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Research article

On the convergence properties of generalized Szász–Kantorovich type operators involving Frobenious–Euler–Simsek-type polynomials

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Abstract: This work focuses on the study of approximation properties of functions by Szász type operators involving Frobenius–Euler–Simsek-type polynomials, which have become more popular recently because of their special characteristics and functional organization. The convergence properties such as uniformly convergence and pointwise convergence in terms of modulus of continuity and Peetre-K functional are investigated with the help of these sequences of operators in depth. This paper also includes the estimation of the error of the approximation of these sequences of operators to some particular class of functions. The estimates are depicted using the Maple scientific computing program and presented in tables.

Keywords: Apostol-type polynomials; Euler polynomials; Korovkin theorem; generating functions; modulus of continuity; rate of convergence
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1. Introduction

Since the groundbreaking works of S. N. Bernstein and P. Chebyshev (among others) were published more than a century ago, approximation theory has developed into a vast area of mathematics with connections to numerous other scientific disciplines. It provides directions for future research and is essential to analyzing numerical methods in the mathematical, physical, medical, engineering, and social sciences.

In 1912, S. N. Bernstein discovered an operator to contribute to the proof of the Weierstrass theorem as follows:

$$B_n(f;x) = \sum_{k=0}^{n} b_{n,k}(x) f\left(\frac{k}{n}\right), \tag{1.1}$$

where $b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, $f \in C[0,1]$, and $x \in [0,1]$ ([1]). Due to the significance of the Bernstein operators, various researchers have discovered their multiple generalizations, as seen in ([2–5]).

The Chebyshev system is crucial in approximation theory. Approximations of complicated functions by simpler functions are computed using Chebyshev polynomials. The function solution in linear systems is made easier by the specific features of these systems, which also inherit numerous approximation qualities (*cf.* [6, 7]).

That being said, discontinuous function approximation is not a good use for traditional Bernstein operators. The construction of traditional Bernstein–Kantorovich operators for Lebesgue-integrable function space is done as follows:

$$K_n(f; x) = (n+1) \sum_{k=0}^{n} b_{n,k}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt, \tag{1.2}$$

where $n \in \mathbb{N}$, $f \in L_1([0,1])$, and $x \in [0,1]$, (see [8, 9]).

Recently, generating functions of special families of polynomials for approximation by positive linear operators have been extensively used by researchers. In 2016, Atakut and Buyukyazıcı defined a generalization of Kantorovich-type operators using Brenke-type polynomials as follows:

$$L_n^{\alpha_n,\beta_n}(f; x) = \frac{\beta_n}{A(1)B(\alpha_n x)} \sum_{k=0}^{\infty} p_k(\alpha_n x) \int_{\frac{k}{\beta_n}}^{\frac{k+1}{\beta_n}} f(t)dt,$$

and investigated their convergence properties [10]. In 2022, Sofyalıoglu and Kanat studied a generalization of Szász–Baskakov operators involving Boas Buck polynomials as below:

$$D_s^*(f; x) = \frac{s-1}{A(1)B(sxH(1))} \sum_{v=0}^{\infty} p_v(sx) \int_0^{\infty} \binom{s+v-1}{v} \frac{t^v}{(1+t)^{s+v}} f(t)dt,$$

and examined some convergence properties, such as the rate of convergence [11]. In 2023, Menekşe Yılmaz gave generalized Kantorovich-type operators, including the generating functions of negative-order Bernoulli-type polynomials in the following equation:

$$\tilde{A}_n(f; x) := n \frac{e^{-nx}}{e-1} \sum_{k=0}^{n} \frac{\tilde{\beta}_k(nx)}{k!} \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)dt,$$

and studied approximation properties of operators such as first-order modulus of continuity, Voronovskaya type, and Grüss–Voronovskaya type asymptotic results [12].

In 2021, Simsek defined Frobenius–Euler–Simsek-type numbers and polynomials and investigated relations and identities between these numbers and polynomials and special numbers and polynomials such as Fubini numbers and polynomials and Bernoulli numbers and polynomials [13]. The Frobenius–Euler–Simsek-type polynomials $\ell_n(x; v)$ are given by the following generating function:

$$F_\ell(x; w, v) := \frac{w^v}{\prod_{j=0}^{v-1} (e^w - j)} e^{wx} = \sum_{n=0}^{\infty} \ell_n(x; v) \frac{w^n}{n!}. \tag{1.3}$$

Putting $x = 0$, we give Frobenius–Euler–Simsek-type numbers in the following equation:

$$F_{\ell}(0; w, v) := \frac{w^v}{\prod_{j=0}^{v-1}(e^w - j)} = \sum_{n=0}^{\infty} \ell_n(0; v)\frac{w^n}{n!}. \tag{1.4}$$

Taking $v = 2$ into the above equation, we have

$$F_{\ell}(x; w, 2) := \frac{w^2}{e^w(e^w - 1)}e^{wx} = \sum_{n=0}^{\infty} \ell_n(x; 2)\frac{w^n}{n!}. \tag{1.5}$$

By the motivation from the definition of Frobenius–Euler–Simsek-type polynomials at (1.5), we consider Kantorovich-type operators at the following equation:

$$\mathbb{E}_n(f; x) := n(e^2 - e)e^{-nx}\sum_{k=0}^{\infty}\frac{\ell_k(nx; 2)}{k!}\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)dt, \tag{1.6}$$

where $\mathbb{E}_n : L_1([0, 1]) \to C_1([0, 1])$ and $f \in L_1([0, 1])$.

2. Preliminieries

It is useful in this context to highlight particular outcomes and define a few terminologies.

Let a sequence of linear positive operators $(L_n)_n$, $L_n : V \to \mathcal{F}[a, b]$, where $\mathcal{F}[a, b]$ is the space of all real-valued functions in the interval [a,b] and V is a linear subspace of $\mathcal{F}[a, b]$. Suppose that $\varphi_0, \varphi_1, \varphi_2 \in V \cap C[a, b]$ forms a Chebychev system on the interval $[a, b]$, if we have

$$\lim_{n\to\infty} L_n(\varphi_j) = \varphi_j,$$

uniformly for $j = 0, 1, 2$, then

$$\lim_{n\to\infty} L_n(f) = f,$$

uniformly, for any $f \in V \cap C[a, b]$ (see [14]).

The theorem of Bohman in [15] is the particular version of the above theorem when $\varphi_j = e_j$, $j = 0, 1, 2$. The monomial functions denoted by e_j are defined to be as:

$$e_j(x) = x^j,$$

where $x \in [a, b]$ and $j \in \mathbb{N} \cup 0$. $e_j(x)$ functions are also called moment functions. Furthermore, the j-order central moment function of the operator L_n is defined as follows:

$$L_n((e_1 - e_0x)^j),$$

(*cf.* [[15, 16]).

In approximation theory, moments and central moments are used to quantify and examine the properties of functions, offering important insights into the performance and quality of approximations. Moments and pivotal moments are important in several ways: Moments are frequently used to calculate approximate formulas or assess how well an approximate fit a target function. Because they shed light on how well an approximation, while accounting for its key features, resembles the original function, central moments are especially helpful in error analysis.

An analytical tool in approximation theory, the modulus of continuity measures how effectively a function or series of functions can be approximated by another function or sequence inside a given interval or domain. A series of approximating functions, like polynomials or trigonometric functions, are evaluated for their ability to approximate functions using the modulus of continuity.

The definition of modulus of continuity is given as:

Definition 2.1. (*cf.* [17]) Let f be a uniformly continuous function on $[0, \infty)$ and $\delta > 0$. The modulus of continuity, $\omega(f, x)$, of the function of f is defined to be

$$\omega(f, \delta) := sup\,|f(x) - f(y)|\,, \tag{2.1}$$

where $x, y \in [0, \infty)$ and $|x - y| < \delta$.

Then for any $\delta > 0$, and for each $x \in [0, \infty)$, the following relation holds

$$|f(x) - f(y)| \leq \omega(f, \delta)\left(\frac{|x - y|}{\delta} + 1\right). \tag{2.2}$$

The second-order continuity module in approximation theory offers a numerical representation of a function's behavior about its first derivative, also known as the gradient. This property explains how the accuracy and quality of approximation approaches are impacted by the quadratic continuity module.

The definition of the second-order modulus of continuity is given below:

Definition 2.2. (*cf.* [18]) The continuity's second-order modulus is given below:

$$\omega^2(f, \delta) = \sup_{0<h\leq\delta}\ \sup_{x\in[0,\infty)} |f(x + h) - 2f(x) + f(x - h)|\,, \tag{2.3}$$

where $f \in C_B[0, \infty)$ and $\delta > 0$.

The Lipschitz class is one of the mathematical tools in approximation theory that offers a framework for comprehending and quantifying the regularity or smoothness of functions. Because of this trait, Lipschitz functions yield good results since the Lipschitz constant also controls the approximation errors.

The definition of the Lipschitz class with order α is given as follows:

Definition 2.3. (*cf.* [19]) $Lip_1(\alpha, K)$, $0 < \alpha \leq 1$, denotes the class of functions that verify the inequality $\omega_1(\phi, \sigma) \leq K\sigma^\alpha$ for all $\sigma > 0$ with positive K. Next, we have

$$\left|E_n^*(\phi; x) - \phi(x)\right| \leq K\sigma_n^\alpha(x). \tag{2.4}$$

A quantitative indicator of the degree to which a linear operator T can approximate a function f in a given function space X while accounting for both the approximation error and the smoothness (or regularity) of the function f is provided by Peetre's K-functional, a fundamental tool for analyzing and comparing various approximation methods and their convergence properties.

The Peetre's K-functional is defined as follows:

Definition 2.4. (*cf.* [20]) Peetre's K-functional is provided in the following equation:

$$K(f; \delta) = inf\{g \in C_B^2[0, \infty) : \|f - g\|_{C_B} + \delta\|g\|_{C_B^2}\}, \tag{2.5}$$

where

$$C_B^2[0,\infty) = \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\},$$

and

$$\|g\|_{C_B^2} := \|g\|_{C_B} + \|g'\|_{C_B} + \|g''\|_{C_B}.$$

The following inequality is obtained between Peetre's K-functional and the second modulus of continuity for any constant M that is independent of f and δ:

$$K(f;\delta) \leq M\{\omega_2(f;\sqrt{\delta}) + min(1,\delta)\|f\|_{C_B}\}. \tag{2.6}$$

A continuous function $f : [a,b] \to \mathbb{R}$ defined concerning a certain dimension (typically h) has an average function known as the Steklov function. The following is the definition of the Steklov function, (see [21]), or $f_h(x)$:

$$f_h(t) = \frac{1}{h}\int_{t-\frac{h}{2}}^{t+\frac{h}{2}} f(u)du.$$

The Steklov function is a function that can be employed to achieve a smoother function by aggregating the values at neighboring points. It finds extensive use in numerical analysis and data processing applications, such as noise reduction and data smoothing.

There is a derivative for the Steklov function f_h.

$$f_h'(t) = \frac{1}{h}\left(f\left(t+\frac{h}{2}\right) - f\left(t-\frac{h}{2}\right)\right),$$

almost everywhere. We also give the following inequalities from [22]:

$$\|f_h - f\|_\infty \leq \frac{3}{4}\omega_2(f;h), \tag{2.7}$$

and

$$\|f_h''\|_\infty \leq \frac{3}{2}\frac{\omega_2(f;h)}{h^2}, \tag{2.8}$$

where function $f \in C[a,b]$ and $h \in \left(0, \frac{b-a}{2}\right)$ are connected via the second-order Steklov function, f_h.

The following definition, which is an enlarged form of the first modulus of the derivative for the class of an arbitrary function, was provided in [15]:

$$\omega_2^d(f;\delta) = \delta\left\{sup\left|\frac{f(x+t)-f(x)}{t}\right| - \left|\frac{f(y+s)-f(x)}{s}\right|, s,t>0, x, x+t, y+s \in [0,1]\right\}, \tag{2.9}$$

where $\{max\{x+t, y+s\} - max\{x,y\} \leq \delta\}$.

Euler-type polynomials are used frequently in the applied sciences and combinatorics, particularly in analytic number theory. For example, T. Kim and D. S. Kim studied degenerate hyperharmonic numbers and investigated some relations and identities for special polynomials such as degenerate Bernoulli, degenerate Euler, degenerate Bell, and degenerate Fubini polynomials [23]. In [24], the authors derived some results involving Frobenius–Euler polynomials and derived some formulas by using umbral calculus. For more information on current work in this area, (see [25–31]). There are linear operators with generating functions of special polynomials in the literature. To integrate positive linear operators with the generating function technique, the purpose of this work is to investigate Kantorovich–Szász-type positive linear operators.

The rest of this research is organized as follows:

In Section 3, we first determine results under the differential operator of Frobenius–Euler–Simsek-type polynomials and derive the moment and central moment functions using the lemmas and definitions from Section 2. Second, we use Korovkin's theorem to demonstrate the uniform convergence of our operator. In Section 4, we investigate several convergence properties of our operator, including modulus of continuity, Lipschitz class, Peetre's K functional, and second-order continuity module. In Section 5, we use the modulus of continuity to compute the operator's convergence rate and provide some numerical examples. In Section 6, we present the findings and conclusions of the study and discuss the uses of these findings and conclusions. In Section 7, our results are summarized and recommendations for future work are presented.

3. Auxilary results

In this chapter, we obtain moment and central moment functions for $\mathbb{E}_n(f;x)$. Additionally, we show that $\mathbb{E}_n(f;x)$ is uniformly convergence by using the Korovkin-Bohman theorem.

By using Eq (1.5), we obtain the following equalities:

$$\frac{d}{dw}F_l(x;w,2) = \frac{we^{w(x-1)}(-xw + e^w(wx - 2w + 2) + w - 2)}{(e^w - 1)^2},$$

and

$$\begin{aligned}\frac{d^2}{dw^2}F_l(x;w,2) &= \frac{e^w(x-1)}{(e^w-1)^3}\Big[- e^w(w^2(2x^2 - 6x + 3) + 4w(2x - 3) + 4) \\ &\quad +w^2(x-1)^2 + e^2w(w^2(x-2)^2 + 4w(x-2) + 2) + 4w(x-1) + 2\Big].\end{aligned}$$

Lemma 3.1. For the operators $\mathbb{E}_n$, one has

$$\mathbb{E}_n(1;x) := 1,$$

$$\mathbb{E}_n(x;x) := x + \frac{e-3}{2n(e-1)},$$

and

$$\mathbb{E}_n(x^2;x) := x^2 + \frac{e-2}{n(e-1)}x + \frac{-5e^2+7e+1}{3(e-1)^2n^2}.$$

Proof. For the proof, we assume $x \to nx$ and $w = 1$.

Let $f(x) = 1$. Applying the definition of $\mathbb{E}_n(f;x)$, we have

$$\mathbb{E}_n(1;x) = n(e^2-e)e^{-nx}\sum_{k=0}^{\infty}\frac{\ell_k(nx;2)}{k!}\int_{\frac{k}{n}}^{\frac{k+1}{n}} dt = n(e^2-e)e^{-nx}\frac{e^{nx}}{n(e^2-e)} = 1.$$

Let $f(x) = x$. The above equation reduces to the following equation:

$$
\begin{aligned}
\mathbb{E}_n(x;x) &= n(e^2-e)e^{-nx}\sum_{k=0}^{\infty}\frac{\ell_k(nx;2)}{k!}\int_{\frac{k}{n}}^{\frac{k+1}{n}} tdt \\
&= \frac{n(e^2-e)}{2n^2}e^{-nx}\sum_{k=0}^{\infty}\frac{\ell_k(nx;2)}{k!}(2k+1) \\
&= \frac{n(e^2-e)}{2n^2}e^{-nx}\left(\frac{e^{nx-1}(enx-nx-1)}{(e-1)^2}+\frac{e^n x}{e^2-e}\right) \\
&= x+\frac{e-3}{2n(e-1)}.
\end{aligned}
$$

Let $f(x) = x^2$, we also obtain

$$
\begin{aligned}
\mathbb{E}_n(x^2;x) &= n(e^2-e)e^{-nx}\sum_{k=0}^{\infty}\frac{\ell_k(nx;2)}{k!}\int_{\frac{k}{n}}^{\frac{k+1}{n}} t^2dt \\
&= \frac{n(e^2-e)}{3n^3}e^{-nx}\sum_{k=0}^{\infty}\frac{\ell_k(nx;2)}{k!}(3k^2+3k+1) \\
&= x^2+\left(\frac{-x}{(e-1)x}+\frac{x}{n}\right)+\left(\frac{1}{3n^2}-\frac{1}{n^2(e-1)}+\frac{-2e^2+4e-1}{n^2(e-1)^2}\right) \\
&= x^2+\frac{e-2}{n(e-1)}x+\frac{-5e^2+7e+1}{3(e-1)^2n^2}.
\end{aligned}
$$

□

Lemma 3.2. Let $x \in [0,\infty)$. For the operators $\mathbb{E}_n$, we obtain the following equalities:

$$\mathbb{E}_n((e_1-e_0x),x) = \frac{e-3}{2n(e-1)}, \tag{3.1}$$

and

$$\mathbb{E}_n((e_1-e_0x)^2,x) = \frac{x}{(e-1)n}+\frac{-5e^2+7e+1}{3(e-1)^2n^2}. \tag{3.2}$$

Proof. Using $\mathbb{E}_n$'s linearity feature, we have ascertained

$$\mathbb{E}_n((e_1-e_0x),x) = \mathbb{E}_n(x,x)-x\mathbb{E}_n(1,x) = \frac{e-3}{2n(e-1)}, \tag{3.3}$$

and

$$\mathbb{E}_n((e_1-e_0x)^2,x) = \mathbb{E}_n(x^2,x)-2x\mathbb{E}_n(x,x)+x^2\mathbb{E}_n(1,x) = \frac{x}{(e-1)n}+\frac{-5e^2+7e+1}{3(e-1)^2n^2}. \tag{3.4}$$

□

With the help of any calculator, the following result is obtained for the 1st and 2nd-order central moment functions.

Corollary 3.3. For all $x \in [0,1]$ and $n \in \mathbb{N}$, we have

$$\mathbb{E}_n((e_1 - e_0 x), x) \leq \kappa_n$$

and

$$\mathbb{E}_n((e_1 - e_0 x)^2, x) \leq \varrho_n,$$

where

$$\kappa_n = \frac{-81}{1000n},$$

and

$$\varrho_n = \frac{581}{1000n} - \frac{191}{100n^2}.$$

Corollary 3.4. The following statements are accurate.

$$\lim_{n\to\infty} n\mathbb{E}_n((e_1 - e_0 x), x) = \frac{e-3}{2n(e-1)},$$

and

$$\lim_{n\to\infty} n\mathbb{E}_n((e_1 - e_0 x)^2, x) = \frac{x}{e-1}.$$

With the help of the moment functions in Lemma 3.1, the uniform convergence of the operator is given in the following theorem using the Korovkin–Bohman theorem.

Theorem 3.5. Let $f \in L_1([0,\infty))$.

$$\lim_{n\to\infty} \mathbb{E}_n(f, x) = f(x) \tag{3.5}$$

uniformly on $C([0,\infty))$.

Proof. Based on Lemma 3.1, it is clear that for any $j = 0, 1, 2$,

$$\lim_{n\to\infty} \mathbb{E}_n(x^j, x) = x^j.$$

According to the Korovkin–Bohman theorem ([15]), easy-to-get operators $\mathbb{E}_n(f, x)$ are uniformly convergent on $C([0,\infty))$. The desired result is obtained.

4. Local approximation characteristics of $\mathbb{E}_n(f; x)$

In this section, we give some convergence results for the operator $\mathbb{E}_n$, such as modulus of continuity, Lipschitz class, Peetre's K functional, and second-order continuity module. We also give estimates for the approximation of our operator with the help of the Steklov function and the extended continuity module for nondifferentiable functions.

Theorem 4.1. Let $f \in C([0,\infty))$ and $x \in [0,\infty)$. The operators $\mathbb{E}_n$ provide

$$|\mathbb{E}_n(f; x) - f(x)| \leq 2\omega_1 \sqrt{\delta_n(x)},$$

where $\delta_n(x) := \mathbb{E}_n((e_1 - e_0 x)^2, x)$.

Proof. Utilizing the linearity characteristic of operators $\mathbb{E}_n$ and Lemma 3.1, we compose

$$|\mathbb{E}_n(f;x)-f(x)| \leq n(e^2-e)e^{-nx}\sum_{k=0}^{\infty}\frac{\ell_k(x;2)}{k!}\int_{\frac{k}{n}}^{\frac{k+1}{n}}|f(t)-f(x)|dt. \tag{4.1}$$

However, we give properties of the first modulus of continuity as below:

$$|f(t)-f(x)| \leq \omega_1(f;|t-x|), \tag{4.2}$$

and

$$\omega_1(f;m\delta) \leq (1+m)\omega_1(f;\delta), m \geq 0. \tag{4.3}$$

With the help of (4.2) and (4.3), we obtain

$$|f(t)-f(x)| \leq \omega_1(f;|t-x|) \leq (1+\delta^{-2}(t-x)^2)\omega_1(f;\delta). \tag{4.4}$$

We should examine two cases here, respectively:

For $|t-x| \leq \delta$, (4.4) is evident.

For $|t-x| \geq \delta$, by taking into account the property (3.2), we have

$$(1+m)\omega_1(f;\delta) \leq (1+m^2)\omega_1(f;\delta), \tag{4.5}$$

where $m = \delta^{-1}(t-x)$ replaces in (4.5).

(4.3) also using (4.4), we have

$$\begin{aligned} |\mathbb{E}_n(f;x)-f(x)| &\leq n(e^2-e)e^{-nx}\sum_{k=0}^{\infty}\frac{\ell_k(x;2)}{k!}\int_{\frac{k}{n}}^{\frac{k+1}{n}}(1+\delta^{-2}(t-x)^2)\omega_1(f;\delta)dt \\ &\leq (\mathbb{E}_n(e_0;x)+\delta^{-2}\mathbb{E}_n((e_1-e_0(x))^2;x))\omega_1(f;\delta), \end{aligned}$$

where $\delta > 0$ and $x \in [0,1]$.

By applying Lemmas 3.1 and 3.2, we obtain

$$|\mathbb{E}_n(f;x)-f(x)| \leq (1+\delta^{-2}\delta_n(x))\omega_1(f;\delta). \tag{4.6}$$

If we choose $\delta = \sqrt{\delta_n}$, the intended outcome is attained. □

Theorem 4.2. Let f be in $Lip_M(\alpha)$. For $x \geq 0$, here are

$$|\mathbb{E}_n(f;x)-f(x)| \leq M\delta^*(x), \tag{4.7}$$

where $\delta^*(x) := \sqrt{\mathbb{E}_n((s-x)^2,x)}$.

Proof. By using $\mathbb{E}_n$'s monotonicity properties, we may derive the following:

$$|\mathbb{E}_n(f;x)-f(x)| \leq M\mathbb{E}_n(|s-x|^{\alpha};x). \tag{4.8}$$

From (4.8), one can be deduced by using the Hölder inequality.

$$|\mathbb{E}_n(f;x)-f(x)| \leq M(\mathbb{E}_n((s-x)^2;x))^{\frac{\alpha}{2}}. \tag{4.9}$$

The proof of the theorem is therefore completed. □

Theorem 4.3. The following statement is true for any $f \in C_B(0,\infty)$ and $x \in (0,\infty)$:

$$|\mathbb{E}_n(f;x) - f(x)| \leq 2K(f;\lambda_n(x)), \tag{4.10}$$

where $\lambda_n(x) = \frac{x}{2n} + \frac{-5e^2+7e+4}{6n^2(e-1)^2} + \frac{e-3}{2n(e-1)}$.

Proof. Let $h \in C_B^2(0,\infty)$. With the aid of Taylor's expansion and the linearity property of $\mathbb{E}_n$ operators, we can

$$\mathbb{E}_n(f;x) - f(x) = f'(x)\mathbb{E}_n n((s-x);x) + \frac{f'(\eta)}{2}\mathbb{E}_n\left((s-x)^2;x\right), \eta \in (x,s). \tag{4.11}$$

Lemma 3.2 allows us to have

$$|\mathbb{E}_n(f;x) - f(x)| \leq \left(\frac{x}{2n} + \frac{(2n+1)(1-e)+(-1+5e-2e^2)}{2n^2(e+1)}\right)\|h\|_{C_B^2(0,\infty)}. \tag{4.12}$$

Conversely, if we utilize Lemma 3.1 and formula (4.12), we acquire at

$$\begin{aligned} |\mathbb{E}_n(f;x) - f(x)| &\leq |\mathbb{E}_n(f-h;x)| + |\mathbb{E}_n(h;x) - h(x)| + |f(x) - h(x)| \\ &\leq 2\|f-h\|_{C_B(0,\infty)} + |\mathbb{E}_n(h;x) - h(x)| \\ &\leq 2(\|f-h\|_{C_B(0,\infty)} + \lambda_n(x)\|h\|_{C_B^2(0,\infty)}). \end{aligned} \tag{4.13}$$

Choosing the infimum on the right side of the equation above yields overall $h \in C_B^2(0,\infty)$. This is the desired outcome:

$$|\mathbb{E}_n(f;x) - f(x)| \leq 2K(f;\lambda_n(x)). \tag{4.14}$$

□

Theorem 4.4. Let $f \in C_B[0,\infty)$. The following inequality is provided:

$$|\mathbb{E}_n(f;x) - f(x)| \leq 2K\{\omega_2(f;\sqrt{\delta}) + min(1,\delta)\|f\|_{C_B(0,\infty)}\}, \tag{4.15}$$

where $\delta = \frac{1}{2}\lambda_n(x)$ and K is a constant that doesn't depend on f and δ.

Proof. Assume that $g \in C_B^2[0,\infty)$. Using Theorem 4.3, we can compose

$$\begin{aligned} |E_n(f;x) - f(x)| &\leq |\mathbb{E}_n((f-g);x)| + |\mathbb{E}_n(g;x) - g(x)| + |g(x) - f(x)| \\ &\leq 2\|f-g\|_{C_B(0,\infty)} + \lambda_n\|g\|_{C_B^2(0,\infty)} \\ &= 2\left(\|f-g\|_{C_B(0,\infty)} + \lambda_n\|g\|_{C_B^2(0,\infty)}\right). \end{aligned}$$

Since the left side of the above inequality is independent on the function $g \in C_B^2[0,\infty)$, we have

$$|E_n(f;x) - f(x)| \leq 2K(f;\delta),$$

where $K(f;\delta)$ is Peetre's K-functional defined by (2.11).

By using the relationship between the second modulus of smoothness and Peetre's K-functional, which is provided by (2.6)

$$|\mathbb{E}_n(f;x) - f(x)| \leq 2K\{\omega_2(f;\ \sqrt{\delta}) + min(1,\delta)\|f\|_{C_B(0,\infty)}\}.$$

Thus, the proof is completed. □

We now use the second-order modulus of continuity to address the rate of convergence of the operators $\mathbb{E}_n(f;x)$ at the following theorem:

Theorem 4.5. Let $f \in C[0,a]$. The following equality is true:

$$|\mathbb{E}_n(f;x) - f(x)| \leq \frac{2}{a}h_n^2\|f\|_\infty + \frac{3}{4}(h_n^2 + 2 + a)\omega_2(f;h_n), \tag{4.16}$$

where $h_n = \left(\mathbb{E}_n(e_1 - e_0x)^2;x)\right)^{\frac{1}{4}}$.

Proof. The Steklov function that corresponds to function f is denoted by f_h. As $\mathbb{E}_n(1;x) = 1$, we are left with

$$|\mathbb{E}_n(f;x) - f(x)| \leq 2\|f - f_h\|_\infty + |\mathbb{E}_n(f;x) - f_h(x)|. \tag{4.17}$$

Starting with [32], we have

$$\|f_h^{'}\|_\infty \leq \frac{2}{a}\|f_h\|_\infty + \frac{a}{2}\|f_h^{''}\|_\infty,$$

and employing (2.8), we provide

$$\begin{aligned}\|f_h^{'}\|_\infty &\leq \frac{2}{a}\|f_h\|_\infty + \frac{3a}{4}\frac{\omega_2(f;h)}{h^2}\\ &\leq \frac{2}{a}\|f\|_\infty + \frac{3a}{4}\frac{\omega_2(f;h)}{h^2}. \end{aligned}\tag{4.18}$$

If we use (4.17) and (4.18) and assume that $f \in C$, we can observe that

$$\begin{aligned}|\mathbb{E}_n(f_h;x) - f_h(x)| &\leq \sqrt{\mathbb{E}_n((e_1 - e_0x)^2, x)}\|f_h^{'}\|_\infty + \frac{1}{2}\mathbb{E}_n((e_1 - e_0x)^2, x)\|f_h^{''}\|_\infty\\ &\leq \left(\frac{2}{a}\|f_h\|_\infty + \frac{3a}{4}\frac{\omega_2(f;h)}{h^2}\right)\sqrt{\mathbb{E}_n((e_1 - e_0x)^2, x)}\\ &+ \frac{3a}{4}\frac{\omega_2(f;h)}{h^2}\mathbb{E}_n((e_1 - e_0x)^2, x)\|f_h^{''}\|_\infty.\end{aligned}$$

With $h = h_n = \left(\mathbb{E}_n((e_1 - e_0x)^2, x)\right)^{\frac{1}{4}}$ selected, it entails that

$$|\mathbb{E}_n(f;x) - f(x)| \leq \frac{2}{a}h_n^2\|f\|_\infty + \frac{3}{4}(h_n^2 + 2 + a)\omega_2(f;h_n). \tag{4.19}$$

The desired inequality is obtained, and the theorem's proof is completed by writing the inequalities (2.7) and (4.19) in (4.17). □

Let f be a non-differentiable function. We obtain an estimate for f at the following theorem:

Theorem 4.6. Let $f \in C([0,1])$. The inequality that follows is true:

$$|\mathbb{E}_n(f;x) - f(x)| \leq \frac{\kappa_n}{\sqrt{\lambda_n}}\omega_1(f;\delta) + \frac{9}{8}\omega_2^d(f;\delta).$$

Proof. By applying Theorem 2.3.7 for $r = 2$ in [15], we have

$$|\mathbb{E}_n(f;x) - f(x)| \leq \delta^{-1}|\mathbb{E}_n((e_1 - e_0 x);x)|\omega_1(f;\delta) + \left[\frac{1}{8}\mathbb{E}_n(1,x) + \delta^{-2}\mathbb{E}_n((e_1 - e_0 x)^2;x)\omega_2^d(f;\delta)\right]. \quad (4.20)$$

If the values of $\mathbb{E}_n(1,x)$, $\mathbb{E}_n((e_1 - e_0 x);x)$, and $\mathbb{E}_n((e_1 - e_0 x)^2;x)$ are substituted in (4.18)

$$|\mathbb{E}_n(f;x) - f(x)| \leq \delta^{-1}\kappa_n\omega_1(f;\delta) + \left[\frac{1}{8} + \delta^{-2}\lambda_n\right]\omega_2^d(f;\delta).$$

By choosing $\delta = \sqrt{\lambda_n}$, we obtain the intended outcome. □

5. Numerical examples

We calculate the error estimate of the approximation speed of the operator for some special functions such as trigonometric, rational, and exponential functions.

Example 5.1. We provide the approximation of $\mathbb{E}_n(f;x)$ to $f(x) = sin(\pi x)$ and its numerical results at Table 1:

Table 1. The error of approximation of the operators $\mathbb{E}_n(f;x)$ to $f(x) = sin(\pi x)$ for $n = 1...7$.

n	**Estimation by** $\omega(f,\delta)$
10	0.2450440082
10^2	0.03536477702
10^3	0.003644664930
10^4	0.0003655467420
10^5	0.00003656547490
10^6	0.000003656655494
10^7	0.0000003656666294

Example 5.2. We show the approximation of $\mathbb{E}_n(f;x)$ to $f(x) = \frac{x}{\sqrt{x^2+1}}$ and its numerical results at Table 2:

Table 2. The error of approximation of the operators $\mathbb{E}_n\,(f;x)$ to $f(x)=\frac{x}{\sqrt{x^2+1}}$ for $n=1...7$.

n	**Estimation by** $\omega(f,\delta)$
10	0.07813671190
10^2	0.01125736652
10^3	0.001160133326
10^4	0.0001163571424
10^5	0.00001163915215
10^6	0.000001163949594
10^7	0.0000001163953032

Example 5.3. We obtain the approximation of $\mathbb{E}_n\,(f;x)$ to $f(x)=x^2e^{-2x}$ and its numerical results at Table 3:

Table 3. The error of approximation of the operators $\mathbb{E}_n\,(f;x)$ to $f(x)=x^2e^{-2x}$ for $n=1...7$.

n	**Estimation by** $\omega(f,\delta)$
10	0.01801480579
10^2	0.002595711044
10^3	0.0002675028378
10^4	0.00002682961434
10^5	0.000002683670778
10^6	0.0000002683779152
10^7	0.00000002687615848

Based on the results of the three examples above, we can say that the error of the approximation speed decreases as the values of n increase. These results show that our operator can be used in applied fields and engineering, etc. instead of trigonometric, rational, and exponential functions.

6. Discussion

These results show that a generalized Kantorovich type Szász operator is obtained by using the generating functions of polynomials of Frobenius–Euler–Simsek-type. Considering that special polynomials with generating functions have important applications in combinatorics, engineering, and statistics, our findings show that generating functions can be presented as solutions to problems in approximation theory. We have shown the uniform convergence of the operator we have defined with the help of the Korovkin–Bohman theorem. We have also presented the convergence properties of the operator with approximation tools such as the continuity module, Lipschtz class, and Peetre's K functional, second-order continuity module. In doing so, we have used the moment and central moment functions of the operator. We think that our operator can be an alternative to trigonometric, exponential, etc. functions used in computational sciences with the help of the properties we have obtained.

7. Conclusions

This paper investigates the convergence properties of Szász–Kantorovich type operators involving the generating functions of Frobenius–Euler–Simsek-type polynomials. The moment and central moment functions that will be used to determine the convergence properties of the obtained operator are obtained. The basic approximation properties of the operator such as approximation speed and approximation error estimation, are examined, and the results are given as theorems. Finally, for trigonometric, rational, and exponential functions, we calculated the approximation error rate of the operator to these special functions with the help of the Maple scientific computing program and presented the results in tabular form.

In future works, the q analogue of Frobenius–Euler–Simsek-type polynomials can be defined, positive linear operators can be constructed with the help of their generating functions, and the convergence properties of these operators can be studied.

Conflict of interest

The author declares no conflict of interest.

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