{\pi}{2n}\right)$$

where α_1 is the biggest modulus of the eigenvalues of A.

Proof. Let A be antisymmetric, $n \geq 2$. Then

$$K_n(A) = \begin{pmatrix} 0 & A & \cdots & A \\ -A & 0 & A & \vdots \\ -A & -A & 0 & \vdots \\ -A & \cdots & -A & 0 \end{pmatrix} \in M_{nr}(\mathbb{R}).$$

We will compute all its eigenvalues. A convenient way to denote $K_n(A)$ is obtained by introducing the antisymmetric matrix

$$T_n = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ -1 & 0 & 1 & \vdots \\ -1 & -1 & 0 & \vdots \\ -1 & \cdots & -1 & 0 \end{pmatrix} \in M_n(\mathbb{R}).$$

Then

$$K_n(A) = T_n \otimes A.$$

If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of T_n and $\mu_1, \ldots \mu_r$ are those of A, then the eigenvalues of $T_n \otimes A$ are all the products $\lambda_i \mu_j$, $i = 1, \ldots, n$, $j = 1, \ldots, r$.

Since A is antisymmetric, there exists P orthogonal such that $P^{-1}AP = R$, as in (3.3) and (3.4). Then the eigenvalues of A are 0 with multiplicity k, $k \geq 0$ and $i\alpha_1, -i\alpha_1, \ldots, i\alpha_s, -i\alpha_s$, with $\alpha_i > 0$ for $i = 1, \ldots, s$, and we can assume $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_s > 0$.

We come to the matrix T_n. Let $v = {}^t(x_1, \ldots, x_n) \in \mathbb{R}^n$ be an eigenvector of T_n, so that $T_n v = \lambda v$ for some $\lambda \in i\mathbb{R}$. Consider the i-th and $(i+1)$-th equations of $T_n v = \lambda v$:

$$-x_1 - x_2 + \cdots - x_{i-1} + x_{i+1} + \cdots + x_n = \lambda x_i,$$
$$-x_1 - x_2 + \cdots - x_i + x_{i+2} + \cdots + x_n = \lambda x_{i+1}.$$

Subtracting gives

$$x_i + x_{i+1} = \lambda(x_i - x_{i+1}). \tag{3.11}$$

Hence $x_{i+1} = \gamma_\lambda x_i$, where $\gamma_\lambda = \frac{\lambda-1}{\lambda+1}$. It follows that $x_k = \gamma_\lambda^{k-1} x_1$ and then

$$v = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 \\ \gamma_\lambda x_1 \\ \vdots \\ \gamma_\lambda^{n-1} x_1 \end{pmatrix}$$

is the form on an eigenvector of T_n. Therefore, $v \neq 0$ is an eigenvector of T_n relative to λ if and only if

$$x_2 + \cdots + x_n = \lambda x_1,$$
$$\gamma_\lambda x_1 + \cdots + \gamma_\lambda^{n-1} x_1 = \lambda x_1,$$
$$x_1 + \gamma_\lambda x_1 + \cdots + \gamma_\lambda^{n-1} x_1 = x_1 + \lambda x_1,$$
$$(1 + \gamma_\lambda + \cdots + \gamma_\lambda^{n-1}) x_1 = (1 + \lambda) x_1.$$

Note that $\gamma_\lambda \neq 1$, hence $v \neq 0$ is an eigenvector of T_n relative to λ if and only if

$$\begin{aligned}(\gamma_\lambda - 1)(1 + \gamma_\lambda + \cdots + \gamma_\lambda^{n-1})x_1 &= (\gamma_\lambda - 1)(1 + \lambda)x_1,\\ (\gamma_\lambda^n - 1)x_1 &= -2x_1,\\ (\gamma_\lambda^n + 1)x_1 &= 0.\end{aligned}$$

It follows that λ is an eigenvalue of T_n if and only if $\gamma_\lambda^n = -1$, i.e., $\gamma_\lambda = e^{i\vartheta}$, $\vartheta = \frac{\pi}{n} + k\frac{2\pi}{n}$, $k = 0, \dots, n-1$. From $\gamma_\lambda = \frac{\lambda-1}{\lambda+1}$, we get

$$\lambda = \frac{1+\gamma_\lambda}{1-\gamma_\lambda} = \frac{1+e^{i\vartheta}}{1-e^{i\vartheta}} = i\cot\frac{\vartheta}{2}.$$

The eigenvalues of T_n are thus

$$\lambda = i\cot\left(\frac{\pi}{2n} + k\frac{\pi}{n}\right), \quad k = 0, \dots, n-1.$$

The maximum is $\cot\left(\frac{\pi}{2n}\right)$, the minimum is

$$\cot\left(\frac{\pi}{2n} + \frac{(n-1)\pi}{n}\right) = -\cot\left(\frac{\pi}{2n}\right).$$

Recalling that the biggest modulus of the eigenvalues of A is α_1, $\alpha_1 > 0$, the maximum eigenvalue of $K_n(A)$ is then

$$\alpha_1 \cot\left(\frac{\pi}{2n}\right) \quad \left(\sim \alpha_1 \frac{2}{\pi} n, \quad \text{as } n \to +\infty\right)$$

and the minimum is

$$-\alpha_1 \cot\left(\frac{\pi}{2n}\right).$$

□

The consequence for the control problem is as follows: This case happens when the level set of the function u is flat.

Theorem 3.14. Let $A = {}^tS(\hat{x}) \in M_r(\mathbb{R})$ be antisymmetric, $A \neq 0$. Then the highest decrease rate of u at $\hat{x}$ for the control system is $-\alpha_1/\pi$, where α_1 is the biggest modulus of the eigenvalues of A. Moreover as $n \to +\infty$ the error for using the best rate with n controls is of the order $|\lambda_1^{(n)}/n + \alpha_1/\pi| \sim \pi/(12n^2)$.

Proof. Since the sequence

$$a_n = -\frac{1}{2n}\cot\left(\frac{\pi}{2n}\right)$$

is strictly decreasing; by Theorem 2.8, we just need to compute $\lim_{n\to+\infty} a_n\alpha_1 = -\alpha_1/\pi$. The highest rate is attained in the limit as $n \to +\infty$. □

Example 3.15. An instance of a control system where an antisymmetric matrix appears is the well-known system of the generators of the Heisenberg group. In this case (1.1) is in dimension 3 and has the following data

$$\sigma(x, y, z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ y & -x \end{pmatrix}.$$

We choose $u(x, y, z) = z$ and want to find a decrease rate of u with respect to the system at a point $(0, 0, z)$. Notice that $\nabla u\ \sigma(0, 0, z) = (0, 0)$ so we can expect a second order decrease rate. As in Section 2, we define

$$A \equiv {}^tS(0, 0, z) = D(\nabla u\ \sigma)\sigma(0, 0, z) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

whose eigenvalues are $\pm i$. Of course, $K_1(A) = A^* = 0$, so we look at

$$K_2(A) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

This matrix has $\lambda_1^{(2)} = -1$ and as an eigenvector $v = (0, 1, 1, 0)$ (the space of eigenvectors has dimension 2). Therefore if in an interval $[0, t]$ with t small we use controls $(0, 1)$ and $(1, 0)$ in consecutive subintervals of equal length we can expect a second order rate

$$\frac{1}{2}\frac{\lambda_2^{(2)}}{2} = -\frac{1}{4}.$$

This is already significant since from the classical formula (2.1), the best rate from a Lie bracket is easily computed to be $-1/8$. Therefore a Lie bracket never gives the best rate of decrease.

We could do better by exploiting $K_n(A)$ with higher n and a control function with $n - 1$ switches instead, because the expected optimal second order rate is $-1/\pi$ as we proved above. Indeed computing $K_3(A)$ we find the minimal eigenvalue $-\sqrt{3}$, which corresponds to a rate $-1/(2\sqrt{3})$, with an eigenvector for instance $(-1/2, \sqrt{3}/2, 1/2, \sqrt{3}/2, 1, 0)$, which gives us the three controls to use in order to achieve that rate for sufficiently small time.

Next we consider the case when $A \in M_r(\mathbb{R})$ is symmetric.

Proposition 3.16. Let $A \in M_r(\mathbb{R})$ be symmetric, $A \neq 0$. Then 0 is always an eigenvalue of $K_n(A)$ for $n \geq 2$, the maximum eigenvalue of $K_n(A)$ is $\max\{0, n\alpha_1\}$ and the minimum is $\min\{0, n\alpha_r\}$, where α_1, α_r are the maximal and minimal eigenvalues of A, respectively.

Proof. Since A is symmetric,

$$K_n(A) = \begin{pmatrix} A & A & \cdots & A \\ A & A & A & \vdots \\ A & A & A & \vdots \\ A & \cdots & A & A \end{pmatrix} \in M_{nr}(\mathbb{R}).$$

In this case, to denote $K_n(A)$, we consider the symmetric matrix

$$R_n = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \vdots \\ 1 & 1 & 1 & \vdots \\ 1 & \cdots & 1 & 1 \end{pmatrix} \in M_n(\mathbb{R}).$$

Then

$$K_n(A) = R_n \otimes A.$$

If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of R_n and $\mu_1, \ldots \mu_r$ are those of A, then the eigenvalues of $R_n \otimes A$ are all the products $\lambda_i \mu_j$, $i = 1, \ldots, n$, $j = 1, \ldots, r$.

Let $\alpha_1, \ldots, \alpha_r$ be the eigenvalues of A, with $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_r$. The eigenvalues of R_n are 0, with multiplicity $n-1$, and n. Therefore the maximum eigenvalue of $K_n(A)$ is $\max\{0, n\alpha_1\}$ and the minimum is $\min\{0, n\alpha_r\}$. □

The consequence of the previous proposition on the control system (1.1) is the following: This case happens when the optimal decrease rate is due to the curvature of the level set of u.

Theorem 3.17. Let $A = {}^tS(\hat{x}) \in M_r(\mathbb{R})$ be symmetric. If A is not positive semidefinite, then u at $\hat{x}$ has a (negative) decrease rate for the control system; the highest decrease rate is $(1/2)\alpha_r$, where α_r is the minimal eigenvalue of A.

Proof. The sequence $\min\{0, n\alpha_r\}/2n = \alpha_r/2$ is constant since $\alpha_r < 0$ by the assumption. □

Example 3.18. The case of a symmetric matrix for the control system comes from Example 2.5, where we found

$$A = {}^tS(0,0,0) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}.$$

Then the optimal second order rate among piecewise constant controls is -1, and it is constant on the number of switches. Any control will lead to the optimal rate in this case.

Another special case that we can explicitly deal with, is when $A = sI + A^e$, A^e is antisymmetric. We omit the details.

Proposition 3.19. Let A be such that $A^* = sI$ is scalar, $s \neq 0$, and $A^e \neq 0$. Let α_1 be the biggest modulus of the eigenvalues of A^e, $\theta \in (0, \pi/2)$ be such that $\cot\theta = \frac{|s|}{\alpha_1}$. Then the extremal eigenvalues of $K_n(A)$, $n \geq 2$ are

$$\lambda_{\max} = \alpha_1 \cot\left(\frac{\theta}{n}\right) > ns, \quad \lambda_{\min} = \alpha_1 \cot\left(\frac{\theta}{n} - \frac{\pi}{n}\right) < 0, \quad \text{if } s > 0,$$

$$\lambda_{\max} = -\alpha_1 \cot\left(\frac{\theta}{n} - \frac{\pi}{n}\right) > 0, \quad \lambda_{\min} = -\alpha_1 \cot\left(\frac{\theta}{n}\right) < ns, \quad \text{if } s < 0.$$

When we apply the previous proposition to control systems, if $s < 0$ then the curvature of the level set helps u to decrease, while the opposite happens if $s > 0$.

Theorem 3.20. Let $A = {}^tS(\hat{x}) \in M_r(\mathbb{R})$ be such that $A^* = sI$ is scalar, $s \neq 0$, and $A^e \neq 0$. Let α_1 be the biggest modulus of the eigenvalues of A^e, $\theta \in (0, \pi/2)$ be such that $\cot\theta = \frac{|s|}{\alpha_1}$. Then the highest decrease rate of u at $\hat{x}$ for the control system is

$$\frac{\alpha_1}{2(\theta - \pi)} < 0, \quad \text{if } s > 0,$$

$$-\frac{\alpha_1}{2\theta} < s/2, \quad \text{if } s < 0.$$

Example 3.21. In $\mathbb{R}^3$ we consider the control system where

$$\sigma(x,y,z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ y & -x \end{pmatrix}$$

and $u(x,y,z) = 2z - x^2 - y^2$. Here we can compute at the points of the z−axis $\nabla u\ \sigma(0,0,z) = (0,0)$,

$$A = {}^tS(0,0,z) = \begin{pmatrix} -2 & 2 \\ -2 & -2 \end{pmatrix} = -2I + \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}.$$

Then $\alpha_1 = 2, s = -2, \theta = \frac{\pi}{4}$. We can also compute $\frac{1}{2}\lambda_1^{(1)} = -1$ and

$$\frac{\lambda_1^{(n)}}{2n} = \frac{-2\cot\left(\frac{\pi}{4n}\right)}{2n} = -\frac{\cot\left(\frac{\pi}{4n}\right)}{n},$$

which is negative and decreasing and whose limit is $-4/\pi < -1$. For $n = 2$ we have $\lambda_1^{(2)}/4 = -\cot(\pi/8)/2$ which is smaller in modulus but close to $-4/\pi$.

4. Conclusions

We considered the problem of optimising the second-order decay rate of a function at a point with respect to the trajectories of a symmetric control system. We defined the second-order decrease rate by fixing any piecewise constant control function in a reference interval. We have demonstrated a formula for the minimum rate of decrease as the infimum of the rates when we allow at most $k-1$ switches in the control function. The k controls used are given by eigenvectors (of appropriate norm) of block matrices constructed recursively from the matrix of the second order Lie derivatives of the function. We have provided a way to explicitly calculate the corresponding optimal control functions through linear algebra methods. We performed explicit full calculations in three cases.

Author contributions

Mauro Costantini: Conceptualization, Formal Analysis, Investigation, Methodology, Validation, Writing – original draft, review & editing; Pierpaolo Soravia: Conceptualization, Formal Analysis, Investigation, Methodology, Validation, Writing – original draft, review & editing. Both authors have read and approved the final version of the manuscript for publication.

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Conflict of interest

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References

1. Z. Artstein, Stabilization with relaxed controls, *Nonlinear Anal.*, **7** (1983), 1163–1173. https://doi.org/10.1016/0362-546X(83)90049-4
2. M. Bardi, M. Falcone, An approximation scheme for the minimum time function, *SIAM J. Control Optim.*, **28** (1990), 950–965. https://doi.org/10.1137/0328053
3. M. Bardi, E. Feleqi, P. Soravia, Regularity of the minimum time and of viscosity solutions of degenerate eikonal equations via generalized Lie brackets, *Set-Valued Var. Anal.*, **29** (2021), 83–108. https://doi.org/10.1007/s11228-020-00539-z
4. R. M. Bianchini, G. Stefani, *Local controllability for analytic families of vector fields*, Rapporto No. 19, Istituto Matematico U. Dini, Firenze, Italy, 1981/82.
5. R. M. Bianchini, G. Stefani, Stati localmente controllabili, *Rend. Sem. Mat. Univ. Pol. Torino*, **42** (1984), 15–23.
6. R. M. Bianchini, G. Stefani, Time-optimal problem and time-optimal map, *Rend. Sem. Mat. Univ. Pol. Torino*, **48** (1990), 401–429.
7. W. L. Chow, Über systeme von linearen partiellen differentialgleichungen erster ordnung, *Math. Ann.*, **117** (1940), 98–105. https://doi.org/10.1007/BF01450011
8. F. H. Clarke, Y. S. Ledyaev, L. Rifford, R. J. Stern, Feedback stabilization and Lyapunov functions, *SIAM J. Control Optim.*, **39** (2000), 25–48. https://doi.org/10.1137/S0363012999352297
9. J. M. Coron, *Control and nonlinearity*, Mathematical Surveys and Monographs, Vol. 136, American Mathematical Society, 2007. https://doi.org/10.1090/surv/136
10. H. Frankowska, An open mapping principle for set-valued maps, *J. Math. Anal. Appl.*, **127** (1987), 172–180. https://doi.org/10.1016/0022-247X(87)90149-1
11. M. Kawski, High-order small-time local controllability, In: H. J. Sussmann, *Nonlinear controllability and optimal control*, New York: Routledge, **133** (1990), 431–467.
12. M. Krastanov, M. Quincampoix, Local small time controllability and attainability of a set for nonlinear control system, *ESAIM: Control Optim. Calc. Var.*, **6** (2001), 499–516. https://doi.org/10.1051/cocv:2001120
13. M. Krastanov, A sufficient condition for small-time local controllability, *SIAM J. Control Optim.*, **48** (2009), 2296–2322. https://doi.org/10.1137/070707117
14. M. Krastanov, High-order variations and small-time local attainability, *Control Cybern.*, **38** (2009), 1411–1427.
15. T. T. Le Thuy, A. Marigonda, Small-time local attainability for a class of control systems with state constraints, *ESAIM: Control Optim. Calc. Var.*, **23** (2017), 1003–1021. https://doi.org/10.1051/cocv/2016022
16. A. Lyapunov, Problème général de la stabilité du mouvement, *Ann. Fac. Sci. Univ. Toulouse*, **9** (1907), 203–474.

17. A. A. Liverovskii, A Hölder condition for Bellman's functions, *Differ. Uravn.*, **13** (1977), 2180–2187.

18. A. Marigonda, Second order conditions for the controllability of nonlinear systems with drift, *Commun. Pure Appl. Anal.*, **5** (2006), 861–885. https://doi.org/10.3934/cpaa.2006.5.861

19. A. Marigonda, S. Rigo, Controllability of some nonlinear systems with drift via generalized curvature properties, *SIAM J. Control Optim.*, **53** (2015), 434–474. https://doi.org/10.1137/130920691

20. M. Motta, F. Rampazzo, Asymptotic controllability and Lyapunov-like functions determined by Lie brackets, *SIAM J. Control Optim.*, **56** (2018), 1508–1534. https://doi.org/10.1137/16M1086947

21. N. N. Petrov, Controllability of autonomous systems, *Differ. Uravn.*, **4** (1968), 606–617.

22. N. N. Petrov, On the bellman function for the time-optimal process problem: PMM vol. 34, $n \overset{\circ}{=}$ 5, 1970, pp. 820–826, *J. Appl. Math. Mech.*, **34** (1970), 785–791. https://doi.org/10.1016/0021-8928(70)90060-2

23. P. Soravia, Hölder continuity of the minimum-time function for C^1-manifold targets, *J. Optim. Theory Appl.*, **75** (1992), 401–421. https://doi.org/10.1007/BF00941476

24. P. Soravia, Some results on second order controllability conditions, *2019 IEEE 58th Conference on Decision and Control (CDC)*, 2019, 1468–1473. https://doi.org/10.1109/CDC40024.2019.9029843

25. P. Soravia, A degenerate elliptic equation for second order controllability of nonlinear systems, *Minimax Theory Appl.*, **5** (2020), 413–437.

26. P. Soravia, The Aronsson equation, Lyapunov functions, and local Lipschitz regularity of the minimum time function, *Abstr. Appl. Anal.*, **2019** (2019), 6417074. https://doi.org/10.1155/2019/6417074

27. P. Soravia, A Hamiltonian approach to small time local attainability of manifolds for nonlinear control systems, *Appl. Math. Optim.*, **88** (2023), 1. https://doi.org/10.1007/s00245-023-09973-5

28. H. J. Sussmann, Lie brackets and local controllability: a sufficient condition for scalar-input systems, *SIAM J. Control Optim.*, **21** (1983), 686–713. https://doi.org/10.1137/0321042

29. H. J. Sussmann, A general theorem on local controllability, *SIAM J. Control Optim.*, **25** (1987), 158–194. https://doi.org/10.1137/0325011